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Explicit Formulae for the Distribution Function of the Sums of n Uniformly Distributed Variables

by

A. M. Ostrowski

American University

April, 1952

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Explicit Formulae for the Distribution Function of the Sums of n uniformly Distributed Variables

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1. Let

(1) $$\xi_1, \ldots, \xi_n$$

be n variables subject to the conditions

(2) $$|\xi_\nu| \leqq \alpha_\nu, \quad \alpha_\nu > 0 \quad (\nu = 1, \ldots, n)$$

and uniformly distributed in the corresponding intervals. The <u>distribution function</u> $F(\sigma)$ of the sum

(3) $$\xi_1 + \ldots + \xi_n ,$$

that is the probability for the inequality

(4) $$\xi_1 + \ldots + \xi_n \leqq \sigma ,$$

has been, as far as I know, given explicitly only in the particularly simple case $\alpha_1 = \alpha_2 = \ldots = \alpha_n$ by Laplace[1]. Indeed its expression becomes extremely cumbersome in the general case if one does not use a convenient symbolism.

2. We use the following notations. By κ_+ we denote

(5) $$\kappa_+ = \frac{\kappa + |\kappa|}{2} = \begin{cases} \kappa, & \kappa \geqq 0 \\ 0, & \kappa < 0 \end{cases} .$$

Further we will use the "displacement operator" S^{η} defined by

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1 Laplace, collected papers vol.7, pp 257-260; cf also J.V.Uspensky, Introduction to Mathematical Probabilities 1937, pp 277-278 and the example 8-10 on page 280; H.Cramer, Mathematical Methods of Statistics 1940, pp 244-246; M.G.Kendall, The Advanced Theory of Statistics, vol I, third edition, 1947, pp.140-145; Feller, Introduction to Probability Theory and its Applications, vol.I, Chap.11, pp.235-236, problems 11. [illegible]

(6) $$S_M^{\tau} f = S^{\tau} f(M) = f(M-\tau).$$

Then our expression for $F(\sigma)$ is

(7) $$F(\sigma) = \frac{1}{2^n n!} \frac{1}{\alpha_1 \ldots \alpha_n} \prod_{\nu=1}^{n} (1 - S^{2\alpha_\nu})(\alpha + \sigma)_+^n, \quad (\alpha = \alpha_1 + \ldots + \alpha_n).$$ [2)]

3. In our proof of (7) we consider, as did Laplace (l.c), the corresponding discrete probability problem. Let for a certain integer m which will tend to infinity, x_ν be the rational fractions

(8) $$x_\nu = \frac{r_\nu}{m}, \quad |r_\nu| \leq m_\nu$$

where the integers r_ν are "uniformly distributed" between $-m_\nu$ and m_ν. We obtain the expressions $F_m(\sigma)$ of the probability for the inequality

(9) $$r_1 + \ldots + r_n \leq \sigma m$$

and get finally $F(\sigma)$ as $\lim F_m(\sigma)$ if $m \to \infty$, $m_\nu \to \infty$ so that

(10) $$\frac{m_\nu}{m} \to \alpha_\nu .$$

4. However when the above notations are used, the standard method of complex integration gives, as was pointed out to me by

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2. The corresponding frequency function is

(7') $$F'(\sigma) = \frac{1}{2^n (n-1)!} \frac{1}{\alpha_1 \ldots \alpha_n} \prod_{\nu=1}^{n} (1 - S^{2\alpha_\nu})(\alpha + \sigma)_+^{n-1};$$

this follows at once from the relations

(7'') $$\frac{d}{d\sigma} (\alpha + \sigma)_+^n = n (\alpha + \sigma)_+^{n-1}, \quad (n = 2, 3, \ldots)$$

(7''') $$\int_a^\beta (\alpha + \sigma)_+^n d\sigma = \frac{1}{n+1} \left(S_\alpha^{-\beta} - S_\alpha^{-a} \right) \alpha_+^{n+1}, \quad (n = 1, 2, \ldots)$$

which ore immediately verified.

Dr. J. H. Curtiss, a very simple proof of (7).

