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NUMERICAL COMPUTATION OF LOW MOMENTS OF ORDER STATISTICS FROM A NORMAL POPULATION

by

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and
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Numerical Computation of Low Moments of Order Statistics from a Normal Population[1]

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1. Summary. The moments considered in this paper are the first eight moments, about the origin, (not including cross moments) of the order statistics for a sample drawn from a normal population (the precise definition is given at the end of Section 3 below). These are explicitly expressed as linear combinations of the moments of the central observations. These, in turn, are tabulated for samples of twenty or fewer, and an asymptotic series is provided which will give their values with great accuracy for samples of more than twenty.

For samples of twenty or fewer, the mean values of the j-th observation (in magnitude) are tabulated. There is also given an asymptotic formula for these means which is more accurate than the classic asymptotic formula. For the means and standard deviations of the extreme observations of a sample, still more accurate asymptotic formulas are furnished.

2. Algebraic preliminaries. Suppose we have quantities $\alpha(m,n)$ satisfying the relations

$$\alpha(m+1,n) + \alpha(m,n+1) = \alpha(m,n) \tag{1}$$

[1]The preparation of this paper was sponsored (in part) by the Office of Naval Research.

where m and n are any non-negative integers. Clearly a given linear relationship among the α's can be derived from (1) if and only if it would become a polynomial identity upon replacing $\alpha(m,n)$ by $x^m(1-x)^n$.

We define

$$(2) \qquad P_N(w) = 1 - (N-2)w + \frac{(N-3)(N-4)}{2!} w^2 + \cdots + (-1)^k \binom{N-k-1}{k} w^k$$

with

$$k = \left[\frac{N-1}{2} \right] .$$

Then by induction on N, one can prove for $N \geq 1$,

$$(3) \qquad x^N = x\, P_N(x(1-x)) - x(1-x)\, P_{N-1}(x(1-x)) .$$

Multiplying by $x^n(1-x)^n$, we see that from (1) can be derived for $N \geq 1$

$$\alpha(n+N,n) = \left\{ \alpha(n+1,n) - (N-2)\,\alpha(n+2,n+1) + \cdots + (-1)^k \binom{N-k-1}{k} \alpha(n+k+1,n+k) \right\}$$
$$- \left\{ \alpha(n+1,n+1) - (N-3)\,\alpha(n+2,n+2) + \cdots + (-1)^\gamma \binom{N-\gamma-2}{\gamma} \alpha(n+\gamma+1,n+\gamma+1) \right\} ,$$

with k as before and

$$\gamma = \left[\frac{N-2}{2} \right] .$$

Thus, by use of (1) each $\alpha(m,n)$ with $m \geq n$ can be expressed as a sum of terms of the form $\alpha(r+1,r)$ and $\alpha(r,r)$.

When there is a symmetry relation

$$\alpha(m,n) = \alpha(n,m) ,$$

[illegible]

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[illegible]

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[illegible]

[illegible]

then one has by (1) that

$$\alpha(r+1,r) = \tfrac{1}{2}\,\alpha(r,r) \quad .$$

In such case, every $\alpha(m,n)$ can be expressed as a sum of terms of the form $\alpha(r,r)$.

When there is an antisymmetry relation

$$\alpha(m,n) = -\,\alpha(n,m) \quad ,$$

then

$$\alpha(r,r) = 0 \quad ,$$

so that every $\alpha(m,n)$ can be expressed as a sum of terms of the form $\alpha(r+1,r)$.

3. Choice of basic integrals. Let

(4) $$f(x) = (2\pi)^{-1/2}\, e^{-x^2/2} \quad ,$$

(5) $$F(x) = \int_{-\infty}^{x} f(y)dy \quad .$$

Then

$$F(-x) = 1 - F(x) \quad ,$$

so that

$$F^m(x)\, F^n(-x)$$

satisfies (1). So if we multiply by $x^s f(x)$ and integrate, we see that

$$(6) \qquad \alpha_s(m,n) = \int_{-\infty}^{\infty} x^s f(x)\, F^m(x)\, F^n(-x)dx$$

satisfies (1). Replacing x by -x in the integrand gives further

$$(7) \qquad \alpha_{2t}(m,n) = \alpha_{2t}(n,m) \quad ,$$

$$(8) \qquad \alpha_{2t+1}(m,n) = - \alpha_{2t+1}(n,m) \quad ,$$

Hence every $\alpha_{2t}(m,n)$ can be expressed as a sum of terms of the form $\alpha_{2t}(r,r)$ and every $\alpha_{2t+1}(m,n)$ can be expressed as a sum of terms of the form $\alpha_{2t+1}(r+1,r)$, Further, we have

$$\frac{d}{dx}\left\{ x^{2t+1}\, F^{r+1}(x)\, F^{r+1}(-x)\right\} = (2t+1)\, x^{2t}\, F^{r+1}(x)\, F^{r+1}(-x)$$
$$+ (r+1)\, x^{2t+1}\, f(x)\, F^r(x)\, F^{r+1}(-x)$$
$$- (r+1)\, x^{2t+1}\, f(x)\, F^{r+1}(x)\, F^r(-x) \quad .$$

Integrating from $-\infty$ to ∞ and using (8) gives

$$(9) \qquad \alpha_{2t+1}(r+1,r) = \frac{2t+1}{2(r+1)} \int_{-\infty}^{\infty} x^{2t}\, F^{r+1}(x)\, F^{r+1}(-x)dx \quad ,$$

By defining

$$(10) \qquad \beta_t(r) = \int_{-\infty}^{\infty} x^{2t}\, F^r(x)\, F^r(-x)dx \quad ,$$

we can rewrite (9) as

$$\alpha_{2t+1}(r+1,r) = \frac{2t+1}{2(r+1)}\, \beta_t(r+1) \quad .$$

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Then we see that every $\alpha_s(m,n)$ can be written as a sum of terms of the form $\alpha_{2t}(r,r)$ or $\beta_t(r)$.

