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NATIONAL BUREAU OF STANDARDS REPORT

1492

ON CERTAIN CHARACTER MATRICES

by

D. H. Lehmer

National Bureau of Standards, Los Angeles, California

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NBS PROJECT
1101-11-5100

February 14, 1952

NBS REPORT
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PREPRINT

*The preparation of this paper was sponsored (in part) by the Office of Naval Research

NBS No. 1492
2-14-52
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For only a very limited class of matrices M is it possible to give explicit formulas for the determinant of M, the general element of M^k, the characteristic roots of M and the inverse of M. Non-trivial instances of such matrices are useful as examples in testing the correctness and efficacy of various matrix computing routines especially when the elements of the matrices are exact integers or rational numbers. The purpose of this note is to indicate a new set of such matrices. Although they arose in connection with a class of exponential sums, they appear to warrant an independent treatment. A discussion of the most general case will not be attempted here. The matrices considered are real and symmetric and of order p-1 where p is an odd prime.

The elements of our matrices are equal to 0, 1 or -1 and are based on what are called real non-principal characters modulo p. These are, for the cases considered, Legendre's symbols for which we use the notation [1]

$$\chi(a) = \left(\frac{a}{p}\right) = \begin{cases} 0 & \text{if } p \text{ divides } a \\ -1 & \text{if the congruence } x^2 \equiv a \pmod p \text{ is impossible} \\ +1 & \text{otherwise} \end{cases}$$

*The preparation of this paper was sponsored (in part) by the Office of Naval Research.

For example if $p = 7$ we have

a	0	1	2	3	4	5	6
$\chi(a)$	0	1	1	-1	1	-1	-1

Besides the simple properties

$$\chi(a)\ \chi(b) = \chi(ab)$$

$$\chi(a + p) = \chi(a)$$

we use the following identities

$$\sum_{k=1}^{p-1} \chi(a+k) = -\chi(a) \tag{1}$$

$$\sum_{k=1}^{p-1} \chi(a+k)\ \chi(b+k) = p\,\delta_a{}^b - 1 - \chi(a)\ \chi(b) \tag{2}$$

where $\delta_a{}^b$ is Kronecker's delta modulo p, that is

$$\delta_a{}^b = \begin{cases} 1 & a \equiv b \pmod p \\ 0 & a \not\equiv b \pmod p \end{cases} .$$

The first of these identities is equivalent to the well known fact that there is a residue for every non-residue modulo p. The second identity may be written

$$\sum_{k=0}^{p-1} \chi(a+k)\ \chi(b+k) = p\,\delta_a{}^b - 1 .$$

This is obvious if $a \equiv b \pmod p$ since $\chi^2(a+k) = 0$ or 1 according as $a + k$ is divisible by p or not. If $a \not\equiv b \pmod p$ we may

set $b - a \equiv r \pmod{p}$ and write the left side as follows

$$\sum_{k=0}^{p-1} \chi(a+k)\ \chi(b+k) = \sum_{h=0}^{p-1} \chi(h)\ \chi(r+h) \quad .$$

If we denote this sum by $S(r)$ and write

$$h \equiv rm \pmod{p}$$

we have

$$S(r) = \sum_{m=0}^{p-1} \chi(rm)\ \chi(rm+r)$$

$$= \chi^2(r) \sum_{m=0}^{p-1} \chi(m)\ \chi(1+m) = S(1) \quad .$$

Hence to show that $S(r) = -1$ we have only to show that

$$\sum_{r=1}^{p-1} S(r) = 1 - p \quad .$$

Now

$$\sum_{r=1}^{p-1} S(r) = \sum_{r=1}^{p-1} \sum_{h=0}^{p-1} \chi(h)\ \chi(r+h)$$

$$= \sum_{h=0}^{p-1} \chi(h) \sum_{r=1}^{p-1} \chi(r+h)$$

$$= - \sum_{h=0}^{p-1} \chi^2(h) = -(p-1)$$

by (1). This proves (2) in all cases. This identity appears to be due to E. Jacobsthal [2].

With these preliminaries we proceed to introduce our general matrix and to give its properties.

Let α be any integer parameter and let M_α be the square matrix of order p-1 whose general element a_{ij} is $\chi(\alpha+i+j)$, then:

I. The inverse M_α^{-1} has for its general element

$$a_{ij}^{(-1)} = \left\{\chi(\alpha+i+j) - \chi(\alpha+i) - \chi(\alpha+j) + \chi(\alpha)\right\}/p .$$

II. The characteristic equation

$$|\lambda I - M_\alpha| = 0$$

of M_α is $(\lambda^2 - p)^{\frac{p-3}{2}}(\lambda^2 + \chi(\alpha)\lambda - 1) = 0$.