5. Another approach to our problem which goes back probably to R. A. Fisher, 1915, is a geometric one. Denote by P the parallelopiped (2) and by P_σ the common part of P with the half plane (4), then we have

(11) $$F(\sigma) = \frac{|P_\sigma|}{|P|}$$

where $|P| = 2^n \alpha_1 \ldots \alpha_n$ is the volume of P and $|P_\sigma|$ is the volume of P_σ. Already in the Laplacian case Kendall (l.c) has used this representation of $F(\sigma)$ and derived from this geometric interpretation another proof of Laplace's formula in using a pretty involved geometric argument. It was pointed out to me by Dr. A. J. Hoffman that in using the expression (11) the formula (7) can be easily verified, and Dr. Hoffman's proof is probably the simplest proof of the formula (7).

6. However it seems that the standard statistical method of complex integral is not easily applied to the following problem of a more special character. Consider the n variables (1) subject to the conditions

(12) $$\max_{\nu=1,\ldots,n} \frac{|\xi_\nu|}{\alpha_\nu} = 1 \quad ;$$

then we want to determine "the probability" $F^*(\sigma)$ for the inequality (4). However the problem is in this form from the point of view of continuous probabilities not even completely determined. It is therefore better to state the corresponding problem concerning discrete probabilities. We consider again for a positive integer m a set of n variables (8) where the integer numerators r_ν run "uniformly distributed" between $-m_\nu$ and m_ν. Let $F^*_m(\sigma)$ be the probability of the inequality (9) under the additional condition

(13) $$\max_{\nu=1,\dots n} \frac{|x_\nu|}{m_\nu} = 1 \;;$$

then $F^*(\sigma)$ is defined as

(14) $$F^*(\sigma) = \lim_{m\to\infty} F^*_m(\sigma) \qquad \left(\frac{m_\nu}{m} \to \alpha_\nu\right).$$

We obtain in this way

(15) $$F^*(\sigma) = \frac{1}{2^n(n-1)!}\,\frac{1}{\alpha_1\cdots\alpha_n\left(\frac{1}{\alpha_1}+\dots+\frac{1}{\alpha_n}\right)}\sum_{\mu=1}^{n}\frac{1+S^{2\alpha_\mu}}{1-S^{2\alpha_\mu}}\prod_{\nu=1}^{n}\left(1-S^{2\alpha_\nu}\right)(\alpha+\sigma)_+^{n-1}.$$

7. This expression can be interpreted in the following way. Let $F^{(\mu)}(\sigma)$ be the probability of the inequality (4) under the conditions (12) and the additional condition $|\xi_\mu| = \alpha_\mu$; in assuming that $\xi_\mu = +\alpha_\mu$ and $\xi_\mu = -\alpha_\mu$ are equally probable we obtain from the formula (7) easily the expression

(16) $$F^{(\mu)}(\sigma) = \frac{\alpha_\mu}{2^n(n-1)!\,\alpha_1\cdots\alpha_n}\,\frac{1+S^{2\alpha_\mu}}{1-S^{2\alpha_\mu}}\prod_{\nu=1}^{n}\left(1+S^{2\alpha_\nu}\right)(\alpha+\sigma)_+^{n-1}$$

and (15) can be written in the form

(17) $$F^*(\sigma) = \sum_{\mu=1}^{n}\omega_\mu F^{(\mu)}(\sigma)\,,$$

(18) $$\omega_\mu = \frac{1/\alpha_\mu}{\frac{1}{\alpha_1}+\dots+\frac{1}{\alpha_n}}\,.$$

8. It might therefore seem that (15) can be obtained by straight forward application of the elementary probability theorems if each ω_μ is interpreted as the probability for the equality sign in (2) being attained just for $v = \mu$. However the "weight coefficients" ω_μ cannot be interpreted in this way since the probability that the equality sign in (2) is assumed at least for both $v = 1$ and $v = 2$ would then be $\geq \omega_1\omega_2$ while the corresponding terms in the expression of $F^*(\sigma)$ obviously tend to zero with $m \to \infty$.

