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SOME RELATIONS AMONG THE BLOCKS OF SYMMETRICAL GROUP DIVISIBLE DESIGNS

by

W. S. CONNOR

U. S. DEPARTMENT OF COMMERCE
NATIONAL BUREAU OF STANDARDS

U. S. DEPARTMENT OF COMMERCE
Charles Sawyer, Secretary

NATIONAL BUREAU OF STANDARDS
A. V. Astin, Acting Director

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FOREWORD

The combinatorial problem which is studied in this paper is of interest in the design of experiments.

The paper considers the way in which the blocks of a symmetrical Group Divisible incomplete block design, with parameters v, r, m, n, λ_1, and λ_2, are connected by common treatments. Structural and characteristic matrices are defined, and a relation among them is exhibited. It is shown that if a solution exists for a given set of parameters which are such that $r \neq \lambda_1$, $r^2 \neq v\lambda_2$, and $r^2 - v\lambda_2$ and $\lambda_1 - \lambda_2$ are relatively prime, then the interchange of blocks with treatments yields a solution which corresponds to the given set of parameters.

J. H. Curtiss
Chief, National Applied
Mathematics Laboratories

A. V. Astin
Acting Director
National Bureau of Standards

SOME RELATIONS AMONG THE BLOCKS OF SYMMETRICAL GROUP DIVISIBLE DESIGNS

W. S. Connor
National Bureau of Standards[1]

1. SUMMARY. It is well known that if every pair of treatments in a symmetrical Balanced incomplete block design occurs in λ blocks, then every two blocks of the design have λ treatments in common. In this paper it will be shown that a somewhat similar property holds for symmetrical Group Divisible designs. In the course of the investigation there will be introduced certain matrices which are of intrinsic interest.

2. INTRODUCTION. Some of the combinatorial properties of Group Divisible incomplete block designs were considered in [1]. Here we shall need the definition of Group Divisible designs and the three classes into which they fall. An incomplete block design with v treatments each replicated r times in b blocks of size k is said to be Group Divisible (GD) if the treatments can be divided into m groups, each with n treatments, so that the treatments belonging to the same group occur together in λ_1 blocks and the treatments belonging to different groups occur together in λ_2 blocks, $\lambda_1 \neq \lambda_2$. The three exhaustive and mutually exclusive classes into which the GD designs fall are as follows:

(a) Singular GD designs characterized by $r - \lambda_1 = 0$;

(b) Semi-regular GD designs characterized by $r - \lambda_1 > 0$,

[1]This work was begun while the author was at the University of North Carolina

$rk - v\lambda_2 = 0$; and

(c) Regular GD designs characterized by $r - \lambda_1 > 0$, $rk - v\lambda_2 > 0$.

In this paper we shall study classes (b) and (c) for the symmetrical case, that is, the case when $r = k$, or equivalently, $b = v$.

3. THE INCIDENCE AND STRUCTURAL MATRICES. In [2] there was defined the structural matrix for Balanced incomplete block designs. We now shall define the incidence matrix, and two structural matrices for GD designs.

Let us consider first the incidence matrix of a GD design,

$$(3.1) \qquad N = \begin{bmatrix} n_{11} \cdots\cdots n_{1b} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ n_{v1} \cdots\cdots n_{vb} \end{bmatrix},$$

where the rows represent treatments, the columns represent blocks, and $n_{ij} = 1$ or 0 according as the i-th treatment does or does not occur in the j-th block. From the conditions satisfied by the design it is easy to see that

$$(3.2) \qquad \sum_{j=1}^{b} n_{ij} = r, \quad (i = 1, \cdots, v),$$

and

$$(3.3) \qquad \sum_{j=1}^{b} n_{ij}n_{uj} = \lambda_1 \text{ or } \lambda_2,$$

according as the i-th and u-th treatments ($i \neq u$) do belong or do not belong to the same group.

Throughout the paper let us adopt the convention that the treatments $n(w-1) + 1$, $n(w-1) + 2$, $\cdots$, nw shall belong to the w-th group, ($w=1, \cdots, m$). Then

$$(3.4) \qquad NN' = \begin{bmatrix} A & B & \cdot & \cdot & \cdot & B \\ B & A & \cdot & \cdot & \cdot & B \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ B & B & \cdot & \cdot & \cdot & A \end{bmatrix},$$

where the elements of the $n \times n$ submatrix A are r in the principal diagonal and λ_1 elsewhere, and the elements of the $n \times n$ submatrix B are λ_2 everywhere. Of course NN' contains $v = mn$ rows and columns.

