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# NATIONAL BUREAU OF STANDARDS REPORT

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SOME RELATIONS AMONG THE BLOCKS OF SYMMETRICAL  
GROUP DIVISIBLE DESIGNS

by

W. S. CONNOR



U. S. DEPARTMENT OF COMMERCE  
NATIONAL BUREAU OF STANDARDS



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1163-11-1107

17 April 1952

1662

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**W. S. Connor**



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## FOREWORD

The combinatorial problem which is studied in this paper is of interest in the design of experiments.

The paper considers the way in which the blocks of a symmetrical Group Divisible incomplete block design, with parameters  $v, r, m, n, \lambda_1$ , and  $\lambda_2$ , are connected by common treatments. Structural and characteristic matrices are defined, and a relation among them is exhibited. It is shown that if a solution exists for a given set of parameters which are such that  $r \neq \lambda_1$ ,  $r^2 \neq v\lambda_2$ , and  $r^2 - v\lambda_2$  and  $\lambda_1 - \lambda_2$  are relatively prime, then the interchange of blocks with treatments yields a solution which corresponds to the given set of parameters.

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Chief, National Applied  
Mathematics Laboratories

A. V. Astin  
Acting Director  
National Bureau of Standards



# SOME RELATIONS AMONG THE BLOCKS OF SYMMETRICAL GROUP DIVISIBLE DESIGNS

W. S. Connor  
National Bureau of Standards<sup>1</sup>

1. SUMMARY. It is well known that if every pair of treatments in a symmetrical Balanced incomplete block design occurs in  $\lambda$  blocks, then every two blocks of the design have  $\lambda$  treatments in common. In this paper it will be shown that a somewhat similar property holds for symmetrical Group Divisible designs. In the course of the investigation there will be introduced certain matrices which are of intrinsic interest.

2. INTRODUCTION. Some of the combinatorial properties of Group Divisible incomplete block designs were considered in [1]. Here we shall need the definition of Group Divisible designs and the three classes into which they fall. An incomplete block design with  $v$  treatments each replicated  $r$  times in  $b$  blocks of size  $k$  is said to be Group Divisible (GD) if the treatments can be divided into  $m$  groups, each with  $n$  treatments, so that the treatments belonging to the same group occur together in  $\lambda_1$  blocks and the treatments belonging to different groups occur together in  $\lambda_2$  blocks,  $\lambda_1 \neq \lambda_2$ . The three exhaustive and mutually exclusive classes into which the GD designs fall are as follows:

- (a) Singular GD designs characterized by  $r - \lambda_1 = 0$ ;
- (b) Semi-regular GD designs characterized by  $r - \lambda_1 > 0$ .

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<sup>1</sup>This work was begun while the author was at the University of North Carolina

THE UNIVERSITY OF CHICAGO  
DIVISION OF THE PHYSICAL SCIENCES

DEPARTMENT OF CHEMISTRY

MEMORANDUM FOR THE RECORD

DATE: [illegible]

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FROM: [illegible]

SUBJECT: [illegible]

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$rk - v\lambda_2 = 0$ ; and

(c) Regular GD designs characterized by  $r - \lambda_1 > 0$ ,  
 $rk - v\lambda_2 > 0$ .

In this paper we shall study classes (b) and (c) for the symmetrical case, that is, the case when  $r = k$ , or equivalently,  $b = v$ .

3. THE INCIDENCE AND STRUCTURAL MATRICES. In [2] there was defined the structural matrix for Balanced incomplete block designs. We now shall define the incidence matrix, and two structural matrices for GD designs.

Let us consider first the incidence matrix of a GD design,

$$(3.1) \quad N = \begin{bmatrix} n_{11} & \cdots & n_{1b} \\ \vdots & & \vdots \\ n_{v1} & \cdots & n_{vb} \end{bmatrix},$$

where the rows represent treatments, the columns represent blocks, and  $n_{ij} = 1$  or  $0$  according as the  $i$ -th treatment does or does not occur in the  $j$ -th block. From the conditions satisfied by the design it is easy to see that

$$(3.2) \quad \sum_{j=1}^b n_{ij} = r, \quad (i = 1, \dots, v),$$

and

$$(3.3) \quad \sum_{j=1}^b n_{ij}n_{uj} = \lambda_1 \text{ or } \lambda_2,$$

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according as the  $i$ -th and  $u$ -th treatments ( $i \neq u$ ) do belong or do not belong to the same group.

Throughout the paper let us adopt the convention that the treatments  $n(w-1) + 1, n(w-1) + 2, \dots, nw$  shall belong to the  $w$ -th group, ( $w=1, \dots, m$ ). Then

$$(3.4) \quad NN' = \begin{bmatrix} A & B & \cdot & \cdot & \cdot & B \\ B & A & \cdot & \cdot & \cdot & B \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ B & B & \cdot & \cdot & \cdot & A \end{bmatrix},$$

where the elements of the  $n \times n$  submatrix  $A$  are  $r$  in the principal diagonal and  $\lambda_1$  elsewhere, and the elements of the  $n \times n$  submatrix  $B$  are  $\lambda_2$  everywhere. Of course  $NN'$  contains  $v = mn$  rows and columns.

