SOME RELATIONS AMONG THE BLOCKS OF SYMMETRICAL
GROUP DIVISIBLE DESIGNS

by

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FOREWORD

The combinatorial problem which is studied in this paper is of interest in the design of experiments.

The paper considers the way in which the blocks of a symmetrical Group Divisible incomplete block design, with parameters \( v, r, m, n, \lambda_1, \) and \( \lambda_2 \), are connected by common treatments. Structural and characteristic matrices are defined, and a relation among them is exhibited. It is shown that if a solution exists for a given set of parameters which are such that \( r \neq \lambda_1 \), \( r^2 \neq v\lambda_2 \), and \( r^2 - v\lambda_2 \) and \( \lambda_1 - \lambda_2 \) are relatively prime, then the interchange of blocks with treatments yields a solution which corresponds to the given set of parameters.

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1. SUMMARY. It is well known that if every pair of treatments in a symmetrical Balanced incomplete block design occurs in $\lambda$ blocks, then every two blocks of the design have $\lambda$ treatments in common. In this paper it will be shown that a somewhat similar property holds for symmetrical Group Divisible designs. In the course of the investigation there will be introduced certain matrices which are of intrinsic interest.

2. INTRODUCTION. Some of the combinatorial properties of Group Divisible incomplete block designs were considered in [1]. Here we shall need the definition of Group Divisible designs and the three classes into which they fall. An incomplete block design with $v$ treatments each replicated $r$ times in $b$ blocks of size $k$ is said to be Group Divisible (GD) if the treatments can be divided into $m$ groups, each with $n$ treatments, so that the treatments belonging to the same group occur together in $\lambda_1$ blocks and the treatments belonging to different groups occur together in $\lambda_2$ blocks, $\lambda_1 \neq \lambda_2$.

The three exhaustive and mutually exclusive classes into which the GD designs fall are as follows:

(a) Singular GD designs characterized by $r - \lambda_1 = 0$;

(b) Semi-regular GD designs characterized by $r - \lambda_2 > 0$.

1This work was begun while the author was at the University of North Carolina.
null
\[ rk - v \lambda_2 = 0; \text{ and} \]
\[ rk - v \lambda_k > 0. \]

(c) Regular GD designs characterized by \( r - \lambda_1 > 0, \)
\[ rk - v \lambda_k > 0. \]

In this paper we shall study classes (b) and (c) for the symmetrical case, that is, the case when \( r = k, \) or equivalently, \( b = v. \)

3. THE INCIDENCE AND STRUCTURAL MATRICES. In [2] there was defined the structural matrix for Balanced incomplete block designs. We now shall define the incidence matrix, and two structural matrices for GD designs.

Let us consider first the incidence matrix of a GD design,

\[
(3.1) \quad N = \begin{bmatrix}
    n_{11} & \cdots & n_{1b} \\
    \vdots & \ddots & \vdots \\
    \vdots & \ddots & \vdots \\
    n_{v1} & \cdots & n_{vb}
\end{bmatrix},
\]

where the rows represent treatments, the columns represent blocks, and \( n_{ij} = 1 \) or 0 according as the \( i \)-th treatment does or does not occur in the \( j \)-th block. From the conditions satisfied by the design it is easy to see that

\[
(3.2) \quad \sum_{j=1}^{b} n_{ij} = r, \quad (i = 1, \ldots, v),
\]

and

\[
(3.3) \quad \sum_{j=1}^{b} n_{ij} \lambda_{uj} = \lambda_1 \text{ or } \lambda_2.
\]
according as the $i$-th and $u$-th treatments ($i \neq u$) do belong or do not belong to the same group.

Throughout the paper let us adopt the convention that the treatments $n(w-1) + 1$, $n(w-1) + 2$, $\ldots$, $nw$ shall belong to the $w$-th group, $(w=1, \ldots, m)$. Then

$$
(3.4) \quad MN' = \begin{bmatrix}
A & B & \cdots & B \\
B & A & \cdots & B \\
\vdots & \vdots & \ddots & \vdots \\
B & B & \cdots & A
\end{bmatrix}
$$

where the elements of the $n \times n$ submatrix $A$ are $r$ in the principal diagonal and $\lambda$, elsewhere, and the elements of the $n \times n$ submatrix $B$ are $\lambda_2$ everywhere. Of course $MN'$ contains $v = mn$ rows and columns.