Now choose any $t \leq b$ blocks of the design. Let the submatrix of N which corresponds to these t blocks be denoted by N_o. Let s be the number of treatments common to the j-th and u-th chosen blocks, ($j, u = 1,2,\cdots,t$). Then the $t \times t$ symmetric matrix

$$(3.5) \qquad S_t^I = N_o' N_o' = (s_{ju})$$

is defined to be the <u>intersection structural matrix of the t chosen blocks</u>. The j-th row or column of S_t corresponds to the

j-th chosen block and the successive elements of the j-th row or column give the number of treatments which this block has in common with the 1st, 2nd, ..., t-th chosen blocks.

We next shall consider another structural matrix. Let s^w_{ju} denote the number of treatments from the w-th group which blocks j and u have in common. Then

$$\sum_{w=1}^{m} s^w_{ju} = s_{ju} \tag{3.6}$$

and

$$\sum_{w=1}^{m} s^w_{jj} = k. \tag{3.7}$$

Now consider the matrix

$$G_t = \begin{bmatrix} s^1_{11} & s^1_{22} & \cdot & \cdot & \cdot & s^1_{tt} \\ s^2_{11} & s^2_{22} & \cdot & \cdot & \cdot & s^2_{tt} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ s^m_{11} & s^m_{22} & \cdot & \cdot & \cdot & s^m_{tt} \end{bmatrix}, \tag{3.8}$$

and the product matrix

$$S^G_t = G_t' G_t \ , \tag{3.9}$$

where the element in the j-th row and u-th column is the sum of

products of the number of treatments which the j-th chosen block and the u-th chosen block contain from each group. We define S_t^G as the group structural matrix of the t chosen blocks.

4. THE CHARACTERISTIC MATRIX. We shall define an analogue of the characteristic matrix which was developed for Balanced incomplete block designs in [2]. For the remainder of the paper, except for the last section, we shall restrict our attention to the regular GD designs.

Let the columns of N be permuted so that the first t columns correspond to the t chosen blocks. Then let the incidence matrix be extended by adjoining t new rows, so that the elements of the j-th adjoined row are zero, except for the j-th which is unity. We thus get

$$(4.1) \qquad N_1 = \begin{bmatrix} & N & \\ I_t & & 0 \end{bmatrix},$$

where
~~which~~ I_t is the identity matrix of order t, and 0 is the t x (b-t) zero matrix. Then

$$(4.2) \qquad N_1 N_1' = \begin{bmatrix} NN' & N_0 \\ N_0' & I_t \end{bmatrix}$$

The evaluation of $|N_1 N_1'|$ leads to

(4.3) $$|N_1N_1'| = (rk)^{-t+1}(r-\lambda_1)^{v-t-m}(rk-v\lambda_2)^{m-t-1}|C_t| ,$$

where the typical element of C_t is

(4.4) $$c_{ju} = (rk-v\lambda_2)(rk\delta_{ju}+\lambda_2k^2) + (\lambda_1-\lambda_2)(rk\sum_{w=1}^{m} s_{jj}^{w}s_{uu}^{w} - n\lambda_2k^2) ,$$

where $\delta_{ju} = (r-\lambda_1-k)$ or $-s_{ju}$, according as $j = u$ or $j \neq u$. The matrix C_t is defined as the <u>characteristic matrix of the t chosen blocks</u>. The j-th row or the j-th column of C_t corresponds to the j-th chosen block of the design.

We observe that the characteristic matrix is related to the two structural matrices as is described in the following theorem.

<u>Theorem</u> 4.1. For the regular GD designs there exists a (1,1) correspondence among the elements of the intersection structural matrix S_t^I, the group structural matrix S_t^G, and the characteristic matrix C_t. This correspondence is given by

$$C_t = rk(rk-v\lambda_2)[(r-\lambda_1)I_t-S_t^I] + rk(\lambda_1-\lambda_2)S_t^G + \lambda_2k^2(r-\lambda_1)E_t ,$$

where I_t is the identity matrix of order t, E_t is the singular

$t \times t$ matrix all of whose elements are unity, and the other quantities are scalars.

