NATIONAL BUREAU OF STANDARDS REPORT

1673

26.4

THE SCATTERING OF LIGHT BY A CONICAL SURFACE

by

C. H. Page

U. S. DEPARTMENT OF COMMERCE
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ABSTRACT

The multiple scattering of light from the boundary surface of a conical region is studied, under the assumption that the scattering follows Lambert's Law. For uniform primary illumination, the increased illumination at the tip is found analytically, and the illumination at other points computed by numerical methods.

Let us consider a conical surface illuminated on its inner face by light or other radiant energy. Let the surface be such that the fraction ϵ of the incident radiation is absorbed, and the remainder scattered according to Lambert's Law; i.e., the scattered energy per unit solid angle is proportional to $\cos\theta$, with θ the angle between the direction of scattering and the normal to the surface.

Now radiation of intensity R per unit area, arriving at the angle θ_1 from the normal, gives the surface illumination $I = R\cos\theta_1$. The total scattered radiation is $(1-\epsilon)R\cos\theta_1$, and the scattered intensity in the direction θ_2 is

$$\frac{(1-\epsilon)R\cos\theta_1\cos\theta_2}{\pi} \text{ per unit solid angle}$$

or

$$\frac{(1-\epsilon)R\cos\theta_1\cos\theta_2}{\pi r^2} = \frac{(1-\epsilon)I\cos\theta_2}{\pi r^2} \text{ per unit area at}$$

distance r.

Suppose that the surface emits radiation at the rate $\beta\epsilon$ per unit area, with β a function of temperature only. Assume this radiation to also follow the cosine distribution law. Then the total power appearing to be emitted from a unit area is

(1) $$E = \beta\epsilon + (1 - \epsilon)I.$$

The power incident per unit area at a point (1) is the summation of the radiated and scattered power from all parts of the surface "visible" from the point in question:

$$I(1) = \int E(2)\, K(1,2)\, d\sigma_2$$

where $d\sigma$ is the element of area at point (2), and

$$K(1,2) = \frac{\cos\theta_1 \cos\theta_2}{\pi r_{12}^2}$$ (see Figure 1)

In addition, we must allow for direct illumination from an external source, making the complete equation

$$I(1) = D(1) + \int E(2)\, K(1,2)\, d\sigma_2 \text{ .}$$

We are interested in two special cases:

I. No external source of illumination and the conical surface at a uniform temperature.

Then $$I(1) = \int E(2)\, K(1,2)\, d\sigma_2$$

or $$E(1) = \beta\epsilon + (1 - \epsilon)\int E(2)\, K(1,2)\, d\sigma_2$$

by using equation (1).

Define the "apparent" blackness of the surface as $\hat{\epsilon} \equiv \frac{E}{\beta}$, giving

$$\hat{\epsilon}(1) = \epsilon + (1 - \epsilon)\int \hat{\epsilon}(2)\, K(1,2)\, d\sigma_2$$

and define the gain of apparent blackness by virtue of the scattering as $G \equiv \hat{\epsilon}/\epsilon$, giving

(3) $$G(1) = 1 + (1 - \epsilon)\int G(2)\, K(1,2)\, d\sigma_2.$$

II. Let the emission be zero and the primary illumination uniform

$$I(1) = D + (1 - \epsilon) \int I(2)\, K(1,2)\, d\sigma_2$$

by substituting (1) into (2).

The gain in illumination by virtue of scattering is $G' \equiv I/D$ and is determined by the equation

$$G'(1) = \frac{I}{D} = 1 + (1 - \epsilon) \int \frac{I(2)}{D}\, K(1,2) d\sigma_2$$

or
$$G'(1) = 1 + (1 - \epsilon) \int G'(2)\, K(1,2)\, d\sigma_2 .$$

Note that this is identical with equation (3), hence G' and G are identical.

The specific problem at hand is to solve this integral equation for a conical surface.

