NATIONAL BUREAU OF STANDARDS REPORT

1565

CONFIDENCE AND TOLERANCE INTERVALS

FOR THE NORMAL DISTRIBUTION

by

Frank Proschan

U. S. DEPARTMENT OF COMMERCE
NATIONAL BUREAU OF STANDARDS

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NBS PROJECT
3011-60-0002

3 April 1952

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FOREWORD

This is the text of an invited address, "On intervals of the form $\bar{X} \pm ks$," presented by F. Proschan of the Statistical Engineering Laboratory (Section 3 of Division 11, Applied Mathematics) at the 111th Annual Meeting of the American Statistical Association, Boston, Massachusetts, 28 December 1951. It will appear in published form at a later date in the Journal of the American Statistical Association.

Confidence and tolerance intervals for the normal distribution are presented for the various cases of known and unknown mean and standard deviation. Practical illustration and interpretation of these intervals are given. Tables are presented permitting a comparison among the intervals. Finally, the relationship between the two types of intervals is described.

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CONFIDENCE AND TOLERANCE INTERVALS
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Frank Proschan

1. Introduction. Discussions of the theory of errors will sometimes state that the mean $\pm$ the probable error will include 50 percent of future observations (assumed normally distributed). This, of course, is true only if the mean and the probable error of the population itself are used. Unfortunately, in most practical problems, one or both of those may not be known. Experimenters who use the sample mean $\pm$ the sample probable error with the expectation that this interval will contain 50 percent of future observations may be seriously deluding themselves.

However it is possible to construct intervals of the type $\bar{\bar{x}} \pm ks$ ($\bar{\bar{x}}$ = sample mean, s = sample standard deviation) which will, on the average, include 50 percent of the population. From this, one is led to a more general consideration of such intervals, and to the uses to which they can be put.

2. Summary. All populations discussed in this paper are normal unless otherwise specified. Let μ, σ refer to the population mean and standard deviation respectively.

Any one of four possible situations may exist: (a) μ, σ both known; (b) μ unknown, σ known; (c) μ known, σ unknown; (d) μ, σ both unknown.

Let m represent either μ or $\bar{\bar{x}}$; let s.d. represent either

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σ or s. Then two important types of assertions may be made about intervals of the form

$$m \mp k \text{ s.d.} \qquad (1)$$

A. Confidence Interval. The probability is γ that the interval (1) contains the population mean (or alternately, the second sample mean).

B. Tolerance Interval. In repeated samples, the proportion, p, of the population contained in (1) is

B.1) a, on the average.

B.2) P, or more, γ of the time.

In this paper, a comparison is made among the values of k appropriate to the respective cases obtained from various combinations of A and B with (a), (b), (c), and (d). Practical illustrations and interpretations are given of these cases.

In addition, details of a proof are given of a result by Wilks (1941) for the case B.1. These details are given because they are suggestive of a general method applicable in such problems. Also a table is presented of values of k for combination B.1(d) where $E(p) = a$ ($a = .50, .75, .90, .95, .99, .999$) and sample size $n = 2(1)30, 40, 60, 120, \infty$.

Finally the relationship between confidence intervals and tolerance intervals is discussed.

3. Confidence Intervals. A chemist makes n determinations of the iron content of a solution. What interval shall he select so that he can assert with 50 percent confidence that the "true" value μ lies within that interval? The distribution of

observations is normal with mean μ).

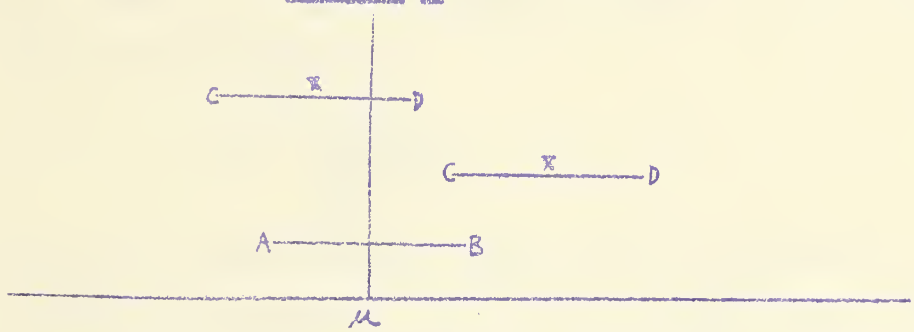
<u>3.1 For the Population Mean, σ Known.</u> First, consider the case where he knows σ. (The determination is of a routine type, for which a great many sets of previous observations are available, from which σ is calculated). In this case

$$\bar{x} \pm \frac{.6745}{\sqrt{n}} \sigma \qquad (2)$$

will contain the "true" value (population mean) 50 percent of the time.