For the particular case when $r = k$, the value of $|N_1N_1'|$ as given by (4.3) reduces to

$$(4.5) \qquad |N_1N_1'| = r^{-2(t-1)}(r-\lambda_1)^{v-t-m}(r^2-v\lambda_2)^{m-t-1}|C_t| \; ,$$

where the typical element of C_t is

$$(4.6) \quad c_{ju} = r^2(r^2-v\lambda_2)(\delta_{ju}+\lambda_2) + r^2(\lambda_1-\lambda_2)\left(\sum_{w=1}^{m} s^w_{jj} s^w_{uu} - n\lambda_2\right).$$

We shall state an analogue of Theorem 3.1 of [2]. The proof is as for that theorem.

<u>Theorem</u> 4.2. If C_t is the characteristic matrix of any set of t blocks chosen from a regular GD design with parameters v,b,r,k,m,n,λ_1, and λ_2, then

(i) $|C_t| \geq 0$ if $t<b-v$,

(ii) $|C_t| = 0$ if $t>b-v$, and

(iii) $r^{-2(t-1)}(r-\lambda_1)^{2v-b-m}(r^2-v\lambda_2)^{m-b+v-1}|C_{b-v}|$

is a perfect integral square.

5. <u>INEQUALITIES ON s_{ju} FOR REGULAR SYMMETRICAL GD DESIGNS</u>. Let $t=1$. Then since the factor outside of $|C_1|$ in (4.5) is positive, it follows from Theorem 4.2 that $|C_1| = 0$. Hence from

(4.6),

$$(5.1)\quad r^2(\lambda_1-\lambda_2)\Big[\sum_{w=1}^{m}(s_{11}^{w})^2 - r^2+v\lambda_2-n\lambda_2\Big] = 0.$$

Since $r^2(\lambda_1-\lambda_2)\neq 0$,

$$(5.2)\qquad \sum_{w=1}^{m}(s_{11}^{w})^2 = r^2-v\lambda_2+n\lambda_2$$

Now let $t = 2$. Since $c_{11} = c_{22} = 0$, it is necessary by Theorem 4.2 that $c_{12} = c_{21} = 0$. Hence from (4.6)

$$(5.3)\qquad s_{12} = \lambda_2 + \frac{e}{(r^2-v\lambda_2)}(\lambda_1-\lambda_2)\ ,$$

where $$e = \sum_{w=1}^{m} s_{11}^{w}s_{22}^{w} - n\lambda_2 .$$

From (5.2) and the observation that $s_{jj}^{w} \geq 0$, $(j=1,2;\ w = 1, \cdots, m)$, it follows that

$$(5.4)\qquad -n\lambda_2 \leq e \leq r^2-v\lambda_2 .$$

From (5.3) and (5.4) we obtain

<u>Theorem</u> 5.1. For a regular symmetrical GD design the number of treatments s_{ju} common to two blocks satisfies the inequalities

$$\lambda_2(r-\lambda_1)\Big/(r^2-v\lambda_2) \leq s_{ju} \leq \lambda_1 ,$$

when $\lambda_1 > \lambda_2$. The inequalities are reversed when $\lambda_1 < \lambda_2$.

6. THE BLOCK STRUCTURE FOR REGULAR SYMMETRICAL GD DESIGNS WHEN $r^2-v\lambda_2$ AND $\lambda_1-\lambda_2$ ARE RELATIVELY PRIME. We need to consider the distribution of the treatments contained in an initial block B_1 among the other blocks. Let n_j be the number of blocks among the remaining (b-1) blocks which has j treatments in common with B_1. Then from the definition of the design we obtain

$$(6.1)\qquad \sum_{j=0}^{k} n_j = b-1 = v-1 \ , \quad \text{and}$$

$$\sum_{j=0}^{k} jn_j = r(k-1) = r(r-1) \ .$$

Also consider $M = \sum_{j=0}^{k} j(j-1)n_j$, which is twice the number of pairs of treatments of B_1 which lie among the other blocks. M is given by

$$(6.2)\qquad M = \sum_{w=1}^{m} s_{11}^{w}(s_{11}^{w}-1)(\lambda_1-1) + \sum_{\substack{x,w=1\\ x\neq w}}^{m} s_{11}^{x}\, s_{11}^{w}(\lambda_2-1) \ .$$