Let the cone have the semi-apex angle α. Locate a point on the cone by its distance s from the apex, and the azimuth angle ϕ from a convenient generator. The cartesian coordinates of a point on the cone are

$$x = s \sin \alpha \cos \phi$$
$$y = s \sin \alpha \sin \phi$$
$$z = s \cos \alpha .$$

The normal at P has the direction cosines

$$\ell = -\cos \alpha \cos \phi$$
$$m = -\cos \alpha \sin \phi$$
$$n = \sin \alpha .$$

The element of area is $d\sigma = s \sin \alpha\, d\phi\, ds$. Without loss of generality let P_1 have the azimuth $\phi_1 = 0$, so that

$$\cos \Theta_1 = -\frac{x_2 - x_1}{r_{12}} \cos \alpha + \frac{z_2 - z_1}{r_{12}} \sin \alpha$$

$$= \frac{s_2 \sin \alpha \cos \alpha (1 - \cos \phi_2)}{r_{12}}$$

and

$$\cos \Theta_2 = \frac{s_1 \sin \alpha \cos \alpha (1 - \cos \phi_2)}{r_{12}}$$

$$K(1,2) = \frac{\cos \Theta_1 \cos \Theta_2}{\pi r_{12}^2} = \frac{\sin^2 \alpha \cos^2 \alpha (1 - \cos \phi_2)^2 s_1 s_2}{\pi r_{12}^4}$$

and

$$r_{12}^2 = s_1^2 + s_2^2 - 2s_1 s_2 \left[\cos^2 \alpha + \sin^2 \alpha \cos \phi_2\right]$$

The basic equation (3) becomes

$$(4) \quad G(s) = 1 + \frac{1 - \epsilon}{\pi} \sin^3 \alpha \cos^2 \alpha \int_0^T \int_0^{2\pi} \frac{G(t)\, st^2 (1 - \cos \phi)^2\, d\phi\, dt}{\left[s^2 + t^2 - 2st(\cos^2 \alpha + \cos \phi \sin^2 \alpha)\right]^2}$$

where T is the slant length of the cone.

The denominator (r_{12}^4) can be written

$$\left[(s - t)^2 + 2st \sin^2 \alpha (1 - \cos \phi)\right]^2$$

Changing the variable from ϕ to $\phi/2$, using $1 - \cos \phi = 2 \sin^2 \phi/2$, puts the angle integration into the form:

$$\int_0^{\pi} \frac{\sin^4 \Theta\, d\Theta}{(a^2 + b \sin^2 \Theta)^2} = \int_{-\pi/2}^{\pi/2} \frac{\sin^2 \Theta\, d\Theta}{(a^2 + b \sin^2 \Theta)^2}$$

A second change of variable to $x = \sin \Theta$ yields

$$(5) \quad \int_{-1}^{1} \frac{x^4\, dx}{(a^2 + bx^2)^2 \sqrt{1 - x^2}}$$

which can be evaluated in the complex plane by the following procedure.

Consider $\int_{-\infty}^{\infty} \frac{z^4 dz}{(a^2 + bz^2)^2 \sqrt{1 - z^2}}$ along the contour of Figure 2.

For $-1 < x < 1$, the integrand is even, but on the remainder of the real axis the integrand is odd because of the branch points. The contribution of the semicircles around ± 1 is zero, making the infinite integral equal to the desired (5). Deforming the contour into an infinite semicircle plus a circle around the pole at $z = ia/\sqrt{b}$ allows straightforward integration, and yields $\pi/b^2 + 2\pi i$ Res at $ia/\sqrt{b}$. Evaluating the residue and substituting for a, b, yields

$$(6) \quad G(s) = 1 + \frac{1-\epsilon}{2} \frac{\cos^2\alpha}{\sin\alpha} \int_0^T G(t) \frac{1}{s} \left\{ 1 - |s-t| \frac{(s-t)^2 + 6st \sin^2\alpha}{\left[(s-t)^2 + 4st \sin^2\alpha\right]^{3/2}} \right\} dt.$$

The expression in braces is a function of the ratio t/s only, and is not affected by interchanging s and t.