This may be seen from the following diagram:

<u>Figure 1</u>



Suppose AB = the interval $\mu \pm \frac{.6745}{\sqrt{n}} \sigma$. Then, since $\bar{x}$ is normally distributed with mean μ, standard deviation $\frac{\sigma}{\sqrt{n}}$, the probability is .50 that $\bar{x}$ will be in $\mu \pm \frac{.6745}{\sqrt{n}} \sigma$. Notice however, that for the interval $\mu \pm \frac{.6745}{\sqrt{n}} \sigma$ to contain $\bar{x}$ is exactly equivalent to the interval CD, $\bar{x} \pm \frac{.6745}{\sqrt{n}} \sigma$, containing μ. Hence, the probability is .50 that $\bar{x} \pm \frac{.6745}{\sqrt{n}} \sigma$ will contain μ.

Values of $k_1 = \frac{.6745}{\sqrt{n}}$ for $n = 2(1)30, 40, 60, 120, \infty$, are presented in table 1, column 1.

To generalize, when the confidence coefficient is γ (instead of .50), the confidence interval for the population mean is

$$\bar{x} \pm \frac{L_{1-\gamma}}{\sqrt{n}} \sigma \tag{3}$$

where

$$\int_{-L_{1-\gamma}}^{L_{1-\gamma}} \frac{1}{\sqrt{2}} e^{-\frac{t^2}{2}} dt = \gamma \quad . \tag{4}$$

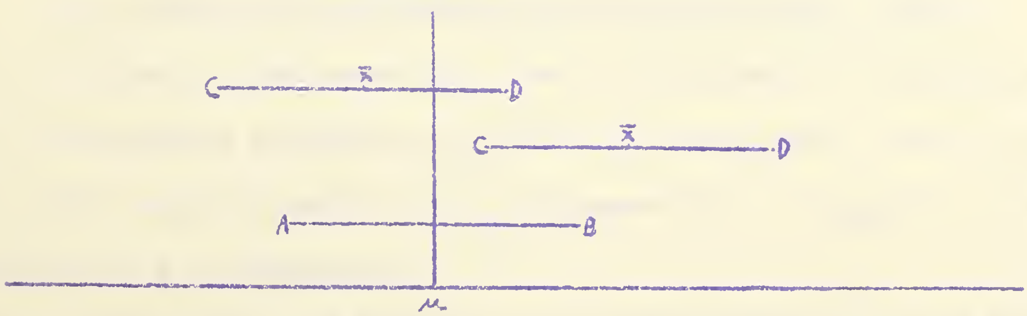
<u>3.2 For the Population Mean, σ Unknown.</u> Consider, now, the case where the only information about σ is in the present sample. Then the interval

$$\bar{x} \pm \frac{t_{.50,n-1}}{\sqrt{n}} s \tag{5}$$

(where $t_{.50,n-1}$ is the Student-t value for n-1 degrees of freedom which is exceeded in absolute value, with probability .50) will, 50 percent of the time, contain the population mean.

The following diagram demonstrates this.

<u>Figure 2</u>



Lay off AB: $\mu \pm \frac{t_{.50,n-1}}{\sqrt{n}} s$ and CD: $\bar{\bar{x}} \pm \frac{t_{.50,n-1}}{\sqrt{n}} s$.

Notice that when $\bar{\bar{x}}$ lies in AB, μ must of necessity lie in CD; and when $\bar{\bar{x}}$ does not lie in AB, μ must fall outside of CD. But the probability of

$$\mu - \frac{t_{.50,n-1}}{\sqrt{n}} s \leq \bar{\bar{x}} \leq \mu + \frac{t_{.50,n-1}}{\sqrt{n}} s \qquad (6)$$

is .50 since $\frac{\bar{\bar{x}} - \mu}{\frac{s}{\sqrt{n}}}$ is distributed as Student's t. Hence the probability that

$$\bar{\bar{x}} - \frac{t_{.50,n-1}}{\sqrt{n}} s \leq \mu \leq \bar{\bar{x}} + \frac{t_{.50,n-1}}{\sqrt{n}} s \qquad (7)$$

is .50. Values of $k_2 = \frac{t_{.50,n-1}}{\sqrt{n}}$ for $n = 2(1)30, 40, 60, 120, \infty$, are presented in table 1, column 2. Comparison of k_1 and k_2 shows $k_2 > k_1$, but as $n \rightarrow \infty$, $k_2 \rightarrow k_1$.