From (3.7) and (5.2), since r=k,

$$(6.3)\qquad \sum_{w=1}^{m} s_{11}^{w}(s_{11}^{w}-1) = (n-1)\lambda_1, \text{ and}$$

$$(6.4)\qquad \sum_{\substack{x,w=1\\ x\neq w}}^{m} s_{11}^{x}s_{11}^{w} = (m-1)n\lambda_2 \ .$$

Hence

$$(6.5)\qquad M = (n-1)(\lambda_1)(\lambda_1-1) + (m-1)(n)(\lambda_2)(\lambda_2-1).$$

Now consider

(6.6) $$B = \sum_{j=0}^{k} (j-\lambda_1)(j-\lambda_2)\, n_j \quad .$$

From (6.1), (6.5), and (6.6) we obtain

(6.7) $$B = 0 \ .$$

Hence the following lemma.

<u>Lemma</u> 6.1. If for a regular symmetrical GD design n_j denotes the number of blocks which have j treatments in common with a given initial block, then

$$B = \sum_{j=0}^{k} n_j (j-\lambda_1)(j-\lambda_2) = 0.$$

Now let $r^2-v\lambda_2$ and $\lambda_1-\lambda_2$ be relatively prime. It follows from (5.3) that s_{12} cannot lie in the open interval (λ_1, λ_2). Then every term of B is positive or zero. But since $B = 0$, every term must be zero. We thus get

<u>Theorem</u> 6.1. If for a regular symmetrical GD design $r^2-v\lambda_2$ and $\lambda_1-\lambda_2$ are relatively prime, then any two blocks have either λ_1 or λ_2 treatments in common.

We further observe that even if $r^2-v\lambda_2$ and $\lambda_1-\lambda_2$ are not relatively prime, it still may not be possible to choose the elements of G_t of (3.8), subject to the restrictions of (3.7) and (5.2), such that s_{ju} is integral but is not λ_1 or λ_2. Consider, for

example, the GD design with parameters v=b=45, r=k=9, m=3, n=15, $\lambda_1=3$, and $\lambda_2=1$. The H.C.F. of $r^2-v\lambda_2$ and $\lambda_1-\lambda_2$ is 2. It is clear that the only positive integers which satisfy (3.7) and (5.2) are 1,1, and 7. But then we must have either $\sum_{w=1}^{m} s_{jj}^{w} s_{uu}^{w}$ = 51 or 15, which correspond respectively to λ_1 and λ_2.

Now assume that the condition of theorem 6.1 is met, or more generally, that positive integers do not exist which meet the restrictions of (3.7), (5.2) and Lemma 6.1 and imply values of s_{ju} other than λ_1 and λ_2. Then from (6.1), we obtain

$$n_{\lambda_1} + n_{\lambda_2} = v-1, \text{ and}$$

(6.8)

$$\lambda_1 n_{\lambda_1} + \lambda_2 n_{\lambda_2} = r(r-1),$$

whence

$$n_{\lambda_1} = n-1, \text{ and}$$

(6.9)

$$n_{\lambda_2} = (m-1)n,$$

so that with respect to any initial block B_1, there are (n-1) other blocks which have λ_1 treatments in common with it, and (m-1)n other blocks which have λ_2 treatments in common with it.

From (5,3) we see that

$$(6.10) \qquad \sum_{w=1}^{m} s_{11}^{w} s_{jj}^{w} = r + (n-1)\lambda_1$$

implies that blocks 1 and j have λ_1 treatments in common, and

conversely. But then from (5.2) and (6.10), it follows that

$$(6.11) \qquad \sum_{w=1}^{m} s^w_{11}\, s^w_{jj} = \sum_{w=1}^{m} (s^w_{11})^2 ,$$

which implies that $s^w_{11} = s^w_{jj}$, $(w=1,\cdots,m;\ j=2,\cdots,b)$. Hence, if blocks B_1 and B_j have λ_1 treatments in common, and blocks B_1 and B_u have λ_1 treatments in common, then B_j and B_u have λ_1 treatments in common. We thus have

Theorem 6.2. If for a regular symmetrical GD design $r^2-v\lambda_2$ and $\lambda_1-\lambda_2$ are relatively prime, then the blocks fall into m groups of n blocks each, which are such that any two blocks from the same group contain λ_1 treatments in common and any two blocks from different groups contain λ_2 treatments in common.