From physical intuition, we would expect that the illumination (hence G) as we approach the apex would be independent of the length of the cone; i.e., the cone appears infinite. This is indeed the case. The cone of infinite length has the solution G = constant = G_o:

$$(7) \qquad G_o = 1 + \frac{1-\epsilon}{2} \frac{\cos^2\alpha}{\sin\alpha} G_o \int_0^\infty f(x)\, dx$$

where $x \equiv t/s$.

This integral can be evaluated by brute force, yielding

$$G_o = 1 + \frac{1-\epsilon}{2} G_o \; 2(1 - \sin^3\alpha)$$

or

$$(8) \qquad G_o = \frac{1}{1 - (1-\epsilon)(1 - \sin^3\alpha)} = \frac{1}{\epsilon + (1-\epsilon)\sin^3\alpha}$$

and

$$\hat{\epsilon}_o = \epsilon G_o = \frac{\epsilon}{\epsilon + (1-\epsilon)\sin^3\alpha} \leq 1.$$

For the finite cone, conveniently scaled to unit slant length

(9) $$G(s) = 1 + \frac{1 - \epsilon}{2} \frac{\cos^2\alpha}{\sin\alpha} \int_0^1 G(t)\, K(s,t)\, dt.$$

Note that $\int_0^{\infty} K(s,t)dt = 2 \frac{\sin\alpha}{\cos^2\alpha}(1 - \sin^3\alpha)$ independent of s.

Now $G > 0$, $K \geq 0$, so

$$\int_0^1 G\, K\, dt \leq G_{max} \int_0^1 K dt \leq G_{max} \int_0^{\infty} K\, dt$$

$$\leq G_{max} \frac{2 \sin\alpha}{\cos^2\alpha}(1 - \sin^3\alpha)$$

or

$$G(s) \leq 1 + (1 - \epsilon)(1 - \sin^3\alpha) G_{max}$$

and

$$G_{max} \leq 1 + (1 - \epsilon)(1 - \sin^3\alpha)\, G_{max}$$

or

$$G_{max} \leq \frac{1}{\epsilon + (1 - \epsilon)\sin^3\alpha} = G_0$$

$$G(s) \leq G(0),\ \epsilon G(s) \leq \epsilon G(0) < 1.$$

Therefore the absorption is nowhere greater than it would be for a perfectly black surface ($\epsilon = 1$), although the intensity of illumination ($I = GD$) may become high as we approach the tip. (As $\epsilon \to 0$, $G_0 \to 1/\sin^3\alpha$).

The straightforward approximation of the integral equation (6 or 9) by a set of simultaneous equations would be achieved by representing the integral by a formula such as Simpson's rule:

(10) $$\int_0^1 G(t)\, K(s,t)dt = \frac{h}{3}\Big\{G(0)\, K(s,0) + 4G(h)\, K(s,h) + 2G(2h)\, K(s,2h) + \cdots + 4G(1 - h)\, K(s, 1 - h) + G(1)\, K(s,1)\Big\},$$

expressing the integral in terms of the unknown values of $G(nh)$. Substitution in (9) yields an equation holding for all s; if we let $s = 0, h, 2h \cdots 1$ successively, we obtain a set of $N = 1 + (1/h)$ simultaneous equations for the N unknowns $G(nh)$; $n = 0 \cdots N - 1$. Unfortunately, this procedure will not lead to correct answers. The Simpson rule integration assumes the integrand to be fitted piecewise by parabolas. The second equation in the set,

for $s = h$, assumes the product $G(t)\,K(h,t)$ to be approximated by a parabola from $t = 0$ to $t = 2h$. The occurrence of $|s - t|$ in the kernel produces a cusp at $s = t$ (Figure 3), hence a parabolic fit for the points $t = 0, h, 2h$ cannot yield a reasonable approximation. This phenomenon is independent of the mesh size h chosen, since the kernel is a function of t/s.