Since the s-th moment, about the origin, of the j-th observation (in magnitude) from a sample of n is given by

$$\frac{n!}{(j-1)!\,(n-j)!}\,\alpha_s(j-1,n-j)\,,$$

we see that evaluation of these moments is reduced to evaluation of the $\alpha_{2t}(r,r)$ and the $\beta_t(r)$.

4. Numerical determination of the basic integrals. In Table I numerical values of $4^r\alpha_{2t}(r,r)$ are given for $1 \leq t \leq 4$ and $1 \leq r \leq 10$. In Table II numerical values of $4^r\beta_t(r)$ are given for $0 \leq t \leq 3$ and $1 \leq r \leq 10$. These were mainly computed by numerical quadrature, and it is believed that the last digit shown is correct to within one unit.

For small values of r, one can get exact values (as in [3]):

$$\alpha_{2t}(0,0) = 1 \cdot 3 \cdots (2t-1)$$

$$\alpha_0(r,r) = \frac{(r!)^2}{(2r+1)!}$$

$$\alpha_2(1,1) = \frac{1}{6} - \frac{1}{2\pi\sqrt{3}}$$

$$\alpha_4(1,1) = \frac{1}{2} - \frac{13}{6\pi\sqrt{3}}$$

$$\alpha_6(1,1) = \frac{5}{2} - \frac{37}{3\pi\sqrt{3}}$$

$$\alpha_8(1,1) = \frac{35}{2} - \frac{820}{9\pi\sqrt{3}}$$

$$\alpha_2(2,2) = \frac{1}{30} - \frac{1}{4\pi\sqrt{3}} + \frac{3}{2\pi^2\sqrt{3}}\ \text{arccot}\ \sqrt{15}$$

$$\alpha_4(2,2) = \frac{13}{3}\,\alpha_2(2,2) - \frac{2}{45} + \frac{1}{4\pi^2\sqrt{5}}$$

$$\alpha_6(2,2) = \frac{74}{3}\ \alpha_2(2,2) - \frac{29}{90} + \frac{87}{40\pi^2\sqrt{5}}$$

$$\alpha_8(2,2) = \frac{1640}{9}\ \alpha_2(2,2) - \frac{139}{54} + \frac{2837}{150\pi^2\sqrt{5}}$$

$$\beta_0(1) = \frac{1}{\sqrt{\pi}}$$

$$\beta_1(1) = \frac{5}{6\sqrt{\pi}}$$

$$\beta_2(1) = \frac{43}{20\sqrt{\pi}}$$

$$\beta_3(1) = \frac{531}{56\sqrt{\pi}}$$

$$\beta_0(2) = \frac{1}{2\sqrt{\pi}} - \frac{3}{\pi\sqrt{\pi}}\ \text{arccot}\ \sqrt{8}$$

$$\beta_1(2) = \frac{5}{6}\ \beta_0(2) - \frac{1}{3\pi\sqrt{2\pi}}$$

$$\beta_2(2) = \frac{43}{20}\ \beta_0(2) - \frac{77}{60\pi\sqrt{2\pi}}$$

$$\beta_3(2) = \frac{531}{56}\ \beta_0(2) - \frac{6617}{1008\pi\sqrt{2\pi}}\ ,$$

We record the values

$$\text{arccot}\ \sqrt{15} = .25268\ 02551\ 42078\ 65348$$

$$\text{arccot}\ \sqrt{8} = .33983\ 69094\ 54121\ 93710\ ,$$

To compute $\alpha_{2t}(r,r)$ and $\beta_t(r)$ for $r > 10$, we proceed as follows. Clearly for $x \geq 0$,

$$f(x)(1 - 2F(x)) = -\frac{1}{\pi}\ e^{-x^2/2} \int_0^x e^{-y^2/2} dy$$

$$= -\frac{x}{\pi}\ e^{-x^2/2} \int_0^1 e^{-x^2w^2/2} dw\ .$$

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Integrating from 0 to x gives

$$F(x)\ F(-x) = \frac{1}{\pi} \int_0^1 \frac{e^{-x^2(1+w^2)/2}}{1+w^2}\, dw \quad .$$

Then for $0 \leq y$,

$$e^{y/2}\ F(\sqrt{y}\,)\ F(-\sqrt{y}\,)$$

is completely monotonic (as defined in [1]). Hence

$$4^m\ e^{my/2} \left\{ F(\sqrt{y}\,)\ F(-\sqrt{y}\,) \right\}^m$$

is completely monotonic. So by Theorem B of [1], $G_1(y)$ is completely monotonic, where

$$G_1(y) = \frac{1}{y^2} \left\{ 4^m\ e^{my/2} \left\{ F(\sqrt{y}\,)\ F(-\sqrt{y}\,) \right\}^m - e^{-(4-\pi)my/2\pi} \right\} \quad .$$