To generalize, when the confidence coefficient is γ (instead of .50), the confidence interval becomes

$$\mu - \frac{t_{1-\gamma,n-1}}{\sqrt{n}} s \leq \bar{\bar{x}} \leq \mu + \frac{t_{1-\gamma,n-1}}{\sqrt{n}} s \qquad (8)$$

3.3 Confidence Interval for Second Sample Mean.

Suppose the chemist who made the iron determinations wishes to set up a confidence interval, not for the true mean, but for the mean $\bar{\bar{x}}_2$, of a second sample of n_2 observations. Suppose as in paragraph 3.2, σ is unknown.

Let us now call the mean of the first sample $\bar{\bar{x}}_1$, and the

size of the first sample n_1. We may set up the statistic

$$t = \frac{\bar{\bar{x}}_1 - \bar{\bar{x}}_2}{s_1\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \qquad (9)$$

The numerator is a normally distributed variable, while the denominator is an independent estimate of the standard deviation of the numerator. Hence the ratio, t, is distributed as Student's t. It follows that the interval

$$\bar{\bar{x}}_1 \pm t_{.50,n_1-1} \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}\ s_1 \qquad (10)$$

will constitute a 50 percent confidence interval for $\bar{\bar{x}}_2$. [1]

What does this mean? It simply means that if pairs of samples of size n_1 and n_2 respectively, with means $\bar{\bar{x}}_{1i}$ and $\bar{\bar{x}}_{2i}$ respectively $(i = 1,2,\ldots,\infty)$, are drawn repeatedly, then for 50 percent of these pairs $\bar{\bar{x}}_{2i}$ will lie in

$$\bar{\bar{x}}_{1i} \pm t_{.50,n_1-1}\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}\ s_1 \qquad (11)$$

It does <u>not</u> mean that if one first sample of size n_1 with mean $\bar{\bar{x}}_1$ is drawn, to be followed by the drawing of a great many "second" samples of size n_2, with means $\bar{\bar{x}}_{2i}$ $(i = 1,2,\ldots,\infty)$, that for 50 percent of the "second" samples $\bar{\bar{x}}_{2i}$ will lie in (11).

When $n_2 = n_1$, the coefficient of s_1 in (11) becomes

$$k_3 = t_{.50,n_1-1}\sqrt{\frac{2}{n_1}} \qquad (12)$$

Values of k_3 for $n_1 = 2(1)30,40,60,120,\infty$ are given in table 1, column 3, for purposes of comparison. Note that $k_3 = \sqrt{2}k_2$, simply.

To generalize (10), if the confidence coefficient is γ (instead of .50), (10) becomes

$$\bar{\bar{x}}_1 \pm t_{1-\gamma, n_1-1} \sqrt{\frac{1}{n_1} + \frac{1}{n_1}}\ s_1 \qquad (13)$$

4. <u>Tolerance Intervals.</u> In paragraph 3, an interval (1) was formed to contain the population mean (with a certain probability). Suppose, now, we are interested in setting up an interval (1) which will contain a certain <u>proportion, p, of the population.</u> Such an interval is known as a tolerance interval.

If either μ or σ is unknown, then the interval (1), containing $\bar{\bar{x}}$ or s, is a random variable. Hence the proportion, p, contained in (1) will be a random variable.

<u>4.1 Expected value of p.</u> In 4.1 we determine k so that in repeated sampling E(p) = a, a constant. In 4.2 we determine k so that in a large series of samples from normal universes a certain proportion γ of the intervals (1) will include p or more of the universe.

<u>4.1.1 μ, σ Known.</u> In this case

$$\mu \pm k\,\sigma \qquad (14)$$

may be used as the tolerance interval. The proportion p contained in (14) is constant, and the appropriate value for specified p may be obtained from a table of normal areas. Thus for p = .50, k = .6745 (listed in table 1, column 4, for purposes of comparison).