9. On the other hand the geometric interpretation in this case is still possible, but not quite obvious. The natural interpretation would be in terms of the (n-1) dimensional surface areas of P and P_σ. However, different (n-1)-dimensional "faces" of these surfaces would have to be taken with the weights, proportional to their areas.

As a matter of fact, it was the problem of the determination of the probabilities $F^*_m(\sigma)$ which arose in the course of a discussion of a certain modification of the relaxation method and was the starting point of this whole investigation. For our immediate purpose the $F^*(\sigma)$ has only the meaning of the limiting values of $F^*_m(\sigma)$.

10. It may further be mentioned that from (11) it follows immediately that $F(\sigma)$ is for $-\alpha \leq \sigma \leq \alpha$ a <u>strictly monotonically increasing</u> function without stationary intervals. The same then follows for the expression (16) and since the weights (18) are constants, we see that $F^*(\sigma)$ is a <u>strictly monotonically increasing</u> function of σ for $-\alpha \leq \sigma \leq \alpha$.

11. Since all essential points of our original proof of (7) occur also in the proof of (15), I do not give explicitly my proof of (7), but reproduce in section 12 with his kind permission the proof due to Dr. A. J. Hoffman and in the sections 13-22 give my proof of (15). It is believed that the devices used in this proof can be employed in other similar problems.

12. As we have obviously in the notation of no. 5 $|P| = 2^n \alpha_1 \cdots \alpha_n$, (7) follows at once if we only show

$$(19) \qquad |P_\sigma| = \frac{1}{n!} \prod_{\nu=1}^{n} (1 - 5^{2\alpha_\nu})(\alpha + \sigma)_+^n \qquad (\alpha = \alpha_1 + \cdots + \alpha_n).$$

For $n = 1$ the right hand side is

$$(20) \qquad (\sigma+\alpha_1)_+ - (\sigma-\alpha_1)_+$$

and in considering separately the cases $\sigma \geqq \alpha_1$, $0 \leqq \sigma < \alpha_1$, $-\alpha_1 < \sigma < 0$ and $\sigma \leqq -\alpha_1$ we see that (20) in each case gives the length of the part of $\langle -\alpha_1, \alpha_1 \rangle$ with $x \leqq \sigma$. We can therefore proceed by induction and assume that (19) is true for $(n-1)$ dimensions. Consider the section $P_\sigma^{(u)}$ of P_σ with the $(n-1)$-dimensional plane $x_n = u$; it is obviously given by the corresponding coordinates $x_1, \ldots, x_{n-1}$ which satisfy then the conditions

$$(20) \qquad |x_1| \leqq \alpha_1, \ldots\ldots, |x_{n-1}| \leqq \alpha_{n-1}, \qquad x_1 + \ldots\ldots + x_{n-1} \leqq \sigma - u;$$

therefore in applying (19) to $P_\sigma^{(u)}$ we have for its $(n-1)$-dimensional volume

$$(21) \qquad |P_\sigma^{(u)}| = \frac{1}{(n-1)!} \prod_{\nu=1}^{n-1} (1 - S^{2\alpha_\nu})(\alpha - \alpha_n + \sigma - u)_+^{n-1}$$

and obtain now

$$(22) \qquad |P_\sigma| = \int_{-\alpha_n}^{\alpha_n} |P_\sigma^{(u)}| \, du = \frac{1}{(n-1)!} \prod_{\nu=1}^{n} (1 - S^{2\alpha_\nu}) \int_{-\alpha_n}^{\alpha_n} (\alpha - \alpha_n + \sigma - u)_+^{n-1} du$$

but the integral in the right hand expression is equal by (7) to

$$-\frac{1}{n}\left[(\alpha - \alpha_n \; \sigma - \alpha_n)_+ - (\alpha - \alpha_n + \sigma + \alpha_n)_+^n\right] = \frac{1}{n}\left(1 - S_\sigma^{2\alpha_n}\right)(\alpha + \sigma)_+^n .$$