As has been indicated above, this theorem could be stated somewhat more generally.

7. THE SEMI-REGULAR CLASS. For this class it $rk - v\lambda_2 = 0$, and ~~was shown in [1] that $s^w_{jj} = K$, a constant, for all w and j.~~ Hence the above theory does not apply. We shall give a simple example which demonstrates for small v that there do sometimes exist solutions in which $s_{ju} \neq \lambda_1$ or λ_2 for some j and u.

Consider the GD design with parameters $v=b=8$, $r=k=4$, $m=4$, $n=2$, $\lambda_1 = 0$, and $\lambda_2 = 2$. One solution is

$$N^{(1)} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix},$$

which has the property that the blocks break up into 4 groups of 2 blocks each, which are such that two blocks in the same group have zero treatments in common and any two blocks from different groups have 2 treatments in common.

Another solution is

$$N^{(2)} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix},$$

which is such that any initial block has 1 treatment in common with each of three blocks, 2 treatments in common with each of three blocks, and 3 treatments in common with one block.

We now shall obtain inequalities for the number of treatments s_{ju} common to any two blocks of a symmetrical semi-regular GD design. Since for a semi-regular GD design, $rk = v\lambda_2$, it follows that $r - \lambda_1 = n(\lambda_2 - \lambda_1)$, from which we obtain the following lemma.

Lemma 7.1. For a semi-regular GD design, it is necessary that $\lambda_2 > \lambda_1$.

Now let $r = k$. Choose any two blocks and let the columns of N be permuted so that the first two columns correspond to the chosen blocks. Then to N affix a new columns, the w-th of which contains $(\lambda_2 - \lambda_1)^{1/2}$ in the rows which correspond to the treatments of the w-th group, $(w=1, \cdots, m)$, and zero elsewhere. Let the augmented matrix be denoted by N_2. Now form

$$(7.1) \qquad N_3 = \begin{bmatrix} & N_2 & \\ I_2 & & 0 \end{bmatrix},$$

where I_2 is the identity matrix of order 2 and 0 is the $2 \times (b+m-2)$ matrix all of whose elements are zero. Then

$$(7.2) \qquad |N_3 N_3'| = (r+v\lambda_2-\lambda_1)^{-1}(r-\lambda_1)^{v-2}|B_2|,$$

where B_2 is a 2×2 matrix with elements

$$(7.3) \qquad \begin{aligned} b_{11} &= b_{22} = (r+v\lambda_2-\lambda_1)(-\lambda_1) + \lambda_2 r^2, \text{ and} \\ b_{12} &= b_{21} = (r+v\lambda_2-\lambda_1)(-s_{12}) + \lambda_2 r^2 . \end{aligned}$$

As for Theorem 4.2 it is necessary that $|N_3 N_3'| \geq 0$, and since the factor outside of $|B_2|$ in (7.2) is positive, it is necessary that $|B_2| \geq 0$. Hence, the following theorem:

Theorem 7.1. For a symmetrical semi-regular GD design, the number of treatments common to two blocks, s_{ju}, satisfies the inequalities

$$1 \leq s_{ju} \leq \frac{2\lambda_2 r^2}{r+v\lambda_2-\lambda_1} - \lambda_1 \;.$$

I wish to express my thanks ~~for~~ to Professor R. C. Bose for suggesting this problem.

REFERENCES

[1] R. C. Bose and W. S. Connor, "Combinatorial Properties of Group Divisible Incomplete Block Designs", Annals of Mathematical Statistics, submitted for publication.

[2] W. S. Connor, Jr., "On the Structure of Balanced Incomplete Block Designs", Annals of Mathematical Statistics, Vol 23, (1952), pp. 57-71.

THE NATIONAL BUREAU OF STANDARDS

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