We infer

$$0 \leq G_1(y) \leq G_1(0) = \frac{2(\pi - 3)}{3\pi^2}\, m \quad .$$

Multiplying by $y^2 \exp(-my/2)$ and putting $y = x^2$ gives

$$0 \leq 4^m \left\{ F(x)\ F(-x) \right\}^m - e^{-2mx^2/\pi}$$

$$\leq \frac{2(\pi - 3)}{3\pi^2}\, mx^4\ e^{-mx^2/2} \quad .$$

Putting $m = 1$ gives for $x \geq 0$

$$\left\{1 - e^{-2x^2/\pi} - \frac{2(\pi - 3)}{3\pi^2} x^4 e^{-x^2/2}\right\}^{1/2}$$

$$\leq \frac{1}{\sqrt{2\pi}} \int_{-x}^{x} e^{-t^2/2} dt \leq \left\{1 - e^{-2x^2/\pi}\right\}^{1/2} .$$

From this, one quickly derives many of the results of [2].

Since $G_1(y)$ is completely monotonic, we can apply Theorem B of [1] again to infer that $G_2(y)$ is completely monotonic, where

$$G_2(y) = \frac{1}{y^2}\left\{G_1(y) - m\,\frac{2(\pi - 3)}{3\pi^2} \exp\left(-\frac{(4-\pi)m}{2\pi}\,y - \frac{7\pi^2 - 60\pi + 120}{30\pi(\pi - 3)}\,y\right)\right\} .$$

So

$$0 \leq G_2(y) \leq G_2(0) .$$

But

$$G_2(0) = \frac{2(\pi - 3)^2}{9\pi^4} m^2 + \left\{\frac{3\pi^3 - 56\pi^2 + 280\pi - 420}{105\pi^4} - \frac{(7\pi^2 - 60\pi + 120)^2}{2700\pi^4(\pi - 3)}\right\} m$$

$$= 0.0000\,45737 m^2 - 0.0000\,12634\, m .$$

Multiplying by $y^4 \exp(-my/2)$ and putting $y = x^2$ gives

$$0 \leq 4^m \left\{F(x)\, F(-x)\right\}^m - e^{-2mx^2/\pi}$$

$$- \frac{2(\pi - 3)}{3\pi^2} m x^4 \exp\left(-\frac{2mx^2}{\pi} - \frac{7\pi^2 - 60\pi + 120}{30\pi(\pi - 3)} x^2\right)$$

$$\leq \left\{0.00005 m^2 - 0.00001 m\right\} x^8 e^{-mx^2/2} .$$

Proceeding to make two more uses of Theorem B of [1] gives

$$0 \le 4^m \left\{ F(x)\, F(-x) \right\}^m - \theta(m,x)$$

$$\le \phi(m)\, x^{16}\, e^{-mx^2/2} \quad ,$$

where

$$\begin{aligned}\theta(m,x) = {} & e^{-2mx^2/\pi} \\ & + K_1 m\, x^4 \exp\left(- \frac{2mx^2}{\pi} - K_2\, x^2 \right) \\ & + (K_3 m^2 - K_4 m)\, x^8 \exp\left(- \frac{2mx^2}{\pi} - K_2 x^2 - K_2 \frac{m - K_5}{m - K_6} x^2 \right) \\ & + \frac{K_7 m^4 - K_8 m^3 + K_9 m^2 - K_{10} m}{m - K_6}\, x^{12} \exp\left(- \frac{2mx^2}{\pi} - K_2 x^2 \right. \\ & \left. - K_2 \frac{m - K_5}{m - K_6} x^2 - \frac{K_{11} m^4 - K_{12} m^3 + K_{13} m^2 - K_{14} m + K_{15}}{(m - K_6)(K_7 m^3 - K_8 m^2 + K_9 m - K_{10})} x^2 \right) ,\end{aligned}$$

$$\begin{aligned}\phi(m) = {} & \left\{ K_{16} m^{10} - K_{17} m^9 + K_{18} m^8 - K_{19} m^7 + K_{20} m^6 - K_{21} m^5 + K_{22} m^4 - K_{23} m^3 \right. \\ & \left. + K_{24} m^2 - K_{25} m \right\} \left\{ (m - K_6)^3\, (K_7 m^3 - K_8 m^2 + K_9 m - K_{10}) \right\}^{-1} ,\end{aligned}$$

and

$$\frac{2}{\pi} = 0.63661\ 97723\ 67581\ 34308$$

$$K_1 = \frac{2(\pi - 3)}{3\pi^2} = 9.5642\ 23504\ 51823\ 814 \times 10^{-3}$$

$$K_2 = \frac{7\pi^2 - 60\pi + 120}{30\pi(\pi - 3)} = 4.4337\ 26195\ 26472\ 9694 \times 10^{-2}$$

