

# NATIONAL BUREAU OF STANDARDS REPORT

1293

INTRODUCTION TO THE THEORY OF STOCHASTIC PROCESSES  
DEPENDING ON A CONTINUOUS PARAMETER

By

Henry B. Mann

Ohio State University



U. S. DEPARTMENT OF COMMERCE  
NATIONAL BUREAU OF STANDARDS



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Contemplator enim, cum solis lumina cumque  
Inserti fundunt radii per opaca domorum,  
Multi minuta modis multis per inane videbis  
Corpora misceri, radiorum lumine in ipso.  
Et velut eterno certamine proelia, pugnas  
Edere turmatim certantia nec dare pausam,  
Conciliis et discidiis exercita crebris.  
Conicere ut possis ex hoc, primordia rerum  
Quale sit in magno iactari semper inani.  
Dumtaxat rerum magnarum parva potest res  
Exemplare dare et vestigia notitiae.  
Hoc etiam magis haec animum te advertere par est  
Corpora, quae in solis radiis turbare videntur  
Quod tales turbae motus quoque materiai  
Significant clandestinos caecosque subesse.  
Multa videbis enim plagis ibi percita caecis  
Commutare viam retroque pulsa reverti  
Nunc huc nunc illuc in cunctas undique partes,  
Titus Lucretius Carus  
De Rerum Natura: Vol. II, Vers 113-130.

Let us observe as brightly the rays of the sun  
Penetrate in streams the darkness of our houses  
Thousands of tiny bodies dancing in space  
Approaching each other and parting in the bright light of the sun.  
As if fighting a battle without pause through the ages,  
Like an army of soldiers restlessly warring,  
They advance and retreat in motion never to cease.  
May you conjecture from this the very nature of matter,  
How it is ceaselessly tossed through the vastness of space.  
Thus a phenomenon small as it seems and of little importance  
Often does indicate things highly important and great.  
Hence it is well worthwhile to observe these bodies  
Whirling and dancing without rest in the sunlight,  
Since such irregular motion of visible bodies  
Is a sure indication of the invisible motion of matter.  
For you can see these bodies constantly changing direction,  
Often reversing their motion all of a sudden  
And propelled by invisible impacts moving this way and that way.



## TABLE OF CONTENTS

Foreword . . . . .	I
Introduction . . . . .	III
<u>Chapter 1. Fundamental Concepts</u> . . . . .	1
1. Random variables	1
2. Convergence	2
3. Stochastic processes	10
4. Convergence in the mean	11
5. Differentiation	14
6. Stochastic processes of second order	15
7. Integration	17
<u>Chapter 2. Special Processes</u> . . . . .	32
1. The fundamental random process	32
2. Further properties of the F.R.P.	37
3. Frictional effects. The Ornstein-Uhlenbeck process	47
<u>Chapter 3. Estimation of Parameters</u> . . . . .	55
1. Estimation of the parameter of the F.R.P.	55
2. F.R.P. with mean value function	57
3. Estimation of parameters for the O.U.P.	64
<u>Chapter 4. The General Differential Process</u> . . . . .	72
<u>Chapter 5. Differential Processes Modified by Mechanical Devices</u> . . . . .	86
1. Filter effect	86
2. Counter data	91
<u>Chapter 6. The Fourier Analysis of Stochastic Processes</u> . . . . .	100
1. General theory	100
2. Trigonometric expansion of the F.R.P.	103
3. Stationary processes	108
4. The mean ergodic theorem	112
Index . . . . .	120





## FOREWORD

The theory of stochastic processes is steadily gaining in importance and the applications are ever widening. Nevertheless, it is at present not easy to study this subject, since the literature, although extensive, is widely scattered.

This situation motivated the National Bureau of Standards to invite Professor Henry B. Mann to give a series of lectures on stochastic processes and to write a monograph on the subject. The lectures were given in the period from March 1949 to June 1949, during which Dr. Mann was a member of the staff of the Bureau's Statistical Engineering Laboratory.

It is well known that the theory of stochastic processes depending on a continuous parameter can be developed in a satisfactory way by studying random functions or by considering probability measures in function space. The author of the present monograph has however adopted a different approach which is similar to the definition of a stochastic process given by E. Slutsky. A random variable is considered to be a symbol with which a distribution function is associated, and a stochastic process is then defined as a set of random variables. This approach leads to a theory which for many practical purposes is equivalent to the direct measure-theoretical approach. It has the advantage that the technicalities of measure theory seem less obtrusive at the outset, although for logical completeness they must enter sooner or later if the theory is to be developed in a well-rounded way.



It is hoped that this modest little volume, written by a distinguished contemporary mathematician, will be useful and interesting in various ways. The argument is addressed uncompromisingly to educated mathematicians, and they will not fail to be impressed by the skillful way in which the author develops the theory from his chosen starting point. The user of time-continuous processes in the applied fields who is not interested in the methods of proof may still appreciate having a number of important definitions and results conveniently gathered here between two covers.

Finally, it is hoped that the publication of the monograph will stimulate further expository efforts in the important field of time-continuous stochastic processes, and that in particular the day will come a little sooner than it otherwise might have, when a comprehensive but readable textbook on this subject, using the measure-theoretical approach, appears in the English language.

J. H. Curtiss

National Applied Mathematics Laboratories  
National Bureau of Standards  
Washington 25, D. C.  
October 1951



INTRODUCTION

The study of stochastic processes is becoming increasingly important in many branches of science and accordingly the mathematical theory of stochastic processes has progressed rapidly during the last two decades. This rapid progress has resulted in a large diversification of notation and terminology which makes it difficult even for a mathematician to inform himself on the subject. It seemed, therefore, advisable to bring together under a unified terminology and notation some of the basic definitions and results of this theory. The viewpoint taken was that of the mathematical statistician, and the stochastic process was accordingly defined as a family of distribution functions satisfying certain consistency relations. It was one of the goals of the present monograph to develop the theory of stochastic processes from this viewpoint with as little appeal to abstract measure theory as possible. In most practical problems information about random variables can be obtained only in terms of their joint distribution function, and it is the opinion of the author that a treatise on stochastic processes will be most useful to the statistician if the definitions, theorems, and proofs are given in these terms. It is in many cases almost impossible to trace a result to one particular author, and it was therefore decided to omit references altogether. This does not mean that the author claims credit for any particular result. To the author's knowledge only theorems 7 of chapter 1 and most of chapter 3 are new. (After completion of chapter 3 the author was informed by H. Rubin that some of the



results of this chapter had previously been obtained by him and L. Savage, but their results were never published.) In his presentation of the theory of stochastic processes, as well as in chapter 4, the author has followed the presentations of M. Loeve given in Paul Levy's book on stochastic processes and in M. Loeve's paper "On sets of probability laws and their limit elements" (University of California Press, 1950), respectively. In the treatment of counter data in chapter 5 the author has used Feller's approach and his masterful presentation in the Centennial Anniversary volume. The treatment of the Ornstein Uhlenbeck process in chapter 2 follows a presentation given by J. L. Doob (Ann. of Math., Vol. 43, No. 2).

My thanks are due to Dr. Eugene Lukacs for his valuable help in preparing the final form of the manuscript and to Mrs. P. Moranda who read the proofs and prepared the index. I also wish to thank Professor M. Loeve for many helpful discussions on the subject.

H. B. MANN

Ohio State University  
May 1951





## Chapter 1

### FUNDAMENTAL CONCEPTS

1. Random variables. We consider a finite or infinite set of symbols  $(x, y, \dots)$  such that to every finite set of symbols  $x_1, \dots, x_n$  there is defined a right continuous distribution function

$$(1.1) \quad F_{x_1, x_2, \dots, x_n}(a_1, a_2, \dots, a_n) = P(x_1 \leq a_1, \dots, x_n \leq a_n)$$

called the probability of the event  $x_1 \leq a_1, \dots, x_n \leq a_n$ .

The distribution functions of the family given by (1.1) satisfy the following equations:

$$(1.2) \quad F_{x_{i_1}, \dots, x_{i_n}}(a_{i_1}, \dots, a_{i_n}) = F_{x_1, \dots, x_n}(a_1, \dots, a_n)$$

where  $i_1, \dots, i_n$  is any permutation of the numbers  $1, 2, \dots, n$ ;

$$(1.3) \quad F_{x_1, \dots, x_n}(a_1, \dots, a_{n-1}, \infty) = F_{x_1, \dots, x_{n-1}}(a_1, \dots, a_{n-1})$$

The symbol  $x_i$  is called a random variable. For every Borel set  $A$  in the  $n$ -dimensional Euclidean space we define the symbol

$P[(x_1, \dots, x_n) \in A]$ , called the probability that the "point"

$(x_1, \dots, x_n)$  lies in  $A$  by the equation

$$P[(x_1, \dots, x_n) \in A] = \int_A dF_{x_1, \dots, x_n}(a_1, \dots, a_n)$$



If  $g(x_1, \dots, x_m)$  is a Borel measurable function, then we can define a new random variable  $g(x_1, \dots, x_m)$  by the equations

$$\begin{aligned} P_{g, y_1, \dots, y_m}(a, b_1, \dots, b_m) &= P(g \leq a, y_1 \leq b_1, \dots, y_m \leq b_m) \\ &= F_{y_1 \dots g \dots y_m}(b_1, \dots, a, \dots, b_m) \end{aligned}$$

We now consider sequences  $\{x_j\}$  of random variables. The notion of convergence of such a sequence can be defined in various ways. In our representation of the theory of stochastic processes we shall however use mainly the following definition.

2. Convergence. A sequence  $\{x_j\}$  will be called convergent if for every  $\varepsilon > 0$ ,  $\eta > 0$  there exists an  $N(\varepsilon, \eta)$  such that

$$(1.4) \quad P(|x_{n+h} - x_n| \geq \varepsilon) \leq \eta$$

for  $n > N(\varepsilon, \eta)$  and all  $h$ . If there exists a random variable  $x$  such that  $\lim_{n \rightarrow \infty} P(|x_n - x| \geq \varepsilon) = 0$  for all  $\varepsilon$  then we shall write

$\text{plim}_{n \rightarrow \infty} x_n = x$  and say that  $\{x_n\}$  converges to  $x$  or that  $x$  is the

probability limit of the sequence  $\{x_n\}$ . The convergence defined above is usually termed convergence in probability. This definition can be extended in an obvious manner to random vectors.

We proceed to formulate an important property of convergent sequences.



Theorem 1.1 Let  $\{x_n\}$  be a sequence of random variables. There exists a random variable  $x$  such that  $\text{plim}_{n \rightarrow \infty} x_n = x$  if and only if the sequence  $\{x_n\}$  converges. Moreover, if  $F_{x_n, y_1, \dots, y_m}$  are the distribution functions of  $x_n$  ( $n=1, 2, \dots$ ) and  $y_1, \dots, y_m$  then

$$\lim_{n \rightarrow \infty} F_{x_n, y_1, \dots, y_m} = F_{x, y_1, \dots, y_m}$$

for all points  $(t, b_1, \dots, b_m)$  for which the function  $F_{x, y_1, \dots, y_m}(t, b_1, \dots, b_m)$  is continuous in  $t$ .

Theorem 1.1 gives a condition for convergence in probability similar to Cauchy's criterion. This condition was first established by E. Slutsky<sup>[1]</sup>. We proceed to prove<sup>[2]</sup> theorem 1.1. As a first step we assume that the sequence  $\{x_n\}$  is convergent and show the existence of a random variable  $x = \text{plim}_{n \rightarrow \infty} x_n$ . In the following we write for abbreviation

$$F_n(c) = F_{x_n, y_1, \dots, y_m}(c, b_1, \dots, b_m)$$

to first prove the following lemma.

[1] Helzer, *ibid.* 3:89 (1925); C. R. Acad. Sci. Paris, 187, 370-372(1928).

[2] The proof of theorem 1.1 may be skipped in a first reading without affecting the understanding of the rest of the monograph.



Lemma 1.1 If  $\delta$  and  $\eta$  are any positive numbers, then for sufficiently large  $n$  and all  $h \geq 0$

$$g_n(c+\delta) + \eta \geq g_{n+h}(c) \geq g_n(c-\delta) - \eta$$

for all  $c$ .

For abbreviation we write for any event  $\mathcal{E}$

$$P_b(\mathcal{E}) = P(\mathcal{E}, x_1 \leq b_1, \dots, x_n \leq b_n),$$

that is particular

$$P_b(x_n \leq c) = g_n(c).$$

then

$$P_b(x_n \leq c + \delta) \geq P_b(x_n \leq c + \delta, x_{n+h} \leq c) \geq P_b(x_{n+h} \leq c, |x_{n+h} - x_n| \leq \delta).$$

Since the set of points  $(x_{n+h}, x_n)$  for which  $|x_{n+h} - x_n| > \delta$

includes the points for which  $|x_{n+h} - x_n| > \delta$  and  $x_{n+h} \leq c$ , we have

$$P_b(x_{n+h} \leq c, |x_{n+h} - x_n| \leq \delta) \geq P_b(x_{n+h} \leq c) - P(|x_{n+h} - x_n| > \delta)$$

hence for sufficiently large  $n$  and all  $h \geq 0$

$$g_n(c+\delta) \geq g_{n+h}(c) - \eta.$$

Similarly

$$g_{n+h}(c) \geq g_n(c-\delta) - \eta$$

and (1.5) follows.

For a sequence of functions  $\{f_n(t)\}$  we shall write  $\lim_{n \rightarrow \infty} f_n(t)$

$f(t)$  if  $\lim_{n \rightarrow \infty} f_n(t) = f(t)$  for every continuity point of  $f(t)$ .





Lemma 1.2 There exists a non-decreasing function  $g(c)$  such that

$$\lim_{n \rightarrow \infty} \epsilon_n(c) = g(c).$$

The functions  $\epsilon_n(c)$  are non decreasing and bounded hence by Helly's theorem<sup>[3]</sup> there exists a subsequence  $\epsilon_{n_1}(c)$  such that

$$(1.6) \quad \lim_{i \rightarrow \infty} \epsilon_{n_1}(c) = g(c)$$

where  $g(c)$  is non-decreasing.

Let  $t$  be a continuity point of  $g(c)$ . Fix  $\eta > 0$  and choose  $\delta$  positive and arbitrarily small and so that  $t+\delta$  and  $t-\delta$  are continuity points of  $g(c)$  and

$$g(t+\delta) - g(t) \leq \eta, \quad g(t) - g(t-\delta) \leq \eta.$$

For sufficiently large  $n_1$  and  $n$  we then have by (1.5)

$$\epsilon_{n_1}(t+\delta) + \eta \geq \epsilon_n(t) \geq \epsilon_{n_1}(t-\delta) - \eta,$$

and therefore

$$g(t+\delta) + \eta \geq \epsilon_n(t) \geq g(t-\delta) - \eta.$$

By choice of  $\delta$  we have

$$(1.7) \quad g(t) + 2\eta \geq \epsilon_n(t) \geq g(t) - 2\eta$$

where  $\eta$  can be made arbitrarily small for sufficiently large  $n$ .

Helly's theorem (see, for instance, D.V. Widder, The Laplace Transform 27) states. If the real non-decreasing functions  $\alpha_n(x)$  and the positive constant  $A$  are such that  $|\alpha_n(x)| < A$  ( $n = 0, 1, 2, \dots$ )  $a \leq x \leq b$ ; then there exists a subsequence  $\{\alpha_{n_1}(x)\}$  of  $\{\alpha_n(x)\}$  and a non-decreasing bounded function  $\alpha(x)$  such that

$$\lim_{i \rightarrow \infty} \alpha_{n_1}(x) = \alpha(x) \quad (a \leq x \leq b)$$



It then follows that

$$\lim_{n \rightarrow \infty} g_n(t) = g(t)$$

and lemma 1.2 is proved.

We now define a symbol  $x$  by the equations

$$\begin{aligned} F_{xy_1 \dots y_m}(t, b_1, \dots, b_m) &= F_{y_1 \dots y_{i-1} xy_1 \dots y_m}(b_1, \dots, b_{i-1}, t, b_1, \dots, b_m) \\ &= g(t + 0) \end{aligned}$$

To prove that  $x$  is a random variable we have to show that  $F_{xy_1 \dots y_m}$

is a distribution function and that

$$(1.8) \quad F_{xy_1 \dots y_m}(t, b_1, \dots, b_{m-1}, \infty) = F_{xy_1 \dots y_{m-1}}(t, b_1, \dots, b_{m-1})$$

and

$$(1.9) \quad \lim_{t \rightarrow \infty} F_{xy_1 \dots y_m}(t, b_1, \dots, b_m) = F_{y_1 \dots y_m}(b_1, b_2, \dots, b_m) .$$

That the interval function corresponding to  $F_{xy_1 \dots y_m}$  is non-negative is obvious since it is a limit of functions  $F_{x_n y_1 \dots y_m}$  with this property. We therefore have merely to show that (1.8) and (1.9) hold and that  $F_{xy_1 \dots y_m}$  tends to zero if any one of its arguments tends to  $-\infty$ .



We have

$$0 \leq F_{x_n y_1 \dots y_m}(t, b_1, \dots, b_m) \leq F_{y_j}(b_j)$$

and hence

$$0 \leq F_{x y_1 \dots y_m}(t, b_1, \dots, b_m) \leq F_{y_j}(b_j)$$

so that

$$\lim_{b_j \rightarrow \infty} F_{x y_1 \dots y_m}(t, b_1, \dots, b_m) = 0$$

Furthermore from (1.5) for arbitrarily small  $\eta$  and all  $t$

$$F_{x_n y_1 \dots y_m}(t + \delta, b_1, \dots, b_m) + \eta \geq F_{x y_1 \dots y_m}(t, b_1, \dots, b_m) \geq 0$$

for sufficiently large  $n$  uniformly in  $t$ . We therefore have

$$\lim_{t \rightarrow -\infty} F_{x y_1 \dots y_m}(t, b_1, \dots, b_m) = 0.$$

Furthermore

$$\begin{aligned} 0 &\leq F_{x_n y_1 \dots y_{m-1}}(t, b_1, \dots, b_{m-1}) - F_{x_n y_1 \dots y_m}(t, b_1, \dots, b_m) \\ &\leq 1 - F_{y_m}(b_m); \end{aligned}$$

if we let first  $n \rightarrow \infty$  and then  $b_m \rightarrow \infty$  we obtain (1.8).

To prove (1.9) we choose a continuity point  $c$  of

$F_{x y_1 \dots y_m}(t, b_1, \dots, b_m)$  so large that for fixed  $\delta$  and  $\eta$

$$F_{x_n y_1 \dots y_m}(c - \delta, b_1, \dots, b_m) = F_{y_1 \dots y_m}(b_1, \dots, b_m) - \eta\theta$$

and also

$$F_{x_n y_1 \dots y_m}(c + \delta, b_1, \dots, b_m) = F_{y_1 \dots y_m}(b_1, \dots, b_m) - \eta\theta'$$

where  $0 < \theta < 1$  and  $0 < \theta' < 1$ .



we have this inequality

$$\begin{aligned} F_{x_1, \dots, x_m}(b_1, \dots, b_m) + \eta &\leq F_{x_n, y_1, \dots, y_m}(a, b_1, \dots, b_m) \\ &\geq F_{y_1, \dots, y_m}(b_1, \dots, b_m) - 2\eta \end{aligned}$$

By letting  $\eta \rightarrow 0$  and considering that  $\eta$  may be chosen arbitrarily small we obtain (1.9).

We proceed to prove that  $\lim_{n \rightarrow \infty} x_n = x$ . We represent

the domain  $|y - x| > \varepsilon$  by the sum of a denumerable number of

intervals whose corner points are continuity points of  $F_{x_n, x}$  for

all  $n$ . To do this we remember that there can be at most a denumerable

number of points with a positive probability in the plane, so

that we can construct an interval netting<sup>[4]</sup> which avoids these

points. Since the set  $|y - x| > \varepsilon$  is open, every point of this set

is contained in one interval of this netting which is entirely con-

tained in the set. Therefore we have merely to take all the intervals

of our netting which is entirely in the set  $|y - x| > \varepsilon$ . Thereby

we proceed to such a  $\eta$  that if  $I_1, I_2$  are two such intervals and

[4] If  $M$  is a set of points in the plane and  $\delta$  is a positive number, the union of all squares of side  $\delta$  which do not contain any point of  $M$  is called a  $\delta$ -netting. The maximum area of any  $\delta$ -netting of  $M$  is called the modulus of the set  $M$ . A sequence  $\{\delta_n\}$  of numbers such that  $\delta_n \rightarrow 0$  is a refinement of  $M$  if the modulus of  $M$  converges to zero is called a netting.





$I_1 \supset I_2$  then only  $I_1$  is chosen for our interval covering.

Let  $I_1, I_2, \dots$  be these intervals and denote by  $P_{x_n y} (I_k)$  the probability that the point  $(x_n, y)$  will fall into the interior of the interval  $I_k$  or on <sup>either</sup> its right <sup>or</sup> ~~and~~ upper boundary. Then for sufficiently large  $n$  and arbitrary  $\eta$

$$(1.11) \quad P(|x_{n+h} - x_n| > \epsilon) = \sum_k P_{x_n x_{n+h}} (I_k) \leq \eta$$

for all  $h$ .

Furthermore

$$P(|x_n - x| > \epsilon) = \sum_k P_{x_n x} (I_k)$$

Both sums converge. From now on consider  $n$  as fixed. Choose  $N$  so that for some  $\eta > 0$

$$\sum_{k=N+1}^{\infty} P_{x_n x} (I_k) < \eta .$$

Next choose  $h$  so that for the first  $N$  intervals

$$|P_{x_n x_{n+h}} (I_k) - P_{x_n x} (I_k)| \leq \frac{\eta}{N} .$$

Then

$$P(|x_n - x| > \epsilon) = \sum_1^{\infty} P_{x_n x} (I_k) \leq \sum_1^{\infty} P_{x_n x_{n+h}} (I_k) + 2\eta \leq 3\eta .$$

Since  $\eta$  was arbitrary  $\lim_{n \rightarrow \infty} P(|x_n - x| > \epsilon) = 0$  or  $\text{plim}_{n \rightarrow \infty} x_n = x$ .

[The relation  $\text{plim}_{n \rightarrow \infty} x_n = x$  also follows from the fact that the characteristic function of  $x_n - x_{n+h}$  converges to the characteristic function of  $x_n - x$ ].

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On the other hand if  $\text{plim}_{n \rightarrow \infty} x_n = x$  then for sufficiently

large  $n$  and arbitrary  $\eta$

$$\begin{aligned} P(|x_{n+h} - x_n| \leq \varepsilon) &\geq P(|x_n - x| \leq \frac{\varepsilon}{2} \text{ and } |x_{n+h} - x| \leq \frac{\varepsilon}{2}) \\ &\geq P(|x_n - x| \leq \frac{\varepsilon}{2}) - P(|x_{n+h} - x| > \frac{\varepsilon}{2}) \geq 1 - \eta. \end{aligned}$$

Hence the sequence  $\{x_n\}$  converges and theorem 1.1 is proved.

