NATIONAL BUREAU OF STANDARDS REPORT

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PROBABILITY INEQUALITIES OF THE
TCHEBYCHEFF TYPE

by

I. Richard Savage

NBS

U. S. DEPARTMENT OF COMMERCE
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NBS

FOREWORD

Inequalities of the Tchebycheff type have been used in the theory of probability and in statistical inference. The number of these inequalities has become quite large, and to facilitate their use this report attempts to present a collection of the most important. Emphasis has been placed on the precise conditions under which the inequalities may be used, and on stating them clearly. The accompanying bibliography has been made as complete as possible.

This digest was prepared as part of a continuing program of research on mathematical statistics and its applications carried out at the National Bureau of Standards under the general supervision of Dr. Churchill Eisenhart, Chief of the Statistical Engineering Laboratory. The Statistical Engineering Laboratory is Section 11.3 of the National Applied Mathematics Laboratories (Division 11, National Bureau of Standards), and is concerned with the development and application of modern statistical methods in the physical sciences and engineering.

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PROBABILITY INEQUALITIES OF THE TCHEBYCHEFF TYPE

I. Richard Savage

INTRODUCTION

Inequalities of the Tchebycheff type give upper bounds for the probability of events of certain types. In particular they give estimates for deviations from the mean in terms of the moments.

A selected collection of inequalities of the Tchebycheff type is given. We have picked them for their diverse nature and for their usefulness in applied and theoretical work.

In the following there will be first a section where the various inequalities will be presented with some notes on their uses and the conditions under which they may be used. In several cases more than one form of the inequality is presented in order to make it easier to work with the inequality. With each inequality we shall tell what random variables it is for; that is, whether the random variable is arbitrary, a sum, or whatever other special conditions are involved, such as the dimension of the random variables, and the necessary moments to be used.

Next tables are given that should be useful for finding what particular probabilities are associated with a specific inequality when all the other data is given, and tables that will tell you what size the variable parameters need to be to get the desired probability. These tables will be found useful in choosing which of the inequalities to use. Some of the inequalities involve several parameters for which tables have not been prepared.

Several examples are given showing how some of these inequalities can be used. These examples show various possible uses, but are by no means exhaustive.

The attached bibliography is complete. References by Uspensky, Introduction to Mathematical Probability, and Fréchet, Recherches théoriques modernes sur la théorie des probabilités, Première livre, are recommended as good surveys of the subject.

It has been found useful to use Tchebycheff inequalities when working with distributions whose functional form is unknown. In many cases this avoids the assumption that the random variables are normally distributed. All that is needed to use these inequalities is good estimates of certain population moments. Sometimes something is required of the functional form of the density function of the distributions that are involved. This is true for inequalities X, XI, and XIIb in the text. However, it is easy to verify whether the necessary conditions are true for the distributions that one is discussing.

In statistical work these inequalities have had several types of use. In working theoretical problems, it is often necessary to use these inequalities, for instance, in proving the weak law of large numbers for binomial distributions. These inequalities are particularly useful for testing hypotheses, and finding confidence intervals for the mean of a distribution if one has some information about the other moments. In industrial work, these inequalities have been used to form "tolerance" sets.

Usually one does not have the true value for all of the parameters that are needed for using these inequalities. But if one has upper bounds for the parameters, that is for the moments, then one can use these inequalities. If one has run a process many times with the same type of material, then one usually has a good idea of the variance, even if the process mean has been shifted, so that in a sense we often know some of the moments, and in this way we can test for the other ones.

One must remember that a sample acts as a population; and therefore, once the moments have been computed for a sample, all of these inequalities will be true for that sample; that is, these inequalities will tell you bounds for the portion of the sample in various parts of its range.

Most of the inequalities presented are for the univariate case. There are several papers that discuss the multivariate case in much more detail; in particular, see Camp, "Generalization to N Dimensions of Inequalities of the Tchebycheff Type;" Lesser, "Inequalities for Multivariate Frequency Distributions;" and Pearson, "On Generalized Tchebycheff Theorems in the Mathematical Theory of Statistics." Most of these have been omitted because they are quite complicated and hard to apply. In particular, for each inequality that we give we tell for which dimension the random variables must be; and this is a clue to deciding which one of the inequalities is applicable to a specific problem.

Several of the inequalities given make special assumptions on the shape of the distributions that are involved. All of these special assumptions say that the distribution has an unique mode. In Narumi's article "On Further Inequalities with Possible Applications to Problems on the Theory of Probability" the opposite case is treated, where the distributions have an unique minimum and increase as you go away from it. This case did not seem to be as important as the other, and therefore it was omitted.

In Winsten's article, "Inequalities in Terms of Mean Range," there are inequalities that involve the ranges for various sample sizes. These inequalities will undoubtedly prove useful in the future; but they are not entirely analogous to the Tchebycheff inequalities, and therefore were omitted.

Inequality I contains many of the other ones as special cases, which is a little surprising since this is the simplest of all the inequalities. This results from the fact that I is true for any positive random variable, X, that has a finite expected value. In particular, Ia is derived from I by replacing X by a sum of random variables. II is obtained by replacing X by a random variable that is of the form of the square of the difference between a random variable and its expected value. One can derive many of the other inequalities in this manner.