Now choose any $t < b$ blocks of the design. Let the submatrix of $N$ which corresponds to these $t$ blocks be denoted by $N_0$. Let $s$ be the number of treatments common to the $j$-th and $u$-th chosen blocks, $(j, u = 1, 2, \ldots, t)$. Then the $t \times t$ symmetric matrix

$$
(3.5) \quad S_t = N_0'M_0 = (s_{ju})
$$

is defined to be the intersection structural matrix of the $t$ chosen blocks. The $j$-th row or column of $S_t$ corresponds to the
j-th chosen block and the successive elements of the j-th row or column give the number of treatments which this block has in common with the 1st, 2nd, ..., t-th chosen blocks.

We next shall consider another structural matrix. Let $s^w_{ju}$ denote the number of treatments from the w-th group which blocks j and u have in common. Then

$$\sum_{w=1}^{m} s^w_{ju} = S_{ju},$$

and

$$\sum_{w=1}^{m} s^w_{jj} = k.$$

Now consider the matrix

$$(3.9) \quad G_t = \begin{bmatrix} 1 & 1 & \cdots & s_{1t} \\ 1 & 1 & \cdots & 2 \\ 2 & 2 & \cdots & s_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ m & m & \cdots & s_{mt} \end{bmatrix}$$

and the product matrix

$$(3.9) \quad S^G_t = G_t G_t,$$

where the element in the j-th row and u-th column is the sum of
products of the number of treatments which the \( j \)-th chosen block and the \( u \)-th chosen block contain from each group. We define \( S^G_t \) as the group structural matrix of the \( t \) chosen blocks.

4. THE CHARACTERISTIC MATRIX. We shall define an analogue of the characteristic matrix which was developed for Balanced incomplete block designs in [2]. For the remainder of the paper, except for the last section, we shall restrict our attention to the regular CD designs.

Let the columns of \( N \) be permuted so that the first \( t \) columns correspond to the \( t \) chosen blocks. Then let the incidence matrix be extended by adjoining \( t \) new rows, so that the elements of the \( j \)-th adjoined row are zero, except for the \( j \)-th which is unity. We thus get

\[
N_1 = \begin{bmatrix}
N & I_t & 0 \\
I_t & 0 & \\
\end{bmatrix}
\]

where \( I_t \) is the identity matrix of order \( t \), and \( 0 \) is the \( t \times (b-t) \) zero matrix. Then

\[
N_1N_1^t = \begin{bmatrix}
NN^t & N_0 & \\
N_0 & I_t & \\
\end{bmatrix}
\]

The evaluation of \( |N_1N_1^t| \) leads to
(4.3) \[ |N_1 N_1^t| = (\text{rk})^{-t+1}(r-\lambda_1)^{v-t-n}(\text{rk}-\nu \lambda_2)^{m-t-1}|C_t|, \]

where the typical element of \( C_t \) is

\[
(4.4) \quad c_{ju} = (\text{rk}-\nu \lambda_2)(\text{rk} \delta_{ju} + \lambda_2 k^2) + (\lambda_1 - \lambda_2)(\text{rk} \sum_{u=1}^{m} \delta_{jj} s_{uw} - n \lambda_2 k^2),
\]

where \( \delta_{ju} = (r-\lambda_1-k) \) or \(-\delta_{ju}\), according as \( j = u \) or \( j \neq u \).

The matrix \( C_t \) is defined as the characteristic matrix of the \( t \) chosen blocks. The \( j \)-th row or the \( j \)-th column of \( C_t \) corresponds to the \( j \)-th chosen block of the design.

We observe that the characteristic matrix is related to the two structural matrices as is described in the following theorem.

**Theorem 4.1.** For the regular GD designs there exists a \((1,1)\) correspondence among the elements of the intersection structural matrix \( S^I_t \), the group structural matrix \( S^G_t \), and the characteristic matrix \( C_t \). This correspondence is given by

\[
C_t = \text{rk}(\text{rk}-\nu \lambda_2) [(r-\lambda_1)I_t - S^I_t] + \text{rk}(\lambda_1 - \lambda_2)S^G_t + \lambda_2 k^2(r-\lambda_1)E_t,
\]

where \( I_t \) is the identity matrix of order \( t \), \( E_t \) is the singular
\[ t \times t \text{ matrix all of whose elements are unity, and the other quantities are scalars.} \]

For the particular case when \( r = k \), the value of \(|N_1N_1^t|\) as given by (4.4) reduces to

\[
(4.5) \quad |N_1N_1^t| = r^{-2(t-1)}(r-\lambda_1)v^{-m}(r^3-v\lambda_3)^{m-t-1} |C_t| ,
\]

where the typical element of \( C_t \) is

\[
(4.6) \quad c_{ju} = r^2(r^2-v\lambda_3)(c_{ju}\lambda_3) + r^2(\lambda_1-\lambda_2)(\sum_{w=1}^{m}s_w^{w-n}\lambda_3) .
\]

We shall state an analogue of Theorem 3.1 of [2]. The proof is as for that theorem.