3. Stochastic processes. A set of random variables  $x_t$  where  $t$  is chosen out of some set of real numbers is called a stochastic process. If the set of indices  $t$  is an interval then the stochastic process is said to depend on a continuous parameter. Such a process is called continuous in  $[a, b]$  if for every sequence  $\{h_i\}$  with

$$\lim_{i \rightarrow \infty} h_i = 0, \quad \text{plim}_{i \rightarrow \infty} x_{t+h_i} = x_t \quad \text{for } a \leq t \leq b.$$

The expression

$$\int_{-\infty}^{+\infty} t dF_y(t) = E(y)$$

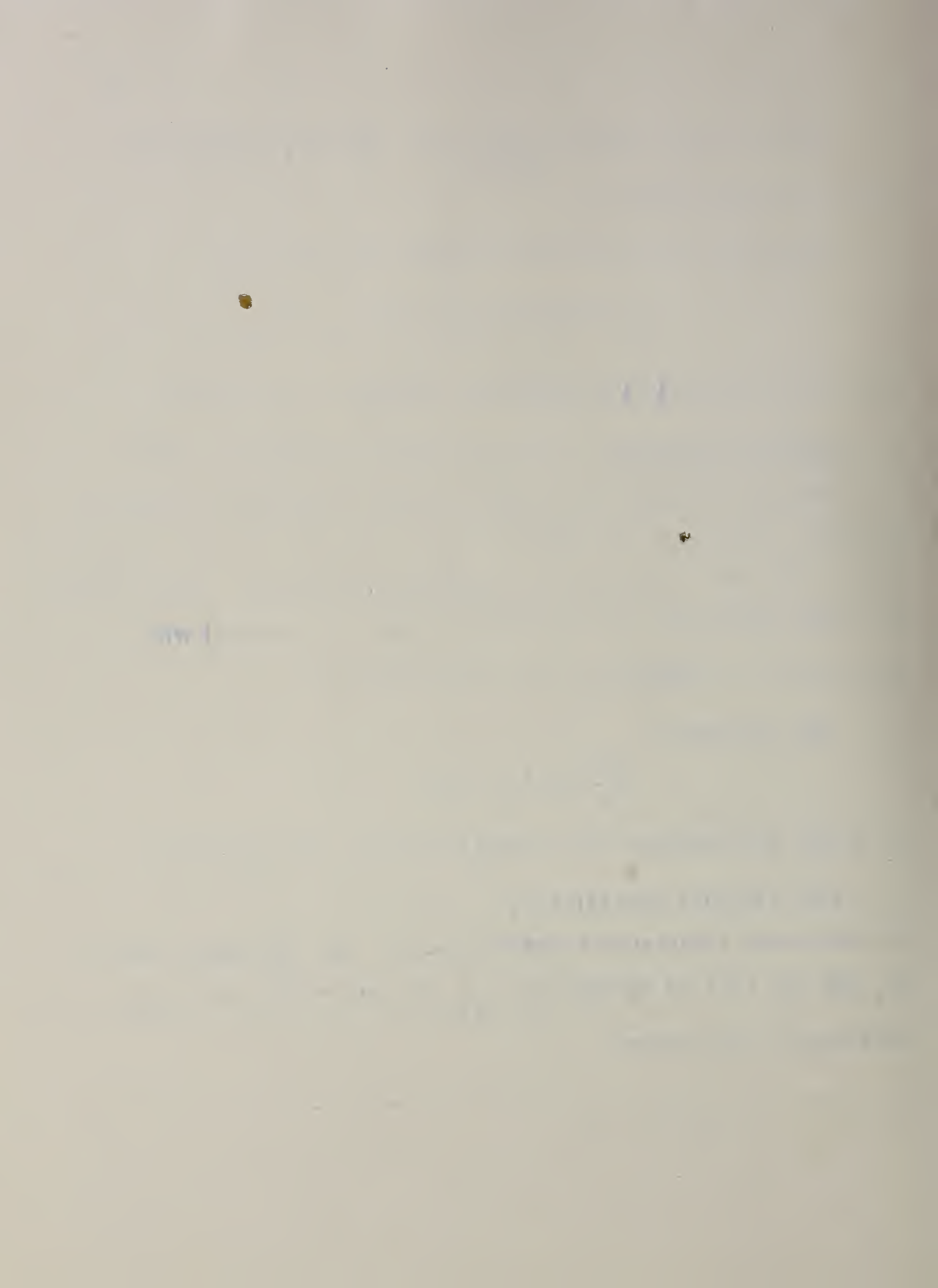
is called the mathematical expectation of  $y$ . The expression

$$E\{[x - E(x)][y - E(y)]\} = \sigma_{xy}$$

is called the covariance between  $x$  and  $y$ . The covariance between

$x_{t_1}$  and  $x_{t_2}$  will be denoted by  $\sigma_{t_1 t_2}$  and called the covariance

function of the process.



4. Convergence in the mean. A sequence of random variables  $\{x_n\}$  is said to converge in the mean to a random variable  $x$  in symbols

$\text{l.i.m.}_{n \rightarrow \infty} x_n = x$  if

$$(1.12) \quad \lim_{n \rightarrow \infty} E(x_n - x)^2 = 0 .$$

We shall now prove several very useful lemmas on convergence in the mean and convergence in probability.

Lemma 1.3. If  $\text{l.i.m.}_{n \rightarrow \infty} x_n = x$  then  $\text{plim}_{n \rightarrow \infty} x_n = x$ .

This follows immediately from Tchebicheff's inequality.

Lemma 1.4. If  $\text{l.i.m.}_{n \rightarrow \infty} x_n = x$  then  $\lim_{n \rightarrow \infty} E(x_n) = E(x)$ .

Since  $E(x_n - x)^2 \leq \epsilon$  for sufficiently large  $n$  we also have

$$\epsilon \geq E(x_n - x)^2 = \sigma_{x_n - x}^2 + [E(x_n - x)]^2 \geq [E(x_n) - E(x)]^2 .$$

Lemma 1.5. If  $\{y_h\}$  is a sequence of non-negative<sup>[5]</sup> random variables and if  $\text{plim}_{h \rightarrow \infty} y_h = y$  and  $E(y_h) \leq M$  then  $E(y) \leq M$ .

Under the conditions of the lemma and in view of theorem 1.1 we have  $\lim_{h \rightarrow \infty} F_h(t) = F(t)$  where  $F_h$  and  $F$  are the cumulative distribution functions of  $y_h$  and  $y$  respectively. Suppose  $E(y) > M$  then there exists a continuity point  $A$  of  $F(t)$  such that  $\int_0^A t dF(t) > M$ .

However  $\int_0^A t dF_h(t) \leq M$  and  $\lim_{h \rightarrow \infty} \int_0^A t dF_h(t) = \int_0^A t dF(t)$ , a contradiction.

[5] I.e., if  $P(y_h < 0) = 0$ .

*[The text in this section is extremely faint and illegible.]*

Lemma 1.5. The sequence  $\{x_n\}$  converges in the mean to a random variable  $x$  if and only if to every  $\varepsilon > 0$  there exists an  $N$  such that

$$(1.13) \quad E(x_m - x_n)^2 \leq \varepsilon \quad \text{for all } m, n \geq N.$$

Suppose first that there exists a random variable  $x$  such that

$\lim_{n \rightarrow \infty} x_n = x$ . Then for sufficiently large  $n$  and  $m$  and arbitrary  $\varepsilon$

$$E(x_n - x)^2 \leq \varepsilon, \quad E(x_m - x)^2 \leq \varepsilon.$$

But

$$E(x_m - x_n)^2 = E(x_m - x)^2 + E(x_n - x)^2 - 2E[(x_m - x)(x_n - x)]$$

and by Schwartz's inequality

$$|E(x_m - x)(x_n - x)| \leq \sqrt{E(x_m - x)^2 E(x_n - x)^2} \leq \varepsilon;$$

hence

$E(x_m - x_n)^2 \leq 4\varepsilon$ . On the other hand from  $E(x_m - x_n)^2 \leq \varepsilon$  it follows

by Tchebicheff's inequality that

$$P(|x_m - x_n| \geq t\sqrt{\varepsilon}) \leq \frac{1}{t^2}.$$

Thus  $\lim_{n \rightarrow \infty} x_n = x$  exists by theorem 1.1. It follows also that

$\lim_{n \rightarrow \infty} (x_m - x_n)^2 = (x_m - x)^2$  and thus by lemma 1.5

$$E(x_m - x)^2 \leq \varepsilon \quad \text{and} \quad \lim_{m \rightarrow \infty} x_m = x.$$

Lemma 1.7. If  $\lim_{n \rightarrow \infty} x_n = x$ ,  $\lim_{n \rightarrow \infty} y_n = y$  and if

$E(x^2)$ ,  $E(y^2)$  exist then  $\lim_{n \rightarrow \infty} E(x_n y_n) = E(xy)$ .





we show first that  $E(x_n^2)$  and  $E(y_n^2)$  are bounded with respect to  $n$ .

By virtue of lemma 1.6 there exists an  $m$  and an  $M$  such that

$$E(x_m - x_n)^2 \leq M \text{ for all } n. \quad [6] \text{ From the almost trivial inequality}$$

$a^2 \leq 2[(a-b)^2 + b^2]$  we see that for all  $n$

$$E(x_n^2) \leq 2[E(x_m - x_n)^2 + E(x_m^2)] \leq 2[M + E(x_m^2)]$$

From this inequality it follows that  $E(x_n^2)$  is bounded for all  $n$ .

Since  $|E(x_n y_n)| \leq \sqrt{E(x_n^2) E(y_n^2)}$  it follows moreover that  $E(x_n y_n)$  exists and furthermore

$$(1.14) \quad |E(x_n y_n - xy)| = |E(x_n(y_n - y) + y(x_n - x))| \\ \leq \sqrt{E(x_n^2) E(y_n - y)^2} + \sqrt{E(y^2) E(x_n - x)^2}$$

$E(y^2)$  exists by lemma 1.3 and  $E(x_n^2)$  is bounded and since  $E(y_n - y)^2$  and  $E(x_n - x)^2$  converge to zero, the right-hand side of (1.14) converges to zero, which proves lemma 1.7. Lemma 1.7 may also be written in the form

$$(1.15) \quad E\left(\lim_{n \rightarrow \infty} (1, i, m, x_n) \mid \lim_{n \rightarrow \infty} (1, i, m, y_n)\right) = \lim_{n \rightarrow \infty} E(x_n y_n)$$

$$(1.15a) \quad E\left(\lim_{n \rightarrow \infty} (1, i, m, x_n) \mid \lim_{n \rightarrow \infty} (1, i, m, y_n)\right) = \lim_{n \rightarrow \infty} \sigma_{x_n y_n}$$

Corollary to lemma 1.7. The sequence  $\{x_n\}$  converges in the mean if and only if  $\lim_{\substack{n \rightarrow \infty \\ m \rightarrow \infty}} E(x_n - x_m)$  exists irrespective of the manner in which  $n$  and  $m$  tend to infinity.

[6] This is seen if we determine  $N$  -- according to lemma 1.6 --, so that  $E(x_m - x_n)^2 \leq \epsilon$  for  $m, n \geq N$  and then take, for a fixed  $m \geq N$ ,  $M = \max\{E(x_m - x_1)^2, E(x_m - x_2)^2, \dots, E(x_m - x_N)^2; \epsilon\}$



From lemma 1.7 it follows immediately that the condition is necessary. To show that the condition is also sufficient we assume that  $\lim_{\substack{n \rightarrow \infty \\ m \rightarrow \infty}} E(x_n x_m)$  exists and is independent of the manner in

which  $n$  and  $m$  go to infinity. Then  $\lim_{\substack{n \rightarrow \infty \\ m \rightarrow \infty}} E(x_n - x_m)^2 = 0$  hence it is

possible to find for every  $\varepsilon > 0$  an  $N = N(\varepsilon)$  such that

$E(x_n - x_m)^2 \leq \varepsilon$  for  $n, m \geq N$ . We see therefore from lemma 1.6 that

$\lim_{n \rightarrow \infty} x_n$  exists. This corollary is due to M. Loève.

5. Differentiation. In order to be able to define derivatives of stochastic processes we have to extend the concepts of limits in probability and limits in the mean. Suppose that for every  $h$  in an interval  $(a, b)$  a random variable  $x_h$  is defined. If for every sequence  $\{h_i\}$  with  $\lim_{i \rightarrow \infty} h_i = a$ ,  $\text{plim}_{i \rightarrow \infty} x_{h_i} = x$  exists then we write

$\text{plim}_{h \rightarrow a} x_h = x$ . In a similar manner we define  $\lim_{h \rightarrow a} x_h = \lim_{i \rightarrow \infty} x_{h_i}$ .

The process  $x_t$  is called differentiable at the point  $t$  if

$\text{plim}_{h \rightarrow 0} \frac{x_{t+h} - x_t}{h} = x'_t$  exists. The stochastic process  $x'_t$  is

called the derivative of  $x_t$ .

In the following we assume  $E(x_t) = 0$ . The modifications of our statements for the case  $E(x_t) \neq 0$  will be obvious.



6. Stochastic processes of second order. A stochastic process  $x_t$  is called of second order if for any values  $t_1, t_2$  the covariance  $\sigma_{t_1 t_2}$  exists. The process  $x_t$  is called differentiable l.i.m. if

$$\text{l.i.m.}_{h \rightarrow 0} \frac{x_{t+h} - x_t}{h} = x'_t \text{ exists.}$$

Theorem 1.2. Necessary and sufficient that the process  $x_t$  is differentiable l.i.m. is that the limit

$$(1.16) \lim_{\substack{h \rightarrow 0 \\ k \rightarrow 0}} \frac{\sigma_{t+h, t+k} - \sigma_{t+h, t} - \sigma_{t, t+k} + \sigma_{t, t}}{hk} = \Sigma_{t, t}$$

exist. The covariance function  $\sigma_{t_1 t_2}$  is then twice differenti-

able and  $\frac{\partial^2 \sigma_{t_1 t_2}}{\partial t_1 \partial t_2} = \frac{\partial^2 \sigma_{t_1 t_2}}{\partial t_2 \partial t_1}$ . Moreover  $x'_t$  is a stochastic process

of second order and its covariance function is  $\frac{\partial^2 \sigma_{t_1 t_2}}{\partial t_1 \partial t_2}$ .

The covariance between  $x_{t^*}$  and  $x'_t$  is given by  $\frac{\partial \sigma_{t^* t}}{\partial t}$ .

Proof: Consider a sequence of difference quotients  $\frac{x_{t+h} - x_t}{h}$ .

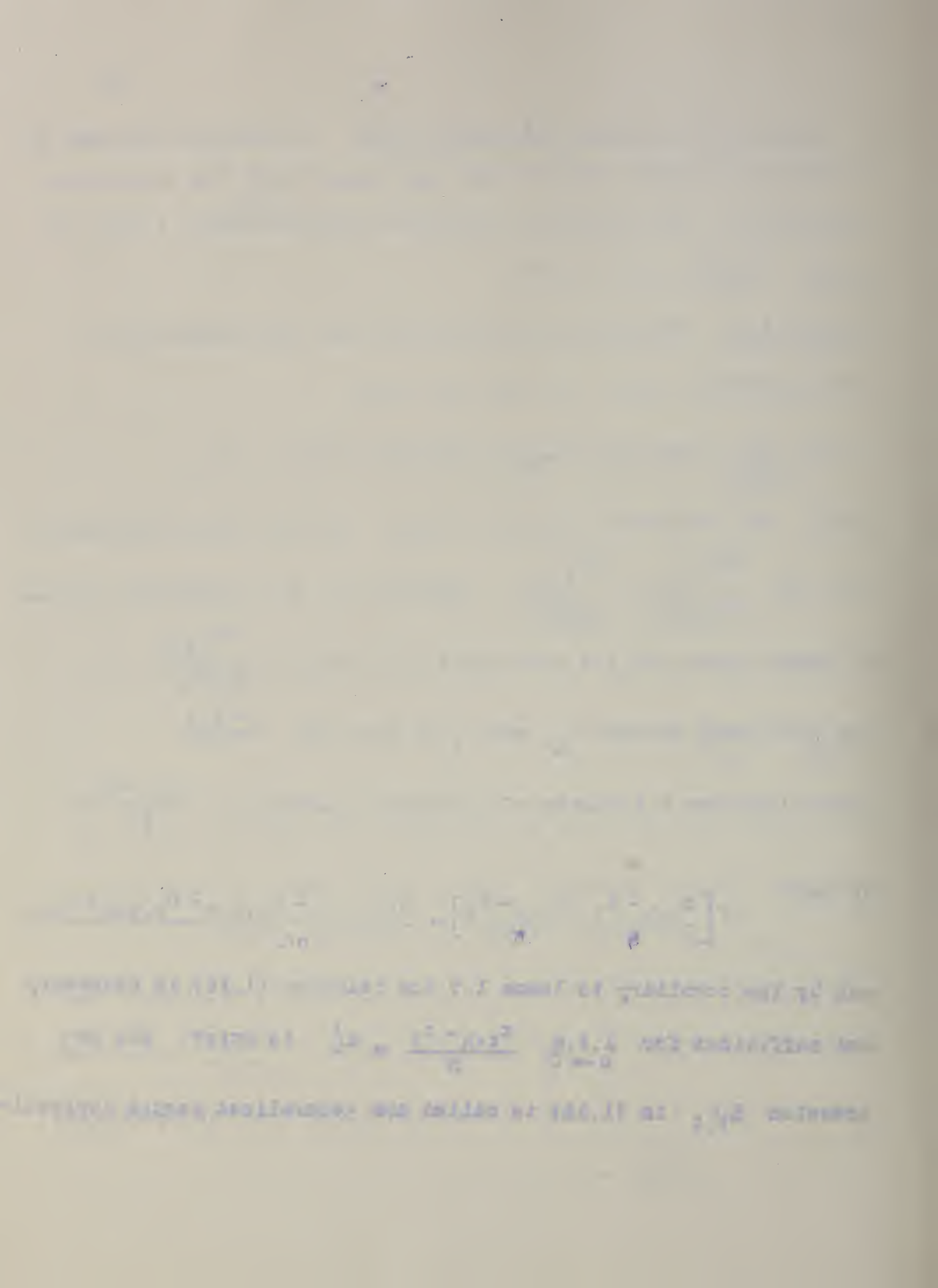
We have

$$E \left[ \frac{x_{t+h} - x_t}{h} \cdot \frac{x_{t+k} - x_t}{K} \right] = \frac{\sigma_{t+h, t+k} - \sigma_{t+h, t} - \sigma_{t, t+k} + \sigma_{t, t}}{hK}$$

and by the corollary to lemma 1.7 the relation (1.16) is necessary

and sufficient for  $\text{l.i.m.}_{h \rightarrow 0} \frac{x_{t+h} - x_t}{h} = x'_t$  to exist. The ex-

pression  $\Sigma_{t, t}$  in (1.16) is called the generalized second derivative.



We moreover have by lemma 1.4  $E(x'_t) = 0$  since  $E(x_t) = E(x_{t+h}) = 0$ . Furthermore by lemma 1.7  $\sigma_{x_t x'_t}$  exists and

$$(1.17) \quad \sigma_{x_t x'_t} = \lim_{h \rightarrow 0} E \left[ x_t \cdot \frac{x_{t+h} - x_t}{h} \right] = \lim_{h \rightarrow 0} \frac{\sigma_{t+h, t^*} - \sigma_{t, t^*}}{h} = \frac{\partial \sigma_{t, t^*}}{\partial t}.$$

Thus  $\frac{\partial \sigma_{t, t^*}}{\partial t}$  exists. It also follows from lemma 1.7 that

$\sigma_{x'_t x'_t}$  exists and

$$\begin{aligned} \sigma_{x'_t x'_t} &= \lim_{\substack{h \rightarrow 0 \\ k \rightarrow 0}} E \left[ \frac{x_{t+h} - x_t}{h} \cdot \frac{x_{t^*+k} - x_{t^*}}{k} \right] \\ &= \lim_{\substack{h \rightarrow 0 \\ k \rightarrow 0}} \frac{\sigma_{t+h, t^*+k} - \sigma_{t, t^*+k} - \sigma_{t+h, t^*} + \sigma_{t, t^*}}{hk} = \Sigma_{t, t^*}. \end{aligned}$$

It easily follows that  $\sigma_{t, t^*}$  is twice differentiable and that

$$\frac{\partial^2 \sigma_{t, t^*}}{\partial t \partial t^*} = \frac{\partial^2 \sigma_{t, t^*}}{\partial t^* \partial t} = \Sigma_{t, t^*}.$$

It is well known that the generalized second derivative of any

function  $f(x, y)$  exists if  $\frac{\partial^2 f}{\partial x \partial y}$  exists and is continuous.

Thus we have

Corollary to theorem 1.2. If  $x_t$  is a stochastic process of second order with covariance function  $\sigma_{t, t^*}$  and if  $\frac{\partial^2 \sigma_{t, t^*}}{\partial t \partial t^*}$  exists and is continuous, ~~at  $t = t^*$~~  then  $x'_t$  exists l.i.m. and its covariance function is  $\frac{\partial^2 \sigma_{t, t^*}}{\partial t \partial t^*} = \Sigma_{t, t^*}$ .





The corollary to theorem 11.1 gives a convenient method to find the covariance function of  $x_t^*$  if  $x_t^*$  exists 1.1.11.

11. Integration. Let  $x_t$  be a stochastic process defined for  $a \leq t \leq b$ . We subdivide the interval from  $a$  to  $b$  into  $n$  parts by means of the points  $a = t_0, t_1, \dots, t_n = b$  and put  $\delta_n = t_1 - t_0 = \delta$ . The number  $\delta$  is called the modulus of the subdivision. Within every interval  $t_{i-1} \leq t \leq t_i$  we choose a value  $t_i^*$  and form the sum

$$(11.18) \quad Y = \sum_{i=1}^n x_{t_i^*} (t_i - t_{i-1})$$

is a random variable. Now consider a sequence  $\{\delta_n\}$  of subdivisions  $S_n$  with moduli  $\delta_n$  such that  $\lim_{n \rightarrow \infty} \delta_n = 0$ . Let  $X_n$  be the random variable corresponding by (11.18) to the subdivision  $S_n$  and some choice of the  $t_i^*$ . If then

$\lim_{n \rightarrow \infty} X_n = X$  exists and is equal for all sequences  $\{\delta_n\}$  with modulus converging to zero and all choices of  $t_i^*$  then  $X$  is called the integral of  $x_t$  and we write

$$(11.19) \quad \int_a^b x_t dt = X$$



Strong continuity. [7] In the following we denote by  $E(\delta, \epsilon, S)$  the event that the relations  $|x_{t_1} - x_{t_k}| \leq \epsilon$  are simultaneously satisfied for all pairs  $(t_1, t_k)$  with  $|t_1 - t_k| < \delta$  and belonging to a finite set  $S$  of points contained in  $[a, b]$ .  $P[E(\delta, \epsilon, S)]$  is then the probability that the inequalities  $|x_{t_1} - x_{t_k}| \leq \epsilon$  are simultaneously fulfilled for all pairs  $(t_1, t_k)$  of a finite set  $S$  of points for which  $|t_1 - t_k| \leq \delta$ .

The process  $x_t$  is called strongly continuous in an interval  $[a, b]$  if to every  $\epsilon$  and  $\eta$  there exists a  $\delta = \delta(\epsilon, \eta)$  such that for every finite set  $S$  of points contained in  $[a, b]$

$$(1.20) \quad P[E(\delta, \epsilon, S)] \geq 1 - \eta.$$

For any stochastic process  $x_t$  consider a set  $S = (t_1, \dots, t_n)$  where  $a \leq t_1 \leq b$  ( $i = 1, \dots, n$ ). We denote by  $M_{abS}$  the largest of the values  $x_{t_1}, \dots, x_{t_n}$ . Let  $\{S_i\}$  be a sequence of subdivisions of the interval  $[a, b]$  whose moduli converge to zero. If  $\text{plim}_{i \rightarrow \infty} M_{abS_i} = M_{ab}$  exists and is the same for all sequences  $\{S_i\}$  whose moduli converge to zero then we shall call  $M_{ab}$  the maximum of  $x_t$  in  $[a, b]$ . The minimum  $m_{ab}$  is similarly defined.

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[7] The ~~definition~~ <sup>concept</sup> is due to P. Lévy.