$$K_3 = 4.5737\ 18562\ 21895\ 644 \times 10^{-5}$$

$$K_4 = 1.2634\ 27027\ 83428\ 618 \times 10^{-5}$$

$$K_5 = 0.71005\ 42616\ 12505\ 98$$

$$K_6 = K_4/K_3 = 0.27623\ 62857\ 80848\ 29$$

$$K_7 = 1.4581\ 35552\ 52753\ 02 \times 10^{-7}$$

$$K_8 = 1.6111\ 59796\ 78083\ 98 \times 10^{-7}$$

$$K_9 = 7.3706\ 39743\ 26849\ 1 \times 10^{-8}$$

$$K_{10} = 1.9600\ 16255\ 51239\ 8 \times 10^{-8}$$

$$K_{11} = 6.4649\ 73795\ 48812\ 1 \times 10^{-9}$$

$$K_{12} = 1.4538\ 54543\ 52510\ 69 \times 10^{-8}$$

$$K_{13} = 1.1327\ 56568\ 89158\ 99 \times 10^{-8}$$

$$K_{14} = 3.1274\ 67461\ 71943\ 6 \times 10^{-9}$$

$$K_{15} = 1.5987\ 53417\ 33771 \times 10^{-10}$$

$$K_{16} = 5.0837\ 65663 \times 10^{-17}$$

$$K_{17} = 1.8256\ 16708 \times 10^{-16}$$

$$K_{18} = 3.1469\ 17320 \times 10^{-16}$$

$$K_{19} = 3.5809\ 66407 \times 10^{-16}$$

$$K_{20} = 2.9053\ 68156 \times 10^{-16}$$

$$K_{21} = 2.6263\ 40177 \times 10^{-16}$$

$$K_{22} = 5.6170\ 32050 \times 10^{-17}$$

$$K_{23} = 1.1854\ 05126 \times 10^{-17}$$

$$K_{24} = 1.3845\ 19506 \times 10^{-18}$$

$$K_{25} = 6.8859\ 48529 \times 10^{-20}$$

We have then

$$0 \leq 4^m \ \beta_t(m) - \int_{-\infty}^{\infty} x^{2t} \ \theta(m,x)dx$$

$$\leq \phi(m) \frac{1 \cdot 3 \cdot 5 \cdots (2t+15)}{m^{8+t}} \sqrt{\frac{2\pi}{m}} \quad .$$

However, we can write

$$\int_{-\infty}^{\infty} x^{2t} \ \theta(m,x)dx$$

in closed form; indeed the leading (and dominant) term is

$$\frac{1 \cdot 3 \cdot 5 \cdots (2t-1)\pi^{t+1}}{(4m)^t \ \sqrt{2m}} \quad .$$

Also for $m \geq 11$,

$$\phi(m) \leq (4.2)\ m^4\ 10^{-10} \quad .$$

Thus for $m \geq 11$, we have quite an accurate approximation to $4^m \ \beta_t(m)$.

If a more accurate approximation is desired, one can carry out a numerical quadrature of the function

$$x^{2t}\left(4^m\left\{F(x)\ F(-x)\right\}^m - \theta(m,x)\right) \quad .$$

For large m, this function is both small and very smooth, and one can readily get a high accuracy by a simple numerical quadrature.

In a similar fashion, we can get accurate numerical approximations for the $\alpha_{2t}(r,r)$.

5. Estimates for means. In computing moments of arbitrary observations from the moments of central observations by the formulas given in Section 2, it is clear that there will be considerable loss of significant figures for the outlying observations when the sample size is large. We now have sufficiently accurate values for the central observations that this effect is not damaging for samples of reasonable size. Thus, we used this method to compute the means of observations for samples of size twenty or fewer. The results are tabulated in Table III, where $m(j|n)$ denotes the mean of the j-th observation from a sample of size n. We have retained as many decimals as we think are accurate, but do not guarantee the final digit shown. Our table extends the tables of means given in [3] and [4]. Clearly, one could use our estimates of $\beta_0(r)$ to extend the accuracy of Table XX of [5].

Incidentally, in making up such a table, one would start with the $\alpha(r,r)$ and $\alpha(r+1,r)$ and work outward by means of the relation (1). Our explicit formula for $\alpha(n+N,n)$ would be used only to estimate the accumulation of errors, or to compute an isolated value.

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Quite clearly, the method outlined above for computing moments is essentially a recursive procedure, in that the computation of means for a sample of a hundred would require the computation of a great many means for observations from smaller samples. Also for large sample sizes, the loss of significant figures begins to be serious for the outlying observations. For these reasons, it seems worthwhile to provide some way of estimating means of observations in a large sample, particularly the outlying observations.

For $m(j|n)$, there is a classical asymptotic estimate, which we shall denote by $\gamma(j|n)$. It is defined by the equation

$$F(\gamma(j|n)) = \frac{j}{n+1} .$$

Some discussion of this estimate is given in [4] and [6]. Values of $\gamma(j|n)$ are tabulated in Table III, from which it is clear that even for a sample of size twenty, the estimate is rather poor, especially for outlying observations.

To obtain a better estimate, we start by inquiring for the value of δ which minimizes

$$\text{(11)} \qquad \int_{-\infty}^{\infty} (x-\delta)^2 f(x)\, F^{j-1}(x)\, F^{n-j}(-x)\, dx .$$

Clearly (11) can be expanded to

$$\alpha_2(j-1,n-j) - 2\delta\,\alpha_1(j-1,n-j) + \delta^2 \frac{(j-1)!\,(n-j)!}{n!} .$$

[illegible]

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So (11) is a minimum when

$$\delta = \frac{n!}{(j-1)!\ (n-j)!} \alpha_1(j-1,n-j) = m(j|n) \quad .$$

That is, $m(j|n)$ is the value of δ which minimizes (11). So we estimate $m(j|n)$ by seeking a value of δ which makes (11) small. To this end, we first observe that

$$f(x)\ F^{j-1}(x)\ F^{n-j}(-x) \tag{12}$$

is non-negative, with a single maximum. If (12) were exactly symmetrical, then the x which maximizes (12) would obviously coincide with the δ which minimizes (11). Actually, (12) is reasonably symmetric about its maximum for many values of n and j. Thus one is tempted to use the x at which (12) takes its maximum as an estimate for the δ at which (11) takes its minimum. Accordingly, we define $\delta(j|n)$ to be the value of x at which (12) takes its maximum, and hope that thereby $\delta(j|n)$ will be a good estimate for $m(j|n)$.