**Theorem 4.2.** If \( C_t \) is the characteristic matrix of any set of \( t \) blocks chosen from a regular GD design with parameters \( v, b, r, k, m, n, \lambda_1, \) and \( \lambda_2 \), then

(i) \(|C_t| \geq 0 \) if \( t < b-v \),

(ii) \(|C_t| = 0 \) if \( t > b-v \), and

(iii) \( r^{-2(t-1)}(r-\lambda_1)^{2v-b-m}(r^3-v\lambda_3)^{m-b-v-1} |C_{b-v}| \)

is a perfect integral square.

5. **INEQUALITIES ON \( a_{ju} \) FOR REGULAR SYMMETRICAL GD DESIGNS.**

Let \( t=1 \). Then since the factor outside of \(|C_1|\) in (4.5) is positive, it follows from Theorem 4.2 that \(|C_1| = 0 \). Hence from
\[(4.6)\]

\[r^2(\lambda_1 - \lambda_2) \left[ \sum_{w=1}^{m} (s_{w}^{11})^2 - r^2 - v \lambda_2 - n \lambda_2 \right] = 0.\]

Since \(r^2(\lambda_1 - \lambda_2) \neq 0\),

\[(5.2)\]

\[\sum_{w=1}^{m} (s_{w}^{11})^2 = r^2 + v \lambda_2 + n \lambda_2\]

Now let \(t = 2\). Since \(c_{11} = c_{22} = 0\), it is necessary by Theorem 4.2 that \(c_{12} = c_{21} = 0\). Hence from (4.6)

\[(5.3)\]

\[s_{12} = \lambda_3 + \frac{e}{(r^2 - v \lambda_3)} (\lambda_1 - \lambda_2),\]

where

\[e = \sum_{w=1}^{m} s_{w}^{11} s_{w}^{22} - n \lambda_2.\]

From (5.3) and the observation that \(s_{w}^{ij} \geq 0\), \((j=1,2; w = 1, \ldots, m)\), it follows that

\[(5.4)\]

\[-n \lambda_2 \leq e \leq r^2 - v \lambda_2.\]

From (5.3) and (5.4) we obtain

**Theorem 5.1.** For a regular symmetrical GD design the number of treatments \(s_{ju}\) common to two blocks satisfies the inequalities

\[\lambda_2 \frac{(r - \lambda_1)}{(r^2 - v \lambda_2)} \leq s_{ju} \leq \lambda_1,\]

when \(\lambda_1 > \lambda_2\). The inequalities are reversed when \(\lambda_1 < \lambda_2\).
6. THE BLOCK STRUCTURE FOR REGULAR SYMMETRICAL GD DESIGNS

WHEN \( r^2 - v \lambda_2 \) AND \( \lambda_1 - \lambda_2 \) ARE RELATIVELY PRIME. We need to consider the distribution of the treatments contained in an initial block \( B_1 \) among the other blocks. Let \( n_j \) be the number of blocks among the remaining \( (b-1) \) blocks which has \( j \) treatments in common with \( B_1 \). Then from the definition of the design we obtain

\[
\frac{k}{j} \sum_{j=0}^{n_j} = b - 1 - v - 1, \quad \text{and} \quad (6.1)
\]

\[
\sum_{j=0}^{k} j n_j = r(k-1) - r(r-1).
\]

Also consider \( M = \sum_{j=0}^{k} j(j-1)n_j \), which is twice the number of pairs of treatments of \( B_1 \) which lie among the other blocks. \( M \) is given by

\[
(6.2) \quad M = \sum_{w=1}^{m} s_{11}^w (s_{11}^w - 1) (\lambda_1 - 1) + \sum_{x, w=1}^{m} s_{11}^x s_{11}^w (\lambda_2 - 1).
\]

From (3.7) and (5.2), since \( r = k \),

\[
(6.3) \quad \sum_{w=1}^{m} s_{11}^w (s_{11}^w - 1) = (n-1) \lambda_1, \quad \text{and}
\]

\[
(6.4) \quad \sum_{x, w=1}^{m} s_{11}^x s_{11}^w = (m-1) n \lambda_2.
\]

Hence

\[
(6.5) \quad M = (n-1)(\lambda_1)(\lambda_1 - 1) + (m-1)(n)(\lambda_2)(\lambda_2 - 1).
\]
Now consider

\[(6.6) \quad B = \sum_{j=0}^{k} (j-\lambda_1)(j-\lambda_2) n_j.\]

From (6.1), (6.5), and (6.6) we obtain

\[(6.7) \quad B = 0.\]

Hence the following lemma.