To simplify the notation we also introduce  $V_{ab} = M_{ab} - m_{ab}$  and  $V_{abS} = M_{abS} - m_{abS}$ . We next derive a criterion for the strong continuity of a process:

Theorem 1.3. A process  $x_t$  is strongly continuous in  $[a, b]$  if and only if

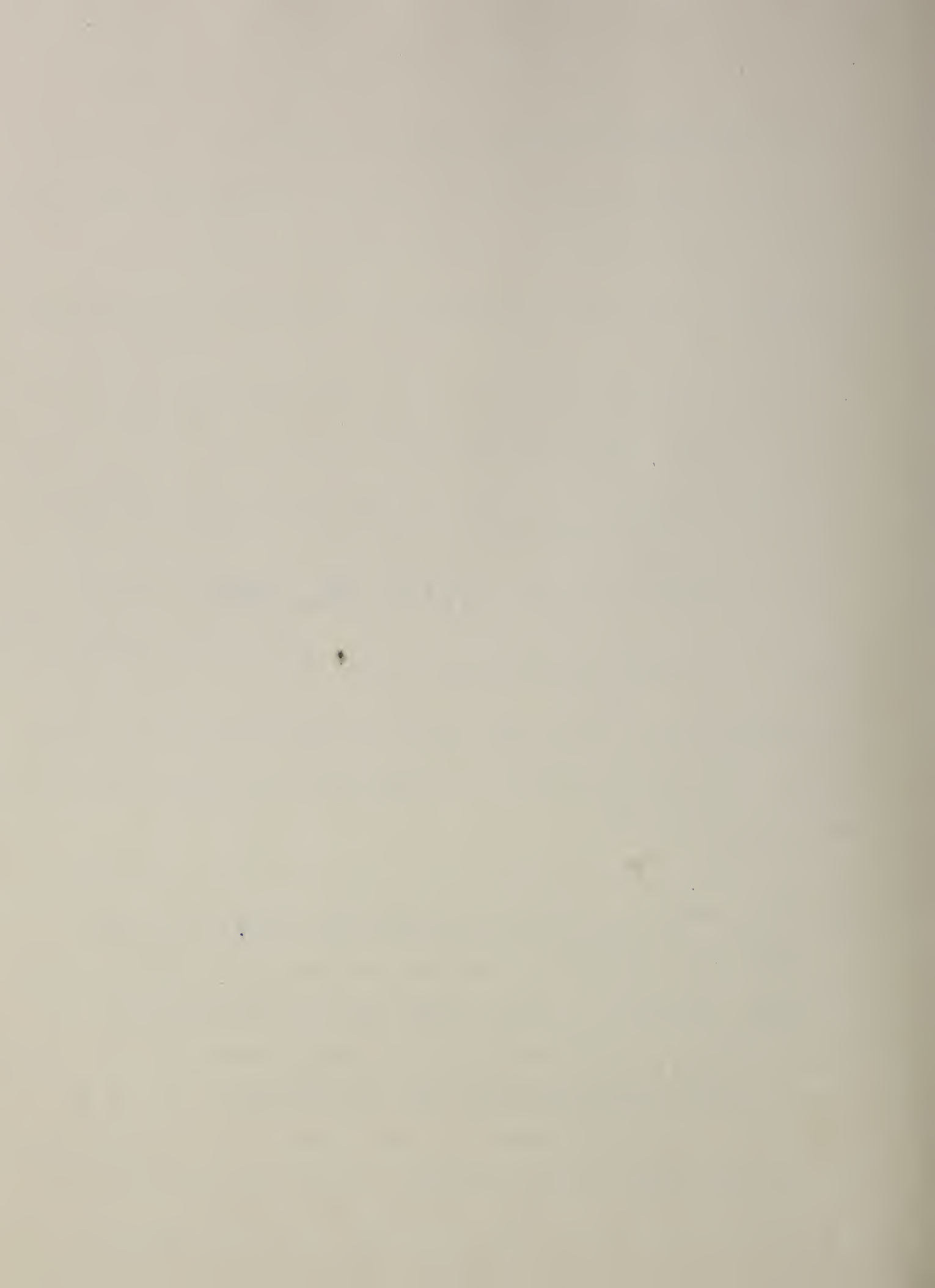
- (i) it possesses a maximum  $M_{tt'}$  and a minimum  $m_{tt'}$  in every subinterval  $[t, t']$  of  $[a, b]$ ;
- (ii) for every  $\epsilon > 0$ ,  $\eta > 0$  there exists a  $\delta$  such that for every subdivision  $S = (a = t_0, t_1, \dots, t_n = b)$  with modulus less than  $\delta$ , it is true that

$$(1.21) \quad P(V_{t_{i-1}t_i} \leq \epsilon; i=1, \dots, n) \geq 1 - \eta.$$

We emphasize that (1.21) means that the probability of the simultaneous fulfillment of all the inequalities  $V_{t_{i-1}t_i} \leq \epsilon$  ( $i=1, \dots, n$ ) must exceed  $1 - \eta$ . [8]

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[8] If  $R_i$  ( $i=1, 2, \dots, n$ ) are  $n$  events then  $P(R_i; i=1, \dots, n)$  means the probability that all  $n$  events occur simultaneously. Thus  $P(R_i; i=1, \dots, n) > k$  means that the probability of the simultaneous occurrence of all  $n$  events exceeds  $k$ ; this should be carefully distinguished from the statement  $P(R_i) > k$ , ( $i=1, \dots, n$ ), which means that the probability of the occurrence of each single event  $R_i$  exceeds  $k$  which does not imply any thing about their joint occurrence. We further emphasize that in conditional probabilities the condition is separated not by a semicolon but by a vertical bar.



We shall use the following:

Lemma 1.8. If  $P(x_n \geq y) = 1$  and  $\text{plim}_{n \rightarrow \infty} x_n = x$  then  $P(x \geq y) = 1$ .

The proof of lemma 1.8 is left to the reader.

Proof of theorem 1.3. Suppose first that conditions (i) and (ii) are fulfilled. Let  $S$  be a finite set of points. We can consider sequences of subdivisions  $\{S_n\}$  such that each  $S_n$  contains  $S$ .

It follows then from lemma 1.8 that

$$(1.22) \quad P(V_{ab} \geq V_{abS}) = 1.$$

Now let  $t'_1, \dots, t'_m$  be the points of  $S$  and consider the sequence  $\{S_n\}$  of subdivisions  $(t_0, \dots, t_n)$  with  $t_k = a + \frac{k(b-a)}{n}$ . For sufficient-

ly large  $n$  and arbitrary  $\epsilon, \eta$  we have by (ii)

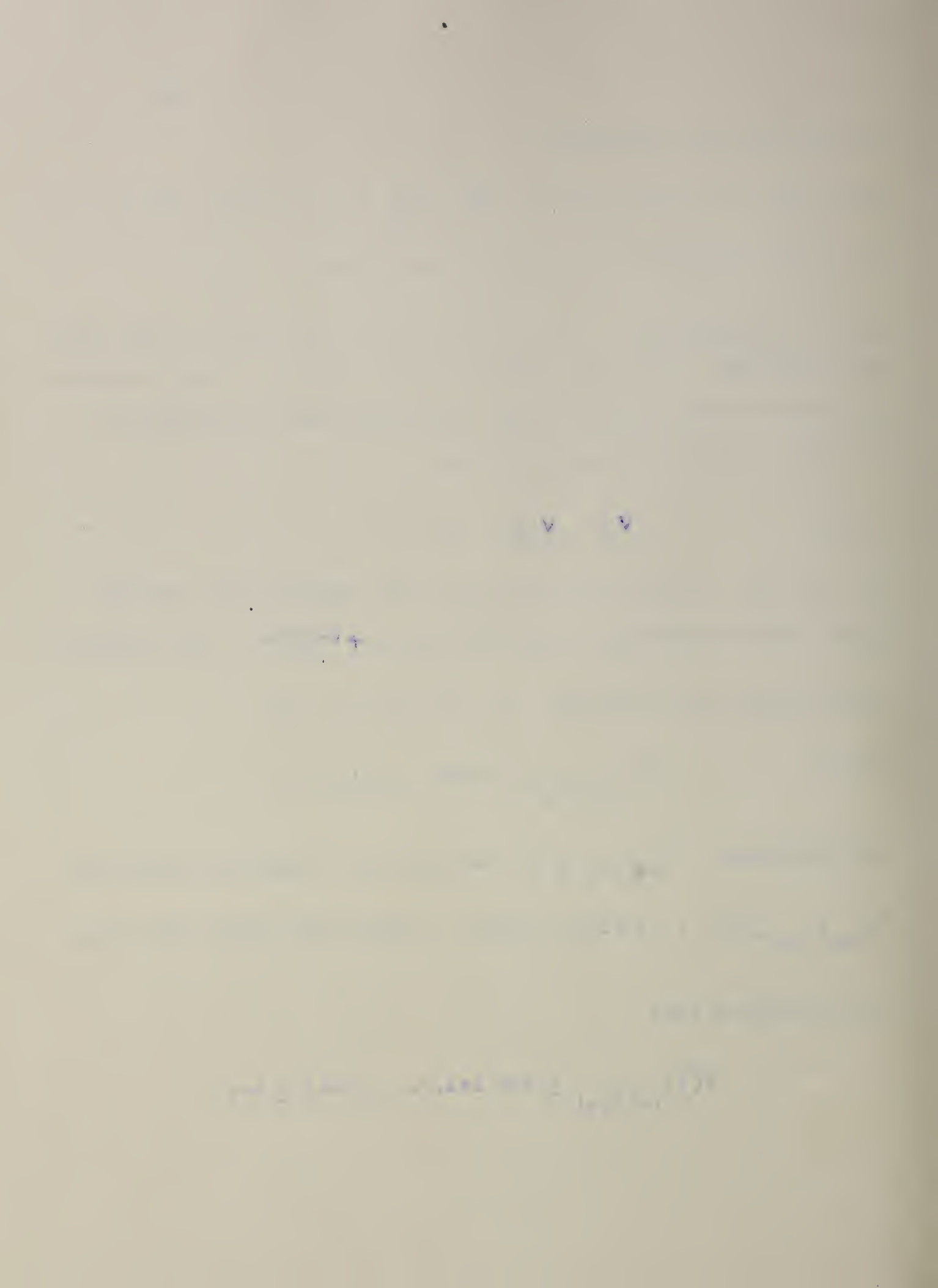
$$(1.23) \quad P(V_{t_{i-1}t_i} \leq \epsilon; i=1, \dots, n) \geq 1-\eta.$$

The relations  $V_{t_{i-1}t_i} \leq \epsilon, i=1, 2, \dots, n$  imply the relations

$V_{t_{i-1}t_{i+1}} \leq 2\epsilon, i=1, 2, \dots, n-1$ . Hence from (1.23) and lemma

1.8 it follows that

$$P(V_{t_{i-1}t_{i+1}} \leq 2\epsilon; i=1, 2, \dots, n-1) \geq 1-\eta.$$





Any two points  $t'_i, t'_k$  with  $|t'_i - t'_k| \leq \frac{1}{n}$  lie together in one of the intervals  $(t_{i-1}, t_{i+1})$ . Hence the above inequality <sup>and (1.22) imply</sup> ~~implies~~

$$(1.24) \quad P \left[ \mathcal{E} \left( \frac{1}{n}, \epsilon, S \right) \right] \geq 1 - \eta .$$

Thus (i) and (ii) imply strong continuity.

We next show that the condition is necessary. We assume that  $x_t$  is strongly continuous and let  $a \leq \bar{a} < \bar{b} \leq b$ . We consider two subdivisions  $S_n$  and  $S_m$  of  $[\bar{a}, \bar{b}]$  both of modulus less than  $\delta$  and the maxima  $M_{\bar{a}\bar{b}S_n}$  and  $M_{\bar{a}\bar{b}S_m}$ . <sup>Let  $S$  consist of the points  $S_n$  and  $S_m$ .</sup> Then the relation

$$|M_{\bar{a}\bar{b}S_n} - M_{\bar{a}\bar{b}S_m}| > \epsilon \text{ implies that for } \text{two points } t, t' \text{ of } S$$

we must have  $|x_t - x_{t'}| > \epsilon$ ,  $|t - t'| \leq \delta$  hence by (1.20) for

sufficiently small  $\delta$  and arbitrary  $\epsilon, \eta$

$$(1.25) \quad P(|M_{\bar{a}\bar{b}S_n} - M_{\bar{a}\bar{b}S_m}| > \epsilon) \leq \eta .$$

On account of theorem 1.1 the relation (1.25) implies that

$\text{plim}_{n \rightarrow \infty} M_{\bar{a}\bar{b}S_n} = M_{\bar{a}\bar{b}}$  exists. In a similar manner it is shown that

$m_{\bar{a}\bar{b}}$  exists so that condition (i) of theorem 1.3 is satisfied.

Now choose  $\delta$  so that for every finite set  $S$  of points

$t_1, \dots, t_n$  and arbitrary  $\epsilon, \eta$

$$(1.26) \quad P \left[ \mathcal{E} \left( \delta, \frac{\epsilon}{2}, S \right) \right] \geq 1 - \eta .$$



Now let  $S = \{a = t_0, t_1, \dots, t_n = b\}$  be any subdivision of modulus less than  $\delta$ ,  $\{S_n\}$  a sequence of subdivisions with moduli converging to zero and containing the points of  $S$ . The relation (1.26) implies

$$(1.27) \quad P(V_{t_{i-1}t_i} S_n \leq \frac{\epsilon}{2}; i=1, \dots, n) \geq 1 - \eta.$$

Now choose  $\epsilon^*$  so that  $\frac{\epsilon}{2} \leq \epsilon^* \leq \epsilon$  and so that  $\epsilon^*$  is a continuity point of the distributions of  $V_{t_{i-1}t_i}$ ;  $i=1, 2, \dots, n$ . It then follows from (1.27)

$$P(V_{t_{i-1}t_i} \leq \epsilon^*; i=1, 2, \dots, n) \geq 1 - \eta.$$

This completes the proof of theorem 1.3.

Theorem 1.4. Let  $x_t$  be a strongly continuous process, then

$$1) \quad X_t = \int_a^t x_\tau d\tau \text{ exists for every } t$$

$$2) \quad x_t = \frac{dX_t}{dt}$$

Proof: By theorem 1.3  $m_{\tau\tau'}, M_{\tau\tau'}$  exist for all pairs  $\tau, \tau'$

and we have for every choice of points  $a = t_0, t_1, t_2, \dots, t_m = t$

and  $t_{i-1} \leq t_i^* \leq t_i (i=1, \dots, n)$

$$(1.28) \quad \sum m_{t_{i-1}t_i} (t_i - t_{i-1}) \leq \sum x_{t_i^*} (t_i - t_{i-1}) \leq \sum M_{t_{i-1}t_i} (t_i - t_{i-1}).$$

To understand this inequality correctly we must remember that

$m_{t_{i-1}t_i}, M_{t_{i-1}t_i}, x_{t_i^*} (i=1, 2, \dots, n)$  are random variables and

that their joint distribution is such that the inequality (1.28)

holds with probability one.

1. The first part of the problem is to show that the function  $f(x) = \frac{1}{x}$  is continuous at  $x = a$ . To do this, we need to show that for any  $\epsilon > 0$ , there exists a  $\delta > 0$  such that if  $|x - a| < \delta$ , then  $|\frac{1}{x} - \frac{1}{a}| < \epsilon$ .

$$\left| \frac{1}{x} - \frac{1}{a} \right| = \left| \frac{a - x}{ax} \right| = \frac{|a - x|}{|ax|}$$

Since  $|a - x| < \delta$ , we have  $\frac{|a - x|}{|ax|} < \frac{\delta}{|ax|}$ . We want this to be less than  $\epsilon$ , so we need  $\frac{\delta}{|ax|} < \epsilon$ , which implies  $\delta < \epsilon |ax|$ . We can choose  $\delta = \min\{\epsilon |ax|, |a|/2\}$  to ensure that  $|ax| > |a|^2/2$ .

3

2. The second part of the problem is to show that the function  $f(x) = \frac{1}{x}$  is not continuous at  $x = 0$ . To do this, we need to show that there exists an  $\epsilon > 0$  such that for any  $\delta > 0$ , there exists an  $x$  such that  $|x - 0| < \delta$  but  $|\frac{1}{x} - \frac{1}{0}| \geq \epsilon$ .

Since  $\frac{1}{0}$  is undefined, we cannot use the standard definition of continuity. Instead, we can show that the function does not have a limit at  $x = 0$ . For any  $M > 0$ , we can find an  $x$  such that  $|x| < \frac{1}{M}$  and  $|\frac{1}{x}| > M$ .

3. The third part of the problem is to show that the function  $f(x) = \frac{1}{x}$  is not continuous at  $x = \infty$ . To do this, we need to show that there exists an  $\epsilon > 0$  such that for any  $M > 0$ , there exists an  $x$  such that  $|x| > M$  but  $|\frac{1}{x} - \frac{1}{\infty}| \geq \epsilon$ .

Since  $\frac{1}{\infty} = 0$ , we need to show that there exists an  $\epsilon > 0$  such that for any  $M > 0$ , there exists an  $x$  such that  $|x| > M$  but  $|\frac{1}{x}| \geq \epsilon$ . We can choose  $\epsilon = \frac{1}{2M}$  and  $x = 2M$ .

4. The fourth part of the problem is to show that the function  $f(x) = \frac{1}{x}$  is not continuous at  $x = -\infty$ . To do this, we need to show that there exists an  $\epsilon > 0$  such that for any  $M > 0$ , there exists an  $x$  such that  $|x| > M$  but  $|\frac{1}{x} - \frac{1}{-\infty}| \geq \epsilon$ .

Since  $\frac{1}{-\infty} = 0$ , we need to show that there exists an  $\epsilon > 0$  such that for any  $M > 0$ , there exists an  $x$  such that  $|x| > M$  but  $|\frac{1}{x}| \geq \epsilon$ . We can choose  $\epsilon = \frac{1}{2M}$  and  $x = -2M$ .

Since the process is strongly continuous we have for any subdivision with sufficiently small modulus  $\delta$

$$P\left[\sum_{t_{i-1}}^{t_i} (t_i - t_{i-1}) - \sum_{t_{i-1}}^{t_i} (t_i - t_{i-1}) \leq \varepsilon(b-a)\right] \geq 1 - \eta.$$

If  $S$  is a subdivision then we call  $Y(S) = \sum_{t_{i-1}}^{t_i} (t_i - t_{i-1})$  the

upper sum and  $y(S) = \sum_{t_{i-1}}^{t_i} (t_i - t_{i-1})$  the lower sum corresponding

to the subdivision  $S$ . We consider now a sequence of subdivisions  $\{S_j\}$  with moduli  $\{\delta_j\}$  such that

$$\lim_{j \rightarrow \infty} \delta_j = 0 \quad \text{and} \quad S_m \subset S_n \quad \text{if} \quad m < n.$$

If  $Y_j = Y(S_j)$  and  $y_j = y(S_j)$  are the corresponding upper and lower sums then

$$Y_j - y_j \geq y_{j+k} - y_j \geq 0$$

and hence for sufficiently large  $n$

$$P\left[0 \leq y_{n+k} - y_n \leq \varepsilon(b-a)\right] \geq 1 - \eta.$$

Hence the sequences of random variables  $\{y_n\}$  and  $\{Y_n\}$  converge

and  $\text{plim}_{n \rightarrow \infty} y_n = \text{plim}_{n \rightarrow \infty} Y_n$ . From here on the proof of the existence

of  $X_t = \int_a^t x_\tau d\tau$  is precisely the same as that of the existence of the ordinary Riemann integral of a continuous function.



It follows also from (1.25) that

$$M_{tt'}(t'-t) \geq \int_t^{t'} x_t dt \geq m_{tt'}(t'-t)$$

Consider now the quotient

$$\frac{X_{t_2} - X_{t_1}}{t_2 - t_1} = \int_{t_1}^{t_2} \frac{x_t dt}{t_2 - t_1}.$$

We have

$$m_{t_1 t_2} \leq \frac{X_{t_2} - X_{t_1}}{t_2 - t_1} \leq M_{t_1 t_2},$$

and

$$m_{t_1 t_2} \leq x_{t_1} \leq M_{t_1 t_2} \quad \text{and since the process is strongly continuous}$$

$$\text{plim}_{t_2 \rightarrow t_1} (M_{t_1 t_2} - m_{t_1 t_2}) = 0. \quad \text{Hence} \quad \text{plim}_{t_2 \rightarrow t_1} \frac{X_{t_2} - X_{t_1}}{t_2 - t_1} = \text{plim}_{t_2 \rightarrow t_1} M_{t_1 t_2} = x_{t_1}$$

and thus

$$\text{plim}_{t_2 \rightarrow t_1} \frac{X_{t_2} - X_{t_1}}{t_2 - t_1} = x_{t_1}.$$

This completes the proof of theorem 1.4.

We shall now consider stochastic processes of second order,

We shall say that

$$X_t = \int_0^t x_t dt \quad \text{exists l.i.m.}$$

if the Riemann sums  $\sum_{t_i} x_{t_i} (t_i - t_{i-1})$  converge in the mean.





Theorem 1.5. Let  $x_t$  be a process with covariance function  $\sigma_{t_1 t_2}$ .

The process is integrable l.i.m. in  $[a, b]$  if and only if for any

$t$  in  $[a, b]$ ,  $\int_a^t \int_a^t \sigma_{t_1 t_2} dt_1 dt_2$  exists. The covariance function

$\Sigma_{t_1 t_2}$  of  $X_t = \int_a^t x_t dt$  is given by

$$(1.29) \quad \Sigma_{t_1 t_2} = \int_a^{t_1} \int_a^{t_2} \sigma_{\tau_1 \tau_2} d\tau_1 d\tau_2.$$

The process  $X_t$  is differentiable l.i.m. and  $X'_t = x_t$  if  $\sigma_{t_1 t_2}$  is continuous.

To prove theorem 1.5 we apply the corollary to lemma 1.7 to a sequence of Riemann sums  $\{\Sigma_n\}$  with moduli going to zero.

We have

$$E(\Sigma_n \Sigma_m) = E\left[\Sigma x_{t_i}^*(t_i - t_{i-1}) \Sigma x_{t_j}^*(t_j - t_{j-1})\right] = \Sigma \Sigma \sigma_{t_i t_j} (t_i - t_{i-1})(t_j - t_{j-1})$$

If  $n$  and  $m$  go to infinity in any manner we have

$$\lim_{\substack{n \rightarrow \infty \\ m \rightarrow \infty}} E(\Sigma_n \Sigma_m) = \int_a^t \int_a^t \sigma_{t_1 t_2} dt_1 dt_2.$$

Thus  $\lim_{n \rightarrow \infty} \Sigma_n = \Sigma$  exists.

