NATIONAL BUREAU OF STANDARDS REPORT

1704

NOTES ON SEDIMENTATION MODELS

By

I. Richard Savage

NBS

U. S. DEPARTMENT OF COMMERCE

NATIONAL BUREAU OF STANDARDS

THE NATIONAL BUREAU OF STANDARDS

The scope of activities of the National Bureau of Standards is suggested in the following listing of the divisions and sections engaged in technical work. In general, each section is engaged in specialized research, development, and engineering in the field indicated by its title. A brief description of the activities, and of the resultant reports and publications, appears on the inside of the back cover of this report.

1. **Electricity.** Resistance Measurements. Inductance and Capacitance. Electrical Instruments. Magnetic Measurements. Electrochemistry.

2. **Optics and Metrology.** Photometry and Colorimetry. Optical Instruments. Photographic Technology. Length. Gage.

3. **Heat and Power.** Temperature Measurements. Thermodynamics. Cryogenics. Engines and Lubrication. Engine Fuels.

4. **Atomic and Radiation Physics.** Spectroscopy. Radiometry. Mass Spectrometry. Physical Electronics. Electron Physics. Atomic Physics. Neutron Measurements. Nuclear Physics. Radioactivity. X-Rays. Betatron. Nucleonic Instrumentation. Radiological Equipment. Atomic Energy Commission Instruments Branch.

5. **Chemistry.** Organic Coatings. Surface Chemistry. Organic Chemistry. Analytical Chemistry. Inorganic Chemistry. Electrodeposition. Gas Chemistry. Physical Chemistry. Thermochemistry. Spectrochemistry. Pure Substances.

6. **Mechanics.** Sound. Mechanical Instruments. Aerodynamics. Engineering Mechanics. Hydraulics. Mass. Capacity, Density, and Fluid Meters.

7. **Organic and Fibrous Materials.** Rubber. Textiles. Paper. Leather. Testing and Specifications. Organic Plastics. Dental Research.

8. **Metallurgy.** Thermal Metallurgy. Chemical Metallurgy. Mechanical Metallurgy. Corrosion.

9. **Mineral Products.** Porcelain and Pottery. Glass. Refractories. Enameled Metals. Building Stone. Concreting Materials. Constitution and Microstructure. Chemistry of Mineral Products.

10. **Building Technology.** Structural Engineering. Fire Protection. Heating and Air Conditioning. Exterior and Interior Coverings. Codes and Specifications.

11. **Applied Mathematics.** Numerical Analysis. Computation. Statistical Engineering. Machine Development.

12. **Electronics.** Engineering Electronics. Electron Tubes. Electronic Computers. Electronic Instrumentation.

13. **Ordnance Development.** Mechanical Research and Development. Electromechanical Fuzes. Technical Services. Missile Fuzing Research. Missile Fuzing Development. Projectile Fuzes. Ordnance Components. Ordnance Tests. Ordnance Research.

14. **Radio Propagation.** Upper Atmosphere Research. Ionospheric Research. Regular Propagation Services. Frequency Utilization Research. Tropospheric Propagation Research. High Frequency Standards. Microwave Standards.

15. **Missile Development.** Missile Engineering. Missile Dynamics. Missile Intelligence. Missile Instrumentation. Technical Services. Combustion.

NATIONAL BUREAU OF STANDARDS REPORT

NBS PROJECT
1103-21-5118

4 June 1952

NBS REPORT
1704

NOTES ON SEDIMENTATION MODELS

by

I. Richard Savage

FOREWORD

This note is concerned with the sedimentation of small spherical particles. In the note it is shown by the use of moment generating functions (Laplace transforms) that there is a fundamental relationship between the distribution of particle sizes and the density of particles (as a function of time).

This mathematical relationship has practical applications for it gives a technique to the experimenter of making inferences about the distribution of particles from measurements on the density of particles. This is important since measurement of density is a much simpler operation than measurement of particle sizes.

This is a technical report on Research in Applications of Mathematical Statistics to Problems of the Chemical Corps carried out in the Statistical Engineering Laboratory of the National Bureau of Standards (National Bureau of Standards Project Number 1103-21-5118) in accordance with War Department Delivery Order Number CD2-2876.

