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COMBINING TOLERANCES

by

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FOREWORD

This report was prepared in the Statistical Engineering Laboratory (Section 11.3, National Bureau of Standards) in response to a request from Mr. Robert S. Hoff of the Projector Fuze Development Section (Section 13.6, National Bureau of Standards).

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Statement of the Problem

Knowing the tolerance limits on the various component parts of an electrical circuit, we wish to determine tolerance limits on the performance of this circuit. More precisely, knowing the means and standard deviations of the distributions of the components, how can we obtain limits which include a certain percent of the distribution of performance? In view of the particular applications to be made, we make the following assumptions:

1. Performance is a linear function of the components.

2. The components are selected at random from their respective distributions.

Summary

The statements that can be made about the expected variability in performance depend on the amount of information available concerning the distribution of performance. The appropriate statements are explained, according as the form of this distribution is (1) unspecified, (2) unimodal and symmetric, and (3) normal.

An important approximation is given when the number of components is large.

Statistical Reasoning

For simplicity, let us assume that the measure of performance, P, depends on two components C_1 and C_2. Let us specify the functional relation as

$$P = a_1C_1 + a_2C_2 ,$$

where a_1 and a_2 are known constants. Let the known means (nominal values) of C_1 and C_2 be M_1 and M_2, respectively, and their standard deviations σ_1 and σ_2, respectively. The mean performance, P_0, is then given by

$$P_0 = a_1M_1 + a_2M_2$$

and the standard deviation of performance, σ_p, by

$$\sigma_p = \sqrt{a_1^2\sigma_1^2 + a_2^2\sigma_2^2} ;$$

the latter relation only holding when C_1 and C_2 are selected at random from their respective distributions.

Knowing P_0 and σ_p, we now wish to know what percent of performances will fall within fixed limits. The sharpness of the results depends on how much additional knowledge we possess concerning the performance distribution.

Case 1: Performance Distribution Unspecified

In this case we know only P_0 and σ_p. Using a form of the "Tchebycheff Inequality" (1), we can make the following statement.

For any given constant K, at least $100(1-\frac{1}{K^2})$ percent of the performance distribution is included in the interval $P_0 \pm K\sigma_p$.

For example, when K=3, we know that the interval of 3 standard deviations on either side of the mean includes at least $100(1-\frac{1}{3^2})$ percent, or 88.9 percent of the distribution.

Case 2: Performance Distribution Unimodal and Symmetric

This case would arise, for example, if both components were drawn at random from the same unimodal symmetric distribution. Using the "Gauss Inequality" (2) the sharper statement can now be made that at least $100(1-\frac{4}{9K^2})$ percent of the performance distribution is included in the interval $P_0 \pm K\sigma_p$.

When K=3, we can now say that the interval of 3 standard deviations on either side of the mean includes more than 95 percent of the distribution.

Case 3: Performance Distribution Normal

This case only arises when both components are drawn at random from normal distributions. We can now make exact statements based on the Table of the Normal Integral (3). Denoting the normal frequency function by f(x) we have the fraction of the performance distribution included in the interval $P_0 \pm K\,\sigma_p$ is

$$\int_{-K}^{K} f(x)\,dx = 1 - 2\int_{-K}^{\infty} f(x)\,dx \,.$$

The latter integral is tabulated in (2) as a function of K.

When K=3, we find that the interval of 3 standard deviations on either side of the mean includes 99.7 percent of the distribution.

Although the above outline deals exclusively with two components, the same argument applies to any finite number.

For example, if

$$P = a_1C_1 - a_2C_2 + a_3C_3$$

we have

$$P_0 = a_1M_1 - a_2M_2 + a_3M_3$$

$$\sigma_p = \sqrt{a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + a_3^2\sigma_3^2}$$

the rest following exactly as before.

Approximation when the Number of Components is Large

As the number of components (each selected at random) increases, the distribution of performance approaches a normal distribution - under general conditions regardless of the type of distribution of the components. Thus we can use Case 3 above to give an approximate answer, the larger the number of components the better the approximation. A rough working rule is to use Case 3 when there are four or more components.