Moreover if  $\{\Sigma'_n\}$  is any other sequence of Riemann sums we put  $\Sigma_{2n}'' = \Sigma_n$ ,  $\Sigma_{2n+1}'' = \Sigma'_n$ . Since l.i.m.  $\Sigma_n''$  exists we must have

$$\Sigma = \lim_{n \rightarrow \infty} \Sigma_n = \lim_{n \rightarrow \infty} \Sigma'_n = \int_a^t x_t dt = X_t.$$



It follows also easily from lemma 1.7 that  $E(X_t X_{t'})$  exists and that

$$(1.29a) \quad E(X_t X_{t'}) = \int_a^t \int_a^{t'} \sigma_{t_1 t_2} dt_1 dt_2$$

we further have

$$Y_{t'} = \frac{X_{t'} - X_t}{t' - t} - X_t = \int_t^{t'} \frac{X_s - X_t}{s - t} ds$$

From (1.29a) it follows that

$$\sigma_{Y_t Y_{t'}} = \frac{1}{(t' - t)^2} \int_t^{t'} \int_t^{t'} (\sigma_{\tau\tau'} - \sigma_{t\tau'} - \sigma_{t\tau} + \sigma_{tt}) d\tau d\tau'$$

If  $\sigma_{\tau\tau'}$  is continuous then by the mean value theorem of integral calculus  $\sigma_{Y_t Y_{t'}}$  becomes arbitrarily small if  $t'$  approaches  $t$ . Thus  $X_t = X_{t'} \text{ l.i.m.}$

Let  $x_t, y_t$  be two stochastic processes, then to every subdivision  $S = (a=t_0, t_1, t_2, \dots, t_n=b)$  we can form Riemann-Stieltjes sums

$$(1.30) \quad X(S) = \sum_{i=1}^n x_{t_{i-1}} (y_{t_i} - y_{t_{i-1}})$$



If now for every sequence  $\{S_n\}$  of subdivisions with modulus  $\epsilon_n$  converging to zero  $\text{plim}_{n \rightarrow \infty} X(S_n)$  exists and is independent of the particular sequence  $\{S_n\}$  and of the choice of points  $t_1^*$  ( $t_{1-1} \leq t_1^* \leq t_1$ ) then we shall write

$$X = \text{plim}_{n \rightarrow \infty} X(S_n) = \int_a^b x_t dy_t$$

We shall call  $X$  the integral of  $x_t$  with respect to  $y_t$ . If the random variables  $X(S_n)$  converge in the mean to  $X$  we shall say that

$$\int_a^b x_t dy_t \text{ exists l.i.m.}$$

Theorem 1.3. Let  $x_t, y_t$  be two independent stochastic processes of second order (that is to say  $x_t$  is independent of  $y_{t'}$  for any  $t$  and any  $t'$ ) with covariance functions  $\sigma_{tt'}, \beta_{tt'}$  respectively.

The integral of  $x_t$  with respect to  $y_t$  exists l.i.m. for every interval  $[a, t]$  contained in  $[a, b]$  if and only if

$$(1.31) \quad \int_a^t \int_a^t \sigma_{t_1 t_2} d\beta_{t_1 t_2}$$

exists. The covariance function of

$$X_t = \int_a^t x_t dy_t$$

is moreover given by

$$\Sigma_{t_1 t_2} = \int_a^{t_1} \int_a^{t_2} \sigma_{\tau_1 \tau_2} d\beta_{\tau_1 \tau_2}.$$



The proof of theorem 1.6 is analogous to that of theorem 1.5 and is left to the reader.

By  $P(E|E')$  we denote the conditional probability that  $E$  will happen provided  $E'$  has happened. [9] <sup>9</sup> We have called a process con-

tinuous if  $\lim_{\tau \rightarrow 0} x_{t+\tau} = x_t$ . A process will be called uniformly continuous in  $[a, b]$  if to every  $\epsilon > 0$ ,  $\eta > 0$  there exists a  $\delta(\epsilon, \eta)$  independent of  $t$  such that

$$P(|x_{t+\tau} - x_t| \leq \epsilon) \geq 1 - \eta \quad \text{for every } |t| \leq a(\epsilon, \eta)$$

Lemma 1.9. If a process is continuous in a closed interval  $[a, b]$  then it is uniformly continuous in  $[a, b]$ .

Proof: Consider a monotone decreasing sequence  $\{\tau_i\}$  with

$\lim_{i \rightarrow \infty} \tau_i = 0$ . For every  $i$  consider the set of points  $t$  for which

$$P(|x_{t+\tau_i} - x_t| > \epsilon) > \eta$$

Assume the lemma to be false, then we can construct a sequence

$t_1, t_2, \dots$  such that for some  $\epsilon > 0$ ,  $\eta > 0$

$$P(|x_{t_1+\tau} - x_{t_1}| > 2\epsilon) > 2\eta$$

for some  $\tau < \tau_1$ . Let  $t$  be an accumulation point of the sequence

$\{t_1\}$ . Then for every  $\delta > 0$  and  $\eta$  we can find a  $t_1$  arbitrarily close

to  $t$  such that  $P(|x_{t_1+\tau} - x_{t_1}| > 2\epsilon) > 2\eta$  for some  $|\tau| < \frac{\delta}{2}$ .

[9] For the concept of conditional probability the reader is referred to Kolmogoroff, Grundbegriffe der Wahrscheinlichkeitsrechnung, Chapter 5, par. 1 and 3.





Choose now  $|t_1 - t| < \frac{\delta}{2}$  and  $t_1$  so close to  $t$  that

$$P(|x_{t_1} - x_t| > \varepsilon) < \eta.$$

Then

$$\begin{aligned} P(|x_{t_1+\tau} - x_t| > \varepsilon) &\geq P(|x_{t_1+\tau} - x_{t_1}| > 2\varepsilon, |x_{t_1} - x_t| \leq \varepsilon) \\ &\geq P(|x_{t_1+\tau} - x_{t_1}| > 2\varepsilon) - P(|x_{t_1} - x_t| > \varepsilon) > \eta. \end{aligned}$$

Hence for arbitrary  $\delta > 0$  and some  $\varepsilon > 0$ ,  $\eta > 0$  there exist values  $\tau < \delta$  such that

$$P(|x_{t+\tau} - x_t| > \varepsilon) > \eta$$

in contradiction with our assumption of continuity of the process  $x_t$ .

We derive next a sufficient condition for the existence of  $M_{ab}$  and  $m_{ab}$ . We have to consider in this connection the event  $\{x_k \leq x_{k-1} \text{ for } k < i, x_k \leq x_{k+1} \text{ for } k > i\}$  and we denote this event by  $A_i$  and state the following

Theorem 1.7. Let  $S = (a \leq t_1 < t_2, \dots, t_n \leq b)$  be a set of points

in  $[a, b]$  and let  $x_t$  be a stochastic process which

- (i) is continuous in  $[a, b]$ ,
- (ii) is such that for sufficiently small  $\tau$  — which is independent of  $S$  —

$$(1.32) \quad P(|x_{t_1+\tau} - x_{t_1}| > \varepsilon | A_1) \leq KP(|x_{t_1+\tau} - x_{t_1}| > \varepsilon)$$

where  $K$  is a constant independent of the choice of  $S$ . Then  $M_{ab}$  and  $m_{ab}$  exist.



Proof: Let  $S_i = (t_1, \dots, t_n)$  and  $S_j = (t'_1, \dots, t'_n)$  be two subdivisions of modulus less than  $\delta$  and put for short  $x_{t_a} = x_a$ ,  $x_{t'_a} = x'_a$ .

To every  $x_k$  we can find an  $x'_{j_k} = y_k$  such that  $t_{j_k} - t_k < \delta$ .

For  $P(A_k) \neq 0$  we thus have on account of Lemma 1.9 for arbitrary  $\eta$  and sufficiently small  $\delta$

$$(1.33) \quad P(A_k, x_k - y_k > \varepsilon) = P(A_k)P(x_k - y_k > \varepsilon | A_k) \leq K\eta P(A_k).$$

If  $P(A_k) = 0$  the inequality (1.33) is also valid.

Denote by  $B_k$  the joint occurrence of the events  $A_k$  and  $x_k - y_k \leq \varepsilon$  and let  $B$  be the event that at least one of the events  $B_k$  occurs. By definition  $A_k$  implies

$$x_k = M_{abS_i} \quad \text{so that } B \text{ implies the existence of some } y_k$$

such that  $y_k \geq M_{abS_i} - \varepsilon$ . From this it follows in turn that

$$M_{abS_j} \geq M_{abS_i} - \varepsilon. \quad \text{Therefore we see that}$$

$$P(M_{abS_j} \geq M_{abS_i} - \varepsilon) \geq P(B) \geq \sum_k P(B_k).$$

From (1.33) we obtain easily

$$P(B_k) = P(A_k)P(x_k - y_k \leq \varepsilon | A_k) \geq (1 - K\eta) P(A_k).$$



The events  $A_k$  exclude each other and exhaust all the possibilities so that by adding these inequalities we obtain

$$\sum_k P(B_k) \geq 1 - K\eta$$

Therefore,

$$(1.34) \quad P(M_{abS_j} \geq M_{abS_1} - \epsilon) \geq 1 - K\eta.$$

Similarly we obtain

$$(1.34a) \quad P(M_{abS_1} \geq M_{abS_j} - \epsilon) \geq 1 - K\eta.$$

Hence

$$(1.35) \quad P(|M_{abS_1} - M_{abS_j}| \leq \epsilon) \geq 1 - 2K\eta.$$

The existence of  $M_{ab}$  follows easily from (1.35) using theorem 1.1 and the existence of  $m_{ab}$  is proved similarly.



## CHAPTER 2

### SPECIAL PROCESSES

#### 1. The fundamental random process.

1 A small particle suspended in a gas is subjected to a continual bombardment by the molecules of this gas. The individual impacts imparted by these molecules are small compared to the mass of the <sup>particle</sup> ~~gas~~ and the number of impacts per second is very large. The impacts are received from all directions and are randomly distributed. Moreover, if we neglect the velocity of the particle itself, which is small compared to the velocity of the molecules, the distribution of these impacts at time  $t$  will be independent of the momentum of the particle at time  $t' \leq t$ . If we denote by  $x_t$  the momentum of the particle, it will therefore be reasonable to assume that  $x_{t+\tau} - x_t$  is independent of  $x_t$  for  $t^* \leq t$ . The motion of the particle is called the Brownian motion.

The momentum of the particle is a special example of a more general type of stochastic processes, called Markoff processes, which satisfy for  $t_1 < t_2 < \dots < t_n < t$  and  $\tau > 0$  the equation

$$P(x_{t+\tau} \leq A | x_{t_1}, \dots, x_{t_n}, x_t) = P(x_{t+\tau} \leq A | x_t) .$$

In words, the conditional distribution of  $x_{t+\tau}$  (where  $\tau > 0$ ), given the values of  $x_{t_1}, \dots, x_{t_n}, x_t$  (where  $t_1 < t_2 < \dots < t_n < t$ ) is the same as the conditional distribution of  $x_{t+\tau}$  given  $x_t$ .





More conversationally speaking, if the present value of  $x_t$  is known the distribution of any future values is independent of the way in which the present value was reached.

In our special case of the motion of a particle suspended in a gas we shall make the following assumptions about its momentum  $x_t$ :

Assumption 1.

$$(2.1) \quad x_{t+\tau} = x_t + \varepsilon_{t,\tau}$$

where  $\varepsilon_{t,\tau}$  is a random variable with mean zero and is independent of  $x_t$  and also of  $\varepsilon_{t',\tau'}$  if the intervals  $(t, t+\tau)$ ,

$(t', t'+\tau')$  do not overlap.

Assumption 2.

The distribution of  $\varepsilon_{t,\tau}$  depends only on  $\tau$ .

Assumption 3.

The variance of  $\varepsilon_{t,\tau}$  exists and is a measurable function of  $\tau$ .

We have for  $\tau = \tau_1 + \tau_2$ ,  $\tau_1 \geq 0$ ,  $\tau_2 \geq 0$

$$\varepsilon_{t,\tau} = \varepsilon_{t,\tau_1} + \varepsilon_{t+\tau_1,\tau_2}$$

and hence

$$(2.2) \quad \sigma_{\tau_1}^2 + \sigma_{\tau_2}^2 = \sigma_{\tau_1 + \tau_2}^2$$

where  $\sigma_{\tau}^2 = \sigma_{\varepsilon_{t,\tau}}^2$ .



From (2.2) and assumption 3 it follows by a well known theorem [10] that

$$(2.3) \quad \sigma_{\tau}^2 = c\tau$$

where  $c$  is some positive constant. Thus  $x_{t+\tau}$  converges to  $x_t$  in the mean with decreasing  $\tau$  and the process is continuous i. i. m. ¶ Suppose further that  $x_0 = 0$ . It follows then from (2.3) and assumption 1 that

$$(2.4) \quad \begin{cases} \sigma_{x_t}^2 = ct \\ \sigma_{x_t x_{t+\tau}} = \sigma\{x_t (x_{t+\tau} - x_t + x_t)\} = \sigma_{x_t}^2 = ct \end{cases}$$

¶ The assumptions 1 to 3 define an important class of Markoff processes, sometimes called differential processes. [11] ¶ In order to define completely our mathematical model for the Brownian motion we must also take account of the fact that we regard the impacts from the molecules as coming in a continuous stream so that large changes of the momentum in a short time interval become much less likely than small ones.

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[10] If a measurable function  $f(x)$  satisfies the functional equation  $f(x+y) = f(x) + f(y)$  then  $f(x) = cx$ . Proof of this theorem may be found in H. Hahn, Theorie der reellen Funktionen, Erster Band, pp. 581-3, J. Springer, Berlin (1921).

[11] we distinguish differential process from "general differential processes" (Chapter 4).



We therefore impose the following additional condition, called the Lindeberg condition, on the distribution functions  $F_{\tau}(a)$  of  $\varepsilon_{t, \tau}$ .

Assumption 4 (Lindeberg condition):

For sufficiently small  $\tau$  and arbitrary  $\rho > 0$ ,  $\eta > 0$

$$(2.5) \quad \int_{|a| > \rho} a^2 dF_{\tau}(a) < \eta \sigma_{\tau}^2.$$

Condition (2.5) may perhaps best be understood if we discuss an important case where it is fulfilled.

Theorem 2.1. If  $\sigma_{\tau}^2 = \int_{-\infty}^{+\infty} a^2 dF_{\tau}(a)$  exists and if  $P\left(\frac{\varepsilon_{t, \tau}}{\sigma_{\tau}} < a\right) = F(a)$  is independent of  $\tau$  then  $\varepsilon_{t, \tau}$  fulfills the Lindeberg condition (2.5).

*Proof.* From the conditions of the theorem we have

$$F_{\tau}(a \sigma_{\tau}) = F(a), \quad F_{\tau}(a) = F(a / \sigma_{\tau}).$$

Hence for arbitrarily small  $\rho > 0$  and  $\eta > 0$

$$\begin{aligned} \int_{|a| > \rho} a^2 dF_{\tau}(a) &= \int_{|a| > \rho} a^2 dF(a / \sigma_{\tau}) = \sigma_{\tau}^2 \int_{|a| > \rho / \sigma_{\tau}} \frac{a^2}{\sigma_{\tau}^2} dF\left(\frac{a}{\sigma_{\tau}}\right) \\ &= \sigma_{\tau}^2 \int_{|y| > \rho / \sigma_{\tau}} y^2 dF(y) \leq \eta \sigma_{\tau}^2 \end{aligned}$$

✓ for sufficiently small  $\tau$  since  $\sigma_{\tau}^2 = \sigma \tau$ .

*the inequality holding*



We next prove

Theorem 2.2. If  $\varepsilon_{t,\tau}$  fulfills the Lindeberg condition then  $\varepsilon_{t,\tau}$  is normally distributed with variance  $\sigma^2 \tau$ .

As long as we consider only the distribution of  $\varepsilon_{t,\tau}$  there will not be any danger of confusion if we write  $\varepsilon_\tau$  for  $\varepsilon_{t,\tau}$ .

We shall use the following

Lemma 2.1: Let  $x_1, x_2, \dots, x_k$  be independent random variables with <sup>mean zero and unit</sup> distribution functions  $F_1, \dots, F_k$  respectively and  $\sigma^2_{(x_1 + \dots + x_k)} = 1$  then to every  $\delta$  there exists a  $\rho$  and an  $\eta$  such that

$$|P(x_1 + \dots + x_k < a) - \int_{-\infty}^a (1/\sqrt{2\pi}) e^{-x^2/2} dx| < \delta$$

whenever for all  $k$

$$\int_{|x| > \rho} x^2 dF_k(x) < \eta \sigma_{x_k}^2$$

A proof of this lemma can be found for instance in Khintchine's "Asymptotische Gesetze der Wahrscheinlichkeitsrechnung", p.3 (Ergebnisse der Mathematik, J. Springer, Berlin 1933).

To prove theorem 2.2 we put  $\varepsilon'_\tau = \varepsilon / \sqrt{\sigma^2 \tau}$ ; then  $\varepsilon'_\tau$  has variance 1. We divide the interval  $0 \leq t \leq \tau$  into  $n$  equal parts and put

$$\varepsilon_i = \{ (x_{t+(i\tau/n)} - x_{t+[(i-1)\tau/n]}) / \sqrt{\sigma^2 \tau} \text{ for } i=1, 2, \dots, n.$$





The  $\epsilon_i$  are independently distributed and  $\epsilon'_\tau = \epsilon_1 + \epsilon_2 + \dots + \epsilon_n$ .

Since the  $\epsilon_i$  fulfill the Lindeberg condition we see from lemma 2.1 that the distribution of  $\epsilon'_\tau$  differs arbitrarily little from the normal distribution with unit variance. Hence  $\epsilon_\tau = \sigma_\tau \epsilon'_\tau$  is normally distributed with variance  $\sigma_\tau^2 = \sigma\tau$ .

The processes defined by assumptions 1, 2, 3, and 4 will be called fundamental random processes (abbreviated F.R.P.).

In the following we shall repeatedly use the fact that the distribution of the limit of a sequence of random variables equals the limit of the distribution functions in all its continuity points.

## 2. Further properties of the F.R.P.

Theorem 2.3. Every F.R.P. is strongly continuous.

Without loss of generality we shall assume  $\sigma = 1$ ; that is,

$E[(x_{t+\tau} - x_t)^2] = \tau$ . We derive next several lemmas needed for the proof of theorem 2.3.

Lemma 2.2. For  $a > 0$  we have

$$(2.5) \quad \int_a^{\infty} e^{-x^2/2} dx \leq e^{-a^2/2} / a.$$

Proof:

$$a \int_a^{\infty} e^{-x^2/2} dx < \int_a^{\infty} x e^{-x^2/2} dx = e^{-a^2/2}.$$

Lemma 2.3. In an interval  $(t, t')$  of length  $\delta$  and for every <sup>finite</sup> set  $S$  of points in this interval

$$(2.7) \quad P(V_{tt'S} \geq K) \leq (8e^{-K^2/8\delta} / \sqrt{2\pi}) \sqrt{\delta}.$$



We add the points  $t$  and  $t'$  to  $S$ . This can only increase the value of  $V_{tt'S} = M_{tt'S} - m_{tt'S}$ . Without loss of generality we may further assume  $x_t = 0$ , since we could otherwise consider the process  $x'_s = x_{s-t}$ . Let  $t_0 = t, t_1, t_2, \dots, t_n = t'$  be the subdivision points of  $S$  and put  $x_i = x_{t_i}$ . Let  $A_i$  ( $i=1, 2, \dots, n$ ) be the event  $\{x_1 < M, \dots, x_{i-1} < M, x_i \geq M\}$ . The events  $A_i$  are exclusive and exhaust all cases for which  $M_{tt'S} \geq M$ .

We have with  $x_0 = 0, M \geq 0$

$$P(A_i, x_n \geq M) = P(A_i) P(x_n \geq M | A_i) = P(A_i) P(x_n - x_1 \geq M - x_1 | A_i) \\ \geq P(A_i) P(x_n - x_1 \geq 0 | x_1 < M, \dots, x_{i-1} < M, x_i \geq M) ;$$

our  
on account of  $\Lambda$  assumptions we know that  $x_n - x_1$  is independently distributed of  $x_1, \dots, x_i$  and has a normal distribution with zero mean and variance  $t_n - t_1$  so that

$$(2.8) \quad P(A_i, x_n \geq M) \geq P(A_i) \int_0^{\infty} \frac{1}{\sqrt{2\pi(t_n - t_1)}} \exp\left[-\frac{1}{2} \frac{(x_n - x_1)^2}{(t_n - t_1)}\right] d(x_n - x_1) \\ = \frac{1}{2} P(A_i) .$$

The events  $\{A_i, x_n \geq M\}$  comprise all cases for which  $x_n \geq M$  and are mutually exclusive. Adding (2.8) over all  $i$  we therefore obtain

$$(2.9) \quad 2P(x_n \geq M) \geq P(M_{tt'S} \geq M) .$$



The left side is by (2.6) smaller than

$$(2.10) \quad (2\sqrt{3}/M\sqrt{2\pi})e^{-M^2/2\delta}$$

The same estimate is obtained also for  $P(m_{tt'S} \leq -M)$ .

Furthermore

$$\begin{aligned} P(V_{tt'S} \geq M) &\leq P(\text{either } M_{tt'S} \geq \frac{M}{2} \text{ or } m_{tt'S} \leq -\frac{M}{2}) \\ &\leq P(M_{tt'S} \geq \frac{M}{2}) + P(m_{tt'S} \leq -\frac{M}{2}) \leq (8\sqrt{2}/M\sqrt{2\pi})e^{-M^2/8\delta}, \end{aligned}$$

which establishes (2.7).

Lemma 2.4. Let  $t' - t = \ell$  and consider subdivisions  $S_n = (t = t_0, t_1, \dots, t_n = t')$  such that  $t_i - t_{i-1} = \ell/n = \delta_n$ . Then there exists for every positive  $\epsilon, \eta$  an  $N$  such that for an arbitrary ~~subset~~ <sup>finite set</sup>  $S$  of points

$$(2.11) \quad P(V_{t_{i-1}t_i} S \leq \epsilon, i=1, \dots, n) \geq 1 - \eta \quad \text{for } n > N.$$

To prove (2.11) we add the points of  $S_n$  to  $S$ . This will at most decrease the probability in (2.11). Since the distribution of  $V_{t_{i-1}t_i} S$  is independent of the distribution of that of

$V_{t_{j-1}t_j} S$  for  $i \neq j$  we have by lemma 2.3

$$\begin{aligned} (2.12) \quad P(V_{t_{i-1}t_i} S \leq \epsilon, i=1, \dots, n) &\geq \left[ 1 - \frac{8 \exp(-n\epsilon^2/8\ell)}{\sqrt{2\pi}\epsilon\sqrt{n}} \sqrt{\ell} \right]^n \\ &\geq (1 - ke^{-nk'})^n \end{aligned}$$



where  $k$  and  $k'$  are <sup>positive</sup> constants independent of  $n$ . Since it is easily seen that  $\lim_{n \rightarrow \infty} (1 - k e^{-nk'})^n = 1$ , lemma 2.4 follows.

Lemma 2.5.  $M_{tt'}$  and  $m_{tt'}$  exist for every interval  $[t, t']$ .

Consider a sequence of subdivisions  $\{S_i\}$  of the interval  $[t, t']$  of modulus  $\{\delta_i\}$  with  $\lim_{i \rightarrow \infty} \delta_i = 0$ . If  $S_i$  and  $S_j$  have both sufficiently small modulus ~~then~~ then in every interval of length  $\delta_n$  of lemma 2.4 there will be at least one point of  $S_i$  and one of  $S_j$ . Hence applying lemma 2.4 to the union of  $S_i$  and  $S_j$  we obtain

$$P(|M_{tt'} S_i - M_{tt'} S_j| > \epsilon) \leq \eta.$$

Thus  $\text{plim}_{i \rightarrow \infty} M_{tt'} S_i = M_{tt'}$  exists for every sequence  $\{S_n\}$  whose moduli converge to zero and is the same for every such sequence. In a similar manner it is possible to prove that  $\text{plim}_{i \rightarrow \infty} m_{tt'} S_i = m_{tt'}$  exists.

To prove theorem 2.3 let  $S$  be any subdivision of modulus  $\leq \delta/2$  and consider

$$(2.13) \quad P(\bigvee_{i=1}^n t_{i-1} t_i \leq \epsilon; i=1, 2, \dots, n).$$

We form a new subdivision with  $\delta/2 \leq t_i - t_{i-1} \leq \delta$  by deleting points of  $S$ . The probability (2.13) for this new subdivision is smaller than that for  $S$ .





By lemma 2.3 we have, since the distribution of  $V_{t_{i-1}t_i}S$  converges to the distribution of  $V_{t_{i-1}t_i}^0$

$$(2.14) \quad P(V_{t_{i-1}t_i} \leq \epsilon; i=1, \dots, n) \geq \frac{[1 - \delta \exp(-\epsilon^2/95)]^{2t/\delta}}{\epsilon \sqrt{2\pi}}$$

The limit on the right of (2.14) is 1 for  $\delta \rightarrow 0$ , which proves theorem 2.3.

