

# NATIONAL BUREAU OF STANDARDS REPORT

NBS Project

NBS Report

1103-11-1107

1699

## A NOTE ON NON-PARAMETRIC METHODS

By

I. Richard Savage



## U. S. DEPARTMENT OF COMMERCE NATIONAL BUREAU OF STANDARDS

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## FORWORD

This report is a synopsis of a talk presented by Mr. Savage entitled "Some univariate aspects of multivariate non-parametric statistics" presented at the "Statistical Engineering Seminar", National Bureau of Standards, 16 November 1951.

Work on this research project is carried out at the National Bureau of Standards under the general supervision of Dr. Churchill Eisenhart, chief of the Statistical Engineering Laboratory. The Statistical Engineering Laboratory is Section 11.3 of the National Applied Mathematics Laboratories (Division 11, National Bureau of Standards) and is concerned with the development and application of modern statistical methods in the physical sciences and engineering.

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## SUMMARY

The main tools of non-parametric statistics are described. It is pointed out that some of these tools do not have natural extensions to multivariate problems. The possibility of finding tests of randomness that are independent of ordinary tests of hypotheses is explained.

## I INTRODUCTION

The object of non-parametric methods is to make probability statements about random variables when "little" is known of the distribution of these random variables. It is hard to give the precise meaning of "little" in the above statement. It refers here to the fact that all we know about the random variables of interest can be phrased in terms of such general function theory concepts as continuity of the distribution functions, the existence of a density function, the possibility of factoring the distribution functions that are involved, etc.

It is interesting to note that most of the work that has been done in a non-parametric manner has been concerned with real-valued (scalar) random variables. Exceptions to

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CHAPTER

The first part of the book is devoted to a study of the  
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this are, the work done on multidimensional tolerance limits [2] and rank correlation [5]. We shall point out that this is a natural limitation of the common tools that have been used for most of this work, i.e. the use of the probability integral transform, of topological invariance, of randomization, and of dichotomization.

## II PROBABILITY INTEGRAL TRANSFORM

The probability integral transform technique says that if  $F(x)$  is the cumulative distribution function (continuous) of the random variable  $X$ , then the new random variable  $Y \equiv F(X)$  has a uniform distribution. This fact has been most useful in achieving by non-parametric methods such results as finding confidence intervals for the median [15], finding confidence intervals for a distribution by the Wald-Wolfowitz method [11], developing the Neyman smooth test [9], etc.

Simpson [10] pointed out that the corresponding transformation for two-dimensional random variables, and consequently for higher dimensions does not have the desired properties. However, in facing a multivariate non-parametric problem it is sometimes possible to consider a real-valued random variable that is a function of the several original random variables, and in this manner the problem can be reduced to one that can be handled. This is essentially what Simpson did in [10].

M. Rosenblatt in his paper ("Limit theorems associated with variants of the von Mises statistic" presented at the

THE PROBABILITIES OF INTEGRAL FUNCTIONS

The probability of integral functions is a subject of considerable interest in the theory of probability. It is a subject which has attracted the attention of many of the greatest mathematicians of the world. The theory of integral functions is a branch of the theory of probability which deals with the probabilities of the values of the variables of a function of several variables. The theory of integral functions is a branch of the theory of probability which deals with the probabilities of the values of the variables of a function of several variables. The theory of integral functions is a branch of the theory of probability which deals with the probabilities of the values of the variables of a function of several variables.



March 21, 1952 contributed paper session of the 51st meeting of the Institute of Mathematical Statistics), has given an analogue to the probability integral transformation which should prove useful in future multivariate, non-parametric studies.

### III TOPOLOGICAL INVARIANCE

If points are placed on a line, and then a topological transformation of the line into itself is made, the order of the points on the line either remains the same or is reversed. Many non-parametric procedures depend on this fact, i.e. the Wald-Wolfowitz test of goodness-of-fit using the total number of runs [12], the Mann test of randomness [7], the Mann-Whitney two-sample test [8], etc.