<u>4.1.2 μ, σ Unknown.</u> Unfortunately in most practical problems μ and σ are not known. Hence $\bar{\bar{x}}$ and s must

be used. How shall be determine k so that in a large series of samples from a normal universe, the average p contained in $\bar{x}_i \pm ks_i$ $(i = 1,2,\ldots,\infty)$ will be a?

A solution was given by Wilks in [8] without giving details of the proof. (For an independent derivation see appendix.) Stated explicitly, let

$$p = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{\bar{x}-ks}^{\bar{x}+ks} e^{-\frac{1}{2}(x-\mu)^2/\sigma^2}\, dx \tag{15}$$

Then

$$E(p) = \int_{-\infty}^{\infty} \int_{0}^{\infty} p\, f(\bar{x},s)\, ds\, d\bar{x} = \frac{\Gamma(n/2)}{\sqrt{\pi(n-1)}\,\Gamma\left(\frac{n-1}{2}\right)} \int_{-t}^{t} \frac{dx}{\left(1+\frac{x^2}{n-1}\right)^{n/2}} \tag{16}$$

$$t = k\sqrt{\frac{n}{n+1}}$$

where $f(\bar{x},s)$ is the joint distribution of $\bar{x}$ and s:

$$f(\bar{x},s) = \frac{\sqrt{n}\,(n-1)^{\frac{n-1}{2}}\, s^{n-2}}{2^{n/2-1}\sigma^n\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)}\, e^{-\frac{\frac{1}{2}[n(\bar{x}-\mu)^2 + (n-1)s^2]}{\sigma^2}} \tag{17}$$

In other words, the tolerance limits which will include, on the average (for repeated samples), a proportion, a, of the normal universe are

$$\bar{x} \pm t_{1-a,n-1}\sqrt{\frac{n+1}{n}}\, s \tag{18}$$

where $t_{1-a,n-1}$ is the value of t for which the integral in (16) is equal to a. Hence

$$k = t_{1-a,n-1}\sqrt{\frac{n+1}{n}} \tag{19}$$

Values of k for $n = 2(1)30, 40, 60, 120, \infty$, and for $a =$ .50, .75, .90, .95, .99, .999 are given in table 2. This table should be of use both to the experimenter and to the quality controller. Table 2 will supplement the values of k given in [3]. An example of the use of table 2 is given:

EXAMPLE: An industrial quality control engineer measures the voltages of a random sample of 30 batteries from his production line. (Production is in statistical control, and the successive battery voltages may be assumed to be random values from a normal universe.) From the sample mean voltage, $\bar{x} = 7.52$, and the sample standard deviation of voltages, $s = .90$, he wishes to estimate tolerance limits that will, on the average, contain 95 percent of the population of batteries. What shall these tolerance limits be?

The tolerance limits will be of the form $\bar{x} \pm ks$. To find k, he enters table 2 with $n = 30$. The value of $k_{.95}$ is given as 2.079. Hence the tolerance limits are:

$$7.52 \pm 2.079(.90) = 7.52 \pm 1.87 = 5.65 \text{ to } 9.39.$$

Notice that $k_{.95} = 2.079$ is larger than the value 1.96 that would be used if μ and σ were available.

For purposes of comparison, values of $k_{.50}$ for $n = 2(1)30, 40, 60, 120, \infty$, are included in table 1, column 5.

One Sided Tolerance Limits. Suppose now the problem is to find the value of k' such that, on the average, the proportion of the normal population less than $\bar{x} + k's$ is a specified value a.

In other words, if

$$p' = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\bar{\bar{x}}+k's} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}} dx \qquad (20)$$

find the value of k' such that

$$E(p') = a \quad . \qquad (21)$$

From the previous proof, it follows that the answer now is:

$$k' = t'_a \sqrt{\frac{n+1}{n}} \qquad (22)$$

where t'_a is the 100 a percentile of the Student-t distribution. Hence to get the answer from table 2, find k_{2a-1}. Then the desired value is

$$k' = k_{2a-1} \qquad (23)$$

A similar result holds if the proportion of the normal population greater than $\bar{\bar{x}} - k's$ is to be a specified value a, on the average.

EXAMPLE: A pilot run of 40 electron tubes is made. For each tube, a certain critical characteristic, x, is measured; for the sample $\bar{\bar{x}} = 12.25$, s = .68. From past experience, it is known that x is normally distributed. What is the value of L such that 99 percent of the population of tubes will, on the average, have a value less than L?