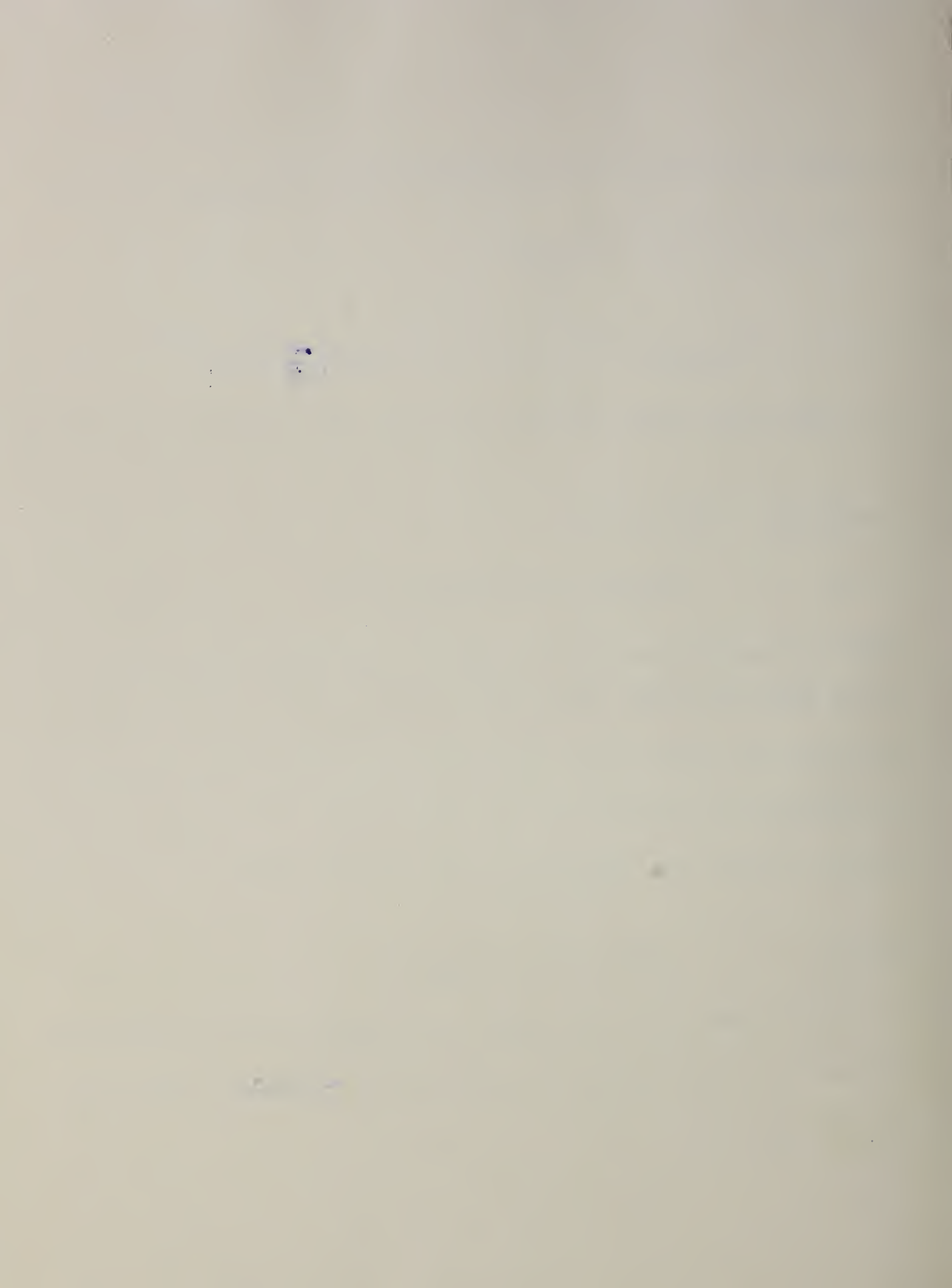
Theorem 2.4. For the F.R.P.

$$(2.15) \quad P(M_{ab} - x_a \geq M) = 2P(x_b - x_a \geq M)$$

*Proof:* We consider the proof of lemma 2.3 and write  $t' = b$ ,  $t = a$  we see from (2.9) that  $2P(x_b \geq M) \geq P(M_{abS_1} \geq M)$ . Here  $S_1$  may be any set of points in the interval  $[a, b]$ . Let  $S_1$  be an element of a sequence of subdivisions  $\{S_i\}$  whose moduli go to zero. According to lemma 2.5  $\text{plim}_{i \rightarrow \infty} M_{abS_i} = M_{ab}$  exists, hence

$$(2.16) \quad 2P(x_b \geq M) \geq P(M_{ab} \geq M)$$

We consider again the events  $A_i$  of lemma 2.3 for a subdivision of  $[a, b]$  into equal parts. We then have for ~~points~~  $x_0$  with  $a > b$  and with  $x_2 = 0$



$$\begin{aligned}
 (2.17) \quad P(A_1, x_0 \geq M) &= P(A_1, x_0 - x_1 \geq 0) + P(A_1, M \leq x_0 < x_1) \\
 &= \frac{1}{2}P(A_1) + P(A_1, M \leq x_0 < x_1) \\
 &\leq \frac{1}{2}P(A_1) + P(A_1, x_{i-1} \leq x_0 < x_1) .
 \end{aligned}$$

Let  $\epsilon$  be an arbitrary positive number, then

$$\begin{aligned}
 (2.18) \quad P(A_1, x_{i-1} \leq x_0 < x_1) &= P(A_1, x_i - x_{i-1} \leq \epsilon, x_{i-1} \leq x_0 < x_1) \\
 &\quad + P(A_1, x_i - x_{i-1} > \epsilon, x_{i-1} \leq x_0 < x_1) \\
 &\leq P(A_1, 0 < x_i - x_{i-1} \leq \epsilon) + P(x_i - x_{i-1} > \epsilon) \\
 &\leq P(A_1, \overset{0 \leq}{x_b - x_0} \leq \epsilon) + P(x_i - x_{i-1} > \epsilon)
 \end{aligned}$$

since  $x_i - x_0$  has larger variance than  $x_b - x_0$  and both are normally distributed with mean zero. Adding the inequalities (2.17) and using

(2.18) we obtain

$$\begin{aligned}
 (2.19) \quad P(x_0 \geq M, M_{ab} \geq M) &\leq \frac{1}{2}P(M_{ab} \geq M) + P(0 \leq x_b - x_0 \leq \epsilon) \\
 &\quad + \sum P(x_i - x_{i-1} > \epsilon) .
 \end{aligned}$$

From (2.7) we see easily that  $\sum P(x_i - x_{i-1} > \epsilon)$  converges to zero for every positive  $\epsilon$  and every sequence  $\{S_n\}$  whose modulus converges to zero. Since  $\epsilon$  was arbitrary we have

$$(2.20) \quad P(x_0 \geq M, M_{ab} \geq M) \leq \frac{1}{2}P(M_{ab} \geq M) .$$



Since  $c$  may be chosen arbitrarily close to  $b$ , it follows from (2.20) that

$$(2.21) \quad 2P(x_b \geq M) \leq P(M_{ab} \geq M) .$$

The inequalities (2.21) and (2.16) together imply theorem 2.4.

Corollary to theorem 2.4. Let  $S_1, S_2, \dots$  be a sequence of subdivisions of the interval  $(t, t+\tau)$  with modulus  $\leq$  converging to zero and consider for each  $S_n = \{t=t_1, \dots, t_n=t+\tau\}$  the probability

$$P_n = P(x_{t_2} - x_{t_1} \leq 0, \dots, x_{t_n} - x_{t_{n-1}} \leq 0) ;$$

then  $\lim_{n \rightarrow \infty} P_n = 0$  .

*Proof:* We have

$$\lim_{n \rightarrow \infty} P_n = P(M_{t, t+\tau} - x_t = 0) = 1 - P(M_{t, t+\tau} - x_t > 0) = 1 - 2P(x_{t+\tau} - x_t > 0) = 0 .$$

Theorem 2.5. If  $\rho \geq 1/2$  ,  $\text{plim}_{\tau \rightarrow 0} (x_{t+\tau} - x_t) / \tau^\rho$  does not exist.

It is zero if  $\rho < 1/2$  .

The proof of this theorem is left to the reader.

Theorem 2.6. Let  $[a, b]$  be any interval and  $\epsilon, \eta$  arbitrary positive numbers and  $\rho < 1/2$  . Then there exists a  $\delta$  such that for every subdivision  $S = \{a=t_0, t_1, \dots, t_n=b\}$  of modulus not exceeding  $\delta$

$$P[V_{t_{i-1}, t_i} / (t_i - t_{i-1})^\rho \leq \epsilon; i=1, \dots, n] \geq 1-\eta$$



For the proof we need the following

Lemma 2.6. For  $\delta_1 > 0$ ,  $\delta_2 > 0$ ,  $k \geq 0$ ,  $k' > 0$ ,  $v > 0$ , and  $\delta_1 + \delta_2$  sufficiently small

$$(2.22) \quad [1 - k \exp(-k'/\delta_1^v)] [1 - k \exp(-k'/\delta_2^v)] \geq 1 - k \exp[-k'/(\delta_1 + \delta_2)^v]$$

Proof of lemma 2.6: The left side of (2.22) is not smaller than

$$1 - k \{ \exp(-k'/\delta_1^v) + \exp(-k'/\delta_2^v) \} .$$

Hence (2.22) is proved if we prove for sufficiently small  $\delta_1$  and  $\delta_2$

$$F(\delta_1, \delta_2) = \exp[-k'/(\delta_1 + \delta_2)^v] - \exp[-k'/\delta_1^v] - \exp[-k'/\delta_2^v] \geq 0$$

We have

$$\lim_{\substack{\delta_1 \rightarrow 0 \\ \delta_2 \rightarrow 0}} F(\delta_1, \delta_2) = 0 \quad \text{and}$$

$$\frac{\partial F}{\partial \delta_1} = vk' \left\{ \frac{\exp[-k'/(\delta_1 + \delta_2)^v]}{(\delta_1 + \delta_2)^{v+1}} - \frac{\exp[-k'/\delta_1^v]}{\delta_1^{v+1}} \right\}$$

$$\frac{\partial F}{\partial \delta_2} = vk' \left\{ \frac{\exp[-k'/(\delta_1 + \delta_2)^v]}{(\delta_1 + \delta_2)^{v+1}} - \frac{\exp[-k'/\delta_2^v]}{\delta_2^{v+1}} \right\}$$





The function  $x^{-v-1} \exp(-k'/x^v)$  is monotonically increasing for sufficiently small  $x$ . We have therefore  $\partial F/\partial \delta_1 > 0$ ,

$\partial F/\partial \delta_2 > 0$  for sufficiently small  $\delta_1 + \delta_2$ . Hence  $F(\delta_1, \delta_2)$  is positive for sufficiently small  $\delta_1 + \delta_2$ .

We proceed to prove theorem 2.6. We have by (2.7) with  $1-2\rho=v$ ,  $\delta_i = t_i - t_{i-1} \leq 1$

$$P \left[ \left( \sum_{i=1}^v t_{i-1} t_i \right) / \delta_i^v \leq \varepsilon \right] \geq 1 - (8\sqrt{\delta_i^v} / \varepsilon / \sqrt{2\pi}) \exp(-\varepsilon^2 / 8\delta_i^v) \\ \geq 1 - k \exp(-k' / \delta_i^v)$$

where  $k$  and  $k'$  are independent of the subdivision.

Thus

$$(2.23) \quad P = P \left[ \frac{\sum_{i=1}^v t_{i-1} t_i}{(t_i - t_{i-1})^v} \leq \varepsilon, i=1, \dots, n \right] \geq \prod_{i=1}^{i=n} \left[ 1 - k \exp\left(-\frac{k}{\delta_i^v}\right) \right]$$

Now let  $S$  have modulus  $\frac{\delta}{2}$  then by lemma 2.6 we may combine the intervals to the right of (2.23) in such a way that all intervals are at least of length  $\frac{\delta}{2}$  and at most of length  $\delta$ . Hence

$$(2.24) \quad P \geq \left[ 1 - k \exp(-k' / \delta^v) \right]^{2(b-a)/\delta}$$

and the right hand side of (2.24) is arbitrarily close to one if  $\delta$  is sufficiently small. This completes the proof of theorem 2.6.



A process is called Gaussian if the joint distribution of  $x_{t_1}, x_{t_2}, \dots, x_{t_n}$  is normal for every choice of  $t_1, t_2, \dots, t_n$ .

We now consider the integral of a F.R.P. and prove

Theorem 2.7. Let  $x_t$  be a F.R.P. with variance  $ct$ . The process  $X_t = \int_0^t x_\tau d\tau$  is a Gaussian process with mean zero and ~~covariance~~ <sup>covariance</sup> function  $\sigma_{tt'} = \frac{c}{2} \max(t, t') [\min(t, t')]^2 - \frac{c}{6} [\min(t, t')]^3$ .

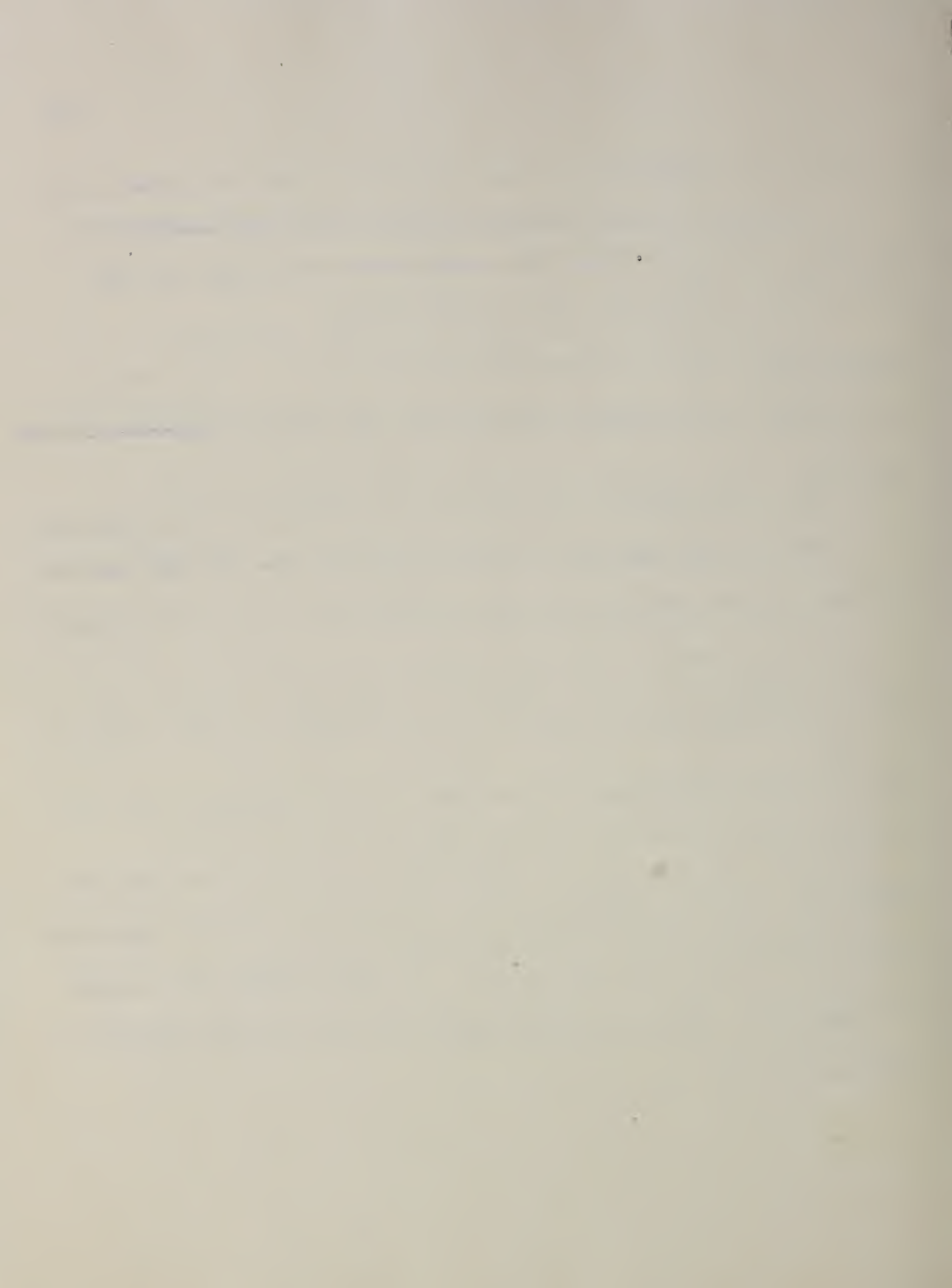
The integral  $X_t$  exists l.i.m. by theorem 1.5. By lemma 1.4 we have  $E(X_t) = 0$  and by theorem 1.5 for  $t' > t$

$$\sigma_{tt'} = c \int_0^t \int_0^{t'} \min(\tau, \tau') d\tau d\tau' = c \int_0^t d\tau \int_0^{\tau} \tau' d\tau' + c \int_0^t \tau d\tau \int_{\tau}^{t'} d\tau' = c \frac{t' t^2}{2} - c \frac{t^3}{6}$$

Each of the approximating Riemann sums is normally distributed and that  $X_t$  itself is Gaussian follows from the following lemma:

Lemma 2.7. Let  $x_n = (x_n^1, x_n^2, \dots, x_n^s)$  be a sequence of normally distributed vectors with mean 0 and assume that  $\lim_{n \rightarrow \infty} \sigma_{x_n^i x_n^j} = \sigma_{ij}$  exists.

If  $\lim_{n \rightarrow \infty} x_n = x$  then  $x$  is normally distributed with mean zero and covariance matrix  $\sigma_{ij}$ .



Proof: The inequality (1.5) may be derived also for vectors if we interpret  $x \leq a$  to mean that the vector  $a-x$  has non-negative components. Lemma 2.7 then follows easily from the fact that for arbitrarily small  $\delta$  and sufficiently large  $n$

$$F_n(a+\delta) + \delta \geq F(a) \geq F_n(a-\delta) - \delta$$

where  $F_n(a)$ ,  $F(a)$  are the distribution functions of  $x_n$  and  $x$  respectively.

### 3. Frictional effects. The Ornstein-Uhlenbeck process.

We have so far in the Brownian motion neglected the effect of the motion of the particle itself on  $\varepsilon_{t,\tau}$ . If the particle has the momentum  $x_t$  then the random impulses will have a mean value proportional to  $x_t$  itself. This leads to the equation

$$(2.25) \quad \begin{cases} x_{t+\tau} = a_\tau x_t + \varepsilon_{t,\tau} \\ a_0 = 1 = \lim_{\tau \rightarrow 0} a_\tau, \quad a_\tau < 1 \text{ for } \tau > 0, \quad E(x_0) = 0 \end{cases}$$

where again  $\varepsilon_{t,\tau}$  is normally distributed with mean value zero and variance  $\sigma_\tau^2$  and is independent of  $x_t$  and of  $\varepsilon_{t',\tau'}$  if the intervals

$(t, t+\tau)$ ,  $(t', t'+\tau')$  do not overlap. we shall further assume that  $x_0$  is normally distributed and that  $a_\tau$  is a measurable function of

$\tau$ .

The first part of the document is a letter from the Secretary of the State to the President, dated January 1, 1865. It contains the following text:

January 1, 1865

Dear Sir: I have the honor to acknowledge the receipt of your letter of the 29th inst. and in reply to inform you that the same has been forwarded to the proper authorities for their consideration.

I am, Sir, very respectfully,  
Your obedient servant,  
John C. Schenck, Secretary of the State.

The second part of the document is a report from the Secretary of the State to the President, dated January 1, 1865. It contains the following text:

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The third part of the document is a report from the Secretary of the State to the President, dated January 1, 1865. It contains the following text:

I am, Sir, very respectfully,  
Your obedient servant,  
John C. Schenck, Secretary of the State.

It follows from (2.25) that

$$(2.26) \quad \begin{cases} x_{t+\tau_1+\tau_2} = a_{\tau_2} a_{\tau_1} x_t + a_{\tau_2} \varepsilon_{t,\tau_1} + \varepsilon_{t+\tau_1,\tau_2} \\ E(x_{t+\tau_1+\tau_2} | x_t) = a_{\tau_1+\tau_2} x_t = a_{\tau_1} a_{\tau_2} x_t \end{cases}$$

Hence  $a_{\tau_1+\tau_2} = a_{\tau_1} a_{\tau_2}$  from which it follows that  $a_{\tau} = e^{-\beta\tau}$

and since  $a_{\tau} < 1$ ,  $a_{\tau} = e^{-\beta\tau}$ ,  $\beta > 0$ . We further have from (2.25)

and (2.26)

$$a_{\tau_2} \varepsilon_{t,\tau_1} + \varepsilon_{t+\tau_1,\tau_2} = \varepsilon_{t,\tau_1+\tau_2}$$

Thus

$$a_{\tau_2}^2 \sigma_{\tau_1}^2 + \sigma_{\tau_2}^2 = \sigma_{\tau_1+\tau_2}^2 = a_{\tau_1}^2 \sigma_{\tau_2}^2 + \sigma_{\tau_1}^2$$

or

$$(\sigma_{\tau_1}^2 / \sigma_{\tau_2}^2) = (1 - a_{\tau_1}^2) / (1 - a_{\tau_2}^2) = (1 - e^{-2\beta\tau_1}) / (1 - e^{-2\beta\tau_2})$$

Therefore

$$(2.27) \quad \sigma_{\tau}^2 = \sigma^2 (1 - e^{-2\beta\tau}), \quad \beta > 0$$

Moreover from (2.25) and our assumption about  $\varepsilon_{t,\tau}$  we have

$$(2.28) \quad \sigma_{x_{t+\tau}}^2 = a_{\tau}^2 \sigma_{x_t}^2 + \sigma^2 (1 - e^{-2\beta\tau})$$

← If  $\tau$  approaches infinity then  $\sigma_{x_{t+\tau}}^2$  approaches  $\sigma^2$ . That is

to say, if the particle has been subjected to these random impacts for a long time then the distribution of its momentum approaches a steady state.

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We shall therefore assume that the process is stationary, that is to say, that the joint distribution of  $x_{t_1}, \dots, x_{t_n}$  is the same as that of  $x_{t_1+h}, \dots, x_{t_n+h}$ . Under this assumption  $\sigma_{x_{t+\tau}}^2 = \sigma_{x_t}^2 = \sigma^2$

and it follows from (2.27) that

$$(2.29) \quad \sigma_{x_t x_{t+\tau}} = a_\tau \sigma^2 = \sigma^2 \exp(-\beta\tau) \text{ for } \tau \geq 0.$$

The  $x_t$  process, satisfying the assumptions listed above, was first considered by L. S. Ornstein and G. E. Uhlenbeck. We will call it the Ornstein-Uhlenbeck process (abbreviated O. U. P.).

We next consider the process given by (2.30).

$$(2.30) \quad \begin{cases} v_t = t^{\frac{1}{2}} x(\log t)/2\beta & \text{for } t > 0 \\ v_t = 0 & \text{for } t \leq 0 \end{cases}$$

We have for  $t' \geq t$

$$(2.31) \quad \begin{cases} E(v_t v_{t'}) = \sigma^2 t \\ E[(v_{t'} - v_t) v_s] = 0 & \text{for } 0 \leq s < t < t' \end{cases}$$

Moreover the  $v_{t_1}, \dots, v_{t_n}$  are jointly normally distributed. Thus

the  $v_t$  process is a F.R.P. From the properties of the  $v_t$  process

we obtain

The first part of the document discusses the importance of maintaining accurate records of all transactions. It emphasizes that every entry should be supported by a valid receipt or invoice. This ensures transparency and allows for easy verification of the data.