Now choose any  $t \leq b$  blocks of the design. Let the submatrix of  $N$  which corresponds to these  $t$  blocks be denoted by  $N_0$ . Let  $s$  be the number of treatments common to the  $j$ -th and  $u$ -th chosen blocks, ( $j, u = 1, 2, \dots, t$ ). Then the  $t \times t$  symmetric matrix

$$(3.5) \quad S_t^I = N_0' N_0 = (s_{ju})$$

is defined to be the intersection structural matrix of the  $t$  chosen blocks. The  $j$ -th row or column of  $S_t$  corresponds to the



$j$ -th chosen block and the successive elements of the  $j$ -th row or column give the number of treatments which this block has in common with the 1st, 2nd, ...,  $t$ -th chosen blocks.

We next shall consider another structural matrix. Let  $s_{ju}^w$  denote the number of treatments from the  $w$ -th group which blocks  $j$  and  $u$  have in common. Then

$$(3.6) \quad \sum_{w=1}^n s_{ju}^w = s_{ju} ,$$

and

$$(3.7) \quad \sum_{w=1}^n s_{jj}^w = k .$$

Now consider the matrix

$$(3.8) \quad G_t = \begin{bmatrix} s_{11}^1 & s_{22}^1 & \cdots & s_{tt}^1 \\ s_{11}^2 & s_{22}^2 & \cdots & s_{tt}^2 \\ \vdots & \vdots & \ddots & \vdots \\ s_{11}^n & s_{22}^n & \cdots & s_{tt}^n \end{bmatrix} ,$$

and the product matrix

$$(3.9) \quad S_t^G = G_t' G_t ,$$

where the element in the  $j$ -th row and  $u$ -th column is the sum of

The first part of the document discusses the general principles of the proposed system, which is designed to be flexible and adaptable to various environments. It highlights the importance of user interaction and the role of the system in providing a seamless experience.

$$\frac{1}{x^2} = x^{-2} \implies \frac{d}{dx} x^{-2} = -2x^{-3} = -\frac{2}{x^3}$$

$$\frac{d}{dx} \frac{1}{x^2} = -\frac{2}{x^3}$$

$$\frac{d}{dx} \frac{1}{x^2} = -\frac{2}{x^3}$$

The second part of the document provides a detailed analysis of the system's performance, including a comparison of different configurations and their impact on overall efficiency.

$$\frac{d}{dx} \frac{1}{x^2} = -\frac{2}{x^3}$$

In conclusion, the proposed system offers a robust and scalable solution for the challenges faced in this domain. Further research and development are needed to optimize its performance and expand its capabilities.

products of the number of treatments which the  $j$ -th chosen block and the  $u$ -th chosen block contain from each group. We define  $S_t^G$  as the group structural matrix of the  $t$  chosen blocks.

4. THE CHARACTERISTIC MATRIX. We shall define an analogue of the characteristic matrix which was developed for Balanced incomplete block designs in [2]. For the remainder of the paper, except for the last section, we shall restrict our attention to the regular GD designs.

Let the columns of  $N$  be permuted so that the first  $t$  columns correspond to the  $t$  chosen blocks. Then let the incidence matrix be extended by adjoining  $t$  new rows, so that the elements of the  $j$ -th adjoined row are zero, except for the  $j$ -th which is unity. We thus get

$$(4.1) \quad N_1 = \begin{bmatrix} N & \\ I_t & 0 \end{bmatrix},$$

where ~~which~~  $I_t$  is the identity matrix of order  $t$ , and  $0$  is the  $t \times (b-t)$  zero matrix. Then

$$(4.2) \quad N_1 N_1' = \begin{bmatrix} NN' & N_0 \\ N_0' & I_t \end{bmatrix}$$

The evaluation of  $|N_1 N_1'|$  leads to

The first part of the document discusses the importance of maintaining accurate records of all transactions. It emphasizes that every entry should be supported by a valid receipt or invoice. The text also mentions the need for regular audits to ensure the integrity of the financial data.

In the second part, the author describes the process of reconciling bank statements with the company's internal records. This involves comparing the dates and amounts of transactions to identify any discrepancies. The document provides a detailed explanation of how to handle such situations, including the steps to take when a discrepancy is found.

The final section of the document outlines the procedures for closing the accounting period. It details the necessary steps to ensure that all accounts are properly balanced and that the financial statements are prepared in accordance with the relevant accounting standards. The text concludes with a summary of the key points discussed throughout the document.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 17 \\ 39 \end{bmatrix} \quad (1)$$

where  $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$  is the coefficient matrix,  $\begin{bmatrix} 5 \\ 6 \end{bmatrix}$  is the constant matrix, and  $\begin{bmatrix} 17 \\ 39 \end{bmatrix}$  is the resulting matrix.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 17 \\ 39 \end{bmatrix} \quad (2)$$

The system of equations (1) and (2) can be solved using the following steps:



$$(4.3) \quad |N_1 N_1'| = (rk)^{-t+1} (r - \lambda_1)^{v-t-n} (rk - v\lambda_2)^{n-t-1} |C_t| ,$$

where the typical element of  $C_t$  is

$$(4.4) \quad c_{ju} = (rk - v\lambda_2) (rk \delta_{ju} + \lambda_2 k^2) \\ + (\lambda_1 - \lambda_2) (rk \sum_{w=1}^m s_{jj}^w s_{uu}^w - n \lambda_2 k^2) ,$$

where  $\delta_{ju} = (r - \lambda_1 - k)$  or  $-\delta_{ju}$ , according as  $j = u$  or  $j \neq u$ .