**Lemma 6.1.** If for a regular symmetrical GD design \(n_j\) denotes the number of blocks which have \(j\) treatments in common with a given initial block, then

\[B = \sum_{j=0}^{k} n_j (j-\lambda_1)(j-\lambda_2) = 0.\]

Now let \(r^2-v\lambda_2\) and \(\lambda_1-\lambda_2\) be relatively prime. It follows from (5.3) that \(\lambda_{12}\) cannot lie in the open interval \((\lambda_1, \lambda_2)\). Then every term of \(B\) is positive or zero. But since \(B = 0\), every term must be zero. We thus get

**Theorem 6.1.** If for a regular symmetrical GD design \(r^2-v\lambda_2\) and \(\lambda_1-\lambda_2\) are relatively prime, then any two blocks have either \(\lambda_1\) or \(\lambda_2\) treatments in common.

We further observe that even if \(r^2-v\lambda_2\) and \(\lambda_1-\lambda_2\) are not relatively prime, it still may not be possible to choose the elements of \(G_\lambda\) of (3.8), subject to the restrictions of (3.7) and (3.2), such that \(\mathbf{a}_j\) is integral. Consider, for
example, the GD design with parameters \( v=b=45 \), \( r=k=9 \), \( m=3 \), \( n=15 \), \( \lambda_1=3 \), and \( \lambda_2=1 \). The H.C.F. of \( r^2-v\lambda_3 \) and \( \lambda_1-\lambda_2 \) is 2. It is clear that the only positive integers which satisfy (3.7) and (5.2) are 1, 1, and 7. But then we must have either \( \sum_{j=1}^{n} w_{j}^2 uu = 51 \) or 15, which correspond respectively to \( \lambda_1 \) and \( \lambda_2 \).

Now assume that the condition of theorem 6.1 is met, or more generally, that positive integers do not exist which meet the restrictions of (3.7), (5.2) and Lemma 6.1 and imply values of \( \mu \) other than \( \lambda_1 \) and \( \lambda_2 \). Then from (6.1), we obtain

\[
\begin{align*}
\lambda_1 n &= v-1, \\
\lambda_2 n &= r(r-1),
\end{align*}
\]

whence

\[
\begin{align*}
n \lambda_1 &= n-1, \\
n \lambda_2 &= (n-1)n,
\end{align*}
\]

so that with respect to any initial block \( B_1 \), there are \( (n-1) \) other blocks which have \( \lambda_1 \) treatments in common with it, and \( (n-1)n \) other blocks which have \( \lambda_2 \) treatments in common with it.

From (5.3) we see that

\[
(6.10) \quad \sum_{j=1}^{n} s_{1j} w_{j}^2 = r \cdot (n-1) \lambda_1
\]

implies that blocks 1 and \( j \) have \( \lambda_1 \) treatments in common, and
conversely. But then from (5.2) and (6.10), it follows that

\[(6.11) \quad \sum_{w=1}^{m} s_{11}^{w} s_{jj}^{w} = \sum_{w=1}^{n} (s_{11}^{w})^2,\]

which implies that \(s_{11}^{w} = s_{jj}^{w}, (w=1, \ldots, n; j=2, \ldots, b).\) Hence, if blocks \(B_j\) and \(B_u\) have \(\lambda_1\) treatments in common, and blocks \(B_l\) and \(B_u\) have \(\lambda_1\) treatments in common, then \(B_j\) and \(B_u\) have \(\lambda_1\) treatments in common. We thus have

**Theorem 6.2.** If for a regular symmetrical GD design \(r^3-v\lambda_3\) and \(\lambda_1-\lambda_2\) are relatively prime, then the blocks fall into \(n\) groups of \(n\) blocks each, which are such that any two blocks from the same group contain \(\lambda_1\) treatments in common and any two blocks from different groups contain \(\lambda_2\) treatments in common.

As has been indicated above, this theorem could be stated somewhat more generally.

7. **THE SEMI-REGULAR CLASS.** For this class it was shown in [1] that \(s_{jj}^{w} = K, a constant, for all w and j. Hence the above theory does not apply. We shall give a simple example which demonstrates for small \(v\) that there do sometimes exist solutions in which \(s_{ju} \neq \lambda_1\) or \(\lambda_2\) for some \(j\) and \(u\).