Furthermore, it is noted that the records should be kept for a minimum of seven years. This is a legal requirement in many jurisdictions and helps in the event of an audit or a dispute.

In addition, the document highlights the need for regular reconciliation. This involves comparing the internal records with the bank statements to identify any discrepancies. If a difference is found, it should be investigated immediately to determine the cause.

Finally, it is advised to use a secure and reliable accounting system. This could be either a physical ledger or a digital software solution. Whichever is chosen, it must be protected from unauthorized access and data loss.

The second part of the document provides a detailed overview of the company's financial performance over the last fiscal year. It includes a summary of the income statement, balance sheet, and cash flow statement.

The income statement shows that the company has achieved a steady increase in revenue, primarily due to the expansion of its product line. However, there has been a corresponding increase in operating expenses, which has resulted in a slight decrease in net profit.

The balance sheet indicates that the company's assets have grown significantly, particularly in the form of property, plant, and equipment. This is a result of the company's investment in new machinery and facilities.

The cash flow statement shows that the company has maintained a positive cash flow throughout the year. This is a positive sign, as it indicates that the company is able to generate enough cash to cover its operating expenses and invest in future growth.

Overall, the company's financial performance is considered satisfactory, given the challenges faced in the current market. The management team is confident that the company is well-positioned to continue its growth in the coming years.

Theorem 2.8. Let  $x_t$  be an O.U.P. Then

- (I)  $x_t$  is continuous;  
 (II)  $x_t$  is not differentiable;  
 (III)  $M_{ab}$  and  $m_{ab}$  exist for the  $x_t$  process and it is strongly continuous;  
 (IV) the following equation — derived from theorem 2.4 — holds for  $M \geq 0$

$$(2.32) \quad P(e^{\beta\tau} x_\tau < M) = 1 - 2 \int_{-\infty}^{\frac{M}{\sigma\sqrt{t}}} (1/\sqrt{2\pi}) \exp(-x^2/2) dx$$

$$-\infty < \tau < (1/2\beta) \log t$$

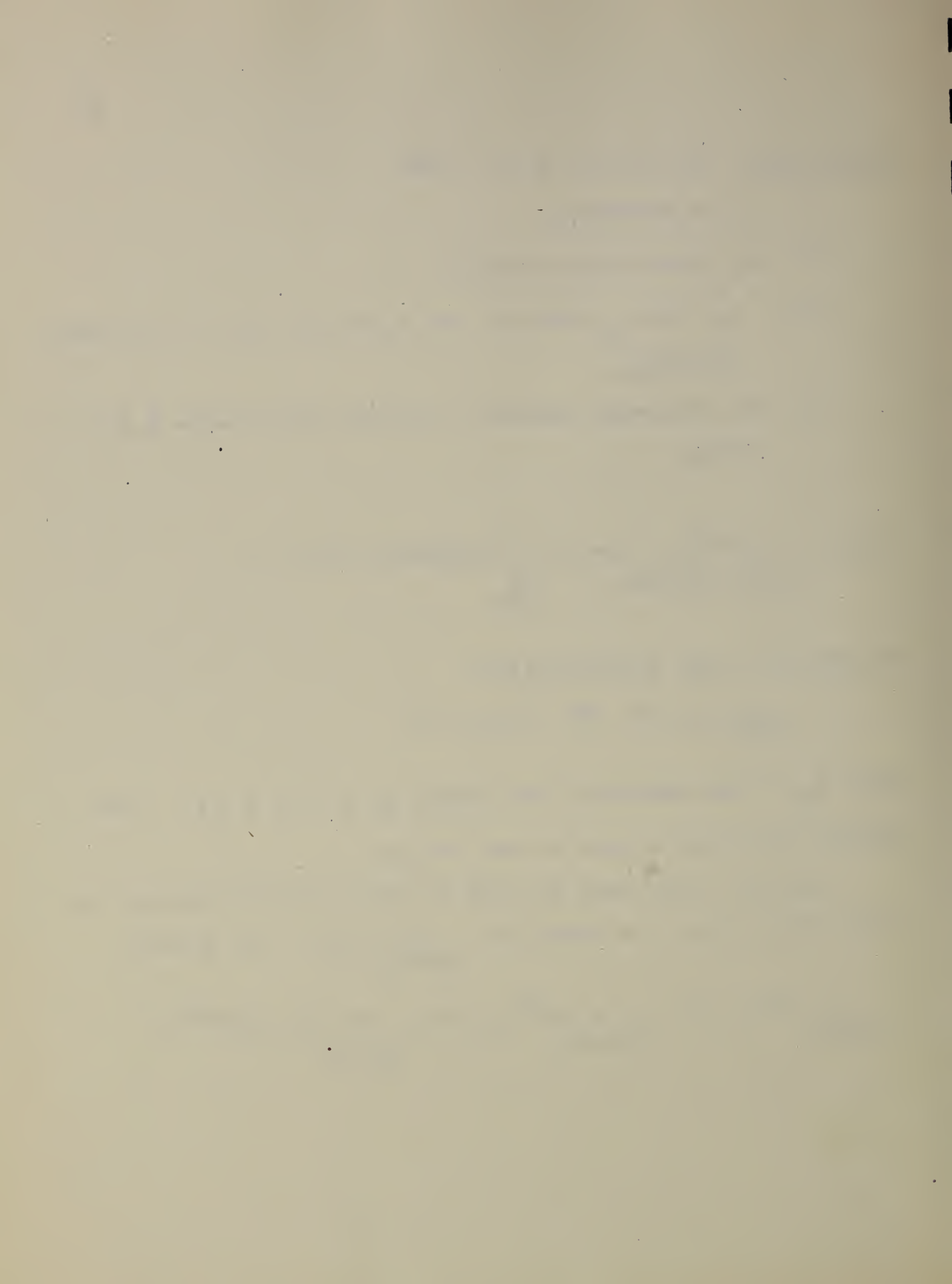
In equation (2.32) we have written

$$P_{a \leq t \leq b} (y_t \leq M) \text{ for } P(M_{ab} \leq M)$$

where  $M_{ab}$  is the maximum of the process  $y_t$  in  $a \leq t \leq b$ . This notation will also be used in what follows.

Equation (2.32) does not seem of great use as it stands, but we can obtain from it a bound for  $P_{\tau_1 \leq \tau \leq \tau_2} (x_\tau < M)$  as follows

$$P_{-\infty < \tau < \tau_1} (e^{\beta\tau} x_\tau \leq M) \leq P_{\tau_1 \leq \tau \leq \tau_2} (e^{\beta\tau} x_\tau \leq M) \leq P_{\tau_1 \leq \tau \leq \tau_2} (x_\tau \leq M e^{-\beta\tau_1});$$



and thus

$$(2.33) \quad P_{\tau_1 \leq \tau \leq \tau_2} (x_\tau \leq M) \geq 1 - 2 \int_{\frac{M - x_0}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \cdot M e^{-\beta(\tau_2 - \tau_1)/\sigma}$$

Theorem 2.9. Let  $x_t$  be an O.U.P. The integral  $X_t = \int_0^t x_\tau d\tau$  of this process exists then l.i.m. and for  $t_2 \geq t_1$ , its covariance function is given by

$$(2.34) \quad R_{t_1 t_2} = \frac{\sigma^2}{\beta^2} \left[ e^{-\beta t_1} + e^{-\beta t_2} + 2\beta t_1 - 1 - e^{-\beta(t_2 - t_1)} \right]$$

Moreover

$$(2.35) \quad B_t = \beta(X_t - X_0) + x_t - x_0$$

is a F.R.P.

*Proof:* For  $t_2 \geq t_1$  we have by theorem 1.5

$$\begin{aligned} R_{t_1 t_2} &= \sigma^2 \int_0^{t_1} \int_0^{t_2} e^{-\beta(\tau_1 - \tau_2)} dx_\tau d\tau = \sigma^2 \int_0^{t_1} \int_0^{\tau_1} e^{-\beta(\tau_1 - \tau_2)} d\tau_1 d\tau_2 \\ &\quad + \sigma^2 \int_0^{t_1} \int_{\tau_1}^{t_2} e^{-\beta(\tau_2 - \tau_1)} d\tau_2 d\tau_1 \\ &= \frac{\sigma^2}{\beta^2} \left[ e^{-\beta t_1} + e^{-\beta t_2} + 2\beta t_1 - 1 - e^{-\beta(t_2 - t_1)} \right]; \end{aligned}$$



and for  $t_1 = t_2$

$$E(X_t - X_0)^2 = \frac{2\sigma^2}{\beta^2}(e^{-\beta t} + \beta t - 1)$$

A straightforward calculation gives for  $t_2 \geq t_1 \geq s_2 \geq s_1$  :

$$\begin{aligned} (2.36) \quad & E[(X_{t_2} - X_{t_1})(X_{s_2} - X_{s_1})] \\ &= \frac{\sigma^2}{\beta} (e^{\beta s_2} - e^{\beta s_1}) (e^{-\beta t_1} - e^{-\beta t_2}) \\ &= -\frac{1}{\beta^2} E[(x_{t_2} - x_{t_1})(x_{s_2} - x_{s_1})] \end{aligned}$$

$$\text{Using } F(X_t, X_s) = E\left(X_t \frac{1}{h} \frac{X_{s+h} - X_s}{h}\right)$$

we obtain from lemma 1.7 and (2.34)

$$(2.37) \quad E(X_t X_s) = \begin{cases} \frac{\sigma^2}{\beta} [2 - e^{-\beta s} - e^{-\beta(t-s)}] & \text{if } t > s \\ \frac{\sigma^2}{\beta} [e^{-\beta(2-t)} - e^{-\beta s}] & \text{if } t \leq s \end{cases}$$

From (2.37) we see easily

$$\begin{aligned} (2.38) \quad & E[(X_{t_2} - X_{t_1})(X_{s_2} - X_{s_1})] \\ &= \frac{\sigma^2}{\beta} (e^{\beta s_2} - e^{\beta s_1}) (e^{-\beta t_1} - e^{-\beta t_2}) \\ &= -E[(X_{s_2} - X_{s_1})(X_{t_2} - X_{t_1})] \end{aligned}$$





*relations*

It is easily seen from the (2.36) and (2.58) that

$$(2.35) \quad B_t = \beta(X_t - X_0) + x_t - x_0$$

has the property that for  $t_2 \geq t_1 \geq s_2 \geq s_1$  the difference

$B_{t_2} - B_{t_1}$  is independent of  $B_{s_2} - B_{s_1}$  and since  $B_t$  is normally dis-

tributed it is a F.R.P.

From (2.35) we have

$$B_t - B_{t'} = \beta(X_t - X_{t'}) + x_t - x_{t'}$$

Thus for every function  $f(t)$  for which the operations indicated below have meaning, we have

$$(2.39) \quad \int_a^t f(t) dx_t = -\beta \int_a^t f(t) x_t dt + \int_a^t f(t) dB_t$$

We may write (2.39) as a stochastic differential equation

$$(2.40) \quad dx_t = -\beta x_t dt + dB_t$$

In the form (2.40) a stochastic differential equation has meaning even if the processes are not differentiable. This interpretation of a stochastic differential equation is due to J.L. Doob. The equation (2.40) may be interpreted as the equation of the motion of a particle ( $x_t$  being its velocity at time  $t$ ) subject to random impacts when the frictional force is proportional to its velocity. One could also interpret  $x_t$  as an electric potential subject to random changes when the decrease in potential is proportional to the potential itself.



(Thus we may consider a condenser which is charged by a randomly fluctuating current and at the same time grounded through a resistance). In short, equation (2.40) describes any situation in which a quantity  $x_t$  is subject to random changes and to a systematic decrease proportional to  $x_t$  itself.



## CHAPTER 3

### ESTIMATION OF PARAMETERS

In the preceding chapter we discussed Markoff processes; we shall now apply our results to obtain estimates of the parameters determining these processes from observations. In our estimating procedures we shall assume that we have at least one curve at our disposal registering the values  $x_t$  for all values  $0 \leq t \leq T$ . Actually it would be sufficient to know  $x_t$  for any dense set in this interval. This procedure may not seem realistic since we never observe the process for every time point. Every method of registering the curve described by  $x_t$  will itself affect  $x_t$  and in particular smooth the path curve of  $x_t$ . Thus what we observe is really a modified process.

However the methods of observation may be so refined as to give us the value of  $x_t$  in a large number of points and at any rate the variances of our estimates if obtained from discrete points may also be computed.

#### 1. Estimation of the parameter of the F.R.P.

The F.R.P. is completely known if the constant  $\frac{E(x_{t+\tau} - x_t)^2}{\tau}$  is known. We first discuss the estimation of the parameter  $\theta$  of a F.R.P.



Theorem 3.1 If  $x_t$  is a F.R.P. and if it is known in a dense set in an arbitrary small interval, then it is possible to estimate the parameter  $c$  with arbitrarily high precision.

*Proof:* Assume that  $N$  observations are taken in the interval  $0 \leq t \leq T$ . Let  $\tau = T/N$  and  $x_{n\tau}$  ( $n = 0, 1, 2, \dots, N$ ) be the sample value at the time  $n\tau$ . Since  $x_t$  is a F.R.P. the variates

$$y_n = \frac{x_{n\tau} - x_{(n-1)\tau}}{\sqrt{\tau}} = \frac{\epsilon_{(n-1)\tau} \cdot \tau}{\sqrt{\tau}} \quad (n=1, 2, \dots, N)$$

are normally and independently distributed with mean zero and variance  $c$ . The maximum likelihood estimate of the variance of  $y_n$  is therefore given by

$$(3.1) \quad \hat{c} = \frac{1}{N} \frac{1}{\tau} \sum_{n=1}^N (x_{n\tau} - x_{(n-1)\tau})^2 = \frac{1}{T} \sum_{n=1}^N (x_{n\tau} - x_{(n-1)\tau})^2$$

We have  $E\hat{c} = c$ . Moreover  $N\hat{c}/c$  has the chi-square distribution with  $N$  degrees of freedom. Its variance is therefore  $2N$ . Hence  $\hat{c}$  has the variance  $2c^2/N$  and this can be made arbitrarily small by taking  $N$  large enough.

Thus if it were possible to observe the process completely in any interval, however small, we could determine  $c$  accurately. Actually however every registering instrument will introduce a time lag and will thus smooth the process. We may infer however from our result that the points for which we read the value of  $x_t$  should be spaced as closely together as possible. That is to say, as close as is consistent with the assumption that the values of  $x_{t+\tau} - x_t$  obtained still represent the actual values supplied by the F.R.P.





2. F.R.P. with mean value function.

More generally we prove

Theorem 3.2 If  $y_t$  is a stochastic process such that  $y_t = x_t + f(t)$  where  $x_t$  is a F.R.P. and  $f(t)$  a function of bounded variation satisfying in  $(0, T)$  a Lipschitz condition  $|\frac{f(t+\tau) - f(t)}{\tau}| \leq M$  for  $\tau$  and if  $y_t$  is known in a dense set of an arbitrarily small interval, some  $M, \Delta$  then it is possible to estimate the parameter  $\sigma$  of the F.R.P.  $x_t$  with arbitrarily high precision.

Proof: Let again  $\tau \equiv T/N$  and consider the sample points  $x_{n\tau}$ , ( $n=0, 1,$

$2, \dots, N$ ). Denote by  $\hat{\sigma} = \frac{1}{N} \sum_{n=1}^N \frac{(y_{n\tau} - y_{(n-1)\tau})^2}{\tau}$ . Then

$$E\left(\frac{\hat{\sigma}}{\sigma}\right)^2 = \frac{1}{N^2} E\left\{ \sum \frac{[x_{n\tau} - x_{(n-1)\tau} + f(n\tau) - f(\overline{n-1}\tau)]^2}{\sigma\tau} \right\}^2$$

Here and in the immediately following formulae the summation is to be extended from  $n=1$  to  $n=N$ . We write also  $\overline{n-1}$  for  $(n-1)$  to simplify such expressions as  $x_{(n-1)\tau} \equiv x_{\overline{n-1}\tau}$ ,  $y_{(n-1)\tau} \equiv y_{\overline{n-1}\tau}$  or  $f[(n-1)\tau] \equiv f(\overline{n-1}\tau)$ . Then

$$E\left(\frac{\hat{\sigma}}{\sigma}\right)^2 = \frac{1}{N^2} E\left\{ \frac{1}{\sigma\tau} \sum (x_{n\tau} - x_{\overline{n-1}\tau})^2 + \frac{2}{\sigma\tau} \sum [f(n\tau) - f(\overline{n-1}\tau)] (x_{n\tau} - x_{\overline{n-1}\tau}) + \frac{1}{\sigma\tau} \sum [f(n\tau) - f(\overline{n-1}\tau)]^2 \right\}^2$$

We expand the right member of this expression; a considerable simplification follows from the assumption that  $x_t$  is a F.R.P., we use in particular the fact that  $\frac{x_{n\tau} - x_{\overline{n-1}\tau}}{\sqrt{\sigma\tau}}$  is normally distributed



with zero mean and unit variance and is independent of  $\frac{x_{m\tau} - x_{(m-1)\tau}}{\sqrt{\sigma\tau}}$

for  $m \neq n$ . Thus we obtain

$$E\left(\frac{\hat{c}}{c}\right)^2 = 1 + \frac{2}{N} + \frac{2}{N} \sum \frac{[f(n\tau) - f((n-1)\tau)]^2}{\sigma\tau} + \frac{4}{N^2} \sum \frac{[f(n\tau) - f((n-1)\tau)]^2}{\sigma\tau} \\ + \frac{1}{N^2} \left\{ \sum \frac{[f(n\tau) - f((n-1)\tau)]^2}{\sigma\tau} \right\}^2,$$

and

$$E\left(\frac{\hat{c}}{c}\right) = 1 + \frac{1}{N} \sum \frac{[f(n\tau) - f((n-1)\tau)]^2}{\sigma\tau}$$

Hence

$$E(\hat{c} - c)^2 = \frac{2\sigma^2}{N} + \frac{4\sigma}{N^2} \sum \frac{[f(n\tau) - f((n-1)\tau)]^2}{\tau} + \frac{1}{N^2} \left\{ \sum \frac{[f(n\tau) - f((n-1)\tau)]^2}{\tau} \right\}^2.$$

Thus  $\hat{c}$  converges in the mean to  $c$  and  $\hat{c} - c$  is stochastically of the order  $1/\sqrt{N}$ . In fact  $E(\hat{c} - c)^2 \leq \frac{2\sigma^2}{N} + \frac{4\sigma}{N^2} MV + \frac{1}{N^2} (MV)^2$ .

Here  $\left| \frac{f(t+\tau) - f(t)}{\tau} \right| \leq M$  and  $V$  is the variation of  $f(t)$  so that also  $V \leq MT$  where  $T$  is the length of the interval.

Thus in estimating the function  $f(t)$  we may assume  $c$  to be known, if we know  $y_t$  in any interval completely.

We shall discuss two examples of the function  $f(t)$ . In the first we assume that

$$f(t) = at$$



(Since we can always consider the process  $y_t - y_0$ , this assumption is identical with the assumption  $f(t) = at + b$ .) We then know that the  $y_{t+\tau} - y_t$  are normally distributed and independent in non-overlapping intervals with mean  $a\tau$  and variance  $\sigma\tau$ . Hence the maximum likelihood estimate of  $a$ , given the values at time  $0, \tau, \dots, N\tau$  becomes

$$\hat{a} = \frac{1}{T} \sum (y_{N\tau} - y_{(N-1)\tau}) = \frac{y_T - y_0}{T}$$

Its variance is

$$\sigma_{\hat{a}}^2 = \frac{\sigma}{T}$$

For the second example we assume that  $f(t)$  is given by

$$f(t) = \sum_{j=1}^m (\alpha_j \cos jt + \beta_j \sin jt)$$

and that the values of  $y_t$  are known in the interval  $(0, 2\pi)$  and that  $y_0 = 0$ . If we just choose the values  $y_0, y_\tau, y_{2\tau}, \dots, y_{N\tau}$  where  $N\tau = 2\pi$ , the maximum likelihood estimates of the  $\alpha_j$  and  $\beta_j$  will be given by those values, which minimize the expression

$$\sum_{i=1}^N (y_{i\tau} - y_{(i-1)\tau} - \sum_{j=1}^m \alpha_j [\cos i j \tau - \cos (i-1) j \tau] - \sum_{j=1}^m \beta_j [\sin i j \tau - \sin (i-1) j \tau])^2$$



Hence the maximum likelihood equations are

$$\begin{aligned}
 (3.2.1) \quad & \sum_{i=1}^n (y_{i\tau} - y_{i-1\tau}) [\cos i k \tau - \cos(i-1)k\tau] \\
 &= \sum_{i=1}^n \sum_{j=1}^m \hat{\alpha}_j [\cos i j \tau - \cos(i-1)j\tau] [\cos i k \tau - \cos(i-1)k\tau] \\
 &+ \sum_{i=1}^n \sum_{j=1}^m \hat{\beta}_j [\sin i j \tau - \sin(i-1)j\tau] [\cos i k \tau - \cos(i-1)k\tau] \\
 & \qquad \qquad \qquad k = 1, \dots, m
 \end{aligned}$$

and

$$\begin{aligned}
 (3.2.2) \quad & \sum_{i=1}^n (y_{i\tau} - y_{i-1\tau}) [\sin i k \tau - \sin(i-1)k\tau] \\
 &= \sum_{i=1}^n \sum_{j=1}^m \hat{\alpha}_j [\cos i j \tau - \cos(i-1)j\tau] [\sin i k \tau - \sin(i-1)k\tau] \\
 &+ \sum_{i=1}^n \sum_{j=1}^m \hat{\beta}_j [\sin i j \tau - \sin(i-1)j\tau] [\sin i k \tau - \sin(i-1)k\tau] \\
 & \qquad \qquad \qquad k = 1, 2, \dots, m
 \end{aligned}$$

If we divide the first of these equations by  $\tau$  and let  $\tau$  go to zero, we obtain because of the orthogonality of the sine and cosine functions

$$-k \int_0^{2\pi} \sin kt \, dy_t = \hat{\alpha}_k k^2 \int_0^{2\pi} \sin^2 kt \, dt = \hat{\alpha}_k \pi k^2$$





The rules of calculation for the integral in the left member of this equation are completely analogous to those for the ordinary Riemann-Stieltjes integral. Integration by parts on the left gives therefore

$$\int_0^{2\pi} y_t \cos kt \, dt = \hat{\alpha}_k \pi$$

and thus

$$(3.3.1) \quad \hat{\alpha}_k = \frac{1}{\pi} \int_0^{2\pi} y_t \cos kt \, dt = -\frac{1}{\pi k} \int_0^{2\pi} \sin kt \, dy_t$$

Similarly, we obtain from (3.2.2)

$$k \int_0^{2\pi} \cos kt \, dy_t = \pi k^2 \hat{\beta}_k$$

Integration by parts gives again

$$(3.3.2) \quad \hat{\beta}_k = \frac{1}{\pi k} (y_{2\pi} - y_0) + \frac{1}{\pi} \int_0^{2\pi} y_t \sin kt \, dt = \frac{1}{\pi k} \int_0^{2\pi} \cos kt \, dy_t$$

The integrals in (3.3.1) and (3.3.2) are to be understood as stochastic limits of Riemann sums, and it is easy to see from the corollary to lemma 1.7 that these Riemann sums converge in the mean. From lemma 1.4 we see therefore

$$E(\hat{\alpha}_k) = \alpha_k, \quad E(\hat{\beta}_k) = \beta_k$$

and similarly from lemma 1.7



$$\sigma_{\hat{\alpha}_k}^2 = \frac{1}{\pi^2 k^2} \lim_{\substack{\delta_1 \rightarrow 0 \\ \delta_j \rightarrow 0}} \sum_1 \sum_j \sin kt_1 \sin kt_j \sigma \{ (y_{t_1} - y_{t_{1-1}})(y_{t_j} - y_{t_{j-1}}) \}$$

where  $\delta_1 = \max |t_1 - t_{1-1}|$  and  $\delta_j = \max |t_j - t_{j-1}|$ .