The matrix  $C_t$  is defined as the characteristic matrix of the  $t$  chosen blocks. The  $j$ -th row or the  $j$ -th column of  $C_t$  corresponds to the  $j$ -th chosen block of the design.

We observe that the characteristic matrix is related to the two structural matrices as is described in the following theorem.

Theorem 4.1. For the regular GD designs there exists a (1,1) correspondence among the elements of the intersection structural matrix  $S_t^I$ , the group structural matrix  $S_t^G$ , and the characteristic matrix  $C_t$ . This correspondence is given by

$$C_t = rk (rk - v\lambda_2) [(r - \lambda_1) I_t - S_t^I] + rk (\lambda_1 - \lambda_2) S_t^G \\ + \lambda_2 k^2 (r - \lambda_1) E_t ,$$

where  $I_t$  is the identity matrix of order  $t$ ,  $E_t$  is the singular

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$t \times t$  matrix all of whose elements are unity, and the other quantities are scalars.

For the particular case when  $r = k$ , the value of  $|N_1 N_1^0|$  as given by (4.3) reduces to

$$(4.5) \quad |N_1 N_1^0| = r^{-2(t-1)} (r - \lambda_1)^{v-t-m} (r^2 - v\lambda_2)^{n-t-1} |C_t|.$$

where the typical element of  $C_t$  is

$$(4.6) \quad c_{ju} = r^2 (r^2 - v\lambda_2) (\delta_{ju} + \lambda_2) + r^2 (\lambda_1 - \lambda_2) \left( \sum_{v=1}^n s_{jj}^v s_{uu}^v - n\lambda_2 \right).$$

We shall state an analogue of Theorem 3.1 of [2]. The proof is as for that theorem.

Theorem 4.2. If  $C_t$  is the characteristic matrix of any set of  $t$  blocks chosen from a regular GD design with parameters  $v, b, r, k, m, n, \lambda_1$ , and  $\lambda_2$ , then

- (i)  $|C_t| \geq 0$  if  $t < b - v$ ,
- (ii)  $|C_t| = 0$  if  $t > b - v$ , and
- (iii)  $r^{-2(t-1)} (r - \lambda_1)^{2v-b-m} (r^2 - v\lambda_2)^{n-b+v-1} |C_{b-v}|$

is a perfect integral square.

##### 5. INEQUALITIES ON $s_{ju}$ FOR REGULAR SYMMETRICAL GD DESIGNS.

Let  $t=1$ . Then since the factor outside of  $|C_1|$  in (4.5) is positive, it follows from Theorem 4.2 that  $|C_1| = 0$ . Hence from

1. The first part of the document discusses the importance of maintaining accurate records of all transactions.

2. It is essential to ensure that all entries are supported by appropriate evidence and are clearly dated.

3. The second part of the document outlines the various methods used to collect and analyze data.

4. These methods include both qualitative and quantitative approaches, each with its own strengths and limitations.

5. The third part of the document provides a detailed overview of the statistical techniques employed.

6. These techniques are used to identify trends, patterns, and relationships within the data set.

7. The fourth part of the document discusses the challenges associated with data collection and analysis.

8. These challenges include issues such as data quality, sample size, and the potential for bias.

9. The fifth part of the document offers practical advice on how to overcome these challenges.

10. This advice includes strategies for improving data quality, increasing sample size, and minimizing bias.

11. The sixth part of the document concludes with a summary of the key findings and recommendations.

12. It is hoped that this document will provide a valuable resource for anyone interested in data analysis.

(4.6),

$$(5.1) \quad r^2(\lambda_1 - \lambda_2) \left[ \sum_{w=1}^m (s_{11}^w)^2 - r^2 - v\lambda_2 - n\lambda_2 \right] = 0.$$

Since  $r^2(\lambda_1 - \lambda_2) \neq 0$ ,

$$(5.2) \quad \sum_{w=1}^m (s_{11}^w)^2 = r^2 - v\lambda_2 + n\lambda_2$$

Now let  $t = 2$ . Since  $c_{11} = c_{22} = 0$ , it is necessary by Theorem 4.2 that  $c_{12} = c_{21} = 0$ . Hence from (4.6)

$$(5.3) \quad s_{12} = \lambda_2 + \frac{e}{(r^2 - v\lambda_2)} (\lambda_1 - \lambda_2),$$

where

$$e = \sum_{w=1}^m s_{11}^w s_{22}^w - n\lambda_2.$$

From (5.2) and the observation that  $s_{jj}^w \geq 0$ , ( $j=1,2$ ;  $w=1, \dots, m$ ), it follows that

$$(5.4) \quad -n\lambda_2 \leq e \leq r^2 - v\lambda_2.$$

From (5.3) and (5.4) we obtain

Theorem 5.1. For a regular symmetrical GD design the number of treatments  $s_{ju}$  common to two blocks satisfies the inequalities

$$\lambda_2 (r - \lambda_1) / (r^2 - v\lambda_2) \leq s_{ju} \leq \lambda_1,$$

when  $\lambda_1 > \lambda_2$ . The inequalities are reversed when  $\lambda_1 < \lambda_2$ .