III. The determinant of M_α is $-(-p)^{(p-3)/2}$.

The result III follows at once from II by setting $\lambda = 0$. The result I, once guessed, may be verified as follows. We have to show that

$$\sum_{k=1}^{p-1} a_{ik}\, a_{kj}^{(-1)} = \delta_i^j .$$

Substituting, we have

$$p^{-1}\sum_{k=1}^{p-1} \chi(\alpha+i+k)\left\{\chi(\alpha+k+j) - \chi(\alpha+k) - \chi(\alpha+j) + \chi(\alpha)\right\}$$

$$= p^{-1}[p\,\delta_i^j - 1 - \chi(\alpha+i)\,\chi(\alpha+j) - p\,\delta_{\alpha+i}^{\alpha} + 1 + \chi(\alpha+i)\,\chi(\alpha)$$

$$- \left\{\chi(\alpha+j) - \chi(\alpha)\right\}(-\chi(\alpha+i))]$$

by (1) and (2). Since $i \not\equiv 0 \pmod p$ the above expression reduces δ_i^j as required.

It remains to prove II. The method of proof given below is not the most direct one that could be given.[1] It has the merit, however, of producing as a byproduct the general element of any power of M_α and this information may be useful in checking routine computations of powers of a given matrix.

If we denote by $a_{ij}^{(k)}$ the general element of M^k we have

$$p\, a_{ij}^{(-1)} = \chi(\alpha + i + j) - \chi(\alpha + i) - \chi(\alpha + j) + \chi(\alpha)$$

$$a_{ij}^{(0)} = \delta_i^{\,j}$$

$$a_{ij}^{(1)} = \chi(\alpha + i + j)$$

$$a_{ij}^{(2)} = p\, \delta_i^{\,j} - \chi(\alpha + i)\, \chi(\alpha + j) - 1$$

the last result following directly from (2). An inspection of even this small sample of $a_{ij}^{(k)}$ together with considerations of symmetry leads one to the conclusion that $a_{ij}^{(k)}$ depends somewhat on the parity of k and that in making an inductive proof of the general form of $a_{ij}^{(k)}$ one should assume that

$$(3) \quad a_{ij}^{(k)} = \Psi_k(i,j) + S_k[\chi(\alpha + i) + \chi(\alpha + j)] + P_k\, \chi(\alpha + i)\, \chi(\alpha + j) + C_k$$

where S_k P_k C_k depend on α and p only and

$$\Psi_k(i,j) = \begin{cases} p^{k/2}\ \delta_i^{\,j} & \text{if k is even} \\ p^{(k-1)/2}\ \chi(\alpha + i + j) & \text{if k is odd .} \end{cases}$$

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[1]An alternative short method of proof was suggested by R. M. Robinson.

At least, (3) is seen to hold for $k = -1, 0, 1, 2$ and we have in fact

$$(4) \qquad \begin{array}{lll} S_o = 0 & P_o = 0 & C_o = 0 \\ S_1 = 0 & P_1 = 0 & C_1 = 0 \\ S_2 = 0 & P_2 = -1 & C_2 = -1 \; . \end{array}$$

That $a_{ij}^{(k)}$ has the form (3) follows at once from (1) and (2) and, moreover, these latter identities give us recurrence relations for the S_k P_k and C_k. In fact if we assume that (3) holds for $k = 2m$ and if we substitute into the relation

$$a_{ij}^{(2m+1)} = \sum_{r=1}^{p-1} a_{ir}\, a_{rj}^{(2m)}$$

we find

$$(5) \qquad \begin{aligned} S_{2m+1} &= - P_{2m} = - \chi(\alpha)\, S_{2m} - C_{2m} \\ P_{2m+1} &= - S_{2m} - \chi(\alpha)\, P_{2m} \\ C_{2m+1} &= - S_{2m} \; . \end{aligned}$$

Similarly we find

$$(6) \qquad \begin{aligned} S_{2m+2} &= - \chi(\alpha)\, S_{2m+1} - C_{2m+1} = - P_{2m+1} \\ P_{2m+2} &= - p^{m} - S_{2m+1} - \chi(\alpha)\, P_{2m+1} \\ C_{2m+2} &= - p^{m} - S_{2m+1} \; . \end{aligned}$$

We have only to solve these difference equations for S_k, P_k, C_k subject to the initial condition (4).