13. In what follows we will have to use the "generalized binomial coefficients"

$$(23) \qquad \begin{bmatrix} \gamma \\ n \end{bmatrix} = \begin{cases} \binom{\gamma}{n}, & \gamma \geqq 0 \\ 0, & \gamma \leqq 0 \end{cases} \qquad n = 1, 2, 3, \ldots .$$

The systematic use of this notation together with the symbolism introduced in no. 2 appears to be very convenient in many combinatorial discussions. Its use would also simplify considerably Laplace's argument (l.c.) for the case $\alpha_1 = \alpha_2 = \ldots = \alpha_n$. We have for any integer $k \geq 2$ and any integer γ

$$(24) \qquad \left[\begin{matrix}\gamma\\ k\end{matrix}\right] - \left[\begin{matrix}\gamma-1\\ k\end{matrix}\right] = \left[\begin{matrix}\gamma-1\\ k-1\end{matrix}\right], \qquad (k = 2, 3, \ldots)$$

as follows at once from (23) for $\gamma \geq 1$ and $\gamma \leq 0$.

In applying (24) repeatedly we have for two integers k_1, k_2, and any integer $k \geq 2$

$$(25) \qquad \sum_{\nu=k_1}^{k_2} \left[\begin{matrix}\nu\\ k-1\end{matrix}\right] = \left[\begin{matrix}k_2+1\\ k\end{matrix}\right] - \left[\begin{matrix}k_1\\ k\end{matrix}\right] = \left(1 - S^{k_2-k_1+1}\right)\left[\begin{matrix}k_2+1\\ k\end{matrix}\right].$$

14. Suppose now that we have for $m \to \infty$ and a variable integer b

$$(26) \qquad \frac{b}{m} \to \beta \qquad (m \to \infty),$$

then

$$(27) \qquad \left[\begin{matrix}b\\ n\end{matrix}\right] \Big/ \frac{m^n}{n!} \to \beta_+^n \qquad (n = 1, 2, \ldots).$$

Indeed if $\beta \leq 0$, then $\frac{1}{m^n}\left[\begin{matrix}b\\ n\end{matrix}\right]$ is either zero or

$$(28) \qquad \frac{1}{n!}\,\frac{b}{m} \cdots \frac{b-n+1}{m} \to 0 .$$

And if $\beta > 0$, the left side in (28) tends to $\frac{\beta^n}{n!}$. More generally if we assume in addition to (26) that for a variable integer ℓ, $\ell/m \to \lambda$, then we have from (27)

(29) $\quad S^{\ell}\left[{k \atop n}\right] / \frac{m^n}{n!} \longrightarrow (\sigma-\lambda)_+^n = S^{\lambda}\beta_+^n \quad , \left(\frac{k}{m}\to\sigma, \frac{\ell}{m}\to\lambda\right).$

15. We have with the notation (23)

(30) $$(1-x)^{-n} = \sum_{\nu=-\infty}^{\infty}\left[{\nu+n-1 \atop n-1}\right] x^{\nu} \qquad (n = 1,2,\ldots).$$

Indeed the right side expression is by (23)

$$\sum_{\nu=0}^{\infty}\binom{\nu+n-1}{n-1}x^{\nu} + \sum_{\nu=-(n-1)}^{-1}\binom{\nu+n-1}{n-1}x^{\nu} =$$
$$= (1-x)^{-n} + x^{1-n}\sum_{\mu=0}^{n-2}\binom{\mu}{n-1}x^{\mu} = (1-x)^{-n} .$$

16. If we have a development

(31) $$P(x) = \sum_{\kappa=-\infty}^{\infty} N(\kappa)x^{\kappa}$$

we have for any integer γ

$$x^{\gamma}P(x) = \sum_{\kappa=-\infty}^{\infty} N(\kappa)\,x^{\kappa+\gamma} = \sum_{\lambda=-\infty}^{\infty} N(\lambda-\gamma)\,x^{\lambda} = \sum_{\kappa=-\infty}^{\infty} S_{\kappa}^{\gamma}\,N(\kappa)\,x^{\kappa}$$

and more generally for any polynomial in x

(32) $$\phi(x)P(x) = \sum_{\kappa=-\infty}^{\infty}\phi(S_{\kappa})N(\kappa)x^{\kappa} .$$