The difficulty can be avoided by using the trapezoidal rule for numerical integration, but the loss of accuracy seems too high a price to pay.

Using Simpson's rule, it appears that we need more divisions in t than in s; hence assuming values of $G(t)$ at (say) 7 points yields (say) 4 equations for $G(s)$. This leaves us with more unknowns than equations, allowing subsidiary conditions. The obvious choice is to determine $G(t)$ at the extra 3 points by interpolation. That is, we set up 4 equations for $G(s)$ at 4 points, express the integral as a 7 point Simpson integration expressing the values of $G(t)$ at the additional intermediate points as linear combinations of the values of $G(t)$ at the additional intermediate points as linear combinations of the values of $G(t)$ at the first 4 points. This procedure can be expressed as the matric equation

$$G = 1 + \lambda\, K\, w\, L\, G \tag{11}$$

where G is the vector whose components are the values of $G(s)$ at the points $s = 0, 1/3, 2/3, 1$; K is the non-square matrix of values of $K(s,t)$ at $s = n/3$ $(n = 0 \cdots 3)$ and $t = m/6$ $(m = 0 \cdots 6)$; w is a diagonal matrix of the Simpson weights; L is an interpolation matrix of seven rows and four columns; and

$$\lambda \equiv \frac{1 - \epsilon}{2}\,\frac{\cos^2 \alpha}{\sin \alpha} .$$

Since the values of $G(s)$ at $s = n/3$ determine a cubic from which the values at $s = m/6$ can be found, we have

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 5/16 & 15/16 & -5/16 & 1/16 \\ 0 & 1 & 0 & 0 \\ -1/16 & 9/16 & 9/16 & -1/16 \\ 0 & 0 & 1 & 0 \\ 1/16 & -5/16 & 15/16 & 5/16 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

For a cone with semi-apex angle 15°, equation (10) is best solved by the minimized iteration technique of Lanczos [1/] and yields the results of Figures 4 and 5 for several values of ϵ.

1/ C. Lanczos, NBS Jour. of Res. 45, p.255 (1950)

For a cone with semi-apex angle 45°, the contribution of multiple scattering falls off rapidly with the order of the scatter, so the simple Liouville-Neumann iteration converges fairly rapidly. This solving procedure yields results shown in Figures 6 and 7.

APPENDIX

It is of some interest to compare the cone with a wedge of infinite transverse length. The equation corresponding to (4) is readily found

$$(12) \qquad G(s) = 1 + \frac{1-\epsilon}{\pi} \sin^2 2\alpha \int_0^T \int_{-\infty}^{\infty} \frac{G(t)\, st\, dy\, dt}{\{s^2 + t^2 - 2st \cos 2\alpha + y^2\}^2}$$

where α is the semi-apex angle of the wedge. Integration with respect to y is straightforward and yields

$$(13) \qquad G(s) = 1 + \frac{1-\epsilon}{2} \sin^2 2\alpha \int_0^T \frac{G(t)\, st\, dt}{\{s^2 + t^2 - 2st \cos 2\alpha\}^{3/2}}$$

The solution at the tip is again found by making $T = \infty$, and is

$$(14) \qquad G_o = \frac{1}{\epsilon + (1-\epsilon)\sin^2 \alpha}$$

Comparison with (8) shows that G_o for the wedge is always less than G_o for the corresponding cone.