In cases where the inequality is given only for the random variable X minus its mean, there are also inequalities for a sample average minus the expected value of that average.

Most of the inequalities are given for the probability of the random variable being greater than or equal to some number; there are opposite inequalities for the event that the random variable is less than or equal to the same number. These are the same expressions, having the inequality reversed within the probability symbol; the 'greater than or equal' symbol being replaced by a 'less than' symbol, and the right-hand side being replaced by one minus the original right-hand side. Also, one may derive a new set of inequalities from the ones given here by interchanging $\leq$ with $<$, and $\geq$ with $>$. If the variables are continuous this adds nothing new.

As given, some of the inequalities are very weak, for the right-hand sides may be greater than one; but a probability is always less than or equal to one, so to get the most out of these inequalities the right-hand sides should be interpreted as the minimum of whatever is there now or one.

If one considers whether most of these inequalities can be improved, that is, can the right-hand sides be replaced by smaller quantities, one will find that this cannot be done. That is, there always is some distribution that satisfies the conditions necessary to use the inequality, and such that the left-hand side equals the right-hand side. Of course this will only occur for certain exceptional cases; but you might have that case. If you know that you do not have that case, then there might be a better inequality that you could use.

In the following, unless otherwise noted, lambda (λ) is any positive number. EX equals the expected value of the random variable, and will be denoted by μ; and if needs be this will be given a subscript. The expectation sign E will be used to denote other expected values depending on the argument that follows it. That is, $E(X-\mu)^2$ will be the variance and will be denoted by σ^2. In general, as far as moments go, we will use the notation of Cramér (1946). The symbol P(A) means the probability of the event A.

I. MARKOV

$$P\left(\frac{X}{\mu} \geq \lambda\right) \leq 1/\lambda \qquad \text{(I)}$$

$$P(X \geq \lambda) \leq \mu/\lambda \qquad \text{(I')}$$

Random Variable: X

Restrictions: X is non-negative, that is, $P(X<0) = 0$

Dimension: One

Moments: $\mu = EX$

References:

Cramér (1946); Fréchet (1937)

Notes:

1. This is a fundamental inequality from which inequalities Ia, II, IIa, IIb, III, IV, V, VI, XII, XIII, XV and XVa may be derived by picking X as a special function of the random variables that are of interest.

2. In using this inequality, note that you need to know only one moment, or you are testing an hypothesis about only one moment.

3. By itself this is rather a weak inequality, for the probability is bounded by $1/\lambda$; this is of course to be expected, since we are only using one moment, and therefore we have very little knowledge about the distribution that is involved, or at least are only using very little of this knowledge.

Ia.

$$P\left(\frac{X}{\mu} \geq \lambda\right) \leq 1/\lambda \qquad \text{(Ia)}$$

$$P\left(\sum_{i=1}^{n} X_i / \sum_{i=1}^{n} \mu_i \geq \lambda\right) \leq 1/\lambda \qquad \text{(Ia')}$$

Random Variable: $X = \sum_{i=1}^{n} X_i$

Restrictions: Each X_i is non-negative

Dimension: Each X_i is one-dimensional, but actually the X_i may be considered as one observation on an n-dimensional random variable, i.e. the X_i may be dependent.

Moments: $\mu_i = EX_i$

$\mu = \sum_{i=1}^{n} \mu_i$

References:

Cramér (1946); Fréchet (1937)

Notes:

1. This inequality is formally the same as I, but shows how I may be used where the random variable that is of interest is actually the sum of several random variables.

2. It is clear how this inequality is derived from the first one, for it is the same as that one, except that the random variable may be written in two ways, that is, either as X or as a sum of X_i.

II. BIENAYMÉ-TCHEBYCHEFF

$$P(|X-\mu| \geq \lambda\sigma) \leq 1/\lambda^2 \qquad \text{(II)}$$

$$P(X \geq \lambda\sigma + \mu \ \underline{\text{or}}\ X \leq \mu - \lambda\sigma) \leq 1/\lambda^2 \qquad \text{(II')}$$

$$P(|X-\mu| \geq \lambda) \leq \sigma^2/\lambda^2 \qquad \text{(II'')}$$

Random Variable: X

Restrictions: None

Dimension: One

Moments: $\sigma^2 = E(X-\mu)^2$

References:

Cramér (1946); Uspensky (1937); Fréchet (1937)

Notes:

1. This is the standard Tchebycheff inequality for one random variable.

2. We now find that the probability is decreasing as $1/\lambda^2$, which means that the probability of large deviations from the mean becomes quite small. It is to be noted that for the normal distribution and for large λ this is actually a very poor estimate of the probability of large deviations, for there the probability of a large deviation is smaller than e to the minus one half λ^2, but for intermediate values this is not a bad approximation.

If one has fairly good estimates of σ, then this inequality may be used for the testing of hypothesis about the mean, and for finding confidence intervals for the mean. In many industrial applications this inequality is used for getting an estimate of how much of the production will be near the mean of the process, where one has a good idea of the variance.