Wolfowitz [16] pointed out that for higher dimensions there was no analogue to this procedure. This is most unfortunate since it has turned out that this is one of the most useful of the non-parametric methods.

It is interesting to note that rank order statistics and symmetric statistics are independent, given a sample of real-valued continuous random variables that are identically and independently distributed [4]. By a rank order statistic is meant a statistic which depends only on the order in which the ranked variables were observed, and thus using the topological invariance, its distribution under random sampling

EXPERIMENTAL PROCEDURE

The first step in the experimental procedure was the preparation of the test specimens. The specimens were prepared by cutting the material into the desired shape and size. The specimens were then subjected to a series of tests to determine their mechanical properties. The first test was a tensile test, in which the specimen was pulled until it broke. The load and displacement were recorded, and the stress-strain curve was plotted. The yield strength, ultimate tensile strength, and elongation at break were determined from this curve. The second test was a compression test, in which the specimen was pushed until it failed. The load and displacement were recorded, and the stress-strain curve was plotted. The yield strength, ultimate compressive strength, and buckling load were determined from this curve. The third test was a shear test, in which the specimen was pulled in opposite directions until it failed. The load and displacement were recorded, and the stress-strain curve was plotted. The yield strength, ultimate shear strength, and shear modulus were determined from this curve. The fourth test was a torsion test, in which the specimen was twisted until it failed. The torque and angle of twist were recorded, and the shear stress-shear strain curve was plotted. The yield strength, ultimate shear strength, and shear modulus were determined from this curve. The fifth test was a fatigue test, in which the specimen was subjected to a cyclic load until it failed. The number of cycles to failure, the mean stress, and the stress range were recorded. The S-N curve was plotted, and the fatigue strength was determined. The sixth test was a creep test, in which the specimen was subjected to a constant load at a constant temperature for a long period of time. The strain versus time curve was plotted, and the primary, secondary, and tertiary creep regions were identified. The creep rate and creep strain were determined from this curve. The seventh test was a fracture toughness test, in which the specimen was loaded until it fractured. The load versus displacement curve was plotted, and the fracture toughness was determined from the area under the curve. The eighth test was a impact test, in which the specimen was struck by a hammer. The energy absorbed during the impact was recorded, and the impact strength was determined. The ninth test was a hardness test, in which the specimen was indented by a diamond tip. The depth of the indentation was measured, and the hardness was determined. The tenth test was a wear test, in which the specimen was rubbed against a counterface. The amount of material lost during the test was measured, and the wear rate was determined. The eleventh test was a corrosion test, in which the specimen was immersed in a corrosive solution. The weight loss of the specimen was measured, and the corrosion rate was determined. The twelfth test was a thermal stability test, in which the specimen was heated to a certain temperature and held there for a certain period of time. The weight loss of the specimen was measured, and the thermal stability was determined. The thirteenth test was a dimensional stability test, in which the specimen was heated to a certain temperature and held there for a certain period of time. The change in dimensions of the specimen was measured, and the dimensional stability was determined. The fourteenth test was a moisture absorption test, in which the specimen was immersed in water. The amount of water absorbed by the specimen was measured, and the moisture absorption was determined. The fifteenth test was a density test, in which the specimen was weighed in air and in water. The density of the specimen was determined from the difference in weights. The sixteenth test was a porosity test, in which the specimen was immersed in a liquid. The amount of liquid absorbed by the specimen was measured, and the porosity was determined. The seventeenth test was a permeability test, in which the specimen was subjected to a pressure differential. The amount of gas that passed through the specimen was measured, and the permeability was determined. The eighteenth test was a diffusion test, in which the specimen was subjected to a concentration gradient. The amount of material that diffused through the specimen was measured, and the diffusion coefficient was determined. The nineteenth test was a diffusion coefficient test, in which the specimen was subjected to a concentration gradient. The amount of material that diffused through the specimen was measured, and the diffusion coefficient was determined. The twentieth test was a diffusion coefficient test, in which the specimen was subjected to a concentration gradient. The amount of material that diffused through the specimen was measured, and the diffusion coefficient was determined.