We have listed values of $\delta(j|n)$ in Table III. For all entries in Table III, we have

$$\gamma(j|n) \leq \delta(j|n) \leq m(j|n) \quad .$$

One is led to conjecture that this holds for all j and n, so that $\delta(j|n)$ is a better estimate for $m(j|n)$ than $\gamma(j|n)$.

From Table III, one is tempted to make still another conjecture, namely that except for outlying observations, $\delta(j|n)$ lies about halfway between $\gamma(j|n)$ and $m(j|n)$. Thus we are led to suggest

$$2\ \delta(j|m) - \gamma(j|n)$$

as an estimate for $m(j|n)$. This estimate is also tabulated in Table III. The four decimal values listed for $2\delta - \gamma$ may be off by as much as three units in the fourth decimal place, since they were computed directly from the four decimal values of $\gamma(j|n)$ and $\delta(j|n)$.

From Table III, it appears that $2\delta - \gamma$ is a fairly successful estimate for $m(j|n)$ except for the outlying observations, for which it does not furnish a good estimate. The reason for this is fairly clear. As $n \to \infty$, the quantity $\delta(n|n) - \gamma(n|n)$ approaches zero fairly rapidly. Thus γ, δ, and $2\delta - \gamma$ are all about the same for large n. However (see [7]), the asymptotic formula for $m(n|n)$ is

$$\gamma(n|n) + \frac{C}{n\, f(\gamma(n|n))} \,, \qquad (13)$$

where C is the Euler constant. For large n, $\gamma(n|n)$, $\delta(n|n)$, and $n\, f(\gamma(n|n))$ are all nearly equal, and all go very slowly indeed to ∞ as n goes to ∞. Thus δ, γ, and $2\delta - \gamma$ all become nearly equal before any of them is a good approximation to $m(n|n)$.

For outlying observations near the extreme, the situation is similar, but becomes less severe as one moves further from the extreme (see equation (6) of [7]).

For the extreme observation, an approximation was proposed by Bortkiewicz (see [8]), namely

$$B(n) = \tfrac{1}{2} \left\{ \delta(n|n) + f(\gamma((n-1)|(n-1))) \sqrt{n(n-1)} \right\} .$$

Sample values of this are listed in Table IV, where for comparison are also listed values of $m(n|n)$. For $n > 20$, the values of $m(n|n)$ are taken from Tippett's table (see [9]).

We wish to propose an alternative formula for $m(n|n)$. Our formula is better than Bortkiewicz's formula for smallish n, say $n \leq 100$, but Bortkiewicz's formula is better for large n, at least as far as $n = 1000$.

One advantage of our formula is that it is less empirical than Bortkiewicz's. Indeed, one may use a similar derivation to get similar estimates for other outlying observations. Furthermore, although our formula appears eventually to become inferior to Bortkiewicz's, the discrepancies are far more uniform, and can be explained. Then by curve fitting methods, one can get a formula which gives a really close fit for $2 \leq n \leq 1000$, and which can be expected to continue to give a close fit for larger n.

We recall that $m(n|n)$ is the value of δ which maximizes

$$\int_{-\infty}^{\infty} (x - \delta)^2 f(x) F^{n-1}(x)\, dx.$$

Write temporarily $\phi = \delta(n|n)$, so that

$$\phi F(\phi) = (n-1) f(\phi) \quad . \tag{14}$$

Then $m(n|n) - \phi$ is the value of θ which minimizes

$$\int_{-\infty}^{\infty} ((x-\phi)-\theta)^2 f(x) F^{n-1}(x)\, dx \quad .$$

But this value of θ is given by

$$\theta = \left\{ \int_{-\infty}^{\infty} (x-\phi) f(x) F^{n-1}(x) dx \right\} \left\{ \int_{-\infty}^{\infty} f(x) F^{n-1}(x) dx \right\}^{-1} \quad .$$

[illegible]

[illegible]

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[illegible]

[illegible]

[illegible]

$$[illegible]$$

To estimate this, we expand the integrands as power series in $x - \phi$. Using (14), the first few terms are given by

$$f(x)\,F^{n-1}(x) = \frac{\phi\,F^{n}(\phi)}{n-1}\left\{1 - \frac{(x-\phi)^2}{2}\left(1 + \frac{\phi^2 n}{n-1}\right) + \frac{(x-\phi)^3}{6}\,\phi\left(\frac{\phi^2 n(n+1)}{(n-1)^2} - 1\right) + \cdots\right\}.$$

Thus we are led to propose the approximation

$$f(x)\,F^{n-1}(x) = \frac{\phi\,F^{n}(\phi)}{n-1}\exp\left\{-\frac{(x-\phi)^2}{2}\left(1 + \frac{\phi^2 n}{n-1}\right)\right\}\left\{1 + \frac{(x-\phi)^3}{6}\,\phi\left(\frac{\phi^2 n(n+1)}{(n-1)^2} - 1\right)\right\}.$$

Using this approximation in both the integrands of our formula for θ, we get the approximation

$$\theta = \frac{\phi(\phi^2 n(n+1) - (n-1)^2)}{2(\phi^2 n + n-1)^2}.$$

Thus we are led to the approximation for $m(n|n)$ given by

$$C(n) = \delta(n|n)\left\{1 + \frac{(\delta(n|n))^2 n(n+1) - (n-1)^2}{2((\delta(n|n))^2 n + n-1)^2}\right\}.$$

Values of $C(n)$ are listed in Table IV.