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6. THE BLOCK STRUCTURE FOR REGULAR SYMMETRICAL GD DESIGNS WHEN  $r^2 - v\lambda_2$  AND  $\lambda_1 - \lambda_2$  ARE RELATIVELY PRIME. We need to consider the distribution of the treatments contained in an initial block  $B_1$  among the other blocks. Let  $n_j$  be the number of blocks among the remaining  $(b-1)$  blocks which has  $j$  treatments in common with  $B_1$ . Then from the definition of the design we obtain

$$(6.1) \quad \sum_{j=0}^k n_j = b-1 = v-1, \text{ and}$$

$$\sum_{j=0}^k j n_j = r(k-1) = r(r-1).$$

Also consider  $M = \sum_{j=0}^k j(j-1)n_j$ , which is twice the number of pairs of treatments of  $B_1$  which lie among the other blocks.

$M$  is given by

$$(6.2) \quad M = \sum_{v=1}^n s_{11}^v (s_{11}^v - 1) (\lambda_1 - 1) + \sum_{\substack{x, w=1 \\ x \neq w}}^n s_{11}^x s_{11}^w (\lambda_2 - 1).$$

From (3.7) and (5.2), since  $r=k$ ,

$$(6.3) \quad \sum_{v=1}^n s_{11}^v (s_{11}^v - 1) = (n-1)\lambda_1, \text{ and}$$

$$(6.4) \quad \sum_{\substack{x, w=1 \\ x \neq w}}^n s_{11}^x s_{11}^w = (n-1)n\lambda_2.$$

Hence

$$(6.5) \quad M = (n-1)(\lambda_1)(\lambda_1 - 1) + (n-1)(n)(\lambda_2)(\lambda_2 - 1).$$

The first step in the derivation of the wave equation is to consider the forces acting on a small element of length  $\Delta x$ . The forces are the tension forces  $T$  at each end and the weight force  $w \Delta x$ . The tension forces are directed along the string, and the weight force is directed vertically downwards. The net force in the horizontal direction is zero, and the net force in the vertical direction is  $T \sin \theta_2 - T \sin \theta_1 - w \Delta x$ .

$$\begin{aligned}
 \sum F_x &= T \cos \theta_2 - T \cos \theta_1 = 0 \\
 \sum F_y &= T \sin \theta_2 - T \sin \theta_1 - w \Delta x = 0
 \end{aligned}$$

For small angles,  $\sin \theta \approx \theta$  and  $\cos \theta \approx 1$ . The horizontal force equation is satisfied. The vertical force equation becomes  $T(\theta_2 - \theta_1) - w \Delta x = 0$ . The slope of the string is  $\theta = \frac{\partial y}{\partial x}$ , so  $\theta_2 - \theta_1 = \frac{\partial^2 y}{\partial x^2} \Delta x$ .

$$T \frac{\partial^2 y}{\partial x^2} \Delta x - w \Delta x = 0 \implies T \frac{\partial^2 y}{\partial x^2} = w$$

This is the wave equation for a string under tension.

$$\frac{\partial^2 y}{\partial x^2} = \frac{w}{T}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 y}{\partial t^2}$$



Now consider

$$(6.6) \quad B = \sum_{j=0}^k (j - \lambda_1)(j - \lambda_2) n_j .$$

From (6.1), (6.5), and (6.6) we obtain

$$(6.7) \quad B = 0 .$$

Hence the following lemma.

Lemma 6.1. If for a regular symmetrical GD design  $n_j$  denotes the number of blocks which have  $j$  treatments in common with a given initial block, then

$$B = \sum_{j=0}^k n_j (j - \lambda_1)(j - \lambda_2) = 0 .$$

Now let  $r^2 - v\lambda_2$  and  $\lambda_1 - \lambda_2$  be relatively prime. It follows from (5.3) that  $s_{12}$  cannot lie in the open interval  $(\lambda_1, \lambda_2)$ . Then every term of  $B$  is positive or zero. But since  $B = 0$ , every term must be zero. We thus get

Theorem 6.1. If for a regular symmetrical GD design  $r^2 - v\lambda_2$  and  $\lambda_1 - \lambda_2$  are relatively prime, then any two blocks have either  $\lambda_1$  or  $\lambda_2$  treatments in common.