We may write

$$L = \bar{\bar{x}} + k's \qquad (24)$$

Then according to (23)

$$k' = k_{2(.99)-1} = k_{.98} \text{ of table 2.}$$

Table 2 yields $k_{.98} = 2.455$. Hence

$$L = 12.25 + 2.455(.68) = 13.92 \quad .$$

<u>4.1.3 μ Unknown, σ Known.</u> In this case an interval of the form

$$\bar{x} \pm k\sigma \tag{25}$$

must be used.

Using the same method as in the proof above, the following result may be derived:

If the expected value E(p) of the proportion, p, of the normal universe contained in (25) is to be a, then

$$k = \sqrt{\frac{n+1}{n}} \quad L_{1-a} \tag{26}$$

where L_{1-a} is the normal curve, (N(0,1)), deviate such that the area between $\pm L_{1-a}$ is a.

For purposes of comparison, k of (26) is given in table 1, column 6, for a = .50, and n = 2(1)30,40,60,120,∞.

<u>4.1.4 μ Known, σ Unknown.</u> In this case the interval

$$\mu \pm ks \tag{27}$$

must be used.

Again using the same method as in the proof above, the appropriate value for k for (27) to include, on the average, a is given by

$$k = t_{1-a, n-1} \tag{28}$$

where $t_{1-a,n-1}$ is the value of t for which the integral in (16)

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is equal to a.

For purposes of comparison, values of k of (28) are given in table 1, column 7, for a = .50 and n = 2(1)30,40,60,120,∞ .

4.2 Confidence Statement About Tolerance Interval.

A number of papers have been written on the problem of confidence statements for tolerance intervals. [2],[3],[6],[7],[8]. The problem may be illustrated as follows:

4.2.1 μ, σ Unknown. Suppose the battery engineer of 4.1.2 asked the following question: What value of k shall I take so that I can be 95 percent confident that $\bar{x} \pm ks$ will include at least 80 percent of my population of batteries?

[3] contains extensive tables of k such that "in a large series of samples for normal universes a certain proportion γ of the intervals $\bar{x} \pm ks$ will include p or more of the universe; γ is called the 'confidence coefficient' since it is a measure of the confidence with which we may assert that a given tolerance range includes at least P of the universe".[3] In these tables γ = .75, .90, .95, .99, .999.

4.2.2 μ Known, σ Unknown. Consider the case where μ is known. Then an interval of the form (27) can be set up to include at least P of the population with confidence γ as follows:

Let us take specific values of P = .80 and γ = .95 for illustrative purposes. We note first that P is monotonic increasing with s (and with s^2). Hence, when s^2 takes on its

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value exceeded 95 percent of the time (call it $s^2_{.95}$), P will take on its value exceeded 95 percent of the time. But

$$s^2_{.95} = \frac{\chi^2_{.95,n-1}}{n-1}\,\sigma^2 \tag{29}$$

Then the appropriate value of k to use in (27) is

$$k = L_{.20}\Big/\sqrt{\chi^2_{.95,n-1}/(n-1)} \tag{30}$$

Values of k for $P = \gamma = .50$ for $n = 2(1)30, 40, 60, 120, \infty$ are given in table 1, column 8, for purposes of comparison.

For general P, γ, if L_{1-P} is defined as in (26), then the appropriate value of k to use in (27) is

$$k = \frac{L_{1-P}}{\sqrt{\chi^2_{\gamma,n-1}/(n-1)}} \tag{31}$$

4.2.3 u Unknown, σ Known. In this case, interval (25) must be used. Let us solve for k when $P = .80$, $\gamma = .95$ to illustrate the reasoning.

We first note that 95 percent of the $\bar{x}$'s lie in $\mu \pm \frac{L_{.05}}{\sqrt{n}}\sigma$, in other words, 95 percent of the $\bar{x} \pm k\sigma$ intervals have their centers inside $\mu \pm \frac{L_{.05}}{\sqrt{n}}\sigma$. Now find k_9 such that

$$\int_{\mu + \frac{L_{.05}}{\sqrt{n}}\sigma - k_9\sigma}^{\mu + \frac{L_{.05}}{\sqrt{n}}\sigma + k_9\sigma} \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-\frac{(t-\mu)^2}{2\sigma^2}}\, dt = .80 \quad . \tag{32}$$

Then 95 percent of the $\bar{x} \pm k_9\sigma$ intervals will contain $P \geq .80$ (namely those intervals for which $\bar{x}$ lies in $\mu \pm \frac{L_{.05}}{\sqrt{n}}\sigma$).