IIa.

$$P(\sqrt{n}\ |\bar{X} - \mu| \geq \lambda\sigma) \leq 1/\lambda^2 \qquad \text{(IIa)}$$

$$P(|\bar{X} - \mu| \geq \lambda) \leq \sigma^2/n(\lambda)^2 \qquad \text{(IIa')}$$

Random Variable: $\bar{X} = \sum_{i=1}^{n} X_i \Big/ n$

Restrictions: If $i \neq j$, then X_i and X_j are uncorrelated;

that is, $E(X_i - EX_i)(X_j - EX_j) = 0$

Dimension: One

Moments: $\mu = EX_i$

$\sigma^2 = E(X_i - \mu)^2$

References:

Cramér (1946); Uspensky (1937); Fréchet (1937)

Notes:

1. This is one of the most useful of the Tchebycheff inequalities. We can use this whenever we have sample averages of identically and independently distributed random variables.

2. This inequality gives us the square root of the sample size law. That is, $|\bar{X} - \mu|$ is of the order of magnitude $1/\sqrt{n}$.

3. The uses of this inequality are much like those of II, but we can also use it in the case where we are working with sample averages.

IIb.

$$P(|X-\mu| \geq \lambda\sigma) \leq 1/\lambda^2 \qquad \text{(IIb)}$$

$$P(|X-\mu| \geq \lambda) \leq \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} \Big/ \lambda^2 \qquad \text{(IIb')}$$

Random Variable: $X = \sum_{i=1}^{n} X_i$

Restrictions: None

Dimension: One

Moments:

$$\mu_i = EX_i \quad , \quad \mu = \sum_{i=1}^{n} \mu_i$$

$$\sigma_{ij} = E(X_i - \mu_i)(X_j - \mu_j)$$

$$\sigma^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij}$$

Reference:

Uspensky (1937)

Notes:

1. This is another form of the Tchebycheff inequality. In this case the random variables forming the sum in which we are interested may have different variances and they may be correlated; we still get the same type of inequality.

2. Actually, if all of the covariances are about of the same size as the variances, then this inequality is likely not to give much better bounds than $1/\lambda$.

3. A little thought will show how to write this inequality in terms of the sample average rather than the sample sum. In the remaining inequalities we will only give the inequality for the sample of one observation, but the way to change it to sample sums or sample averages will have become clear from the various versions that we have given of these first two inequalities.

4. Another point of interest is the fact that although the inequalities often require exact estimates of certain moments before they can be used, it is possible to get similar inequalities by substituting upper bounds for the moments that are involved. For instance, in this inequality we might not actually know the value of σ, but from previous experience we might know that it cannot under any circumstances be greater than, say, δ. In this case, if we use δ instead of σ, then the inequality will not be as good as the given one, but still may prove useful. This technique may with some care be used for all of the later inequalities that we give, as well as the earlier ones.

III. PEARSON

$$P\left(\frac{|X-\mu|}{\beta_r^{1/r}} \geq \lambda\right) \leq 1/\lambda^r \tag{III}$$

$$P(|X-\mu| \geq \lambda') \leq \beta_r/(\lambda')^r \tag{III'}$$

$$P\left(\frac{|X-\mu|}{\sigma} \geq \lambda''\right) \leq \frac{\beta_r}{\sigma^r(\lambda'')^r} \tag{III''}$$

Random Variable: X

Restrictions: None

Dimension: One

Moments: $\mu = EX$

$\sigma^2 = E(X-\mu)^2$

$\beta_r = E|X-\mu|^r$

References:

Pearson (1919); Narumi (1923); Fréchet (1937)

Notes:

1. In order to use this inequality we need an absolute moment of the random variable.

2. If we know several of the moments and wish an inequality for a particular λ, then we should pick that moment that makes the right-hand side of III' the smallest for that particular λ.

IV. BIRNBAUM, RAYMOND and ZUCKERMAN

$$P\left(\sum_{i=1}^{n} \frac{(X_i - \mu_i)^2}{\sigma_i^2\, t_i^2} \geq \lambda^2\right) \leq 1/\lambda^2 \sum_{i=1}^{n} 1/t_i^2 \qquad \text{(IV)}$$

$$P\left(\sum_{i=1}^{n} \frac{(X_i - \mu_i)^2}{t_i^2} \geq (\lambda')^2\right) \leq \frac{\sum_{i=1}^{n} \frac{\sigma_i^2}{t_i^2}}{(\lambda')^2} \qquad \text{(IV')}$$

Random Variable: $X = (X_1, \ldots, X_n)$

Restrictions: The t_i are arbitrary positive constants
Dimension: n
Moments: $\mu_i = EX_i$

$$\sigma_i^2 = E(X_i - \mu_i)^2$$

Reference:

Birnbaum, Raymond and Zuckerman (1947)

Notes:

1. This inequality gives a geometrical result. That is, it gives the probability of the deviations from the various means that are involved to all lie within a hyperellipse. And of course if the σt's are all equal, this becomes a hypersphere.

2. From the first comment it is clear that this sort of inequality could be used for bombing problems, etc.

3. Although none of the covariances occur in this inequality, it is applicable when the covariances are not equal to zero. The next inequality will show how the information about the covariances may be used in order to get a better result.

