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A THEOREM ON CONVEX CONES WITH APPLICATIONS TO LINEAR INEQUALITIES

by

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National Bureau of Standards, Los Angeles, California

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A Theorem on Convex Cones with Applications to Linear Inequalities*

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1. INTRODUCTION

This note is concerned with the convex cone associated with the two systems of linear inequalities

(1) $\sum_{j=1}^{n} a_{ij} x_j \geqq 0$, $i = 1, 2, \cdots, m$, and

(2) $\sum_{j=1}^{n} a_{ij} x_j \geqslant 0$, $i = 1, 2, \cdots, m$, where the symbol $\geqslant$

demands that the inequality ($>$) hold for at least one value of i. For brevity these systems will be written (1) $Ax \geqq 0$ and (2) $Ax \geqslant 0$.

Interpreting $(a_{i1}, \cdots, a_{in})$ as a vector a_i in E_n, with initial point at the origin, we denote by A the convex cone generated by these m vectors and by A^* the polar cone, the vectors of which give the solutions of (1). A cone will be considered null if it contains only $(0, 0, \cdots, 0)$.

The purpose of this paper is to show that in general $A \cdot A^* \neq 0$ and to characterize the cases in which A and A^* do not intersect. Some applications are then made to obtain theorems on the existence

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of, and methods of obtaining, solutions of (1) and (2).

II. THE MAIN THEOREM

Before proving the main theorem, it should be remarked that the proof given does not require that the cone A be generated by finitely many vectors but the applications to be made do require this. The theorem, of which a special case has been stated in [2][1], can be formulated as follows:

<u>Theorem 2.1</u>. <u>If</u> A^* <u>is non-null and if some vector of</u> A <u>is not perpendicular to all the vectors of</u> A^*, <u>then</u> $A \cdot A^* \neq 0$.

<u>Proof</u>. Let x be a vector of A which is not perpendicular to all the vectors of A^*. If x is not in A^* (otherwise the theorem holds immediately) let y be that vector of A^*, of unit length, closest to x. In other words, among the unit vectors of A^*, y makes the inner product $x \cdot y$ a maximum. In the hyperplane orthogonal to y let u be the unit vector farthest from x, that is, minimizing $x \cdot u$.

Now u is a vector of A. This follows since it is a vector of the polar cone of A^* and since $(A^*)^* = A$. To see that u is in $(A^*)^*$ we remark that if A^* contained a vector z making an obtuse angle with u, the plane containing y and z would contain a unit vector of A^* closer to x than y.

Letting $\gamma = \frac{1}{x \cdot y}$, the vector $y - \gamma x$ is orthogonal to y, since y is a unit vector. Among the unit vectors orthogonal to y, u

[1]Numbers in brackets refer to the bibliography.

minimizes $x \cdot u$ and hence maximizes $y - \gamma x$. Thus $y - \gamma x$ has the same direction as u, and $y - \gamma x = ku$, $k > 0$.

Then $y = \gamma x + ku$, γ and k both positive, and y is a vector in $A \cdot A^*$.

If A is not contained in an E_{n-1}, the conditions of the theorem are satisfied unless A^* is null. If A is contained irreducibly in an E_r, $r < n$, and $A \cdot A^* = 0$, then A is equal to the E_r, since no vector of A^* lies in the E_r. Thus we can characterize those cones with $A \cdot A^* = 0$ as follows:

Theorem 2.2. $A \cdot A^* = 0$ if and only if A is an E_r and A^* an orthogonal E_{n-r}.

III. SOME EXISTENCE THEOREMS

In this section the foregoing discussion is applied to obtain theorems on the existence of solutions for a system of linear inequalities. A vector y is called positive if $y_i > 0$ for each i, non-negative if $y_i \geqq 0$ for each i and $y_i > 0$ for some i.

Theorem 3.1. The system (2) has a solution if and only if $A \cdot A^* \neq 0$.

The proof being immediate from Theorem 2.2, we state the same result algebraically.

Corollary: If (2) has a solution it has one of the form $x = A'y$, where y is non-negative and A' is the transpose of the matrix A.