The case of $\chi(\alpha) = 0$ is much simpler than $\chi(\alpha) = \pm 1$. In fact the above relations then become simply

$$-S_{k+1} = C_k = P_k$$

$$C_{2m+1} = C_{2m-1}$$

$$C_{2m} = -p^m + C_{2m} .$$

The solution is easily seen to be

$$S_{2m} = 0 \qquad P_{2m} = C_{2m} = -(p^m - 1)/(p-1)$$

$$S_{2m+1} = (p^m - 1)/(p-1) \qquad P_{2m+1} = C_{2m+1} = 0 .$$

Hence if

$$a_{ij} = \chi(i + j)$$

then

$$\text{(7)} \qquad a_{ij}^{(2m)} = p^m \delta_i^j - \frac{p^m - 1}{p-1} \left\{ \chi(ij) + 1 \right\}$$

$$\text{(8)} \qquad a_{ij}^{(2m+1)} = p^m \chi(i+j) + \frac{p^m - 1}{p-1} \left\{ \chi(i) + \chi(j) \right\} .$$

In case $\chi(\alpha) \neq 0$ the solution of the systems (5) and (6) involves the Fibonacci numbers

$$U_n = (A^n - B^n)/(A - B)$$

where

$$A = \tfrac{1}{2}(1 + \sqrt{5}) , \quad B = \tfrac{1}{2}(1 - \sqrt{5}) .$$

Early values of U_n are given by the following table

n	-2	-1	0	1	2	3	4	5	6	7	8
U_n	-1	1	0	1	1	2	3	5	8	13	21

and in general

$$U_{n+1} = U_n + U_{n-1} \text{ .}$$

In what follows we shall need to refer also to the companion sequence

$$V_n = A^n + B^n = U_{n+1} + U_{n-1} \text{ .} \tag{9}$$

The solution of (5) and (6) may be given as follows

$$S_k = - P_{k-1}$$

$$\text{(10)} \quad \begin{aligned} P_{2m+1} &= \chi(\alpha) \left\{ p^{m+1} - p\, U_{2m+2} + U_{2m} \right\} / (p^2 - 3p + 1) \\ C_{2m+1} &= \chi(\alpha) \left\{ p^{m} - p\, U_{2m} + U_{2m-2} \right\} / (p^2 - 3p + 1) \end{aligned}$$

$$\text{(11)} \quad \begin{aligned} P_{2m} &= - \left\{ p^{m+1} - p^m - p\, U_{2m+1} + U_{2m-1} \right\} / (p^2 - 3p + 1) \\ C_{2m} &= - \left\{ p^{m+1} - 2p^m - p\, U_{2m-1} + U_{2m-3} \right\} / (p^2 - 3p + 1) \text{ .} \end{aligned}$$

The fact that these values satisfy the difference equations (5), (6) and the initial conditions (4) may be verified without using anything more complicated than the fact that

$$U_n = 3U_{n-2} - U_{n-4} \text{ .}$$

Of course the expressions (10) and (11) are polynomials in p.

If (10) and (11) are substituted into (3) we obtain expressions for $a_{ij}^{(2m)}$ and $a_{ij}^{(2m+1)}$.

For example

$$(12)\quad \begin{aligned} a_{ij}^{(5)} &= p^2\, \chi(\alpha + i+j) + (p+2) \left\{\chi(\alpha+i) + \chi(\alpha+j)\right\} \\ &\quad + (p+3)\, \chi(\alpha)\, \chi(\alpha+i)\, \chi(\alpha+j) + \chi(\alpha) \;. \end{aligned}$$

We note that this does not become (8), with $m = 2$, if we set $\alpha = 0$. We now consider the trace of $M_\alpha^{\ k}$, that is the sum $\Sigma_{j=1}^{p-1}\, a_{jj}^{(k)}$ which is also the sum of the k-th powers of the characteristic roots of M_α. The case of $\chi(\alpha) = 0$ is simple. From (7) and (8) we have

$$\sum_{j=1}^{p-1} a_{jj}^{(2m)} = (p-1)\, p^m - 2(p^m - 1) = (p-3)\, p^m + 2$$

$$\sum_{j=1}^{p-1} a_{jj}^{(2m+1)} = 0 \;.$$

Hence in both cases

$$\Sigma\, a_{jj}^{(k)} = \frac{p-3}{2}\, (p^{\frac{1}{2}})^k + \frac{p-3}{2}\, (-p^{\frac{1}{2}})^k + (-1)^k + 1^k \;.$$

Since this holds for $k = 1, 2, 3, \ldots$ the characteristic roots of M_0 must be those of the equation

$$(13)\qquad (\lambda^2 - p)^{(p-3)/2} (\lambda^2 - 1) = 0 \;.$$

This proves II in case $\chi(\alpha) = 0$.