V. BERGE

$$P\left(\text{either } \frac{|X_i - \mu_i|}{\sigma_1} \geq \lambda \text{ or } \frac{|X_2 - \mu_2|}{\sigma_2} \geq \lambda\right) \leq \frac{1 + \sqrt{1-\rho^2}}{\lambda^2} \qquad (V)$$

$$P\left(\max\left\{\frac{|X_1 - \mu_1|}{\sigma_1}, \frac{|X_2 - \mu_2|}{\sigma_2}\right\} \geq \lambda\right) \leq \frac{1 + \sqrt{1-\rho^2}}{\lambda^2} \qquad (V')$$

Random Variable: $X = (X_1, X_2)$

Restrictions: None

Dimension: Two

Moments: $\mu_i = EX_i$

$\sigma_i^2 = E(X_i - \mu_i)^2$

$\sigma_{1,2} = E(X_1 - \mu_1)(V_2 - \mu_2)$

$\rho = \frac{\sigma_{1,2}}{\sigma_1 \sigma_2}$

References:

Berge (1938)

Notes:

1. This inequality tells us the probability of falling outside of a rectangle about the means for a bivariate sample.

2. This inequality uses the fact that the random variables may be dependent, and therefore in order to apply this inequality one needs actually to have some knowledge about the correlation.

3. In this case if we have no knowledge about the correlation, then the worst that can happen as far as the value of the right-hand side goes is for the correlation to be equal to zero, and then we still get a fairly good result. And the best that can happen is for the absolute value of the correlation to be one, and in that case we get the best inequality of this type.

VI. GUTTMAN

$$P\left[(\bar{X}-\mu)^2 \geq \frac{S^2}{n-1} + \sigma^2\sqrt{\frac{2(\lambda^2-1)}{n(n-1)}}\right] \leq 1/\lambda^2 \qquad \text{(VI)}$$

$$\lambda \geq 1$$

Random Variable: $\bar{X} = \dfrac{\sum_{i=1}^{n} X_i}{n}$

Restrictions: X_i are identically and independently distributed

Dimension: One

Moments: $\mu = EX_i$

$\sigma^2 = E(X_i - \mu)^2$

and define

$$S^2 = \frac{\sum_{i=1}^{n} (X_i-\bar{X})^2}{n}$$

References:

Guttman (1948); Midzuno (1950)

Notes:

1. This inequality is applicable whenever IIa can be used; however, this one takes more computing than that one does, and that is its only disadvantage. If we compare those two inequalities, the only way they differ is by the quantity on the right of the inequality sign within the probability statement. For this inequality that quantity is a random variable, and it is not for IIa. It turns out that the expected size of this random variable is much smaller than that of the quantity that occurs in IIa, and therefore it would seem to be a much better inequality to use than IIa.

VII. KOLMOGOROV

$$P(T \geq \sqrt{n}\,\sigma\lambda) \leq 1/\lambda^2 \qquad \text{(VII)}$$

Random Variable: $S_i = \sum_{j=1}^{i} (X_j - \mu_j)$; $i = 1,2,\ldots,n$

Let $T = \max_{1 \leq i \leq n} [|S_1|,\ldots,|S_i|,\ldots|S_n|]$

Restrictions: X_i and X_j independent and identical

Dimension: One

Moments: $\mu_i = EX_i$

$\sigma^2 = E(X_i - \mu_i)^2$

References:

Uspensky (1937); Fréchet (1937); Kolmogoroff (1928)

Notes:

1. This inequality can be used whenever IIa is applicable, although the use of this one is a little different than that. A typical use of this inequality is in the extreme-value type of situation. For instance, if we are putting together an assembly we might ask what is the probability that the cumulative error ever exceeds a certain quantity, and this inequality would give the answer.

VIII. USPENSKY

$$P(X - u \leq \lambda) \leq \frac{\sigma^2}{\sigma^2+\lambda^2} \quad \text{if } \lambda < 0 \qquad \text{(VIII)}$$

$$\leq 1 - \frac{\sigma^2}{\sigma^2+\lambda^2} \quad \text{if } \lambda \geq 0$$

Random Variable: X

Restrictions: None

Dimension: One

Moments: $\mu = EX$

$\sigma^2 = E(X-\mu)^2$

References:

Uspensky (1937)

Notes:

1. This inequality is applicable whenever inequality IIa can be used.

2. In this case we are interested in one-sided alternatives, that is we are thinking of the case where large positive deviations from the mean would be the things that we wish to detect. This occurs, for instance, whenever one is using one-sided confidence intervals or one-sided test regions.