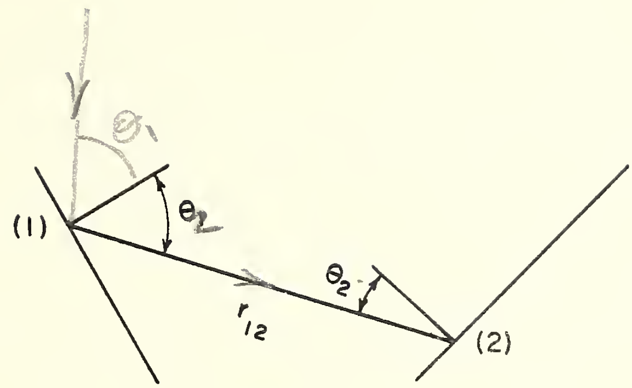

FIG. 1

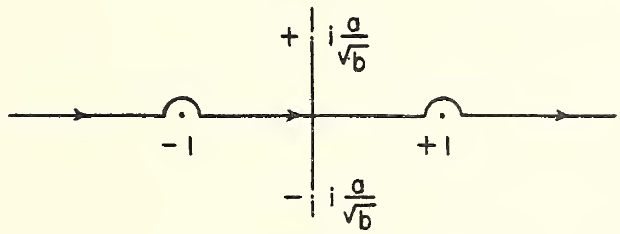

FIG. 2

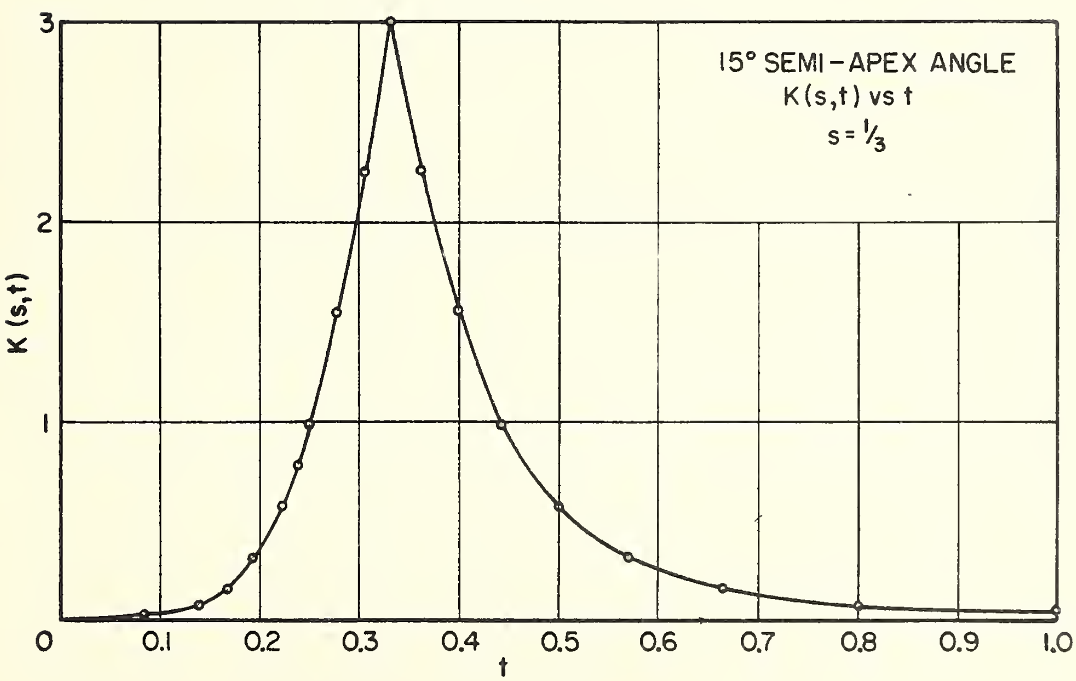

FIG. 3

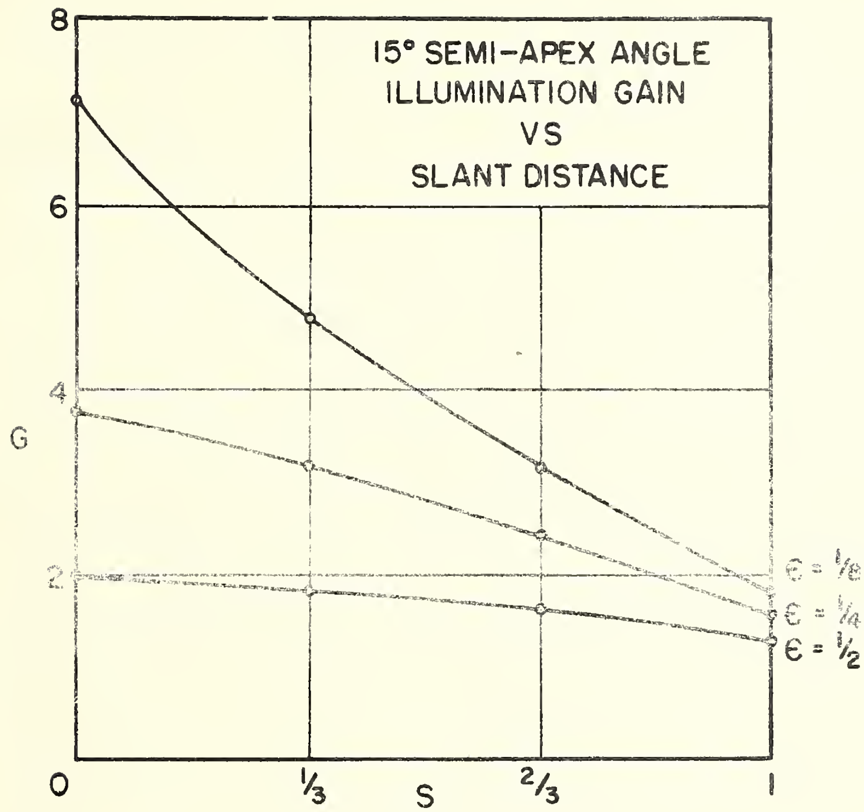
8
6
4
2
0
G
15° SEMI-APEX ANGLE
ILLUMINATION GAIN