For $\chi(\alpha) \neq 0$ we find from (3)

$$\sum_{j=1}^{p-1} a_{jj}^{(2m)} = (p-1)\, p^m - 2\, \chi(a)\, S_{2m} + (p-2)\, P_{2m} + (p-1)\, C_{2m}$$

$$\sum_{j=1}^{p-1} a_{jj}^{(2m+1)} = -\, \chi(a) p^m - 2\, \chi(a) S_{2m+1} + (p-2) P_{2m+1} + (p-1) C_{2m+1} \;.$$

If we substitute from (10) and (11) and simplify we obtain

$$\sum_{j=1}^{p-1} a_{jj}^{(2m)} = (p-3)p^m + U_{2m+1} + U_{2m+1}$$

$$= (p-3)p^m + V_{2m}$$

$$\sum_{j=1}^{p-1} a_{jj}^{(2m+1)} = - \chi(\alpha)(U_{2m+2} + U_{2m})$$

$$= - \chi(\alpha)\, V_{2m+1} \quad .$$

Recalling the definition (9) of V_n we may write in general

$$\sum_{j=1}^{p-1} a_{jj}^{(k)} = \tfrac{1}{2}(p-3)(p^{\frac{1}{2}})^k + \tfrac{1}{2}(p-3)(-p^{\frac{1}{2}})^k + (-\chi(\alpha)A)^k + (-\chi(\alpha)B)^k \; .$$

This shows that the characteristic roots of M_α consist of $\frac{1}{2}(p-3)$ roots $p^{\frac{1}{2}}$, $\frac{1}{2}(p-3)$ roots $-p^{\frac{1}{2}}$, $-\chi(\alpha)A$ and $-\chi(\alpha)B$. Since

$$A + B = 1 \; , \qquad A\,B = -1 \; ,$$

the characteristic equation of M_α is

$$(\lambda^2 - p)^{(p-3)/2}\,(\lambda^2 + \chi(\alpha)\lambda - 1) = 0 \quad .$$

Although this was derived on the assumption that $\chi(\alpha) \neq 0$, if we set $\chi(\alpha) = 0$ we do obtain (13). This completes the proof of II.

Concerning the latent vectors of M_α, the reader may verify that, corresponding to a root ρ of

$$\lambda^2 + \chi(\alpha)\lambda - 1 = 0$$

there is the latent vector whose j-th component is

$$\chi(\alpha + j) - \chi(\alpha) - \rho \text{ .}$$

The following example will illustrate the foregoing. We take $p = 7$, $\alpha = 3$; then our matrix M_3 is

$$M_3 = \begin{pmatrix} -1 & -1 & 0 & 1 & 1 & -1 \\ -1 & 0 & 1 & 1 & -1 & 1 \\ 0 & 1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 0 \\ -1 & 1 & -1 & -1 & 0 & 1 \end{pmatrix}$$

whose determinant is -49. If the elements of M^{-1} are each multiplied by 7 we obtain

$$\begin{pmatrix} -4 & -2 & -1 & -1 & -2 & -4 \\ -2 & 1 & 2 & 1 & -2 & 0 \\ -1 & 2 & 2 & -1 & 0 & -2 \\ -1 & 1 & -1 & 0 & -3 & -3 \\ -2 & -2 & 0 & -3 & -4 & -3 \\ -4 & 0 & -2 & -3 & -3 & -2 \end{pmatrix}$$

By (12) we have, for example,

$$M_3^{\,5} = \begin{pmatrix} -42 & -40 & 9 & 57 & 56 & -42 \\ -40 & -29 & 20 & 39 & -40 & 58 \\ 9 & 20 & 20 & -59 & 58 & -40 \\ 57 & 39 & -59 & 48 & -41 & -41 \\ 56 & -40 & 58 & -41 & -42 & 7 \\ -42 & 58 & -40 & -41 & 7 & 56 \end{pmatrix}$$

February 26, 1952.

References

[1] See for example E. Landau, Elementare Zahlentheorie New York, Chelsea, 1946, p. 83-87.

[2] E. Jacobstahl, "Über die Darstellung der Primzahlen der Form 4n + 1 als Summe zweier Quadrate," Jour. für. Math. v. 132, 1907, pp. 238-245.

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