

# **NATIONAL BUREAU OF STANDARDS REPORT**

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**A PROPERTY OF STRONGLY CONTINUOUS PROCESSES**

by

**Eugene Lukacs**



**U. S. DEPARTMENT OF COMMERCE  
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Eugene Lukacs  
National Bureau of Standards

## 1. Introduction.

The purpose of this report is to give a condition which assures that the increments of a stochastic process  $y(t)$  are normally distributed. This condition is of interest in connection with the study of the fundamental random process. The fundamental random process - also called the Wiener process - affords a particularly simple model for certain phenomena. One of the properties of the fundamental random process is that its increments are normally and independently distributed (for a definition see [4] or [5]). In the present report it is shown that the normality of the increments follows from a certain continuity property.

## 2. Definitions and statement of the theorem.

In this section we give the definitions necessary to formulate the theorem.

The increment of a stochastic process  $y(t)$  over the time interval  $(t, t + \gamma)$  is the random variable  $y(t + \gamma) - y(t)$ .

A stochastic process is said to be a process with independent increments if the increments over non-overlapping time intervals are completely independent of each other.





A process  $y(t)$  is said to be strongly continuous [5] in the interval  $[a, b]$  if to every  $\epsilon > 0$  and  $\eta > 0$  there exists a  $\delta = \delta(\epsilon, \eta)$  such that for every finite set  $S$  of points contained in  $[a, b]$

$$(1) \quad P[\mathcal{G}(\delta, \epsilon, S)] \geq 1 - \eta.$$

Here  $\mathcal{G}(\delta, \epsilon, S)$  is the event that the inequalities  $|y(t_i) - y(t_k)| \leq \epsilon$  are simultaneously satisfied for all pairs  $(t_i, t_k)$  with  $|t_i - t_k| < \delta$  belonging to a finite set  $S$  of points contained in  $[a, b]$ .

The symbol  $P[\dots]$  stands here and in the following for the probability of the event  $[\dots]$  in the brackets.

We are now in a position to formulate the condition mentioned in the introductory section.

Theorem: Let  $y(t)$  be a stochastic process and assume that

- (i)  $y(t)$  is a process with independent increments
- (ii)  $y(t)$  is strongly continuous in the interval  $[a, b]$ .

Then  $y(b) - y(a)$  is normally distributed.

A theorem of this type is due to P. Levy [3] (theorem 16, 3). However it might be of interest to give a different proof using the notation and terminology of H. B. Mann's monograph [5].

### 3. Khintchine's theorem.

For the proof we need a theorem of A. Khintchine which gives conditions for the convergence to the normal distribution. This theorem is given in a book by Khintchine<sup>[2]</sup>, published in the Russian language. The following formulation was taken from a paper by B. V. Gnedenko [1] which is available in an English translation.

We consider a sequence of sequences  $X_{n1}, X_{n2}, \dots, X_{nk_n}$  ( $n=1, 2, \dots$  ad inf) of random variables which are independent within each sequence. The random variables  $X_{nk}$  are said to be infinitesimal if for any  $\epsilon > 0$  the relation  $\lim_{n \rightarrow \infty} P[|X_{nk}| > \epsilon] = 0$  uniformly in  $k$  ( $1 \leq k \leq k_n$ ).





We denote by  $F_{n_k}(x)$  the distribution function of the random variable  $X_{nk}$  and state Khinchin's theorem: If the distributions of the sums

$$(2) T_n = X_{n1} + X_{n2} + \dots + X_{nk_n}$$

of independent (within each sequence) infinitesimal random variables  $X_{nk}(1 \leq k \leq k_n)$  converge to a limiting distribution, then the necessary and sufficient condition for the limiting distribution to be normal is that for any  $\epsilon > 0$

$$(3) \lim_{n \rightarrow \infty} \sum_{k=1}^{k_n} \int_{|x| \geq \epsilon} dF_{n_k}(x) = 0.$$

For the proof the reader is referred to Gnedenko's paper [1].

#### 4. Proof of the theorem stated in section 2

We consider a sequence  $\{S_n\}$  of subdivisions of the interval  $[a, b]$ .