17. We consider now n integers ($n \geqq 2$) $m_1, m_2, \ldots, m_n$ and denote by C_k the number of representations of the integer k as the sum

(33) $\quad (C_k) \quad k = x_1 + \ldots + x_n \quad (|x_1| \leqq m_1, \ldots, |x_n| \leqq m_n)$

of n integers x_ν satisfying the indicated conditions. Then we can write ($M = m_1 + \ldots + m_n$):

$$\sum_{\kappa=-\infty}^{\infty} C_{\kappa}x^{\kappa} = \left(x^{-m_1} + x^{-m_1+1} + \ldots + x^{m_1}\right)\ldots\left(x^{-m_n} + x^{-m_n+1} + \ldots + x^{m_n}\right) =$$
$$= x^{-M}\prod_{\nu=1}^{n}\left(1 + x + \ldots + x^{2m_\nu}\right) = x^{-M}\prod_{\nu=1}^{n}\left(\frac{1-x^{2m_\nu+1}}{1-x}\right) ,$$

$$(34)\quad \sum_{k=-\infty}^{\infty} C_k x^k = x^{-M}(1-x)^n \prod_{\nu=1}^{n} (1-x^{2m_\nu+1});$$

the right side expression is by (30) and (32)

$$x^{-M}\prod_{\nu=1}^{n}(1-x^{2m_\nu+1})\sum_{k=-\infty}^{\infty}\begin{bmatrix}k+n-1\\ n-1\end{bmatrix}x^k =$$

$$= \sum_{k=-\infty}^{\infty} S_k^{-M}\prod_{\nu=1}^{n}(1-S_k^{2m_\nu+1})\begin{bmatrix}k+n-1\\ n-1\end{bmatrix}x^k;$$

then we obtain

$$(35)\quad C_k = \prod_{\nu=1}^{n}(1-S_k^{2m_\nu+1})\begin{bmatrix}M+k+n-1\\ n-1\end{bmatrix} \equiv \prod_{\nu=1}^{n}(1-S_M^{2m_\nu+1})\begin{bmatrix}M+k+n-1\\ n-1\end{bmatrix}.$$

18. From (35) we have again for two integers k_1, k_2, $k_1 < k_2$:

$$\sum_{k=k_1}^{k_2} C_k = \sum_{k=k_1}^{k_2}\prod_{\nu=1}^{n}(1-S_M^{2m_\nu+1})\begin{bmatrix}M+k+n-1\\ n-1\end{bmatrix} = \prod_{\nu=1}^{n}(1-S_M^{2m_\nu+1})\sum_{k=k_1}^{k_2}\begin{bmatrix}M+k+n-1\\ n-1\end{bmatrix}$$

and since the right hand sum is by (25)

$$\begin{bmatrix}M+k_2+n\\ n\end{bmatrix}-\begin{bmatrix}M+k_1+n-1\\ n\end{bmatrix} = (1-S_M^{k_2-k_1+1})\begin{bmatrix}M+k_2+n\\ n\end{bmatrix};$$

$$(36)\quad \sum_{k=k_1}^{k_2} C_k = (1-S_M^{k_2-k_1+1})\prod_{\nu=1}^{n}(1-S_M^{2m_\nu+1})\begin{bmatrix}M+k_2+n\\ n\end{bmatrix}.$$

19. We denote now by C'_k the number of representations of k as the sum of n integers x_ν satisfying the conditions $|x_\nu| < m_\nu$ $(\nu = 1,\dots,n)$. Obviously we obtain C'_k from C_k in replacing each m_ν by $m_\nu - 1$ (and M by M-n), then we obtain from (35) and (36) easily

$$C'_k = \prod_{\nu=1}^{n}(1-S_M^{2m_\nu-1})\begin{bmatrix}M+k-1\\ n-1\end{bmatrix};$$

$$(37)\quad \sum_{k=k_1}^{k_2} C'_k = (1-S_M^{k_2-k_1+1})\prod_{\nu=1}^{n}(1-S_M^{2m_\nu-1})\begin{bmatrix}M+k_2\\ n\end{bmatrix}.$$