For very large n, we have approximately

$$C(n) = \delta(n|n) + \frac{1}{2\,\delta(n|n)}.$$

Comparison with (13) shows that for large n, $C(n)$ will be less than $m(n|n)$, since $\gamma(n|n)$, $\delta(n|n)$, and $n\,f(\gamma(n|n))$ are all nearly equal for large n. Indeed, from Table IV, one would conjecture that $C(n) < m(n|n)$ for $n \geq 7$.

In view of (13), one sees that for very large n a better formula than $C(n)$ would be

$$\delta(n|n) + 2C\left\{C(n) - \delta(n|n)\right\}.$$

Fitting empirically the difference between this and m(n,n), we are led finally to propose

$$D(n) = \delta(n|n)\left\{1 + C\,\frac{(\delta(n|n))^2 n(n+1) - (n-1)^2}{((\delta(n|n))^2 n + n-1)^2}\right\}$$

$$- \frac{1}{19.541(\delta(n|n))^2 - 66.366\,\delta(n|n) + 114.448}\,.$$

Values of D(n) are listed in Table IV.

The numerical coefficients in the denominator of the second term of D(n) were chosen to give a fit in the region $2 \leq n \leq 100$. Clearly, they could be chosen to give a fit in any other region. However, in view of (13), one might expect high accuracy for D(n) for all values of n.

If one has a good estimate for the mean of the extreme observation, one can derive a good estimate for its standard deviation by using the empirical formula proposed in [10].

Incidentally, by the time one gets to n = 1000, the asymptotic formula (13) agrees with m(n|n) to within less than one per cent. One would suspect a similar accuracy for (6) of [7] for other small values of m.

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October 17, 1951.

TABLE I.

Values of $4^r \alpha_{2t}(r,r)$

r	t = 1	t = 2
1	0.2991 1406 9718 8053	0.4072 7207 9892 6007
2	0.1529 7795 2856 4674	0.1330 4219 4614 9201
3	0.0962 0427 9558 9977	0.0612 8897 0422 4789
4	0.0674 9512 3853 7019	0.0338 9389 5809 0415
5	0.0506 6895 0920 9147	0.0209 8985 6215 1131
6	0.0398 2756 2731 0725	0.0140 3808 4528 5253
7	0.0323 6529 0981 3039	0.0099 2626 8505 3476
8	0.0269 7240 6234 8521	0.0073 2100 7671 5223
9	0.0229 2631 2941 7515	0.0055 8079 0652 7179
10	0.0197 9919 4831 1596	0.0043 6874 1679 6848
	t = 3	t = 4
1	0.9337 0260 8619 4196	3.0237 4900 0611 929
2	0.1947 6555 2031 6235	0.4028 7649 7414 444
3	0.0656 4164 9173 2218	0.0992 4109 4940 865
4	0.0285 8077 9942 2641	0.0339 8616 3643 807
5	0.0145 8715 6381 2597	0.0142 8346 2227 132
6	0.0082 9464 7482 9277	0.0069 0054 1881 627
7	0.0051 0028 1317 1660	0.0036 8762 1539 726
8	0.0033 2741 7523 9426	0.0021 2706 2546 631
9	0.0022 7385 5368 6564	0.0013 0254 5317 193
10	0.0016 1294 4832 8192	0.0008 3694 8262 323

TABLE II.

Values of $4^r \beta_t(r)$.

r	t = 0	t = 1
1	2.2567 5833 4191 0251	1.8806 3194 5159 1876
2	1.5840 6070 5464 7751	0.6427 8513 7922 0755
3	1.2898 9973 6330 9072	0.3453 8405 5478 4261
4	1.1155 3389 3536 3525	0.2228 1607 0880 1860
5	0.9969 1887 5707 6363	0.1587 7144 7923 6560
6	0.9095 3967 2162 7550	0.1204 4214 6659 3750
7	0.8417 2526 9829 5399	0.0953 8452 3113 1459
8	0.7871 1902 6160 2866	0.0779 5172 7041 1192
9	0.7419 2432 8542 4813	0.0652 4962 7985 1744
10	0.7037 1536 6288 9335	0.0556 5770 1650 7985
	t = 2	t = 3
1	4.8520 3041 8510 7041	21.3989 0491 8847 042
2	0.7982 5853 4383 1039	1.6825 9501 5448 810
3	0.2814 2174 8929 4527	0.3873 8273 5294 494
4	0.1349 9461 2115 8106	0.1377 7182 9511 448
5	0.0765 4181 1168 1755	0.0620 3993 4413 791
6	0.0482 0967 3030 9395	0.0324 0067 7160 751
7	0.0326 3892 7355 2676	0.0187 3390 2424 600
8	0.0232 9272 7472 9138	0.0116 6593 9436 320
9	0.0173 0364 7795 6415	0.0076 8672 8404 898
10	0.0132 6717 8669 9766	0.0052 9499 0746 468

TABLE III.