For the sake of simplicity we let  $S_n$  be the subdivision ~~of~~ X

$S_n = (t_0^n, t_1^n, t_2^n, \dots, t_n^n)$  of  $[a, b]$  into  $n$  equal parts, that is we put

$$t_j^n = a + (b - a) j/n \quad (j = 0, 1, 2, \dots, n). \text{ We write}$$

$$x_{n,j} = y(t_j^n) - y(t_{j-1}^n) \text{ for } j = 1, 2, \dots, n \text{ and show first that the}$$

random variables  $x_{n,j}$  are infinitesimal. x(i,c)

By assumption (ii) the process  $y(t)$  is strongly continuous in  $[a, b]$ .

It is therefore possible to determine for every  $\epsilon > 0$  and  $\eta > 0$  a  $\delta = \delta(\epsilon, \eta)$

such that for every subdivision  $S_n$

$$(4) P[\xi(\delta, \epsilon, S_n)] \geq 1 - \eta$$

We denote by  $P[|x_{n,j}| \leq \epsilon : j = 1, 2, \dots, n]$  the probability that

the  $n$  inequalities  $|x_{n,j}| \leq \epsilon (j = 1, 2, \dots, n)$  hold simultaneously (\*)

(\*) If  $R_i (i=1, 2, \dots, n)$  are  $n$  events then  $P[R_i; i=1, 2, \dots, n]$  means the probability that all  $n$  events occur simultaneously. Thus  $P[R_i; i=1, 2, \dots, n] \geq 1 - \eta$  means that the probability of the simultaneous occurrence of all  $n$  events is at least equal to  $1 - \eta$ . This should be carefully distinguished from the statement  $P[R_i] \geq 1 - \eta (i=1, 2, \dots, n)$  which means that the probability of the occurrence of each single event is at least equal to  $1 - \eta$ ; this statement does not imply anything about the probability of the joint occurrence of the  $n$  events.



We next choose a number  $N = N(\xi, \eta)$  such that  $N \geq \frac{(b-a)}{\eta(\xi, \eta)}$ .

For any  $n \geq N$  the event  $\mathcal{C}(\xi, \eta, S_n)$  implies that the  $\eta$  (inequalities

$|x_{n,j}| \leq \xi$  ( $j=1, 2, \dots, n$ ) are simultaneously satisfied. We conclude then from (1) that

$$(5) 1 - \eta \leq P[|x_{n,j}| \leq \xi; j=1, 2, \dots, n] \text{ and also}$$

$$(6) 1 - \eta \leq P[|x_{n,j}| \leq \xi] \text{ for } j=1, 2, \dots, n, \text{ if only } n \geq N.$$

For every  $\xi > 0$  and  $\eta > 0$  it is therefore possible to find an  $N = N(\xi, \eta)$  such that for  $n \geq N$

$$(7) P[|x_{n,j}| > \xi] \leq \eta \text{ for } j=1, 2, \dots, n.$$

This shows that  $P[|x_{n,j}| > \xi]$  converges (uniformly in  $j$ ) to zero as  $n \rightarrow \infty$ , or in other words the  $x_{n,j}$  are infinitesimal random variables.

We consider the sequence of random variables

$$(8) T_n = x_{n,1} + x_{n,2} + \dots + x_{n,n}$$

We have shown that  $T_n$  is the sum of infinitesimal random variables

which are by assumption independent (within each sequence). Since

$$T_n = \sum_{j=1}^n x_{n,j} = \sum_{j=1}^n [y(t_j^n) - y(t_{j-1}^n)] = y(t_n^n) - y(t_0^n) = y(b) - y(a)$$

We see that the limiting distribution of the  $T_n$  is the distribution of the random variable  $y(b) - y(a)$ .

We have already shown that it is possible to find for every  $\xi > 0$  and  $\eta > 0$  a

$$N = N(\xi, \eta) \text{ such that } P[|x_{n,j}| \leq \xi; j=1, \dots, n] \geq 1 - \eta$$

for  $n \geq N$ .