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It follows that the interval

$$\bar{\bar{x}} \pm k_9\sigma \tag{33}$$

will contain .80 or more of the population, .95 of the time.

Values of k_9 for $P = \gamma = .50$ are given in table 1, column 9, for $n = 2(1)30,40,60,120,\infty$.

For general P, γ, k is found from

$$\int_{\mu + \frac{L_{1-\gamma}}{\sqrt{n}} - k\sigma}^{\mu + \frac{L_{1-\gamma}}{\sqrt{n}} + k\sigma} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(t-\mu)^2}{2\sigma^2}} \, dt = P \tag{34}$$

where $L_{1-\gamma}$ is defined as in (26).

5. <u>Relationship Between Confidence Intervals and Tolerance Intervals.</u> There is a very interesting relationship between confidence intervals and tolerance intervals that can be illustrated by the following example:

Suppose as in paragraph 3.3 we wanted to find a confidence interval for the mean of a second sample. But now let $n_2 = 1$. In other words, we will now be finding a confidence interval for a single future observation. According to the result in paragraph 3.3, our answer is

$$\bar{\bar{x}}_1 \pm t_{1-a,n_1-1}\sqrt{\frac{1}{n_1} + \frac{1}{1}}\ s_1 = \bar{\bar{x}} \pm t_{1-a,n-1}\sqrt{\frac{n_1+1}{n_1}}\ s_1 \tag{35}$$

where a is the confidence coefficient.

What does (35) mean? One way of looking at it is that if repeatedly a sample of size n_1 is first drawn and then a second sample of one item is drawn, then a of the time the single item

will lie in the corresponding interval (35). But a little thought shows that this is exactly equivalent to stating that in repeated samples of size n_1, the average proportion, P, of the population contained in (35) is a. In other words, confidence limits with confidence coefficient a for a second sample of size 1 are identical with tolerance limits that will include a proportion, a, on the average. This is confirmed by the fact that (35) is identical with (18).

The above is an illustration of a theorem by Paulson [5]:

"If confidence limits $U_1(x_1,\ldots,x_n)$ and $U_2(x_1,\ldots,x_n)$ on a probability level $= \alpha_0$ are determined for g, a function of a future sample of k observations, [with distribution $\psi(g)$], and $p = \int_{U_1}^{U_2} \psi(g)dg$, then $E(p) = \alpha_0$."

The proof is now given, because it is short and instructive:

"Let $\psi(g)dg$ and $\phi(U_1,U_2)dU_1dU_2$ denote the distribution of g and U_1,U_2 respectively. Then by the definition of expected value

$$E(p) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\int_{U_1}^{U_2} \psi(g)dg\right]\phi(U_1,U_2)dU_1dU_2 \quad (36)$$

This triple integral is however exactly the probability that g will lie between U_1 and U_2, which by the nature of confidence limits must equal α_0."

In the illustration given above, g corresponds to the value of the single future observation, and k = 1.

Similarly we can check the results of paragraphs 4.1.3 and 4.1.4 by the use of Paulson's theorem.

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Mathematical Proof of (16). The details of the proof (independently derived by I. R. Savage of the Statistical Engineering Laboratory, National Bureau of Standards) of (16) are given, since the method is a suggestive one:

By an appropriate linear transformation, the problem may be reduced to that of finding

$$E(p) = C_1 \int_0^{\infty}\int_{-\infty}^{\infty}\int_{\bar{\bar{x}}-ks}^{\bar{\bar{x}}+ks} e^{-\frac{1}{2}t^2}\,dt\; s^{n-2}\, e^{-\frac{1}{2}[n\bar{\bar{x}}^2+(n-1)s^2]}d\bar{\bar{x}}ds \quad (36)$$

where C_1 is a constant free of k. In the following, C_1 = constant free of k.