Since the increments of  $y_t$  in non-overlapping intervals are independent of each other we have

$$\sigma_{\hat{\alpha}_k}^2 = \frac{1}{\pi^2 k^2} \lim_{\delta_1 \rightarrow 0} \sum_1 \sigma(t_1 - t_{1-1}) \sin^2 kt_1 = \frac{\sigma}{\pi^2 k^2} \int_0^{2\pi} \sin^2 kt \, dt$$

so that

$$(3.8) \quad \sigma_{\hat{\alpha}_k}^2 = \frac{\sigma}{\pi k^2}$$

and similarly

$$(3.9) \quad \sigma_{\hat{\beta}_k}^2 = \frac{\sigma}{\pi k^2}$$

Further

$$(3.10) \quad \begin{aligned} \sigma_{\hat{\alpha}_k \hat{\alpha}_l} &= \sigma_{\hat{\beta}_k \hat{\beta}_l} = 0 \quad \text{for } k \neq l \\ \sigma_{\hat{\alpha}_k \hat{\beta}_l} &= 0 \end{aligned}$$

From lemma 2.6 we see moreover that  $\hat{\alpha}_k$  and  $\hat{\beta}_k$  are normally distributed.



Suppose now that we take observations in the interval  $(0, T)$  so that we have

$$(3.11) \quad f(t) = \sum [\alpha_n \cos 2\pi n \frac{t}{T} + \beta_n \sin 2\pi n \frac{t}{T}]$$

We put  $\tau = \frac{2\pi t}{T}$  and  $y_\tau^* = y_t$ . Then

$$\text{Var} \left\{ \frac{(y_{\tau+K}^* - y_\tau^*)}{\sqrt{K}} \right\} = c^* = \text{Var} \left\{ \frac{(y_{t+(KT/2\pi)} - y_t)}{\sqrt{K}} \right\} = \frac{Tc}{2\pi}$$

The maximum likelihood estimates  $\hat{\alpha}_k$  and  $\hat{\beta}_k$  then become

$$\hat{\alpha}_k = -\frac{1}{\pi k} \int_0^T \sin(2\pi k \frac{t}{T}) dy_t$$

$$\hat{\beta}_k = \frac{1}{\pi k} \int_0^T \cos(2\pi k \frac{t}{T}) dy_t$$

and

$$\sigma_{\hat{\alpha}_k}^2 = \sigma_{\hat{\beta}_k}^2 = \frac{Tc}{2\pi^2 k^2}, \quad \sigma_{\hat{\alpha}_k \hat{\beta}_l} = 0, \quad \sigma_{\hat{\alpha}_k \hat{\alpha}_l} = \sigma_{\hat{\beta}_k \hat{\beta}_l} = 0 \text{ for } k \neq l.$$

Thus a confidence region for the  $\alpha_k, \beta_l$  is given by

$$\chi^2 = \sum \frac{2\pi^2 k^2}{\sigma^2 T} [(\hat{\alpha}_k - \alpha_k)^2 + (\hat{\beta}_k - \beta_k)^2] \leq M,$$



where the sum runs over those terms  $(\hat{\alpha}_k - \alpha_k)^2$ ,  $(\hat{\beta}_k - \beta_k)^2$  which are not zero by assumption, and  $\chi^2$  has the  $\chi^2$ -distribution with the number of degrees of freedom equal to the number of terms in the sums on the right. The estimates  $\hat{\alpha}_k$ ,  $\hat{\beta}_k$  are consistent in the following sense. Suppose  $f(t)$  is given by (3.11) and we observe  $y_t$  in the interval  $VT$  where  $V$  is an integer. We then have

$$f(t) = \sum_{n=1}^m \left[ \alpha_n \cos 2\pi \frac{(nV)t}{VT} + \beta_n \sin 2\pi \frac{(nV)t}{VT} \right]$$

so that

$$\sigma_{\hat{\alpha}_k}^2 = \sigma_{\hat{\beta}_k}^2 = \frac{VT\sigma}{2\pi^2 V^2 k^2} = \frac{T\sigma}{2\pi^2 V k^2}$$

and thus  $\text{plim}_{V \rightarrow \infty} \hat{\alpha}_k = \alpha_k$ . Similarly,  $\text{plim}_{V \rightarrow \infty} \hat{\beta}_k = \beta_k$ .

### 3. Estimation of parameters for the O.U.P.

We now turn to the discussion of the O.U.P. given by (2.25)

and prove

Theorem 3.3 If  $x_t$  is an O.U.P. determined by the two parameters  $\beta$  and  $\sigma^2$  and if the values of  $x_t$  are known in a dense set in any interval  $0 \leq t \leq T$ , then it is possible to determine  $\sigma^2 \beta$  with arbitrarily high precision.

*Proof:* We form with  $N\tau = T$

$$(3.12) \quad D = \frac{1}{N} \sum \frac{(x_{n\tau} - x_{(n-1)\tau})^2}{\tau}$$

$$\text{We have} \quad E(D) = \frac{2\sigma^2}{\tau} (1 - a_\tau) = 2\sigma^2 \frac{(1 - e^{-\beta\tau})}{\tau}.$$

For  $\tau \rightarrow 0$  this converges to  $2\sigma^2 \beta$ .





We now compute the variance of  $D$ . For this purpose we shall need the value of  $E(x_t x_{t'} x_{t''} x_{t'''})$  with  $t \leq t' \leq t'' \leq t'''$ .

We have for  $t \leq t' \leq t'' \leq t'''$  on replacing  $x_{t''}$  by

$a_{t''-t'} x_{t'} + \varepsilon_{t', t''-t'}$  and analogous transformations

$$E(x_t x_{t'} x_{t''} x_{t'''}) = a_{t''-t'} E(x_t x_{t'} x_{t''}^2)$$

$$\begin{aligned} E(x_t x_{t'} x_{t''}^2) &= E[x_t x_{t'} (a_{t''-t'} x_{t'} + \varepsilon_{t', t''-t'})^2] \\ &= a_{t''-t'}^2 E(x_t x_{t'}^3) + a_{t''-t'} (1 - a_{t''-t'}^2) \sigma^4 \end{aligned}$$

and

$$\begin{aligned} E(x_t x_{t'}^3) &= E[x_t (a_{t''-t'} x_{t'} + \varepsilon_{t', t''-t'})^3] \\ &= a_{t''-t'}^3 E(x_t^4) + 3a_{t''-t'} E(x_t^2) \sigma^2 (1 - a_{t''-t'}^2) \\ &= \sigma^4 [3a_{t''-t'}^3 + 3a_{t''-t'} - 3a_{t''-t'}^3] = 3a_{t''-t'} \sigma^4 \end{aligned}$$

Thus

$$\begin{aligned} E(x_t x_{t'} x_{t''}^2) &= \sigma^4 [3a_{t''-t'}^2 a_{t''-t'} + a_{t''-t'} (1 - a_{t''-t'}^2)] \\ &= \sigma^4 (a_{t''-t'} + 2a_{t''-t'}^2 a_{t''-t'}) \end{aligned}$$

and

$$\begin{aligned} E(x_t x_{t'} x_{t''} x_{t'''}) &= \sigma^4 (a_{t''-t'} a_{t'''} + 2a_{t''-t'} a_{t'''}^2 a_{t''-t'}) \\ &= \sigma^4 (a_{t''-t'} a_{t'''} + 2a_{t''-t'} a_{t'''}^2) \end{aligned}$$

or

$$13) \quad E(x_t x_{t'} x_{t''} x_{t'''}) = \sigma^4 \left\{ \exp[-\beta(t''' - t'' + t' - t)] + 2 \exp[-\beta(t''' + t'' - t' - t)] \right\}$$



For  $n < m$  we find easily

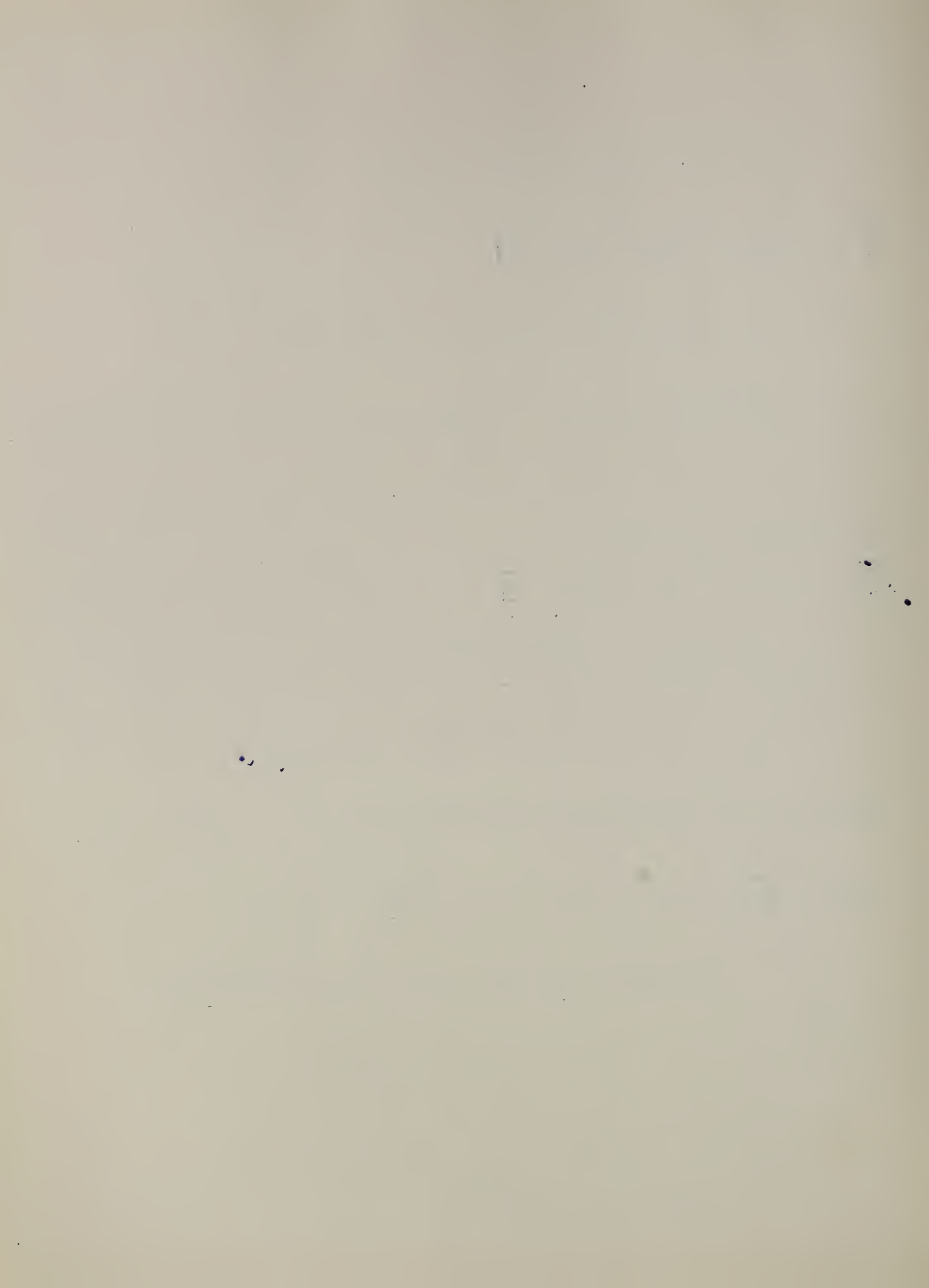
$$\begin{aligned}
 & E[(x_{n\tau} - x_{n-1\tau})^2 (x_{m\tau} - x_{m-1\tau})^2] \\
 &= E\{(x_{n\tau} - x_{n-1\tau})^2 [(a_\tau - 1)x_{m-1\tau} + \epsilon_{m-1\tau, \tau}]^2\} \\
 &= (a_\tau - 1)^2 [E(x_{n\tau}^2 x_{m-1\tau}^2) - 2E(x_{n-1\tau} x_{n\tau} x_{m-1\tau}^2) + E(x_{n-1\tau}^2 x_{m-1\tau}^2) \\
 &\quad + 2\sigma^4(1 + a_\tau)].
 \end{aligned}$$

From (3.13) we see then

$$\begin{aligned}
 & E[(x_{n\tau} - x_{n-1\tau})^2 (x_{m\tau} - x_{m-1\tau})^2] \\
 &= (a_\tau - 1)^2 \sigma^4 \{1 + 2e^{-2\beta(m-n-1)\tau} - 2e^{-\beta\tau} - 4e^{-\beta\tau[2(m-n)-1]} \\
 &\quad + 1 + 2e^{-2\beta(m-n)\tau} + 2 + 2a_\tau\}.
 \end{aligned}$$

Since  $a_\tau = e^{-\beta\tau}$  we finally have for  $m > n$

$$\begin{aligned}
 (3.14) \quad & E[(x_{n\tau} - x_{n-1\tau})^2 (x_{m\tau} - x_{m-1\tau})^2] \\
 &= 2\sigma^4(1 - a_\tau)^2 [2 + (1 - a_\tau)^2 e^{-2\beta(m-n-1)\tau}].
 \end{aligned}$$



Further

$$\begin{aligned}
 E(x_{n\tau} - x_{\overline{n-1}\tau})^4 &= E[(a_\tau - 1)x_{\overline{n-1}\tau} + \varepsilon_{\overline{n-1}\tau, \tau}]^4 \\
 &= (a_\tau - 1)^4 E(x_{\overline{n-1}\tau}^4) + 6(a_\tau - 1)^2 E(x_{\overline{n-1}\tau}^2)(1 - a_\tau^2)\sigma^2 + 3\sigma^4(1 - a_\tau^2)^2 \\
 &= 3\sigma^4 [(a_\tau - 1)^4 + 2(a_\tau - 1)^2(1 - a_\tau^2) + (1 - a_\tau^2)^2] \\
 &= 3\sigma^4(1 - a_\tau)^2 [(1 - a_\tau)^2 + 2(1 - a_\tau^2) + (1 + a_\tau)^2]
 \end{aligned}$$

that is

$$(3.15) \quad E(x_{n\tau} - x_{\overline{n-1}\tau})^4 = 12\sigma^4(1 - a_\tau)^2.$$

From (3.12), (3.14), and (3.15) we have

$$E(D^2) = \frac{\sigma^4}{N^2\tau^2} (1 - a_\tau)^2 \left[ 12N + 4N(N-1) + 4(1 - a_\tau)^2 \sum_{n=1}^{N-1} \sum_{m=n+1}^N e^{-2\beta(m-n-1)\tau} \right]$$

so that

$$\sigma_D^2 = \frac{4\sigma^4(1 - a_\tau)^2}{N^2\tau^2} \left[ 2N + (1 - a_\tau)^2 \sum_{n=1}^{N-1} \sum_{m=n+1}^N e^{-2\beta(m-n-1)\tau} \right].$$

We now compute

$$A = \sum_{n=1}^{N-1} \sum_{m=n+1}^N (a_\tau^2)^{m-n-1} = \sum_{n=1}^{N-1} \sum_{m=n}^{N-1} (a_\tau^2)^{m-n}.$$



By using the formula for the sum of a geometric series it is easily seen that

$$\sum_{n=1}^{N-1} \sum_{m=n}^{N-1} b^{m-n} = \frac{N}{1-b} - \frac{1-b^N}{(1-b)^2}$$

hence

$$A = \frac{N}{1-a_\tau^2} - \frac{1-a_\tau^{2N}}{(1-a_\tau^2)^2}$$

We substitute this in the expression for  $\sigma_D^2$  and obtain

$$\sigma_D^2 = \frac{4\sigma^4(1-a_\tau)^2}{N^2\tau^2} \left[ 2N + \frac{N(1-a_\tau)}{1+a_\tau} - \frac{1-a_\tau^{2N}}{(1+a_\tau)^2} \right]$$

or

$$(3.16) \quad \sigma_D^2 = \frac{4\sigma^4(1-a_\tau)^2}{N^2\tau^2} \left[ 2N + \frac{N-1-Na_\tau^2+a_\tau^{2N}}{(1+a_\tau)^2} \right]$$

Equation (3.16) shows that  $\sigma_D^2$  can be made arbitrarily small by making  $N$  large enough. In fact

$$\lim_{N \rightarrow \infty} \frac{\sigma_D^2}{N} = 8\sigma^4\beta^2 \quad \text{and} \quad \lim_{N \rightarrow \infty} E(D) = 2\beta\sigma^2;$$

also

$$E(D - 2\beta\sigma^2)^2 = \sigma_D^2 + 4\sigma^4 \left( \frac{1-e^{-\beta\tau}}{\tau} - \beta \right)^2$$





and

$$E[\sqrt{N}(D - 2\beta\sigma^2)] = 2\sigma^2\sqrt{N} \left[ \frac{1 - e^{-\beta\tau}}{\tau} - \beta \right]$$

and therefore since  $\tau = T/N$

$$\lim_{N \rightarrow \infty} E[\sqrt{N}(D - 2\beta\sigma^2)] = 0$$

We proceed to prove that the limit distribution of  $\sqrt{N}(D - 2\beta\sigma^2)$  is normal. This may be seen as follows;

$$\sqrt{N} D = \frac{1}{\sqrt{N}} \sum_{n=1}^N \frac{(x_{n\tau} - x_{(n-1)\tau})^2}{\tau} = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \frac{[(a_\tau - 1)x_{n\tau} + \epsilon_{n\tau, \tau}]^2}{\tau}$$

or

$$(3.17) \quad \sqrt{N} D = \frac{1}{\sqrt{N}} \frac{(a_\tau - 1)^2}{\tau} \sum_0^{N-1} x_{n\tau}^2 + \frac{2}{\sqrt{N}} \frac{(a_\tau - 1)}{\tau} \sum_0^{N-1} x_{n\tau} \epsilon_{n\tau, \tau} + \frac{1}{\sqrt{N}} \sum_0^{N-1} \frac{\epsilon_{n\tau, \tau}^2}{\tau}.$$

The last sum is a sum of independently distributed variables all with the same distribution and converges to the normal distribution by lemma 2.1. Thus the normality of the limit distribution will be proved if we can prove that the first two sums in (3.17) converge to zero.



We therefore put

$$(3.18) \quad \Sigma_1 = \frac{1}{\sqrt{N}} \frac{(a_\tau - 1)^2}{\tau} \sum_0^{N-1} x_{n\tau}^2 \quad \text{and} \quad \Sigma_2 = \frac{2}{\sqrt{N}} \frac{a_\tau - 1}{\tau} \sum_0^{N-1} x_{n\tau} \varepsilon_{n\tau, \tau}.$$

We have by (3.13)

$$E(\Sigma_1^2) = \frac{\sigma^4}{N} \frac{(a_\tau - 1)^4}{\tau^2} \left[ N^2 + 2N + 4 \sum_0^{N-2} \sum_{n+1}^{N-1} e^{-2\beta(m-n)\tau} \right].$$

The double sum in the bracket can be easily determined and we have

$$E(\Sigma_1^2) = \frac{\sigma^4 (a_\tau - 1)^4}{\tau^2} \left[ N + \frac{2(1+a_\tau^2)}{1-a_\tau^2} - \frac{4a_\tau^2}{N} \frac{1-e^{-2\beta\tau N}}{(1-a_\tau^2)^2} \right].$$

Clearly  $\lim_{N \rightarrow \infty} E(\Sigma_1^2) = 0$ .

Further

$$E(\Sigma_2^2) = \frac{4}{N} \frac{(a_\tau - 1)^2}{\tau^2} \sum_0^{N-1} E(x_{n\tau}^2 \varepsilon_{n\tau, \tau}^2) = \frac{4(a_\tau - 1)^2}{\tau^2} \sigma^4 (1-a_\tau^2)$$

and therefore  $\lim_{N \rightarrow \infty} E(\Sigma_2^2) = 0$ .

Therefore  $\lim_{N \rightarrow \infty} \text{l.i.m.} \Sigma_1 = \lim_{N \rightarrow \infty} \text{l.i.m.} \Sigma_2 = 0$  so that

$$\lim_{N \rightarrow \infty} \text{l.i.m.} \left( \sqrt{N} D - \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \frac{\varepsilon_{n\tau, \tau}}{\tau} \right) = 0,$$



and since the second term on the left is normally distributed with mean  $2\beta\sigma^2\sqrt{N}$  it follows that  $\sqrt{N}(D - 2\beta\sigma^2)$  is in the limit normally distributed with mean zero and variance  $8\sigma^4\beta^2$ .

To estimate  $\sigma^2$  separately one might use the estimate

$$\hat{\sigma}^2 = \frac{1}{T} \int_0^T x_t^2 dt$$

Its variance is given by

$$\frac{1}{T^2} \int_0^T \int_0^T E(x_t^2 x_{t'}^2) dt dt' - \sigma^4$$

$$= \frac{2\sigma^4}{T^2} \left\{ \int_0^T \int_0^t e^{-2\beta(t-t')} dt' dt + \int_0^T \int_t^T e^{-2\beta(t'-t)} dt' dt \right\}$$

$$= \frac{2\sigma^4}{\beta T} + \frac{\sigma^4(e^{-2\beta T} - 1)}{\beta^2 T^2}$$

If  $\beta T$  is large enough compared with  $\sigma^4$  this comparatively simple estimate may be quite satisfactory.



## CHAPTER 4

### THE GENERAL DIFFERENTIAL PROCESS

We shall consider processes  $x_t$  with the following properties:

(1)  $x_t$  is a continuous process (not necessarily strongly continuous);

(2) Let  $x_{t+\tau} - x_t = \epsilon_{t,\tau}$ . The random variables

$\epsilon_{t_1, \tau_1}, \epsilon_{t_2, \tau_2}, \dots, \epsilon_{t_n, \tau_n}$  are completely

independent of each other if the intervals

$(t_1, t_1 + \tau_1), (t_2, t_2 + \tau_2), \dots, (t_n, t_n + \tau_n)$  do not

overlap;

(3) The distribution of  $\epsilon_{t,\tau}$  is independent of  $t$ .

Processes satisfying these three conditions will be called *general* differential processes. In this chapter we shall find a general expression giving the distributions of  $x_{t+\tau} - x_t$  for all possible

*general* differential processes. Processes satisfying equation (2.1) and assumptions 1, 2, 3, of chapter 2 are differential processes of second order and we discussed in chapter 2 the special case where  $x_{t+\tau} - x_t$  is normally distributed.

We shall first discuss another special case in which the increments  $x_{t+\tau} - x_t$  have a discontinuous distribution. A practical example for such a process is, for instance, the total of insurance claims raised against an insurance company as a result of randomly distributed accidents. An important special case, fundamental also for the understanding of the more general problem,





is that in which  $x_0$  increases a randomly distributed number of times within every time interval but each time by the same amount. We shall call such an increase a shot.

We make the following assumptions:

(1) The probability  $p_\tau^{(k)}$  that  $k$  shots will occur during the interval  $(t, t+\tau)$  is independent of  $t$  and of the number of shots that have occurred up to and including the time  $t$ .

(2) The probability  $q_\tau^{(2)}$  that more than one shot will occur in a time interval of length  $\tau$  is of smaller order than  $\tau$ . In symbols,

$$q_\tau^{(2)} = o(\tau) \quad \text{or} \quad \lim_{\tau \rightarrow 0} (q_\tau^{(2)}/\tau) = 0.$$

(3)  $p_\tau^{(k)}$  is a measurable function of  $\tau$ .