VS
SLANT DISTANCE
ϵ = 1/8
ϵ = 1/4
ϵ = 1/2
1/3
2/3
1
S

FIG. 4

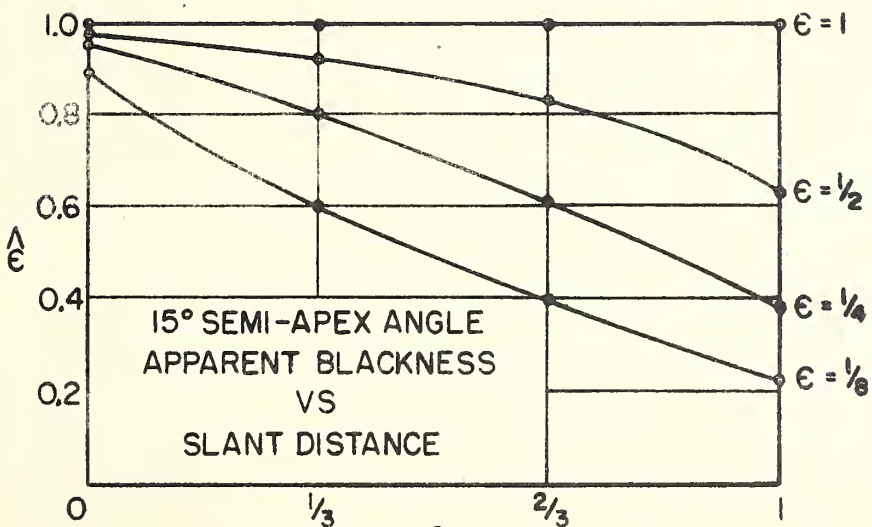
1.0
0.8
0.6
0.4
0.2
0
ϵ̂
ϵ = 1
ϵ = 1/2
ϵ = 1/4
ϵ = 1/8
15° SEMI-APEX ANGLE
APPARENT BLACKNESS
VS
SLANT DISTANCE
1/3
2/3
1
S