The conditions for differentiating under the integral hold. Hence we have

$$\frac{\partial E}{\partial k} = C_1 \int_0^{\infty}\int_{-\infty}^{\infty}\left[se^{-\frac{1}{2}(\bar{\bar{x}}+ks)^2}+se^{-\frac{1}{2}(\bar{\bar{x}}-ks)^2}\right] s^{n-2}e^{-\frac{1}{2}[n\bar{\bar{x}}^2+(n-1)s^2]}\, d\bar{\bar{x}}ds \quad (37)$$

$$= C_1 \int_0^{\infty}\int_{-\infty}^{\infty}\left[e^{-\frac{1}{2}(\sqrt{n+1}\,\bar{\bar{x}}+\frac{ks}{\sqrt{n+1}})^2+(n-1+k^2\frac{n}{n+1})s^2}\right] s^{n-1}d\bar{\bar{x}}ds$$

$$+ C_1 \int_0^{\infty}\int_{-\infty}^{\infty}\left[e^{-\frac{1}{2}(\sqrt{n+1}\,\bar{\bar{x}}-\frac{ks}{\sqrt{n+1}})^2+(n-1+k^2\frac{n}{n+1})s^2}\right] s^{n-1}d\bar{\bar{x}}ds \quad (38)$$

Let $u = \sqrt{n+1}\,\bar{\bar{x}} + \frac{ks}{\sqrt{n+1}}$ in the first integral and $= \sqrt{n+1}\,\bar{\bar{x}} - \frac{ks}{\sqrt{n+1}}$ in the second. Then

$$\frac{\delta E}{\delta k} = C_1 \int_0^\infty \int_{-\infty}^\infty e^{-\frac{1}{2}u^2} \frac{du}{\sqrt{n+1}} s^{n-1} e^{-\frac{1}{2}(n-1+k^2 \frac{n}{n+1})s^2} du\, ds$$

$$+ C_1 \int_0^\infty \int_{-\infty}^\infty e^{-\frac{1}{2}u^2} \frac{du}{\sqrt{n+1}} s^{n-1} e^{-\frac{1}{2}(n-1+k^2 \frac{n}{n+1})s^2} du\, ds \qquad (39)$$

or

$$\frac{\delta E}{\delta k} = C_2 \int_0^\infty s^{n-1} e^{-\frac{1}{2}(n-1 + k^2 \frac{n}{n+1})s^2} ds \qquad (40)$$

Let $y = \frac{1}{2}(n-1 + k^2 \frac{n}{n+1})s^2$. Then

$$\frac{\delta E}{\delta k} = C_2 \int_0^\infty 2^{\frac{n}{2}-1} y^{\frac{n}{2}-1} e^{-y} / (n-1 + k^2 \frac{n}{n+1})^{\frac{n}{2}} dy \qquad (41)$$

$$= C_3 \frac{1}{(n-1 + k^2 \frac{n}{n+1})^{\frac{n}{2}}} \qquad (42)$$

Hence

$$E(p) = C_3 \int_{-k}^{k} \frac{dk}{(n-1 + k^2 \frac{n}{n+1})^{\frac{n}{2}}} \qquad (43)$$

Now let

$$t = k\sqrt{\frac{n}{n+1}} \qquad (44)$$

so that

$$E(p) = C_4 \int_{-t}^{t} \frac{dt}{(n-1 + t^2)^{\frac{n}{2}}} \qquad (45)$$

$$= C_5 \int_{-t}^{t} dt/(1+t^2/(n-1))^{\frac{n}{2}} \tag{46}$$

But the integrand is the well known Student-t density gunction. Now when $k = \infty$, $E(p) = 1$. Hence C_5 must be identical with the constant of the Student-t distribution. (16) follows.

Q.E.D.