We further observe that even if  $r^2 - v\lambda_2$  and  $\lambda_1 - \lambda_2$  are not relatively prime, it still may not be possible to choose the elements of  $G_t$  of (3.8), subject to the restrictions of (3.7) and (5.2), such that  $s_{ju}$  is integral, *but is not  $\lambda_1$  or  $\lambda_2$ .* Consider, for

$$y = \frac{1}{2}x^2 - 3x + \frac{1}{2}$$

Find the vertex of the parabola.

$$x = 3, y = -\frac{5}{2}$$

Write the equation of the parabola in vertex form.

$$y - \frac{1}{2}x^2 + 3x - \frac{1}{2} = 0$$

Complete the square for the x terms. Add the same amount to both sides of the equation.

$$y - \frac{1}{2}x^2 + 3x - \frac{1}{2} + \frac{9}{2} = \frac{9}{2}$$

$$y - \frac{1}{2}x^2 + 3x + \frac{9}{2} - \frac{1}{2} = \frac{9}{2} + \frac{1}{2}$$

Simplify the equation. Add the same amount to both sides of the equation.

$$y - \frac{1}{2}x^2 + 3x + \frac{9}{2} - \frac{1}{2} + \frac{1}{2} = \frac{9}{2} + \frac{1}{2} + \frac{1}{2}$$

$$y - \frac{1}{2}x^2 + 3x + \frac{9}{2} = \frac{11}{2}$$

$$y - \frac{1}{2}x^2 + 3x + \frac{9}{2} - \frac{9}{2} = \frac{11}{2} - \frac{9}{2}$$

$$y - \frac{1}{2}x^2 + 3x = \frac{2}{2}$$

$$y - \frac{1}{2}x^2 + 3x - 3x = \frac{2}{2} - 3x$$

$$y - \frac{1}{2}x^2 = \frac{2}{2} - 3x$$

$$y - \frac{1}{2}x^2 + \frac{1}{2}x^2 = \frac{2}{2} - 3x + \frac{1}{2}x^2$$

$$y = \frac{1}{2}x^2 - 3x + \frac{1}{2}$$

$$y = \frac{1}{2}(x - 3)^2 - \frac{5}{2}$$

example, the GD design with parameters  $v=b=15$ ,  $r=k=9$ ,  $m=3$ ,  $n=15$ ,  $\lambda_1=3$ , and  $\lambda_2=1$ . The H.C.F. of  $r^2-v\lambda_2$  and  $\lambda_1-\lambda_2$  is 2. It is clear that the only positive integers which satisfy (3.7) and (5.2) are 1, 1, and 7. But then we must have either  $\sum_{j=1}^m s_{jj}^w s_{uu}^w = 51$  or 15, which correspond respectively to  $\lambda_1$  and  $\lambda_2$ .

Now assume that the condition of theorem 6.1 is met, or more generally, that positive integers do not exist which meet the restrictions of (3.7), (5.2) and Lemma 6.1 and imply values of  $s_{ju}$  other than  $\lambda_1$  and  $\lambda_2$ . Then from (6.1), we obtain

$$(6.8) \quad \begin{aligned} n \lambda_1 + n \lambda_2 &= v-1, \text{ and} \\ \lambda_1 n \lambda_1 + \lambda_2 n \lambda_2 &= r(r-1), \end{aligned}$$

whence

$$(6.9) \quad \begin{aligned} n \lambda_1 &= n-1, \text{ and} \\ n \lambda_2 &= (n-1)n, \end{aligned}$$

so that with respect to any initial block  $B_1$ , there are  $(n-1)$  other blocks which have  $\lambda_1$  treatments in common with it, and  $(n-1)n$  other blocks which have  $\lambda_2$  treatments in common with it.

From (5.3) we see that

$$(6.10) \quad \sum_{j=1}^m s_{11}^w s_{jj}^w = r + (n-1)\lambda_1$$

implies that blocks  $i$  and  $j$  have  $\lambda_1$  treatments in common, and

The first part of the paper discusses the general theory of the
 subject. It is shown that the general theory of the subject
 is based on the following principles:

$$\begin{aligned}
 & \text{The first principle is that the} \\
 & \text{general theory of the subject} \\
 & \text{is based on the following} \\
 & \text{principles:} \\
 & \text{The second principle is that} \\
 & \text{the general theory of the} \\
 & \text{subject is based on the} \\
 & \text{following principles:} \\
 & \text{The third principle is that} \\
 & \text{the general theory of the} \\
 & \text{subject is based on the} \\
 & \text{following principles:}
 \end{aligned}$$

The second part of the paper discusses the application of the
 general theory to the special case of the subject. It is shown
 that the general theory of the subject can be applied to the
 special case of the subject in the following way:

conversely. But then from (5.2) and (6.10), it follows that

$$(6.11) \quad \sum_{w=1}^n s_{11}^w s_{jj}^w = \sum_{w=1}^n (s_{11}^w)^2,$$

which implies that  $s_{11}^w = s_{jj}^w$ , ( $w=1, \dots, n$ ;  $j=2, \dots, b$ ). Hence, if blocks  $B_1$  and  $B_j$  have  $\lambda_1$  treatments in common, and blocks  $B_1$  and  $B_u$  have  $\lambda_1$  treatments in common, then  $B_j$  and  $B_u$  have  $\lambda_1$  treatments in common. We thus have

Theorem 6.2. If for a regular symmetrical GD design  $r^2 - v\lambda_2$  and  $\lambda_1 - \lambda_2$  are relatively prime, then the blocks fall into  $n$  groups of  $n$  blocks each, which are such that any two blocks from the same group contain  $\lambda_1$  treatments in common and any two blocks from different groups contain  $\lambda_2$  treatments in common.