Illustrative Example

Let us consider a circuit whose performance can be expressed in terms of the four capacitors C_1, C_2, C_3, and C_4 by the equation

$$P = C_1 + C_2 + C_3 + C_4$$

(In this case $a_1 = a_2 = a_3 = a_4 = 1$)

Suppose that the capacitors originally had capacitances that followed a normal distribution with mean (nominal) value 180 μμf and standard deviation 14 μμf, but later the supply was screened and all capacitors beyond ± 15 percent of the nominal value were removed. If four capacitors are now drawn at random and the circuit constructed, what tolerance limits can be obtained on performance?

Notice that the distribution of components in this case is a "truncated normal" distribution. Its mean is

$$\mu = 180 \quad .$$

To find its standard deviation, we note that the point of truncation is (.15) (180) or 27 μμf from the nominal value of 180. This is $\frac{27}{14}$, or 1.93 times the st. dev. of the original normal distribution. In terms of this factor, the desired st. dev. is

$$\sigma = \sqrt{\frac{\int_{-1.93}^{1.93} x^2 f(x)dx}{\int_{-1.93}^{1.93} f(x)dx}} \text{ x st. dev. of the original normal}$$

The Table of Incomplete Normal Moment Functions (4) gives:

$$\int_{-1.93}^{1.93} x^2 f(x)dx = .708$$

The Table of the Normal Integral gives

$$\int_{-1.93}^{1.93} f(x)dx = .946$$

Therefore

$$\sigma = \sqrt{\frac{.708}{.946}}\,(14)$$
$$= 12.1$$

It follows that the mean and standard deviation of performance are

$$P_0 = 4(180)$$
$$= 720$$
$$\sigma_p = \sqrt{4(12.1)^2}$$
$$= 24.2$$

With no additional information, we would have (following Case 1):

The interval $P_0 \pm 3.1\ \sigma_p$ includes at least 90 percent of the distribution of performance.

The interval $P_0 \pm 4.5\ \sigma_p$ includes at least 95 percent of the distribution of performance.

Converting these to percentage statements (noting that σ_p is 3.4 percent of P_0):

At least 90 percent of the distribution falls within 10.5 percent of the nominal performance.

At least 95 percent of the distribution falls within 15.3 percent of the nominal performance.

If we knew only that the performance distribution was unimodal and symmetric we would have (following Case 2):

The interval $P_0 \pm 2.1\ \sigma_p$ includes at least 90 percent of the distribution.

The interval $P_0 \pm 3\ \sigma_p$ includes at least 90 percent of the distribution.

or

At least 90 percent of the distribution falls within 7.1 percent of the nominal performance.
At least 95 percent of the distribution falls within 10.2 percent of the nominal performance.

Fortunately we know, in addition, that the distribution of performance is approximately normal. Hence we can make the following approximate statements based on a Table of the Normal Integral.*

The interval $P_0 \pm 1.65\ \sigma_p$ includes 90 percent of the distribution.
The interval $P_0 \pm 1.96\ \sigma_p$ " 95 " " " " .
The interval $P_0 \pm 2.58\ \sigma_p$ " 99 " " " " .

90 percent of the distribution falls within 5.6 percent of the nominal performance.
95 percent of the distribution falls within 6.7 percent of the nominal performance.
99 percent of the distribution falls within 8.8 percent of the nominal performance.

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* Using this approximation, the percent of the distribution included is in error by less than 1.5 percent when there are four components. For more details, see (5).

Thus, although the capacitances vary up to 15 percent of their nominal value, virtually all performances fall within 8.8 percent of the nominal performance.

If no screening had been done, we would have

$$M = 180$$

$$\sigma = 14 \quad .$$

In this case, the mean and st. dev. of performance are

$$P_0 = 720$$

$$\sigma_p = \sqrt{4(14)^2}$$

$$= 28 \quad .$$

Since the distribution of performance is exactly normal, we can make the following precise statements:

90 percent of the distribution falls within 6.4 percent of the nominal performance.

95 percent of the distribution falls within 7.6 percent of the nominal performance.

99 percent of the distribution falls within 10 percent of the nominal performance.

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