J. H. Curtiss
Chief, National Applied
Mathematics Laboratories

A. V. Astin
Acting Director
National Bureau of Standards

NOTES ON SEDIMENTATION MODELS

By

I. Richard Savage

Summary: This note is concerned with the following sedimentation experiment. A mist is allowed to settle within a closed tank, while being stirred sufficiently to maintain a homogeneous mixture.

Let f(x) be the density of particles of radius x, and let N(t) be the total density of particles at time t. It is shown by the use of the theory of the Laplace transform, that f(x) uniquely determines N(t), and conversely. Further the transform theory gives an explicit method for finding the relationship. One of the results of this work is that one can determine f(x) by simply making a series of density measurements to determine N(t). If f(x) were found directly it would be necessary to measure sizes of small particles which is a much more difficult job than measuring densities. Another result is that in fitting curves to observed functions f(x) and N(t), any knowledge of properties of one of them can be used also in the fitting of the other.

Finally the note contains several pairs of functions f(x) and N(t), that can be used in the analysis of this type of experiment.

Introduction: In this note we shall be interested in the sedimentation of small spheres, and as an added complication the spheres may also die. The objective is to show the mathematical relation between the distribution of particle sizes at the beginning of an experiment and the number of particles at various times.

Stoke's Law: Fundamental to any discussion of sedimentation of small spherical particles in air is Stoke's Law. The law states that small particles fall in air at a constant velocity. Quoting from the 24th Edition of the Handbook of Chemistry and Physics pp. 1872-1873 "Stoke's Law gives the rate of fall of a small sphere in a viscous fluid. When a small sphere falls under the action of gravity through a viscous medium it ultimately acquires a constant velocity,

$$V = \frac{2gx^2(d_1 - d_2)}{9\eta} \quad , \; d_1 > d_2 \quad ,$$

where x is the radius of the sphere, d_1 and d_2 the densities of the sphere and the medium respectively, and η the coefficient of viscosity. V will be in cm. per sec. if g is in cm. per sec.^{2}, x in cm. d_1 and d_2 in g per cm.^{3} and η in dyne-sec. per cm.^{2} or poises."

The coefficient of viscosity η measures the "internal friction" of the air. η has dimensions of mass $\div$ (length x time). Consider two infinite parallel planes, one of which

is moving in its own plane with constant velocity. Then the air between the planes will eventually arrive at a steady state. Here let F be the force acting on a unit area of one of the planes, v the velocity of air between the planes and x a direction perpendicular to the planes, then η is defined as follows:

$$F = - \eta \frac{dv}{dx} \quad .$$

Model: We shall assume that at the beginning of the experiment a chamber contains spherical particles differing only in size. In particular $f(x)N_0$ is the density of particles of radius x. N_0 is chosed so that $\Sigma f(x) = 1$. $N(t;x)$ is the density of the particles in the chamber at time t of size x. It is assumed that there is enough stirring of the air to keep a homogeneous mixture in the chamber, but not enough stirring to distrub the particles that have settled according to Stoke's Law.

Let p(x) be the probability of the death of a particle of radius x in a unit of time.

Differential Equation:

$$\frac{dN(t;x)}{dt} = - p(x)N(t;x) - v_x AN(t;x)/V$$
$$= - N(t;x)[p(x)+v_x A/V] \quad .$$

Here v_x is the velocity of particles of radius x as given by Stoke's Law, A is the horizontal surface of the chamber and V the volume. This differential equaltion is essentially

the same as that for radioactive decay.