Consider the GD design with parameters \(v=b=8, r=k=4, m=4, n=2, \lambda_1=0, and \lambda_2=2\). One solution is
which has the property that the blocks break up into 4 groups of 3 blocks each, which are such that two blocks in the same group have zero treatments in common and any two blocks from different groups have 3 treatments in common.

Another solution is

\[
\eta(2) = \begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 & 0
\end{bmatrix}
\]

which is such that any initial block has 1 treatment in common with each of the three blocks, 3 treatments in common with each of three blocks, and 3 treatments in common with one block.

We now shall obtain inequalities for the number of treatments \( r \) common to any two blocks of a symmetrical semi-regular CD design. Since for a semi-regular CD design, \( \varphi = \lambda_{\text{sym}} \), it follows that \( r - \lambda_{\text{sym}} = n(\lambda_{\text{sym}} - \lambda_{\text{sym}}) \), from which we obtain the following lemma.
Lemma 7.1. For a semi-regular GD design, it is necessary that $\lambda_2 > \lambda_1$.

Now let $r = k$. Choose any two blocks and let the columns of $N$ be permuted so that the first two columns correspond to the chosen blocks. Then to $N$ affix a new column, the $w$-th of which contains $(\lambda_2 - \lambda_1)^{1/2}$ in the rows which correspond to the treatments of the $w$-th group, $(s=1, \ldots, m)$, and zero elsewhere. Let the augmented matrix be denoted by $N_3$. Now form

$$N_3 = \begin{bmatrix} N_2 \\ I_3 \\ 0 \end{bmatrix},$$

where $I_3$ is the identity matrix of order 3 and 0 is the $2 \times (b-a-3)$ matrix all of whose elements are zero. Then

$$(7.3) \quad |N_3N_3^t| = (r+\lambda_2-\lambda_1)^{-1}(r-\lambda_1)^{v-3}|\delta_2|,$$

where $\delta_2$ is a $2 \times 2$ matrix with elements

$$(7.3) \quad b_{11} = b_{22} = (r+\lambda_2-\lambda_1)(-\lambda_1) + \lambda_2x^2, \quad \text{and}$$

$$b_{12} = b_{21} = (r+\lambda_2-\lambda_1)(-\mu_2) + \lambda_1x^2.$$

As for Theorem 4.2 it is necessary that $|N_3N_3^t| \geq \delta_2$ and since the factor outside of $|\delta_2|$ in (7.2) is positive, it is necessary that $|\delta_2| \geq 0$. Hence, the following theorem:
Theorem 7.1. For a symmetrical semi-regular GD design, the number of treatments common to two blocks, $s_{ju}$, satisfies the inequalities

$$1 \leq s_{ju} \leq \frac{2\lambda_2 a^2}{r \cdot v \lambda_3 - \lambda_1} - \lambda_1.$$

I wish to express my thanks to Professor R. C. Bose for suggesting this problem.
REFERENCES


THE NATIONAL BUREAU OF STANDARDS

Functions and Activities

The National Bureau of Standards is the principal agency of the Federal Government for fundamental and applied research in physics, mathematics, chemistry, and engineering. Its activities range from the determination of physical constants and properties of materials, the development and maintenance of the national standards of measurement in the physical sciences, and the development of methods and instruments of measurement, to the development of special devices for the military and civilian agencies of the Government. The work includes basic and applied research, development, engineering, instrumentation, testing, evaluation, calibration services, and various scientific and technical advisory services. A major portion of the NBS work is performed for other government agencies, particularly the Department of Defense and the Atomic Energy Commission. The functions of the National Bureau of Standards are set forth in the Act of Congress, March 3, 1901, as amended by Congress in Public Law 619, 1950. The scope of activities is suggested in the listing of divisions and sections on the inside of the front cover.

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The results of the Bureau's work take the form of either actual equipment and devices or published papers and reports. Reports are issued to the sponsoring agency of a particular project or program. Published papers appear either in the Bureau's own series of publications or in the journals of professional and scientific societies. The Bureau itself publishes three monthly periodicals, available from the Government Printing Office: the Journal of Research, which presents complete papers reporting technical investigations; the Technical News Bulletin, which presents summary and preliminary reports on work in progress; and Basic Radio Propagation Predictions, which provides data for determining the best frequencies to use for radio communications throughout the world. There are also five series of nonperiodical publications: the Applied Mathematics Series, Circulars, Handbooks, Building Materials and Structures Reports, and Miscellaneous Publications.

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