If we denote now by D_k the number of representations of k as the sum of n integers x_ν satisfying the condition

$$(38) \qquad \operatorname*{Max}_{\nu} \frac{|x_\nu|}{m_\nu} = 1$$

we have obviously $D_k = C_k - C'_k$ and from (36) and (37)

$$(39) \quad \sum_{k_1}^{k_2} D_k = \left(1 - S_M^{k_2 - k_1 + 1}\right)\left\{\prod_{\nu=1}^{n}\left(1 - S_M^{2m_\nu + 1}\right) - S_M^n \prod_{\nu=1}^{n}\left(1 - S_M^{2m_\nu - 1}\right)\right\}\begin{bmatrix} M + k_2 + n \\ n \end{bmatrix}.$$

20. In order to obtain now the probability $F^*_m(\sigma)$ of no. 6 we obtain the number of all "favorable cases" in replacing in the right hand expression of (39) k_1 by an arbitrary negative number $\leq -M$ and k_2 by the greater integer $k = [\sigma m]$ contained in σm. Then we obtain

$$(40) \quad N_m(\sigma) = \left\{\prod_{\nu=1}^{n}\left(1 - S^{2m_\nu + 1}\right) - S^n \prod_{\nu=1}^{n}\left(1 - S^{2m_\nu - 1}\right)\right\}\begin{bmatrix} M + k + n \\ n \end{bmatrix}.$$

The number of "all possible cases" is obviously

$$(41) \qquad \Delta_m = \prod_{\nu=1}^{n}(2m_\nu + 1) - \prod_{\nu=1}^{n}(2m_\nu - 1);$$

by dividing we obtain

$$(42) \quad F^*_m(\sigma) = \frac{N_m(\sigma)}{\Delta_m} = \frac{1}{\prod_{\nu=1}^{n}(2m_\nu + 1) - \prod_{\nu=1}^{n}(2m_\nu - 1)}\left\{\prod_{\nu=1}^{n}\left(1 - S_M^{2m_\nu + 1}\right) - S_M^n \prod_{\nu=1}^{n}\left(1 - S_M^{2m_\nu - 1}\right)\right\}\begin{bmatrix} M + k + n \\ n \end{bmatrix}$$

We have obtained now the solution of the discrete probability problem formulated in no. 6.

21. The expression $N_m(\sigma)$ can be brought into a form which is useful as well for a geometric interpretation as for the passage to the limit. We can obviously write for the expression between the braces in (40)

$$(43)\quad \{\ \} = \sum_{\mu=1}^{n} \prod_{\nu=1}^{\mu-1} (1-s^{2m_\nu+1})\, s^{n-\mu} \prod_{\nu=\mu+1}^{n} (1-s^{2m_\nu-1}) \left\{(1-s^{2m_\mu+1}) - s(1-s^{2m_\mu-1})\right\}$$

but we have identically

$$(1-s^{2m_\mu+1}) - s(1-s^{2m_\mu-1}) = (1+s^{2m_\mu})(1-s)$$

and (43) becomes

$$(44)\quad \{\ \} = \sum_{\mu=1}^{n} (1+s^{2m_\mu}) \prod_{\nu=1}^{\mu-1} (1-s^{2m_\nu+1}) \prod_{\nu=\mu+1}^{n} (1-s^{2m_\nu-1})\, s^{n-\mu}(1-s)$$

on the other hand we have by (24)

$$s^{n-\mu}(1-s)\begin{bmatrix} M+K+n \\ n \end{bmatrix} = \begin{bmatrix} M+K+\mu-1 \\ n-1 \end{bmatrix}$$

and so (40) finally becomes

$$(45)\quad N_m(\sigma) = \sum_{\mu=1}^{n} (1+s^{2m_\mu}) \prod_{\nu=1}^{\mu-1} (1-s^{2m_\nu+1}) \prod_{\nu=\mu+1}^{n} (1-s^{2m_\nu-1}) \begin{bmatrix} M+K+\mu-1 \\ n-1 \end{bmatrix}$$