Solution:

$$N(t;x) = f(x)N_0\, e^{-t[p(x)+v_xA/V]} \quad .$$

Let N(t) be the density of particles of all kinds at time t, this is the observale quantity. Clearly $N(t) = \Sigma N(t;x)$ where the summation is over all possible x, and if x can vary continuously the sum of course becomes an integral, which form we shall work with:

$$\frac{N(t)}{N_0} = \int_0^\infty e^{-t[p(x)+v_xA/V]} f(x)dx \quad .$$

Using the v_x given by Stoke's Law we get:

$$\frac{N(t)}{N_0} = \int_0^\infty e^{-t[p(x)+2gx^2A(d_1-d_2)/9\eta V]} dx$$

Interpretation: The solution $\frac{N(t)}{N_0}$ can be given a simple probability interpretation for it is the moment generating function of the random variable:

$$p(x) + 2gx^2A(d_1-d_2)/9\eta V \quad .$$

(It should be remembered $d_1 - d_2 > 0$.) If we set $2gA(d_1-d_2)/9\eta V = \lambda$, and if p(x) is constant (p) then the result may be simply expressed as:

$$\frac{N(t)}{N_0} = e^{-pt} \, E e^{-tx^2 \lambda} \quad .$$

If p is known then $\frac{N(t)}{N_0}$ determines f(x) by inversion formulas for moment generating functions (Laplace transform). And of course if two experiments can be performed one with p known and the other with the p unknown, the unknown p can be found without using the inversion formula.

Methodology: In many experiments it is possible to make $p = 0$. In these cases complete knowledge of $\frac{N(t)}{N_0}$ determines f(x) and conversely. This duality is useful for if we have apriori information of f(x) the observed $\frac{N(t)}{N_0}$ will confirm this information and also determine any unknown parameters in f(x). Likewise an observed $\frac{N(t)}{N_0}$ will imply precisely what the f(x) was. Consequently in the next section we shall give examples of f(x) and the corresponding N(t). The examples of f(x) used there are ones proposed by other writers in this field (except example 4).

Experimentally this duality is useful for if one has an idea of the nature of f(x) then one can select (with good chance of success) the function to fit the observed $\frac{N(t)}{N_0}$.

Of course this duality has the great advantage of relating in a definite manner the functions f(x) and $\frac{N(t)}{N_0}$, and in particular it is possible using this method to test hypotheses about

the function $f(x)$ without measuring x, but simply by making a series of density determinations.

Examples:

1. Γ (distribution)

Boudillon
Studies in Air Hygiene
London (1948)

$$f(x) = \frac{2}{2^{n/2}\sigma^{n}\Gamma(\frac{n}{2})} x^{n-1} e^{-x^2/2\sigma^2}$$

Here

$$\frac{N(t)}{N_0} = \frac{1}{(1+2\lambda\sigma^2 t)^{n/2}} \quad .$$

This distribution has the interesting property that $N(t;x)$ has the same fuctional form as $f(x)$.

2. Weibull (distribution)

Roslin, P. and Rammler, E. J.
Journal of the Institute of Fuel
7, 29 (1933)

$$f(x) = ab\, x^{b-1} e^{-ax^b} \qquad (a,b,x) > 0 \quad .$$

Here

$$\frac{N(t)}{N_0} = \sum_{r=0}^{\infty} \frac{a^{-2r/b}\Gamma(\frac{2r}{b}+1)}{r!} (-\lambda t)^r \quad .$$

3. Logarithmic (distribution)

Hect, T., and Choate, S. B.
Journal of the Franklin Institute
207 369 (1929)

$$f(x) = \frac{1}{\sigma x\sqrt{2\pi}} \quad e^{(\log x-m)^2/2\sigma^2} \qquad (\sigma,x) > 0$$

and $$\frac{N(t)}{N_0} = \sum_{i=o}^{\infty} e^{2im + 2i^2\sigma^2} \frac{(-\lambda t)^i}{i!} \quad .$$

The examples two and three have the desireable property that if say radius has the distribution of the example then also mass and surface area have this same distribution.

4. A distribution with finite range, and a mode at the extreme.

Let $f(x) = (n+1)a^{n+1}x^n$

$$0 \leq x \leq 1/a$$

$$= 0 \qquad x < 0 \text{ or } x > 1/a$$

$$a > 0$$

Here

$$\frac{N(t)}{N_0} = \frac{n+1}{2} \frac{a^{n+1}}{(\lambda t)^{\frac{n+1}{2}}} \int_0^{\lambda t/a^2} y^{\frac{n-1}{2}} e^{-y} \, dy \quad .$$

$$= \frac{n+1}{2}\left(\frac{a^2}{\lambda t}\right)^{\frac{n+1}{2}} \Gamma_{\frac{\lambda t}{a^2}}\left(\frac{n+1}{2}\right) \quad ,$$

where $\Gamma_z(m)$ is the incomplete gamma function.