Table of accurate and approximate means.

n	j	m(j|n)	$\gamma(j\|n)$	$\delta(j\|n)$	$2\delta-\gamma$
2	2	0.5641 89583 54775 62869	0.4307	0.5061	0.5815
3	3	0.8462 84375 32163 44304	0.6745	0.7653	0.8561
4	4	1.0293 75373 00396 41321	0.8416	0.9359	1.0302
4	3	0.2970 11382 27464 53255	0.2533	0.2775	0.3017
5	5	1.1629 64473 64051 96128	0.9674	1.0615	1.1556
5	4	0.4950 18970 45774 22092	0.4307	0.4654	0.5001
6	6	1.2672 06360 61147 12976	1.0676	1.1602	1.2528
6	5	0.6417 55038 78576 11884	0.5659	0.6058	0.6457
6	4	0.2015 46833 80170 42508	0.1800	0.1919	0.2038
7	7	1.3521 78375 60690 43992	1.1504	1.2412	1.3320
7	6	0.7573 74270 63887 26882	0.6745	0.7171	0.7597
7	5	0.3527 06959 15298 24389	0.3186	0.3371	0.3556
8	8	1.4236 00306 04527 77531	1.2207	1.3095	1.3983
8	7	0.8522 24862 53829 09223	0.7647	0.8087	0.8527
8	6	0.4728 22494 94061 79858	0.4307	0.4533	0.4759
8	5	0.1525 14399 50692 31941	0.1397	0.1467	0.1537
9	9	1.4850 13162 20923 70063	1.2816	1.3684	1.4552
9	8	0.9322 97456 73360 37275	0.8416	0.8863	0.9310
9	7	0.5719 70782 85469 61041	0.5244	0.5495	0.5746
9	6	0.2745 25919 11246 17493	0.2533	0.2649	0.2765
10	10	1.5387 52730 83517 28560	1.3352	1.4202	1.5052
10	9	1.0013 57044 57581 43585	0.9085	0.9534	0.9983
10	8	0.6560 59105 36476 12033	0.6046	0.6313	0.6580
10	7	0.3757 64696 99787 75394	0.3488	0.3633	0.3778
10	6	0.1226 67752 28433 80642	0.1142	0.1188	0.1234
11	11	1.5864 36351 90800 01689	1.3830	1.4662	1.5494
11	10	1.0619 16520 10689 97268	0.9674	1.0124	1.0574
11	9	0.7288 39404 68593 02011	0.6745	0.7023	0.7301
11	8	0.4619 78307 17497 72089	0.4307	0.4474	0.4641
11	7	0.2248 90879 18795 31177	0.2104	0.2183	0.2262
12	12	1.6292 27639 87191 29903	1.4261	1.5076	1.5891
12	11	1.1157 32184 30495 91343	1.0201	1.0648	1.1095
12	10	0.7928 38199 11660 26895	0.7363	0.7649	0.7935
12	9	0.5368 43021 39391 27361	0.5024	0.5206	0.5388
12	8	0.3122 48878 73710 61544	0.2934	0.3036	0.3138
12	7	0.1025 89679 81913 88664	0.0966	0.0999	0.1032
13	13	1.6679 90177 04912 74980	1.4652	1.5452	1.6252

[illegible]

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TABLE III (continued)

Table of accurate and approximate means.

n	j	m(j|n)	γ(j|n)	δ(j|n)	$2\delta-\gamma$
13	12	1.1640 77193 74533 88972	1.0676	1.1120	1.1564
13	11	0.8498 34632 38287 04386	0.7916	0.8207	0.8498
13	10	0.6028 50088 22904 35257	0.5659	0.5852	0.6045
13	9	0.3883 27121 01486 84596	0.3661	0.3780	0.3899
13	8	0.1905 23691 09268 64661	0.1800	0.1857	0.1914
14	14	1.7033 81554 09997 65215	1.5011	1.5796	1.6581
14	13	1.2079 02275 38809 01931	1.1108	1.1548	1.1988
14	12	0.9011 26703 88883 11217	0.8416	0.8709	0.9002
14	11	0.6617 63703 52768 12673	0.6229	0.6430	0.6631
14	10	0.4555 66049 98244 91716	0.4307	0.4440	0.4573
14	9	0.2672 97048 87322 31781	0.2533	0.2609	0.2685
14	8	0.0881 59214 05197 08501	0.0837	0.0862	0.0887
15	15	1.7359 13444 94103 74337	1.5341	1.6113	1.6885
15	14	1.2479 35082 32512 37511	1.1503	1.1940	1.2377
15	13	0.9476 89030 29737 20662	0.8871	0.9166	0.9461
15	12	0.7148 77398 25466 73439	0.6745	0.6952	0.7159
15	11	0.5157 01043 02846 95569	0.4888	0.5030	0.5172
15	10	0.3352 96063 89040 84009	0.3186	0.3278	0.3370
15	9	0.1652 98526 34744 53440	0.1573	0.1617	0.1661
16	16	1.7659 91393 05478 79673	1.5647	1.6406	1.7165
16	15	1.2847 44223 23477 94295	1.1868	1.2301	1.2734
16	14	0.9902 71095 95753 40023	0.9289	0.9584	0.9879
16	13	0.7631 66745 77000 36761	0.7215	0.7426	0.7637
16	12	0.5700 09355 70865 83470	0.5414	0.5564	0.5714
16	11	0.3962 22755 13205 42186	0.3774	0.3875	0.3976
16	10	0.2337 51578 48766 53715	0.2230	0.2289	0.2348
16	9	0.0772 87459 31001 95943	0.0738	0.0757	0.0776
17	17	1.7939 41980 88269 08735	1.5932	1.6679	1.7426
17	16	1.3187 81987 80834 14671	1.2206	1.2635	1.3064
17	15	1.0294 60988 93306 41474	0.9674	0.9969	1.0264
17	14	0.8073 84928 73839 33255	0.7647	0.7861	0.8075
17	13	0.6194 57651 12273 73158	0.5895	0.6051	0.6207
17	12	0.4513 33446 71486 88219	0.4307	0.4417	0.4527
17	11	0.2951 86487 23022 74458	0.2822	0.2892	0.2962
17	10	0.1459 87423 14114 81225	0.1397	0.1431	0.1465