is non-parametric. Examples of such are the longest run above the median [11], and the statistics used by Mann [7] to test for trend. By a symmetric statistic is meant a symmetric function of the sample observations. The median, the mean, etc., of the sample etc, are symmetric statistics. The proof of the independence theorem for rank order statistics and symmetric statistics and its extension to several samples is simple. The independence of rank order statistics and symmetric statistics is useful in situations where we wish to test the hypothesis that we have a random sample from a population with some specific property; for example, the hypothesis that we have a random sample from a population with median equal to zero. Ordinarily in this type of testing problem all of the emphasis is put on testing whether the median is equal to zero; but in many situations it would also be important to test whether the sample was random. This method allows us to test both parts of the hypothesis with tests which are independent of each other when the null hypothesis is true. Thus it is possible to get known significance levels [14].

#### IV RANDOMIZATION

The method of randomization is based on the fact that under the assumption of random sampling we can find the exact conditional distribution for any statistic once we have made the set of observations [12].

The following is a summary of the results of the experiments conducted on the effect of the concentration of the solution on the rate of reaction. The results are given in the following table:

Concentration of Solution	Rate of Reaction
0.1 M	0.05
0.2 M	0.10
0.3 M	0.15
0.4 M	0.20
0.5 M	0.25

It is seen from the above table that the rate of reaction increases with the concentration of the solution. This is due to the fact that the number of molecules of the reactants per unit volume increases with the concentration of the solution. As a result, the frequency of collisions between the molecules increases, and hence the rate of reaction increases.

EXPERIMENTAL PROCEDURE

The experiment was conducted in a series of steps. First, a known volume of the reactants was measured and placed in a reaction vessel. The reaction was then initiated by the addition of a catalyst. The time taken for the reaction to complete was measured. This procedure was repeated for different concentrations of the reactants. The results are given in the table above.

In this method there is no problem in extending the technique to the multivariate case. Here the main problem consists in the excessive computation required. Thus we would have to make a separate table giving the distribution for each sample drawn. It does not seem possible to give a simple technique for making exact probability statements using this method, although the asymptotic theory has been worked out in detail [3], for the univariate case.

#### V DICHOTOMIZATION

Many problems where non-parametric methods are called for can be so transformed that all of the variables involved have binomial distributions. Typical of such problems is the testing of whether the median of a continuous distribution is equal to zero. Here one classifies <sup>the</sup> observations as positive or negative; then one proceeds by using the fact that the number of positive and the number of negative observations have a binomial distribution.

By changing one's problems so that they only involve binomial variates one quickly has the required distribution theory. This method has been used by Lehmann [6] for attacking multivariate problems. Perhaps the biggest drawback in using this method is that for some problems it appears to be very inefficient [6].

It will not be surprising to find that the  
method of the multivariate case is more general  
than that of the univariate case. The method  
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### V. DISCUSSION

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1. Introduction  
The purpose of this study is to investigate the effects of the proposed changes on the system's performance. The study is organized as follows:

2. Methodology  
The methodology used in this study is a combination of qualitative and quantitative methods. The data was collected through interviews, surveys, and experiments.

3. Results  
The results of the study show that the proposed changes have a significant impact on the system's performance. The performance metrics improved significantly after the implementation of the changes.

4. Conclusion  
The study concludes that the proposed changes are effective in improving the system's performance. The results of the study provide valuable insights into the effects of the changes on the system's performance.

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June 3, 1952

The first of these is the fact that the  
 government has been unable to raise  
 sufficient funds to meet its obligations.  
 This is due to a number of reasons,  
 including the fact that the government  
 has been unable to attract foreign  
 investment and has had to rely on  
 the sale of its assets to meet its  
 needs.

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