Bibliographic Note: General references to experimental and theoretical material for this type of problem can be found in W. E. Ranze (AEC SO-1000; April 30, 1950) and Boudillon (Studies In Air Hygiene, London, 1948). The Laplace transform is treated extensively in Widder (The Laplace Transform, Princeton, 1946).

ADDENDUM

In this report the following densities have been used: N_0, the number of particles per unit volume; $N(t,x)$, the number of particles with radius x per unit volume at time t; and $N(t)$, the total number of particles per unit volume at time t. In some applications the observable quantities are not associated with counts of particles but rather with masses. In this case we define the following set of densities:

$$M_0 = N_0 \int_0^{\infty} x^3 f(x)\, dx$$

$$M(t;x) = N(t;x)x^3$$

$$M(t) = \int_0^{\infty} N(t;x)x^3 dx = N_0\, e^{-pt}\, Ex^3\, e^{-\lambda t x^2}$$

Here M_0 is proportional to the mass of the particles per unit volume at the beginning of the experiment, $M(t;x)$ is proportional to the mass of particles of radius x per unit volume at time t, and $M(t)$ is proportional to the total mass of particles per unit volume at time t.

When the M functions are observed, instead of the N functions, the theory is slightly modified. In this case, the convenient ratio to use is $M(t)/M_0$. This ratio is the moment generating function of the random variable λx^2, (assuming $p = 0$) where x has the probability density function

$x^3f(x)/M_0$. Consequently, by the use of inversion theory we can find the function $f(x)$ from the observed ratio $N(t)/M_0$.

In the case where the original probability density function, $f(x)$, is of the Γ type, the resulting distribution is also of this type differing from the original by having three more degrees of freedom.

THE NATIONAL BUREAU OF STANDARDS

Functions and Activities

The functions of the National Bureau of Standards are set forth in the Act of Congress, March 3, 1901, as amended by Congress in Public Law 619, 1950. These include the development and maintenance of the national standards of measurement and the provision of means and methods for making measurements consistent with these standards; the determination of physical constants and properties of materials; the development of methods and instruments for testing materials, devices, and structures; advisory services to Government Agencies on scientific and technical problems; invention and development of devices to serve special needs of the Government; and the development of standard practices, codes, and specifications. The work includes basic and applied research, development, engineering, instrumentation, testing, evaluation, calibration services and various consultation and information services. A major portion of the Bureau's work is performed for other Government Agencies, particularly the Department of Defense and the Atomic Energy Commission. The scope of activities is suggested by the listing of divisions and sections on the inside of the front cover.

Reports and Publications

The results of the Bureau's work take the form of either actual equipment and devices or published papers and reports. Reports are issued to the sponsoring agency of a particular project or program. Published papers appear either in the Bureau's own series of publications or in the journals of professional and scientific societies. The Bureau itself publishes three monthly periodicals, available from the Government Printing Office: The Journal of Research, which presents complete papers reporting technical investigations; the Technical News Bulletin, which presents summary and preliminary reports on work in progress; and Basic Radio Propagation Predictions, which provides data for determining the best frequencies to use for radio communications throughout the world. There are also five series of nonperiodical publications: The Applied Mathematics Series, Circulars, Handbooks, Building Materials and Structures Reports, and Miscellaneous Publications.

Information on the Bureau's publications can be found in NBS Circular 460, Publications of the National Bureau of Standards ($1.00). Information on calibration services and fees can be found in NBS Circular 483, Testing by the National Bureau of Standards (25 cents). Both are available from the Government Printing Office. Inquiries regarding the Bureau's reports and publications should be addressed to the Office of Scientific Publications, National Bureau of Standards, Washington 25, D. C.

NBS