As has been indicated above, this theorem could be stated somewhat more generally.

7. THE SEMI-REGULAR CLASS. For this class it ~~was shown~~  <sup>$rk - v\lambda_2 = 0$ , and</sup> in [1] that  ~~$s_{jj}^w = k$ , a constant, for all  $w$  and  $j$ .~~ Hence the above theory does not apply. We shall give a simple example which demonstrates for small  $v$  that there do sometimes exist solutions in which  $s_{ju} \neq \lambda_1$  or  $\lambda_2$  for some  $j$  and  $u$ .

Consider the GD design with parameters  $v=b=8$ ,  $r=k=4$ ,  $n=4$ ,  $n=2$ ,  $\lambda_1 = 0$ , and  $\lambda_2 = 2$ . One solution is

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$$\frac{1}{2} \left( \frac{1}{2} + \frac{1}{2} \right) = \frac{1}{2}$$

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$$N^{(1)} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

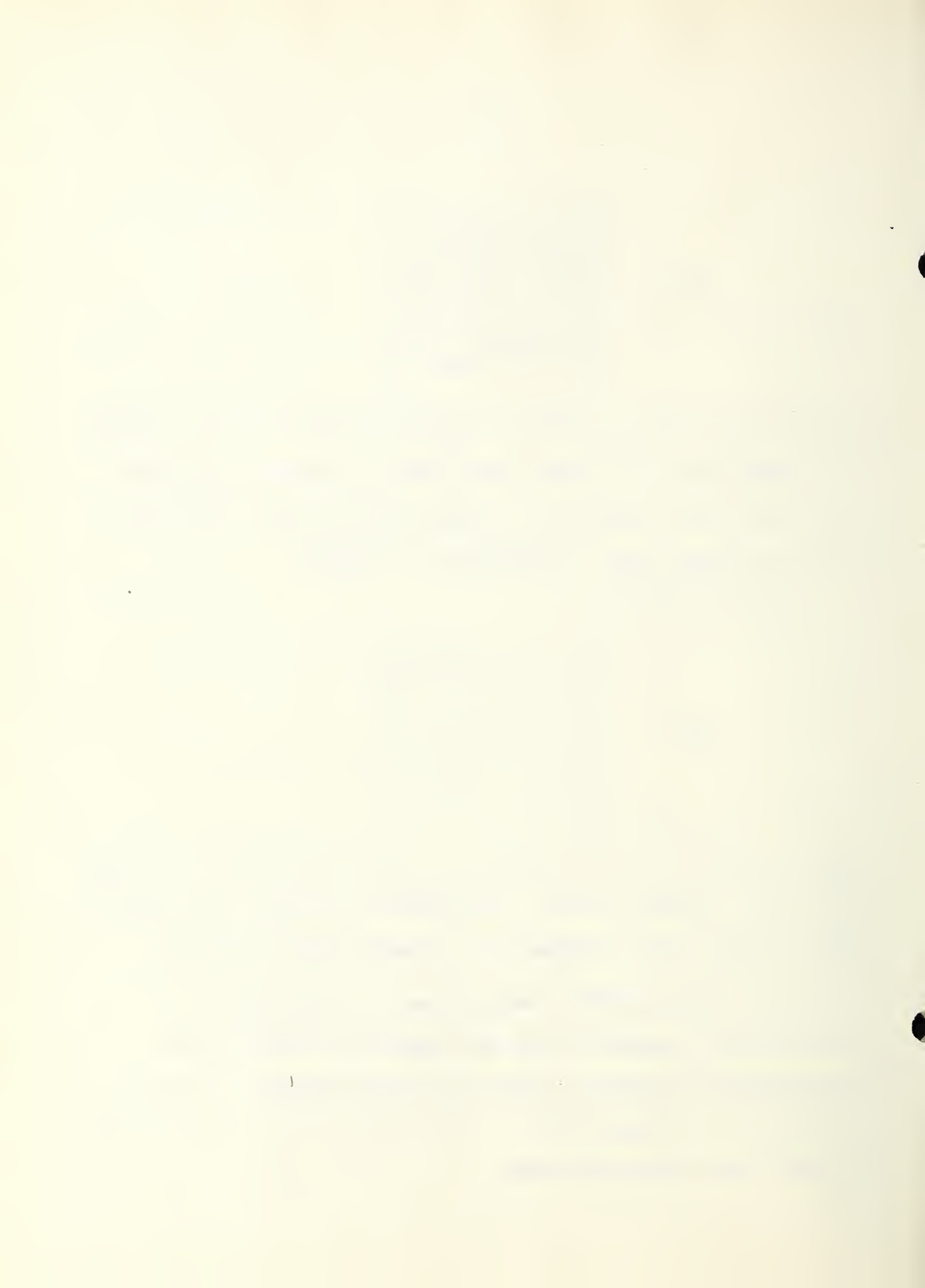
which has the property that the blocks break up into 4 groups of 2 blocks each, which are such that two blocks in the same group have zero treatments in common and any two blocks from different groups have 2 treatments in common.

Another solution is

$$N^{(2)} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

which is such that any initial block has 1 treatment in common with each of three blocks, 2 treatments in common with each of three blocks, and 3 treatments in common with one block.