3. The derivation of this inequality essentially depends on the Schwartz inequality.

IX. WALD-VON MISES

$$P(|X-u| \geq t) \geq \frac{\beta_r - t^r}{t'^r - t^r} \quad \text{if} \quad t \leq t^0 \qquad \text{(IX)}$$

$$\geq 0 \quad \text{if} \quad t^0 \leq t \leq t_0$$

$$\geq 0 \quad \text{if} \quad t_0 \leq t$$

$$P(|X-\mu| < t) \geq 0 \quad \text{if} \quad t \leq t^0 \qquad \text{(IX')}$$

$$\geq \frac{t^r - \beta_r}{t^r} \quad \text{if} \quad t^0 \leq t \leq t_0$$

$$\geq \frac{\beta_r - t^r}{t'^r - t^r} \quad \text{if} \quad t_0 \leq t$$

Random Variable: X

Restrictions: None

Dimension: One

Moments: $\mu = EX$

$$\beta_r = E|X-\mu|^r$$

$$\beta_s = E|X-\mu|^s$$

Let $s > r$, then

$$|t_0|^{s-r} = \frac{\beta_s}{\beta_r}$$

$$|t^0|^r = \beta_r$$

$$\frac{\beta_r - t^r}{t'^r - t^r} = \frac{\beta_s - t^s}{t'^s - t^s}$$

This defines t_0, t^0, and t' in terms of β_r, β_s, and t.

References:

Wald (1938); von Mises (1939)

Notes:

1. This sort of inequality is to be used where we have two absolute moments at our disposal and wish to get the most out of them. The same type of problem also arises when we have more than two moments, and the answer is given in Wald (1939).

2. These inequalities are harder to use, and it is doubtful that one will use them in practice.

X. GAUSS (CAMP-MEIDELL)

$$P(|X-\mu_0| \geq \lambda\tau) \leq \frac{4}{9\lambda^2} \tag{X}$$

$$P(|X-\mu| \geq \lambda\sigma) \leq \frac{4}{9} \cdot \frac{1+s^2}{(\lambda-|s|)^2} \quad \text{if } \lambda > |s| \tag{X'}$$

Random Variable: X

Restrictions: X has a density function with one mode, μ_0

Dimension: One

Moments: $\mu = EX$

$$\sigma^2 = E(X-\mu)^2$$

$$\tau^2 = \sigma^2 + (\mu-\mu_0)^2$$

$$|s| = \left|\frac{\mu-\mu_0}{\sigma}\right|$$

References:

Cramér (1946); Narumi (1923); Fréchet (1937)

Notes:

1. As far as having knowledge of the necessary moments to use this inequality, its application requires the same knowledge needed for use of inequality II. It is also necessary to know or have an estimate of the mode; but since that is usually the same as the mean (always for symmetric distributions), this is not much of a handicap.

2. In practice many distributions have this type of behavior, decreasing as you go away from the mode, particularly the normal, log normal, χ^2 for enough degrees of freedom, and the t-distribution.

3. If this decreasing property actually is true, then this is a better inequality to use than II, for it essentially divides the bound by 4/9.

XI. NARUMI, GAUSS (CAMP-MEIDELL)

$$P(|X-\mu_0| \geq \lambda\gamma_r) \leq 1 - \frac{r+1}{rb}\left(1 - \frac{1}{b^r r}\right) \quad \text{if } 0 \leq \lambda \leq \frac{rb}{b+1} \qquad \text{(XI)}$$

$$\leq \frac{1}{b^r} \quad \text{if } \frac{rb}{r+1} \leq \lambda \leq b$$

$$\leq \frac{1}{\lambda^r} \quad \text{if } b \leq \lambda$$

$$P(|X-\mu_0| \geq \lambda\gamma_r) \leq 1 - \frac{\lambda}{(1+r)^{1/r}} \quad \text{if } \lambda \leq \frac{r}{(1+r)^{1-1/r}} \qquad \text{(XI')}$$

$$\leq \left(\frac{r}{r+1}\right)^r(\lambda)^{-r} \quad \text{if } \frac{r}{(1+r)^{1-1/r}} \leq \lambda$$

Random Variable: X

Restrictions: X has a density function $f(x)$. $f(x)$ has an unique maximum in the interval (μ_0-b, μ_0+b) at μ_0 and $b > 0$; use (XI) if b is finite, and (XI') if b is infinite.

Dimension: One

Moments: $(\gamma_r)^r = E|X-\mu_0|^r$

References:

Narumi (1923); Fréchet (1937)

Notes:

1. In these inequalities we have used absolute moments about the modes of distributions, instead of the ordinary procedure of using them about the means.

2. This case is likely to occur with the b equal to infinity, that is the case where there is only one mode; however, cases where the interval of a decreasing distribution density function is finite occur sometimes when we are sure that the distribution behaves in a nice manner for small values but that there might be little isolated humps of probability at the extremes.

3. Actually, in most cases we would probably use inequality (X), which is a special case of these inequalities; this is why it has been given by itself.

XII. PEEK

$$P(|X-\mu| \geq t\sigma) \leq \frac{1-\rho^2}{t^2-2t\rho+1} \qquad \text{(XII)}$$

Random Variable: X

Restrictions: None

Dimension: One

Moments: $\mu = EX$

$\sigma^2 = E(X-\mu)^2$

$\delta = E|X-m|$

$\rho = \delta/\sigma$

References:

Peek (1933)

Notes:

1. This inequality is much like that of II, except that here one needs to know δ, the mean deviation. If one has this additional information, this is a better inequality to use.