Theorem 3.2. In order that (2) have a solution, it is necessary and sufficient that the system $AA'y \geqq 0$ have a non-negative solution.

Proof: First, if $AA'y \geq 0$ has any solution at all, the relation $x = A'y$ gives a solution of $Ax \geq 0$. On the other hand, if $Ax \geq 0$ has a solution, the corollary to Theorem 3.1 states that it has one of the form $x = A'y$, y non-negative, and this y is a solution of $AA'y \geq 0$.

It should be observed that this theorem is a strengthening of the result, in [1], that $Ax \geq 0$ and $AA'y \geq 0$ are either both consistent or both inconsistent.

Corollary: *If* B *is a symmetric positive semi-definite matrix,* $By \geq 0$ *has a solution if and only if it has a non-negative one.*

Corollary: *A sufficient condition that* (1) *have a solution is that* $AA'y \geq 0$ *have a (non-negative) solution.*

Corollary: *If the rank of* A *is* n, *a necessary condition that* (1) *have a non-trivial solution is that* $AA'y \geq 0$ *have a (non-negative) solution.*

In [3] a necessary and sufficient condition is given that a system of type (2) be solvable. Using Theorem 2 of [3] we can state the following theorem.

Theorem 3.3. $AA'y \geq 0$ *(and hence* (2)) *has a solution if and only if* $AA'y = 0$ *has no positive solution.*

In [4] a method is given for determining whether or not a system of homogeneous linear equations has a positive solution. In the form given, however, the method does not seem feasible for purposes of computation.

An obvious necessary condition that $AA'y = 0$ have a positive solution is that for any collection of rows of AA', the m column

sums be all zero or have a pair of terms of opposite sign. Hence a sufficient condition that $AA'y \geq 0$ (and hence (2)) have a solution is that for some collection of rows of AA', the m column sums form a non-negative set.

Still another form can be given to the problem of determining whether or not $AA'y \geq 0$ is solvable. Let us write $AA' = B = (b_{ij})$ and denote the rank of B by r. We can suppose that the r-th order principal minor R, in the upper left-hand corner, is non-singular and, for the purpose of solving $By \geq 0$, we can assume its determinant positive. For $1 \leq j \leq r$ and $r + 1 \leq k \leq m$, let β_{jk} denote the negative of the determinant of the matrix obtained from R by substituting the first r elements of the k-th column of B for the j-th column of R. If we assign values arbitrarily to $y_{r+1}, \cdots, y_m$ and solve $By = 0$ for $y_1, \cdots, y_r$, we obtain, for $j = 1, 2, \cdots, r$, $y_j = \frac{1}{|R|} [\beta_{jr+1} y_{r+1} + \cdots + \beta_{jm} y_m]$. Thus the problem reduces to that of finding positive solutions of the system:

$$\begin{array}{l} \beta_{1r+1} y_{r+1} + \cdots + \beta_{1m} y_m > 0 \\ \vdots \\ \beta_{rr+1} y_{r+1} + \cdots + \beta_{rm} y_m > 0 \end{array} .$$

In Theorem D5, p. 50, of [5], a criterion for the existence of positive solutions of this system is given.

IV. COMPUTATION OF SOLUTIONS

Theorem 2.1 suggests a method for computing a solution of a system of inequalities, assuming a solution exists. We begin by normalizing the rows of the matrix A, that is, by making

$\sum_{j=1}^{n} a_{ij}^2 = 1$ for each i. We then simply generate a set of unit vectors dense in the set of unit vectors of A, and hence come arbitrarily close to some vector of $A \cdot A^*$, if $A \cdot A^* \neq 0$. This can be done by adjoining to the vectors corresponding to the rows of A the mid-vector of each pair. If no solutions are obtained at the first step, continue the process on the larger set of vectors. The normalized midvector of the vectors a and b has for its k-th component $(a_k + b_k)/\sqrt{2(1+a \cdot b)}$. The computation of one midpoint presents no problem but the number of such computations becomes very large as we repeat the process. It would therefore be of considerable value to know how to find only essential midpoints.

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