We now shall obtain inequalities for the number of treatments  $s_{ju}$  common to any two blocks of a symmetrical semi-regular GD design. Since for a semi-regular GD design,  $rk = r\lambda_2$ , it follows that  $r - \lambda_1 = n(\lambda_2 - \lambda_1)$ , from which we obtain the following lemma.





Lemma 7.1. For a semi-regular CD design, it is necessary that  $\lambda_2 > \lambda_1$ .

Now let  $r = k$ . Choose any two blocks and let the columns of  $N$  be permuted so that the first two columns correspond to the chosen blocks. Then to  $N$  affix a new column, the  $v$ -th of which contains  $(\lambda_2 - \lambda_1)^{1/2}$  in the rows which correspond to the treatments of the  $v$ -th group, ( $v=1, \dots, m$ ), and zero elsewhere. Let the augmented matrix be denoted by  $N_2$ . Now form

$$(7.1) \quad N_2 = \begin{bmatrix} & N_2 \\ I_2 & 0 \end{bmatrix},$$

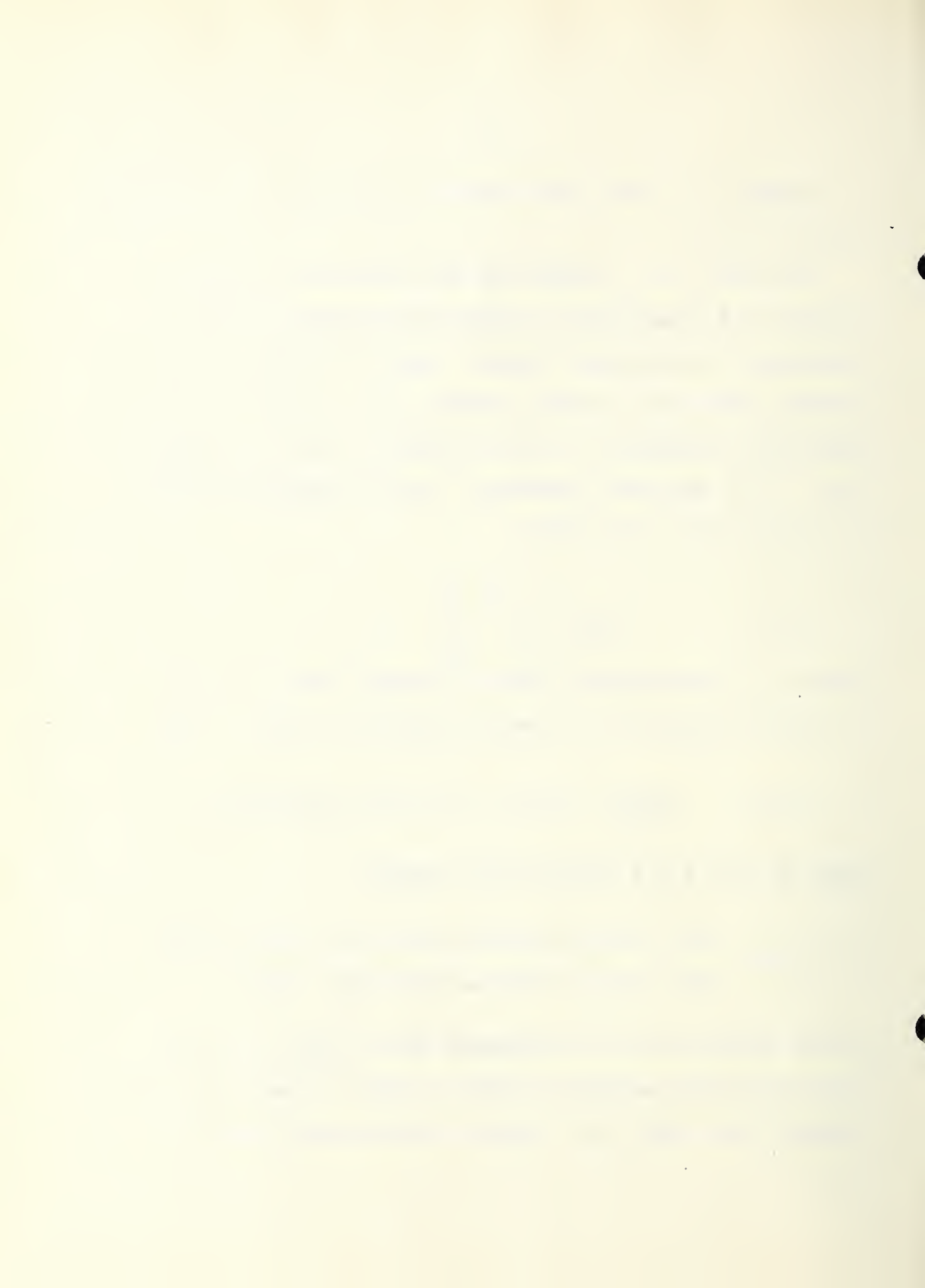
where  $I_2$  is the identity matrix of order 2 and 0 is the  $2 \times (b+2-3)$  matrix all of whose elements are zero. Then

$$(7.2) \quad |N_2 N_2'| = (r+v\lambda_2 - \lambda_1)^{-1} (r - \lambda_1)^{v-3} |B_2|,$$

where  $B_2$  is a  $2 \times 2$  matrix with elements

$$(7.3) \quad \begin{aligned} b_{11} &= b_{22} = (r+v\lambda_2 - \lambda_1)(-\lambda_1) + \lambda_2 r^2, \text{ and} \\ b_{12} &= b_{21} = (r+v\lambda_2 - \lambda_1)(-B_{12}) + \lambda_1 r^2. \end{aligned}$$