XIIa.

$$P(|X-\mu| \geq t\sigma) \leq \frac{4}{9}\,\frac{1-\rho^2}{(t-\rho)^2} \qquad \text{(XIIa)}$$

Random Variable: X

Restrictions: X has a density function whose only mode is its mean

Dimension: One

Moments: $\mu = EX$

$\sigma^2 = E(X-\mu)^2$

$\delta = E|X-\mu|$

$\rho = \delta/\sigma$

References:

Peek (1933)

Notes:

1. This is an improvement over (X), but can only be used if one has an estimate of the mean deviation.

XIII. CANTELLI

$$P(|X-\mu| \geq \lambda) \leq \frac{\beta_r}{\lambda^r} \quad \text{if } \lambda^r \leq \frac{\beta_{2r}}{\beta_r} \qquad \text{(XIII)}$$

$$\leq \frac{\beta_{2r} - \beta_r^2}{(\lambda^r - \beta_r)^2 + \beta_{2r} - \beta_r^2} \quad \text{if } \frac{\beta_{2r}}{\beta_r} \leq \lambda^r$$

Random Variable: X

Restrictions: None

Dimension: One

Moments: $\mu = EX$

$\beta_{2r} = E|X-\mu|^{2r}$

$\beta_r = E|X-\mu|^r$

References:

Cantelli (1910); Peek (1933); Fréchet (1937)

Notes:

1. This inequality has probably no great practical use, but it has several special cases that might be found of use.

2. Of course in this case we need information about two moments.

XIV. BERNSTEIN

$$P(|X-\mu| \geq \lambda) \leq 2e^{-\lambda^2/(2\sigma^2+2C\lambda)} \qquad \text{(XIV)}$$

Random Variable: $X = \Sigma X_i$

Restrictions: X_i and X_j are independent

Dimension: One

Moments: $\mu_i = EX_i$, $\mu = \Sigma\mu_i$

$\sigma_i^2 = E(X_i-\mu_i)^2$, $\sigma^2 = \Sigma\sigma_i^2$

$E(X_i-\mu_i)^s \leq \dfrac{\sigma_i^2 s!\ C^{s-2}}{2}$ for all integers s and

some constant C. $(C > 0)$.

References:

Acceptance Sampling (1950); Craig (1933); Fréchet (1937); Uspensky (1937)

Notes:

1. This is a very nice inequality in the sense that the remainder term goes to zero very rapidly.

2. The one difficulty in applying this inequality is that essentially one has need of knowing all of the moments of the distribution that is at hand, or at least one has to know an upper bound for them; the next inequality treats the latter case.

XIVa.

$$P(|X-\mu| \geq \lambda) \leq 2e^{-\lambda^2/(2\sigma^2+\frac{2}{3}m\lambda)} \qquad \text{(XIVa)}$$

Random Variable: $X = \Sigma X_i$

Restrictions: $P(|X_i - \mu_i| > m) = 0$, X_i and X_j are independent

Dimension: One

Moments: $\mu_i = EX$, $\mu = \Sigma\mu_i$

$\sigma_i^2 = E(X_i - \mu_i)^2$, $\sigma^2 = \Sigma\sigma_i^2$

References:

Fréchet (1937); Uspensky (1937)

Notes:

1. This is the case where the distribution is bounded; and as a result we have an estimate for all of the moments, and therefore can use the previous inequality. If λ is fairly large, this is one of the best inequalities given in this paper for this situation.

XIVb.

$$P(|X-\mu| \geq \sigma\lambda) \leq e^{-\lambda^2/2} \qquad \text{(XIVb)}$$

Random Variable: $X = \Sigma X_i$

Restrictions: X is symmetrical about μ , i.e.:

$$E(X-\mu)^{2i+1} = 0 \quad ; \quad X_i, X_j \text{ are independent}$$

Dimension: One

Moments: $\mu_i = EX_i$, $\mu = \Sigma\mu_i$

$$E(X_i-\mu_i)^{2r} \leq \left(\frac{\sigma_i^2}{2}\right)^k \frac{(2k!)}{k!} \qquad \text{(r an integer)}$$

$$\sigma_i^2 = E(X_i-\mu_i)^2 \quad , \quad \sigma^2 = \Sigma\sigma_i^2$$

References:

Uspensky (1937)

Notes:

1. It is not too uncommon to deal with symmetrical distributions, and therefore from that standpoint this inequality can be used. However, the condition on the moments is rather stringent. This condition is true for the normal distribution.

THE TABLES

Essentially Tables I and II are inverses of each other, column by column. The columns of the tables are associated with the inequalities as follows:

Column	Inequalities
1	(I), (Ia), (Ia')
2	(II), (II'), (IIa), (IIb), (VI), (VII)
3	(III with r = 4)
4	(X)
5	(XI' for r = 4)
6	(VIII)
7	(XIV), (XIVa), (XIVb)

The first table answers the question how large the probability is associated with a specific "deviation" λ. The second table gives the "deviation" associated with specific probabilities P.

An example is now given showing how to use these tables. Suppose we have a sample of nine independent observations from a distribution whose variance is four units squared. (1) What is the probability that the sample average is more than one unit larger than the population mean? (2) How far above the population mean could the ninety percent point of the distribution of sample means be? (3) If we knew the population had an unique mode at its mean, could these results be improved? Ans: It is clear that for questions (1) and (2) we need inequality VIII, and for (3) we need X'.