Since the increments  $x_{n,j}$  are independently distributed [by assumption (i)]

it follows that also

$$(9) P[|x_{n,j}| \leq \xi; j=1, 2, \dots, k] \geq 1 - \eta$$

where  $\xi$  and  $\eta$  are arbitrary positive numbers and  $n \geq N$  while  $k$  is an integer less than  $n$ .





For any subdivision  $S_n$  of  $[ab]$  into  $n$  equal parts we introduce the random variable

$$(10) \quad M_{abS_n} = \max [ |x_{n,1}|, |x_{n,2}|, \dots, |x_{n,n}| ]$$

The statement that the  $n$  inequalities  $|x_{n,j}| \leq \xi$  for  $j=1,2,\dots,n$  hold simultaneously is equivalent to the statement  $M_{abS_n} \leq \xi$ .

Therefore

$$P[M_{abS_n} \leq \xi] = P[|x_{n,j}| \leq \xi; j=1,2,\dots,n] = 1-\eta$$

and also

$$(11) \quad P[M_{abS_n} > \xi] \leq \eta,$$

for all  $\xi > 0$  and  $\eta > 0$ , provided that  $n \geq N$ .

We introduce the following  $n$  events:

$B_1^{(n)}$  is the event that the inequality  $|x_{n,1}| > \xi$  holds.

$B_j^{(n)}$  (for  $j=2,3,\dots,n$ ) is the event that the  $j$  inequalities

$|x_{n,1}| \leq \xi, |x_{n,2}| \leq \xi, \dots, |x_{n,j-1}| \leq \xi, |x_{n,j}| > \xi$  hold simultaneously. The events  $B_1^{(n)}, B_2^{(n)}, \dots, B_n^{(n)}$

are mutually exclusive and exhaust all the cases for which  $M_{abS_n} > \xi$ .

Therefore we see from (11)

$$(12) \quad \eta \geq P[M_{abS_n} > \xi] = \sum_{j=1}^n P[B_j^{(n)}]$$

for all  $\xi > 0$  and  $\eta > 0$ , provided that  $n \geq N$ .

Since the random variables  $x_{n,1}, \dots, x_{n,n}$  are completely independent we have for  $j=2,\dots,n$

$$P[B_j^{(n)}] = P[|x_{n,k}| \leq \xi; k=1,2,\dots,(j-1)] P[|x_{n,j}| > \xi].$$

It follows therefore from (9) that

$$(13) \quad P[B_j^{(n)}] \geq (1-\eta) P[|x_{n,j}| > \xi] \quad (j=2,\dots,n)$$

for any  $\xi > 0$  and  $\eta > 0$  if only  $n \geq N$  so that

$$(14) \quad \sum_{j=1}^n P[B_j^{(n)}] \geq (1-\eta) \sum_{j=1}^n P[|x_{n,j}| > \xi]$$

for any  $\xi > 0$  and  $\eta > 0$  provided that  $n \geq N$ .

From (12) and (14) we see that





$$(15) \quad \eta/(1-\eta) \approx \sum_{j=1}^n P[|x_{n,j}| > \xi]$$

for any  $\xi > 0, \eta > 0$  if  $n \in N$ . Hence we conclude that

$$\lim_{n \rightarrow \infty} \sum_{j=1}^n P[|x_{n,j}| > \xi] = 0 \quad \text{or}$$

$$(16) \quad \lim_{n \rightarrow \infty} \sum_{j=1}^n \int_{|x| > \xi} dF_{n,j}(x) = 0$$

But (16) is exactly Khinchine's condition (3) so that we have shown that

the limiting distribution of the  $T_n$ , that is the distribution of  $y(b) - y(a)$  is normal.



5. References.

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- 2 . A. Khintchine, Limiting distributions for sums of independent random variables. ( in Russian) Moscow - Leningrad (1938).
- 3 . Paul Levy Processus stochastiques et mouvement Brownien. Gauthiers Villars, Paris 1948
- 4 . H. B. Mann Estimation of parameters in certain stochastic processes. Sankhya vol. 11, 97-106, (1951)
- 5 . H. B. Mann Introduction to the theory of stochastic processes depending on a continuous parameter.  
N.B.S. report #1293 (1951) to be published as AMS 24.



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