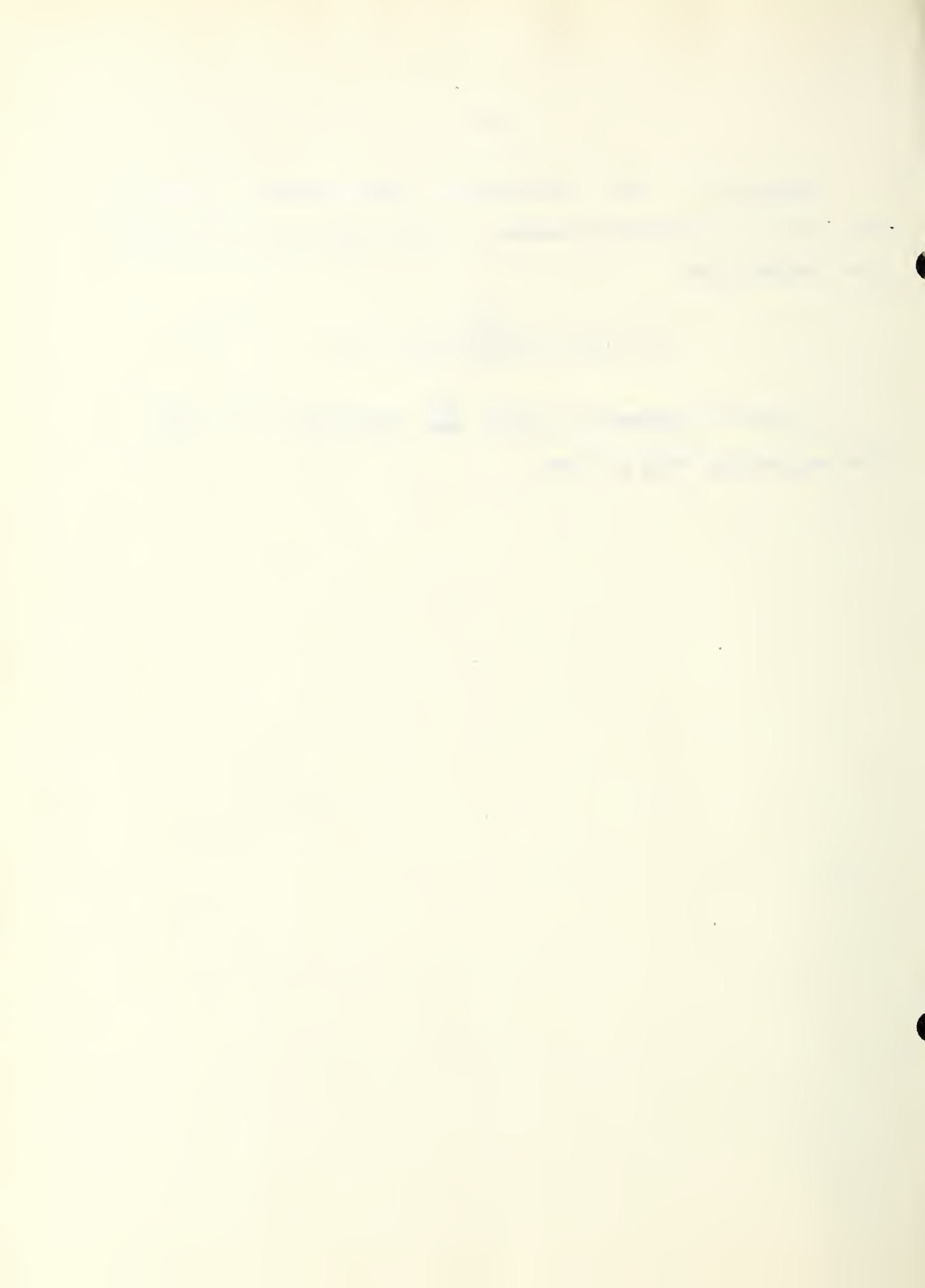
As for Theorem 4.2 it is necessary that  $|N_2 N_2'| \geq 0$ , and since the factor outside of  $|B_2|$  in (7.2) is positive, it is necessary that  $|B_2| \geq 0$ . Hence, the following theorem:



Theorem 7.1. For a symmetrical semi-regular GD design, the number of treatments common to two blocks,  $s_{ju}$ , satisfies the inequalities

$$1 \leq s_{ju} \leq \frac{2\lambda_2 r^2}{r+v\lambda_2 - \lambda_1} - \lambda_1 .$$

I wish to express my thanks ~~for~~<sup>to</sup> Professor R. C. Bose for suggesting this problem.



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