Let us first evaluate the necessary constants. In particular X is a sample average and therefore $\sigma^2 = 4/9$. Also in X' we need s = 0 and $\sigma = 2/3$.

To answer (1) we note $\lambda = 1$ and we need to evaluate

$$\frac{\sigma^2}{\sigma^2 + \lambda^2} = \frac{1}{1 + 9/4} \; .$$

Column (6) Table I for $\lambda = 3/2 = 1.5$ shows $\frac{1}{1+\lambda^2} = .3077$ and this is the answer. The answer to (2) is in Table II, column (6); for P = .10 gives an answer of 3 units for λ/σ, thus $\lambda = 3\sigma = 2$. The corresponding results using X'' which makes use of the unique mode but not the one-sidedness, using $\lambda = 1$ and P = .80 we have - using column (4) - the answer .198, and (with interpolation) we have that $\frac{\lambda''}{\sigma} = 1.5$, or $\lambda'' = 1$, respectively.

Thus for this problem it seems better to use inequality X'' than VIII.

These tables will facilitate choosing which inequality to use when several are available, by comparing the associated probabilities (deviations) with the deviation (probability) of interest, thus making it possible to choose the inequality that gives the smallest probability (deviation) for the problem at hand.

EXAMPLES

EXAMPLE (1). Assuming that all soldiers are between sixty and seventy-eight inches tall, what is the probability that the average height of five hundred soldiers is more than one inch away from the average height of all soldiers?

TABLE I

PROBABILITY ASSOCIATED WITH DEVIATION λ

Probability associated with / λ	(1) $\frac{1}{\lambda}$	(2) $\frac{1}{\lambda^2}$	(3) $\frac{1}{\lambda^4}$	(4) $\frac{4}{9\lambda^2}$	(5) $\left(\frac{4}{5}\right)^4 \frac{1}{\lambda^4}$	(6) $\frac{1}{1+\lambda^2}$	(7) $e^{-\lambda^2/2}$
1.0	1.000	1.000	1.000	.444	.4096	.5000	.607
1.5	.667	.444	.1975+	.198	.0809	.3077	.325
2.0	.500	.250	.0625	.111	.0256	.2000	.135
2.5	.400	.160	.0256	.071	.0105-	.1379	.044
3.0	.333	.111	.0123	.049	.0050	.1000	.011
3.5	.286	.082	.0067	.036	.0027	.0755-	.002
4.0	.250	.062	.0039	.028	.0016	.0588	.000
4.5	.222	.049	.0024	.022	.0010	.0471	
5.0	.200	.040	.0016	.018	.0007	.0385-	
5.5	.182	.033	.0011	.015-	.0005-	.0320	
6.0	.167	.028	.0008	.012	.0003	.0270	
6.5	.154	.024	.0006	.011	.0002	.0231	.000

TABLE II

DEVIATION ASSOCIATED WITH PROBABILITY P

Deviation associated with / P	(1) $\frac{1}{P}$	(2) $\frac{1}{\sqrt{P}}$	(3) $\frac{1}{P^{1/4}}$	(4) $\frac{2}{3\sqrt{P}}$	(5) $\frac{4}{5P^{1/4}}$	(6) $\sqrt{\frac{1}{P}-1}$	(7) $\sqrt{2 \log_e \frac{1}{P}}$
.99	1.0101	1.0050	1.0025	.6700	.802	.1005-	.1418
.95	1.0526	1.0260	1.0129	.6840	.810	.2293	.3202
.9	1.1111	1.0541	1.0267	.7027	.821	.3333	.4590
.75	1.3333	1.1547	1.0746	.7698	.860	.5773	.7585
.50	2.	1.4142	1.1892	.9428	.951	1.	1.1774
.25	4.	2.	1.4142	1.3333	1.131	1.7321	1.6651
.10	10.	3.1623	1.7783	2.1082	1.423	3.	2.1460
.05	20.	4.4721	2.1147	2.9814	1.692	4.3589	2.4477
.01	100.	10.	3.1623	6.6667	2.530	9.9499	3.0348
.001	1000.	31.6228	5.6234	21.0818	4.500	31.6070	3.7169

Solution: Although we have a finite population, it is safe to assume that the measurements in our sample are independent. The largest possible variance occurs if half the soliers have height sixty inches, and half have height seventy-eight inches, in which case the variance is eighty-one inches squared. First apply inequality IIa. Here $\lambda' = 1$ inch, $\sigma^2 = 81$ inches squared, and $n = 500$. Thus the answer is $\frac{\sigma^2}{n\lambda'^2} = \frac{81}{500} = .162$. We may also apply inequality XIVa. Here $\lambda = 500$ inches, $\sigma^2 = 500 \cdot 81$ inches, and $m = 18$ inches. The probability is .11, and thus for this example we see that XIV gives more precise results than II.

EXAMPLE (2). In the course of deposit and withdrawal transactions, such as money in a bank, or radioactive material in a hospital, one often wishes to control the absolute error. That is, in a sequence of, say, one hundred transactions (a day's activity) one does not want one's books to differ from one's assets, at any time, by more than some fixed amount, say 1,000 units. Let us assume that the variance due to errors of measuring and of counting for each transaction is 400 units squared (this value being obtained by previous experience). The question then naturally arises: what is an upper bound for the probability of having an accumulated error of more than 1,000 units at any time during the day?