TABLE I

Factors for confidence and tolerance intervals

Sample Size n	k_1	k_2	k_3	k_4	kk_5	k_6	k_7	k_8	k_9
2	.477	.707	1.000	.674	1.225	.826	1.000	1.000	.754
3	.389	.471	.666	.674	.942	.779	.816	.810	.727
4	.337	.382	.541	.674	.855	.754	.765	.759	.714
5	.302	.331	.469	.674	.812	.739	.741	.736	.706
6	.275	.297	.420	.674	.785	.729	.727	.723	.700
7	.255	.271	.384	.674	.768	.721	.718	.714	.697
8	.238	.251	.356	.674	.754	.715	.711	.708	.694
9	.225	.235	.333	.674	.744	.711	.706	.704	.692
10	.213	.222	.314	.674	.737	.707	.703	.701	.690
11	.203	.211	.299	.674	.731	.704	.700	.698	.688
12	.195	.201	.285	.674	.725	.702	.697	.698	.687
13	.187	.193	.273	.674	.721	.700	.695	.694	.686
14	.180	.185	.262	.674	.718	.698	.694	.692	.686
15	.174	.179	.253	.674	.715	.697	.692	.691	.685
16	.169	.173	.244	.674	.712	.695	.691	.690	.684
17	.164	.167	.237	.674	.710	.694	.690	.689	.684
18	.159	.162	.230	.674	.708	.693	.689	.688	.683
19	.155	.158	.223	.674	.706	.692	.688	.687	.683
20	.151	.154	.218	.674	.705	.691	.688	.687	.682
21	.147	.150	.212	.674	.703	.690	.687	.686	.682
22	.144	.146	.207	.674	.701	.690	.686	.685	.681
23	.141	.143	.202	.674	.701	.689	.686	.685	.681
24	.138	.140	.198	.674	.699	.688	.685	.684	.681
25	.135	.137	.194	.674	.699	.688	.685	.684	.681
26	.132	.134	.190	.674	.697	.687	.684	.684	.680
27	.130	.132	.186	.674	.697	.687	.685	.683	.680
28	.127	.129	.183	.674	.696	.686	.684	.683	.680
29	.125	.127	.179	.674	.695	.686	.683	.683	.680
30	.123	.125	.176	.674	.694	.686	.683	.682	.680
40	.107	.108	.152	.674	.689	.683	.681	.680	.678
60	.087	.088	.124	.674	.685	.680	.679	.678	.677
120	.062	.062	.087	.674	.680	.677	.677	.676	.676
∞	0	0	0	.674	.674	.674	.674	.674	.674
For xplanation ee paragraph	3.1	3.2	3.3	4.1.1	4.1.2	4.1.3	4.1.4	4.2.2	4.2.3

TABLE II

Factors for tolerance intervals.

Sample Size n	$k_{.50}$	$k_{.75}$	$k_{.90}$	$k_{.95}$	$k_{.98}$	$k_{.99}$	$k_{.999}$
2	1.225	2.957	7.733	15.562	38.973	77.964	779.699
3	.942	1.852	3.372	4.969	8.042	11.460	36.486
4	.855	1.591	2.631	3.558	5.077	6.530	14.469
5	.812	1.473	2.335	3.041	4.105	5.043	9.432
6	.785	1.405	2.176	2.777	3.635	4.355	7.409
7	.768	1.361	2.077	2.616	3.360	3.963	6.370
8	.754	1.330	2.010	2.508	3.180	3.711	5.733
9	.744	1.307	1.961	2.431	3.053	3.536	5.314
10	.737	1.290	1.922	2.372	2.959	3.409	5.014
11	.731	1.276	1.893	2.327	2.887	3.310	4.791
12	.725	1.264	1.869	2.291	2.829	3.233	4.618
13	.721	1.255	2.849	2.261	2.782	3.170	4.481
14	.718	1.246	1.833	2.236	2.743	3.118	4.369
15	.715	1.239	1.819	2.215	2.710	3.075	4.276
16	.712	1.234	1.807	2.197	2.682	3.038	4.198
17	.710	1.228	1.797	2.181	2.658	3.006	4.131
18	.708	1.224	1.786	2.168	2.637	2.977	4.074
19	.706	1.220	1.779	2.156	2.618	2.953	4.024
20	.705	1.216	1.772	2.145	2.602	2.932	3.979
21	.703	1.213	1.766	2.135	2.587	2.912	3.941
22	.701	1.210	1.760	2.127	2.575	2.895	3.905
23	.701	1.207	1.754	2.119	2.562	2.880	3.874
24	.699	1.205	1.749	2.112	2.552	2.865	3.845
25	.699	1.202	1.745	2.105	2.541	2.852	3.819
26	.697	1.200	1.741	2.099	2.532	2.840	3.796
27	.697	1.198	1.737	2.094	2.524	2.830	3.775
28	.696	1.197	1.733	2.088	2.517	2.820	3.755
29	.695	1.195	1.730	2.083	2.509	2.810	3.737
30	.694	1.193	1.727	2.079	2.503	2.802	3.719
40	.689	1.182	1.706	2.047	2.455	2.741	3.602
60	.685	1.171	1.636	2.017	2.411	2.684	3.492
120	.680	1.161	1.665	1.988	2.368	2.628	3.388
∞	.674	1.150	1.645	1.960	2.326	2.576	3.291

Let $p = \int_{\bar{x} - k_a s}^{\bar{x} + k_a s} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-u)^2}{2\sigma^2}} dx$. The value of k_a given in the table is such that $E(p) = a$ in repeated sampling. (See par. 4.1.2).

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