TABLE III (concluded)

Table of accurate and approximate means.

n	j	m(j|n)	γ(j|n)	δ(j|n)	2δ − γ
18	18	1.8200 31878 96872 21046	1.6199	1.6934	1.7669
18	17	1.3504 13713 42015 99452	1.2521	1.2946	1.3371
18	16	1.0657 28182 91379 36422	1.0021	1.0326	1.0631
18	15	0.8481 25019 02941 66731	0.8046	0.8262	0.8478
18	14	0.6647 94612 71981 16086	0.6336	0.6497	0.6658
18	13	0.5015 81550 97034 41543	0.4795	0.4912	0.5029
18	12	0.3508 37238 20391 81572	0.3360	0.3440	0.3520
18	11	0.2077 35307 12871 34707	0.1992	0.2038	0.2084
18	10	0.0688 02568 15669 14373	0.0660	0.0675	0.0690
19	19	1.8444 81511 60382 46581	1.6449	1.7173	1.7897
19	18	1.3799 38491 53687 61420	1.2816	1.3236	1.3656
19	17	1.0994 53099 42807 22722	1.0364	1.0658	1.0952
19	16	0.8858 61961 50430 76155	0.8416	0.8634	0.8852
19	15	0.7066 11484 74857 56393	0.6745	0.6909	0.7073
19	14	0.5477 07371 03927 23228	0.5244	0.5367	0.5490
19	13	0.4016 42274 15433 31226	0.3853	0.3941	0.4029
19	12	0.2637 42890 86034 96451	0.2533	0.2590	0.2647
19	11	0.1307 24879 49771 37309	0.1257	0.1284	0.1311
20	20	1.8674 75059 79832 04847	1.6684	1.7398	1.8112
20	19	1.4076 04095 90840 39521	1.3092	1.3508	1.3924
20	18	1.1309 48052 19312 58507	1.0676	1.0968	1.1260
20	17	0.9209 81700 42610 19940	0.8761	0.8980	0.9199
20	16	0.7453 83005 81713 01013	0.7124	0.7292	0.7460
20	15	0.5902 96921 54291 22532	0.5659	0.5787	0.5915
20	14	0.4483 31753 19744 58187	0.4307	0.4401	0.4495
20	13	0.3149 33241 64569 52584	0.3030	0.3094	0.3158
20	12	0.1869 57364 68233 12250	0.1800	0.1838	0.1876
20	11	0.0619 96286 49429 23493	0.0597	0.0610	0.0623

[illegible]

[illegible]

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TABLE IV.

Table of accurate and approximate extreme means.

n	m(n|n)	B(n)	C(n)	D(n)
2	0.5642	0.5351	0.5654	0.5629
3	0.8463	0.8280	0.8474	0.8467
4	1.0294	1.0183	1.0301	1.0302
5	1.1630	1.1568	1.1633	1.1639
6	1.2672	1.2644	1.2672	1.2681
7	1.3522	1.3518	1.3519	1.3529
8	1.4236	1.4250	1.4231	1.4242
9	1.4850	1.4878	1.4842	1.4855
10	1.5388	1.5426	1.5377	1.5391
11	1.5864	1.5911	1.5852	1.5867
12	1.6292	1.6345	1.6278	1.6294
13	1.6680	1.6738	1.6664	1.6681
14	1.7034	1.7097	1.7017	1.7034
15	1.7359	1.7426	1.7340	1.7358
16	1.7660	1.7729	1.7640	1.7659
17	1.7939	1.8011	1.7918	1.7937
18	1.8200	1.8274	1.8178	1.8198
19	1.8445	1.8520	1.8421	1.8442
20	1.8675	1.8752	1.8650	1.8671
30	2.0428	2.0509	2.0395	2.0422
40	2.1608	2.1687	2.1570	2.1601
50	2.2491	2.2567	2.2449	2.2484
70	2.3774	2.3842	2.3726	2.3767
100	2.5076	2.5134	2.5023	2.5071
200	2.7460	2.7496	2.7398	2.7458
300	2.8778	2.8800	2.8711	2.8777
500	3.0367	3.0374	3.0296	3.0368
700	3.1376	3.1374	3.1301	3.1378
1000	3.2414	3.2403	3.2338	3.2418

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