Solution: Inequality VII is suited for this problem. Here $n = 100$, $\sigma = 20$, and $\sqrt{n}\sigma\lambda = 1{,}000$. Thus $\lambda = \frac{1{,}000}{10 \cdot 20} = \frac{1{,}000}{200} = 5$, and $1/\lambda^2 = 1/25 = .04$. If instead of 100 transactions there had been 400, we would have $n = 400$, $\sigma = 20$,

$\sqrt{n}\sigma\lambda = 1,000$, $\lambda = \frac{1,000}{20 \cdot 20} = \frac{1,000}{400} = 2.5$; and the resulting probability is at most $\left(\frac{1}{2.5}\right)^2 = \frac{1}{6.25} = .161$.

EXAMPLE (3). From previous experience let us assume that the correlation between two variables (height and weight, rainfall and crop yield) is at least .8. If a sample of twenty-five is made on this bivariate distribution, what is an upper bound for the deviations from the population means that will not be exceeded more than ten percent of the time, where deviations are measured in standard units?

Solution: Here we can use inequality V. First we must solve the following equation:

$$.10 = \frac{1 + \sqrt{1 - .64}}{\lambda^2} \quad \text{or} \quad .10 = \frac{1 + \sqrt{.36}}{\lambda^2} ,$$

$$\lambda^2 = \frac{1 + .6}{.1} = \frac{1.6}{.1} = 16 \quad ; \quad \lambda = 4 \quad .$$

Thus $P\left(\frac{|\bar{x} - \mu|}{\sigma_{\bar{x}}} \geq 4 \text{ or } \frac{|\bar{y} - \mu|}{\sigma_{\bar{y}}} \geq 4\right) \leq .1$,

or $P\left(\frac{|\bar{x} - \mu|}{\sigma_x} \geq \frac{4}{5} \text{ or } \frac{|\bar{y} - \mu|}{\sigma_y} \geq \frac{4}{5}\right) \leq .1$.

Thus both sample means are within .80 standard units of their respective sample averages with probability .9.

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ERRATA for

Probability Inequalities of the Tchebycheff Type

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i	11b*	IX(pages 23 and 24 deleted from this report)	
ii	3b	Next tables	Next, tables
ii	1b	is	are
iii	9t	The attached bibliography is complete.	The attached bibliography is as complete as possible
iii	7b	is	are
1	12b	XV	XIV
1	11b	XVa	XIVa
9	5t	Dimension: One	Dimension: n
10		Delete note 2	
15	4b	V_2	X_2
21	1t	u	μ
21	2t	$\leq 1 - \frac{\sigma^2}{\sigma^2 + \lambda^2}$	$\geq 1 - \frac{\sigma^2}{4\lambda^2}$
25	1b	divides	multiplies
29	9b	$\delta = E\|X - m\|$	$\delta = E\|X - \mu\|$
42	9t	X''	X''
42	11t	$\lambda = 1$	$\sigma\lambda = 1$
42	11t	$p = .80$	$p = .50$
42	12t	Delete (with interpolation)	

* The letter b or t following a line number indicates counting from bottom or top of page respectively.

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Page	Line	From	To
42	12t	$\frac{\lambda''}{\sigma} = 1.5$	$\frac{\lambda}{\sigma} = 2.1032$
42	13t	$\lambda'' = 1$	$\lambda = 1.41$
42	15t	X''	X^{2}
47	4t	soliers	soldiers
47	6t	$\lambda' = 1$	$\lambda = 1$
48	3b	$\frac{\lvert\bar{y}-\mu\rvert}{\sigma_{\bar{y}}} \leq \frac{4}{5}$	$\frac{\lvert\bar{y}-\mu\rvert}{\sigma_{\bar{y}}} \geq \frac{4}{5}$
53	9b	Add pp. 182-207	

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Reports and Publications

The results of the Bureau's work take the form of either actual equipment and devices or published papers and reports. Reports are issued to the sponsoring agency of a particular project or program. Published papers appear either in the Bureau's own series of publications or in the journals of professional and scientific societies. The Bureau itself publishes three monthly periodicals, available from the Government Printing Office: The Journal of Research, which presents complete papers reporting technical investigations; the Technical News Bulletin, which presents summary and preliminary reports on work in progress; and Basic Radio Propagation Predictions, which provides data for determining the best frequencies to use for radio communications throughout the world. There are also five series of nonperiodical publications: The Applied Mathematics Series, Circulars, Handbooks, Building Materials and Structures Reports, and Miscellaneous Publications.

Information on the Bureau's publications can be found in NBS Circular 460, Publications of the National Bureau of Standards ($1.00). Information on calibration services and fees can be found in NBS Circular 483, Testing by the National Bureau of Standards (25 cents). Both are available from the Government Printing Office. Inquiries regarding the Bureau's reports and publications should be addressed to the Office of Scientific Publications, National Bureau of Standards, Washington 25, D. C.

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