FIG.5

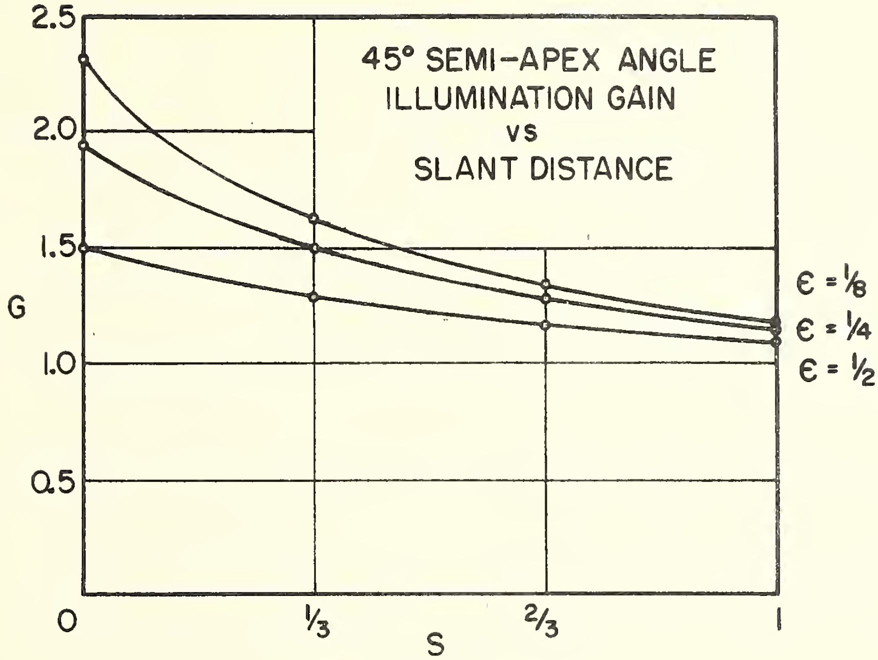
2.5
2.0
1.5
1.0
0.5
G
45° SEMI-APEX ANGLE
ILLUMINATION GAIN
VS
SLANT DISTANCE
ϵ = 1/8
ϵ = 1/4
ϵ = 1/2
0
1/3
2/3
1
S

FIG. 6

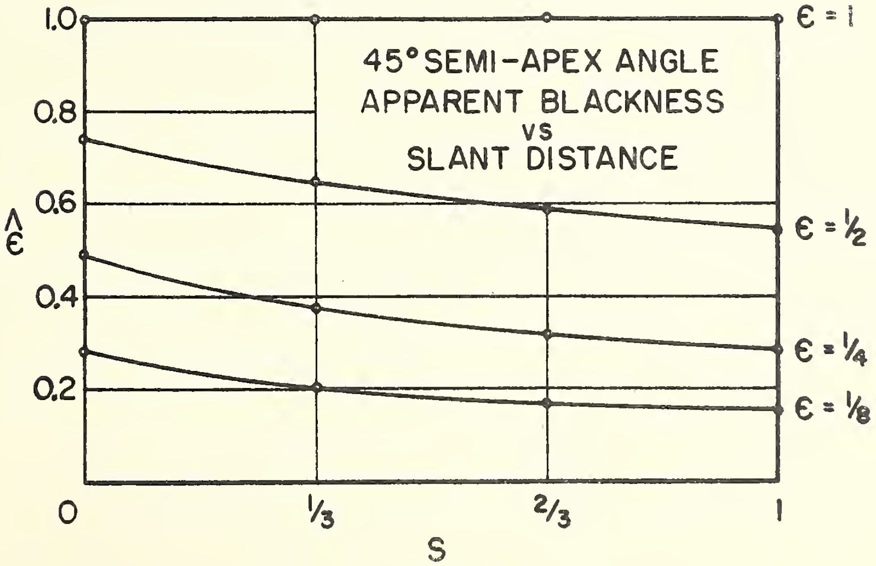
1.0
0.8
0.6
0.4
0.2
ϵ̂
45° SEMI-APEX ANGLE
APPARENT BLACKNESS
VS
SLANT DISTANCE
ϵ = 1
ϵ = 1/2
ϵ = 1/4
ϵ = 1/8
0
1/3
2/3
1
S

FIG. 7

THE NATIONAL BUREAU OF STANDARDS

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