22. Assume now

$$(46)\quad \frac{m_\nu}{m} \to \alpha_\nu, \quad \frac{M}{m} \to \alpha_1 + \dots + \alpha_n \equiv \alpha, \quad m \to \infty.$$

To obtain now the limit of the expression (42) we divide both $N_m(\sigma)$ and Δ_m by m^{n-1}. For the denominator we obtain

$$\frac{\Delta_m}{m^{n-1}} = m\left\{\prod_{\nu=1}^{n}\left(2\frac{m_\nu}{m} + \frac{1}{m}\right) - \prod_{\nu=1}^{n}\left(2\frac{m_\nu}{m} - \frac{1}{m}\right)\right\}$$

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* This transformation is obtained in specializing in the appropriate way the identity

$$\prod_{\nu=1}^{n} A_\nu - \prod_{\nu=1}^{n} B_\nu = \sum_{\mu=1}^{n}\left\{\prod_{\nu=1}^{\mu} A_\nu \prod_{\nu=\mu+1}^{n} B_\nu - \prod_{\nu=1}^{\mu-1} A_\nu \prod_{\nu=\mu}^{n} B_\nu\right\} =$$

$$= \sum_{\mu=1}^{n} \prod_{\nu=1}^{\mu-1} A_\nu \prod_{\nu=\mu+1}^{n} B_\nu (A_\mu - B_\mu).$$

and this tends to

$$(47) \qquad 2^n \alpha_1 \ldots \alpha_n \left(\frac{1}{\alpha_1} + \ldots + \frac{1}{\alpha_n}\right) .$$

2. . In the numerator we have

$$(48) \qquad \frac{N_m(\sigma)}{m^{n-1}} = \sum_{\mu=1}^{n} \frac{1}{m^{n-1}} \left(1 + S^{2m_\mu}\right) \prod_{\nu=1}^{\mu-1} \left(1 - S^{2m_\nu+1}\right) \prod_{\nu=\mu+1}^{n} \left(1 - S^{2m_\nu-1}\right) \begin{bmatrix} M+K+\mu- \\ n \end{bmatrix}$$

in developing the term corresponding to a value of μ we obtain

$$(49) \qquad \frac{1}{m^{n-1}} \left(1 + S^{2m_\mu}\right) \prod_{\nu=1}^{\mu-1} \left(1 - S^{2m_\nu+1}\right) \prod_{\nu=\mu+1}^{n} \left(1 - S^{2m_\nu-1}\right) \begin{bmatrix} M+K+\mu-1 \\ n-1 \end{bmatrix} =$$

$$= \sum \frac{\pm 1}{m^{n-1}} S^{2(m_\mu + m_\nu + \ldots) + a} \begin{bmatrix} M+K+\mu-1 \\ n- \end{bmatrix} ,$$

where a is an integer, $-n < a < n$. The general term of the right hand side converges by the formulae (29) and (46) to

$$\pm \frac{1}{(n-1)!} S^{2(\alpha_\mu + \alpha_\nu + \ldots)} (\alpha + \sigma)_+^{n-1}$$

and the limit of (49) is therefore

$$\frac{1}{(n-1)!} \sum \pm S^{2(\alpha_\mu + \alpha_\nu + \ldots)} (\alpha+\sigma)_+^{n-1} = \frac{1}{(n-1)!} \left(1 + S^{2\alpha_\mu}\right) \prod_{\nu=1}^{\mu-1} \left(1 - S^{2\alpha_\nu}\right) \prod_{\nu=\mu+1}^{n} \left(1 - S^{2\alpha_\nu}\right) (\alpha+\sigma)_+^{n-1}$$

we obtain finally

$$(50) \qquad \frac{N_m(\sigma)}{m^{n-1}} \rightarrow \frac{1}{(n-1)!} \sum_{\mu=1}^{n} \frac{1 + S^{2\alpha_\mu}}{1 - S^{2\alpha_\mu}} \prod_{\nu=1}^{n} \left(1 - S^{2\alpha_\nu}\right) (\alpha + \sigma)_+^{n-1} .$$

The formula (15) follows immediately.

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