We clearly have, if  $\tau_1 + \tau_2 = \tau$

$$(4.1) \quad p_\tau^{(0)} = p_{\tau_1}^{(0)} p_{\tau_2}^{(0)}.$$

From (4.1) and the measurability of  $p_\tau^{(0)}$  it follows that  $p_\tau^{(0)} = e^{a\tau}$  and since  $p_\tau^{(0)} \leq 1$  we have  $a \leq 0$ . Moreover, if there are any shots to be expected we must have  $a < 0$ ,  $a = -\mu$  where  $\mu > 0$ . Thus

$$(4.2) \quad p_\tau^{(0)} = e^{-\mu\tau}, \quad \mu > 0.$$



We now divide the interval  $(t, t+\tau)$  into  $N$  parts. Then for sufficiently large  $N$  the probability that two shots will occur in any of the intervals can be made arbitrarily small so that if  $p_{\tau}^{(k)}$  denotes the probability that  $k$  shots will occur during the time interval  $\tau$

$$(4.3) \quad p_{\tau}^{(k)} = \binom{N}{k} [1 - \exp(-\frac{\mu\tau}{N})]^k \exp[-\frac{\mu\tau}{N} (N-k)] + o(1).$$

For  $N \rightarrow \infty$  we then have

$$(4.4) \quad p_{\tau}^{(k)} = \frac{e^{-\mu\tau} (\mu\tau)^k}{k!}$$

The distribution (4.4) is the Poisson distribution, its mean and variance are both equal to  $\mu\tau$ .

We next consider a situation in which the assumptions (1), (2), (3) regarding  $p_{\tau}^{(k)}$  hold but where the increment of  $x_t$  at each shot varies and has itself a probability distribution  $\phi(x) = \phi_1(x)$  and we shall also assume that the increases in different shots are independent of each other. If  $\phi_k(x)$  is the distribution of the sum of  $k$  independent random variables, each with distribution  $\phi(x)$ , then the distribution of the total increase provided that  $k$  shots have occurred is  $\phi_k(x)$ . Thus the distribution of  $x_{t+\tau} - x_t$  is given by



$$(4.5) \quad P(x_{t+\tau} - x_t \leq A) = \sum_{k=0}^{\infty} \frac{e^{-\mu\tau} (\mu\tau)^k}{k!} \phi_k(A)$$

$$\text{with } \phi_0(A) \begin{cases} 0 & \text{for } A < 0 \\ 1 & \text{for } A \geq 0 \end{cases}$$

In the following we shall need the characteristic function [abbreviated c.f.]

$$(4.6) \quad f_{\tau}(s) = E\{\exp[is(x_{t+\tau} - x_t)]\}$$

of the distribution (4.5). An easy calculation gives, if  $g(s)$  denotes the c.f. of  $\phi(x)$

$$(4.7) \quad f_{\tau}(s) = \sum_{k=0}^{\infty} \frac{e^{-\mu\tau} (\mu\tau)^k}{k!} [g(s)]^k = \exp\{\mu\tau[g(s) - 1]\}.$$

The distribution (4.5) is called the generalized Poisson distribution.

We return now to the general differential process. We have

$$E_{t,\tau} = E_{t, \frac{\tau}{n}} + E_{t+\frac{\tau}{n}, \frac{\tau}{n}} + \dots + E_{t+\frac{n-1}{n}\tau, \frac{\tau}{n}}$$

Hence, if  $\phi_{\tau}(s)$  denotes the c.f. of  $E_{t_0, \tau}$  we have

$$(4.8) \quad \phi_{\tau}(s) = [\phi_{\frac{\tau}{n}}(s)]^n.$$

From the continuity of  $x_t$  it follows that  $\lim_{\Delta\tau \rightarrow 0} \phi_{t+\Delta\tau}(s) = \phi_t(s)$ .

Thus (4.8) implies

$$(4.9) \quad \phi_{\tau}(s) = [\phi_1(s)]^{\tau}.$$



On the other hand every family of distribution functions  $H_{\tau}(x)$  whose characteristic functions satisfy equation (4.9) is the distribution function of the increment of some differential process. Hence the general form of a differential process will be found if we find the general form of characteristic functions  $\phi(s)$  that satisfy the condition that for every  $\tau \geq 0$ ,  $[\phi(s)]^{\tau}$  is a c.f. A distribution law whose c.f. satisfies this condition is called an infinitely divisible law (abbreviated, i.d.l.).

Our main result will be the following:

Theorem 4.1. [12] Let  $\psi(s) = \log \phi(s)$ . The function  $\phi(s)$  is c.f. of an infinitely divisible law if and only if

$$(4.10) \quad \psi(s) = isa + \int_{-\infty}^{+\infty} \left( e^{isx} - 1 - \frac{isx}{1+x^2} \right) \frac{1+x^2}{x^2} dG(x)$$

where  $a$  is real and  $G(x)$  non-decreasing and bounded and the integrand is defined by continuity to be  $-\frac{s^2}{2}$  for  $x=0$ .

Fundamental for the proof of this theorem is the powerful continuity theorem of P. Lévy.

Continuity Theorem: Let  $\{F_n(x)\}$  be a sequence of distribution functions,  $\{f_n(s)\}$  the corresponding sequence of c.f.'s. The sequence  $\{F_n(x)\}$  converges to a distribution function  $F(x)$  if and only if  $f_n(s)$  converges to a function  $f(s)$  continuous for  $s=0$ .

---

[12] Theorem 4.1 is due to P. Lévy (see, for instance, his "Théorie de l'addition des variables aléatoires", Gauthiers Villars Paris, 1937, p. 180). The following elegant proof is due to M. Loeve, University of California Publ. in Stat., vol. 1, No. 5, 53-88 (1950).





For a proof the reader is referred to H. Cramér, *Mathematical Methods of Statistics*, 10.4.

We shall need this theorem in the following, slightly more general, form.

Corollary to the Continuity Theorem: Let  $\{F_n(x)\}$  be a sequence of bounded monotone functions,  $F_n(-\infty) = 0$  and let  $\{f_n(s)\}$  be the sequence of their Fourier transforms

$$(4.11) \quad f_n(s) = \int_{-\infty}^{+\infty} e^{ixs} dF_n(x).$$

The sequence  $\{F_n(x)\}$  converges to a bounded monotonic function  $F(x)$  and  $\lim_{n \rightarrow \infty} [F_n(\infty) - F_n(-\infty)] = F(\infty) - F(-\infty)$  if and only if the sequence  $\{f_n(s)\}$  converges to a function  $f(s)$  continuous at  $s=0$ .

The corollary follows easily from the continuity theorem in observing that

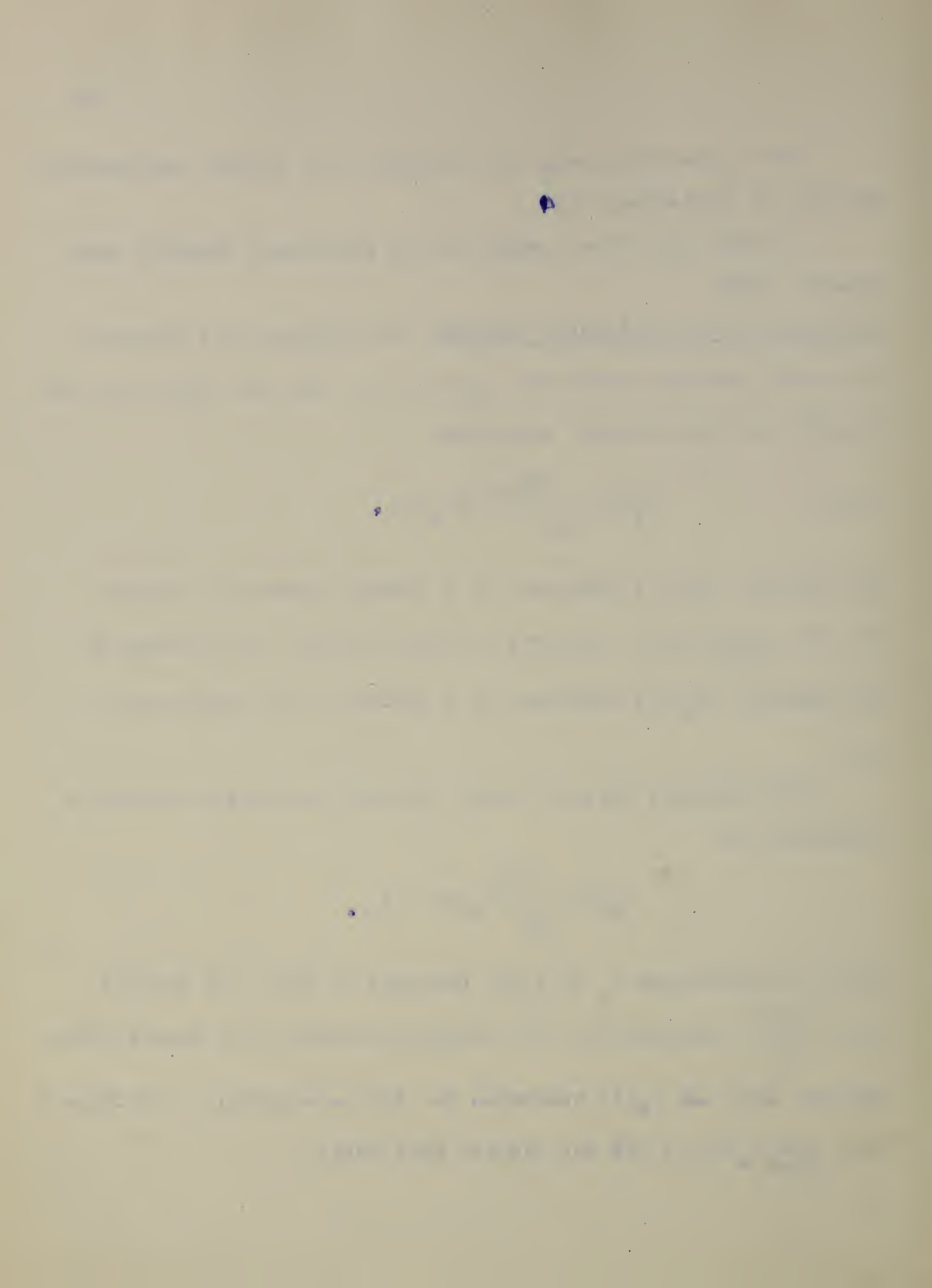
$$f_n(0) = \int_{-\infty}^{+\infty} dF_n(x) = V_n.$$

Hence the variations  $V_n$  of  $F_n(x)$  converge to  $f(0)$ . If  $f(0) \neq 0$

then  $\frac{F_n(x)}{V_n}$  converges by the continuity theorem to a distribution

function  $H(x)$  and  $F_n(x)$  converges to  $F(x) = H(x)f(0)$ . If  $f(0) = 0$

then  $\lim_{n \rightarrow \infty} F_n(x) = 0$  and the theorem also holds.



We shall also need the

Helly-Bray theorem: ~~Let~~ Let  $\{F_n(x)\}$  be a sequence of distribution functions and  $\lim_{n \rightarrow \infty} F_n(x) = F(x)$ . Let further  $g(x)$  be everywhere continuous and assume that to every  $\epsilon > 0$  there exists an  $A$  such that  $\int_{|x| \geq A} |g(x)| dF_n(x) < \epsilon$ . Then

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{+\infty} g(x) dF_n(x) = \int_{-\infty}^{+\infty} g(x) dF(x).$$

For a proof of this theorem the reader is referred to H. Cramér, op. cit., p. 74.

It is easy to verify that the conditions of the Helly-Bray theorem are satisfied if

$$\lim_{n \rightarrow \infty} [F_n(\infty) - F_n(-\infty)] = F(\infty) - F(-\infty)$$

and if  $g(x)$  is bounded.

We shall first show that every function  $\psi(s)$  given by (4.10) is the logarithm of the c.f. of an i.d.l. For this it will be sufficient to show that every  $\psi(s)$  given by (4.10) is the logarithm of a c.f. since it is obvious that with  $\psi(s)$  also  $\frac{\psi(z)}{n}$  satisfies (4.10).

[13] The theorem still holds if the functions  $F_n(x)$  are uniformly bounded monotonic functions and if one or both of the limits of integration is infinite.



To see this, consider first

$$(4.12) \quad I_{\varepsilon}(s) = \int_{\varepsilon < |x| < \frac{1}{\varepsilon}} \left( e^{isx} - 1 - \frac{isx}{1+x^2} \frac{1+x^2}{x^2} \right) dG(x)$$

with  $0 < \varepsilon < 1$ .

$I_{\varepsilon}(s)$  may be written as the limit of Riemann-Stieltjes sums

$$(4.13) \quad S_k = \sum_k \left[ \lambda_k (e^{isx_k} - 1) + is\mu_k \right]$$

with

$$\lambda_k = \frac{1+x_k^2}{x_k^2} [G(x_{k+1}) - G(x_k)]$$

$$\mu_k = \frac{1}{x_k} [G(x_{k+1}) - G(x_k)].$$

It is easy to verify that  $e^{isu}$  is the c.f. of the distribution of a random variable which equals  $u$  with probability one. Hence on account of (4.7) we see that

$$\lambda_k (e^{isx_k} - 1) + is\mu_k$$

is logarithm of a c.f. and so consequently is  $S_k$ . Thus  $I_{\varepsilon}(s)$  is a limit of logarithms of c.f.'s. Also  $I_{\varepsilon}(s)$  is obviously continuous at  $s=0$ . By Lévy's continuity theorem  $I_{\varepsilon}(s)$  is thus the logarithm of the c.f. of a distribution function. By the same theorem also



$$I_0(s) = \lim_{\varepsilon \rightarrow 0} I_\varepsilon(s) = \int_{|x| > 0} (e^{isx} - 1 - \frac{isx}{1+x^2}) \frac{1+x^2}{x^2} dG(x)$$

is the logarithm of a c.f.

$$\text{But } \psi(s) = I_0(s) + isa - \frac{s^2}{2}[G(0+) - G(0-)]$$

is obtained by adding to  $I_0(s) + isa$  the logarithm of the c.f. of a normal distribution. Thus  $\psi(s)$  is the logarithm of a c.f.

Thus the equation (4.10) is sufficient for  $\psi(s)$  to be the logarithm of a c.f. of an i.d.l.

To prove also the necessity of (4.10) we need several lemmas.

Lemma 4.1.  $G(x)$  and  $a$  in (4.10) are uniquely determined by  $\psi(s)$ .

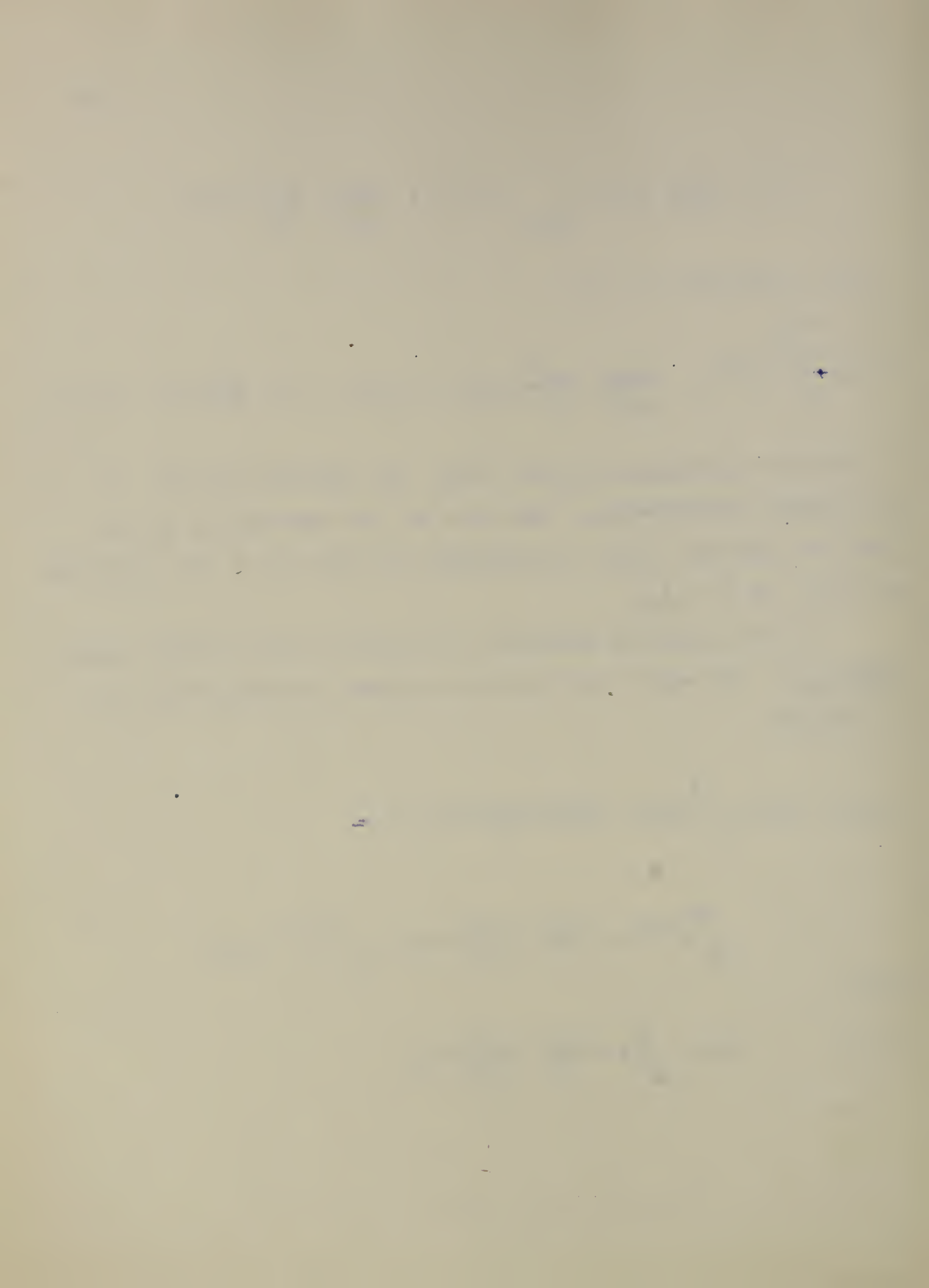
We consider

$$(4.14) \quad e(s) = \int_0^1 \left[ \psi(s) - \frac{\psi(s+h) - \psi(s-h)}{2} \right] dh =$$

$$\int_{-\infty}^{+\infty} e^{isx} \left(1 - \frac{\sin x}{x}\right) \frac{1+x^2}{x^2} dG(x) = \int_{-\infty}^{+\infty} e^{isx} d\phi(x)$$

where

$$(4.15) \quad \phi(x) = \int_{-\infty}^x \left(1 - \frac{\sin y}{y}\right) \frac{1+y^2}{y^2} dG(y)$$





It is easy to verify that  $(1 - \frac{\sin y}{y}) \frac{1+y^2}{y^2}$  is bounded above and below by positive constants. Thus  $\phi(x)$  is monotone and of bounded variation. The Fourier inversion formula determines uniquely  $\phi(x)$  given  $e(s)$  and thus also

$$(4.13) \quad G(x) = \int_{-\infty}^x \frac{y^2}{1+y^2} \left(1 - \frac{\sin y}{y}\right)^{-1} d\phi(y)$$

Finally  $a$  is determined from (4.10).

Lemma 4.2. If  $\psi_n(s)$  determined by  $G_n(x)$  and by  $a_n$  converges uniformly in every finite interval to a function  $b(s)$  continuous at the origin then  $\lim_{n \rightarrow \infty} G_n(x) = G(x)$  and  $\lim_{n \rightarrow \infty} a_n = a$  exist and  $b(s) = \psi(s)$  is determined by  $a$  and  $G(x)$  by equation (4.10).

Proof: From P. Lévy's continuity theorem it follows that  $e^{b(s)}$  is a c.f. hence everywhere continuous. Thus also  $b(s)$ , its logarithm. Therefore,  $e_n(s)$ , defined by  $\psi_n(s)$  by means of (4.14) converges to a continuous function. By the corollary to the Continuity Theorem it then follows that  $\lim_{n \rightarrow \infty} \phi_n(x) = \phi(x)$  exists. Moreover  $\phi(x)$  is non-decreasing and bounded. It follows from the Helly-Bray theorem that

$$\begin{aligned} \lim_{n \rightarrow \infty} G_n(x) &= \lim_{n \rightarrow \infty} \int_{-\infty}^x \left(1 - \frac{\sin y}{y}\right)^{-1} \frac{y^2}{1+y^2} d\phi_n(y) \\ &= \int_{-\infty}^x \left(1 - \frac{\sin y}{y}\right)^{-1} \frac{y^2}{1+y^2} d\phi(y) = G(x) \end{aligned}$$



since the integrand is bounded. It further follows, also from the Helly-Bray theorem, that

$$\begin{aligned} \lim_{n \rightarrow \infty} I_n(s) &= \lim_{n \rightarrow \infty} \int_{-\infty}^{+\infty} \left( e^{isx} - 1 - \frac{isx}{1+x^2} \right) \frac{1+x^2}{x^2} dG_n(x) \\ &= \int_{-\infty}^{+\infty} \left( e^{isx} - 1 - \frac{isx}{1+x^2} \right) \frac{1+x^2}{x^2} dG(x) = I(s) \end{aligned}$$

Finally it follows from the convergence of  $\psi_n(s)$  and  $I_n(s)$  that also the sequence  $\{a_n\}$  must converge and thus  $b(s) = \psi(s)$ .

The converse of lemma 4.2 follows immediately from the Helly-Bray theorem.

Lemma 4.3. The c.f.  $\phi(s)$  of an i.d.l. is everywhere different from zero.

Consider  $|\phi(s)|^{(1/n)}$ . We have  $\lim_{n \rightarrow \infty} [\phi(s)]^{(1/n)} = w(s)$

where  $w(s) = 1$  for  $\phi(s) \neq 0$  and  $w(s) = 0$  for  $\phi(s) = 0$ . By Lévy's theorem  $w(s)$  is a c.f. Since  $w(s) = 1$  for  $s = 0$  and  $w(s)$  is continuous being a c.f. we must have  $w(s) = 1$  everywhere.

Thus  $\phi(s) \neq 0$  everywhere.

Lemma 4.4.  $\log x = \lim_{n \rightarrow \infty} n[x^{(1/n)} - 1]$ ,  $x > 0$ . Lemma 4.4 follows

immediately from the rule of de l'Hospital.

Lemma 4.5. If  $\phi(s)$  is the c.f. of an i.d.l. then there exists a sequence of functions  $\psi_n(x)$  given by (4.10) such that

$$\log \phi(s) = \lim_{n \rightarrow \infty} \psi_n(s) \dots$$



Proof. We have by lemma 4.4

$$\log \phi(s) = \lim_{n \rightarrow \infty} n \{ [\phi(s)]^{(1/n)} - 1 \} = \lim_{n \rightarrow \infty} \psi_n(s)$$

uniformly in every finite interval of  $s$  since  $\phi(s) \neq 0$  with

$$a_n = n \int_{-\infty}^{+\infty} \frac{y}{1+y^2} dF_n(y), \quad G_n(x) = n \int_{-\infty}^x \frac{y^2}{1+y^2} dF_n(y).$$

Here  $F(x)$  is the d.f. whose c.f. is  $[\phi(s)]^{(1/n)}$ .

Proof of the necessity of (4.10): By lemma 4.3  $\phi(s) \neq 0$ , hence  $\log \phi(s)$  is defined everywhere and continuous. Moreover

$\log \phi(s) = \lim_{n \rightarrow \infty} \psi_n(s)$  uniformly in  $s$  with  $\psi_n(s)$  given by (4.10).

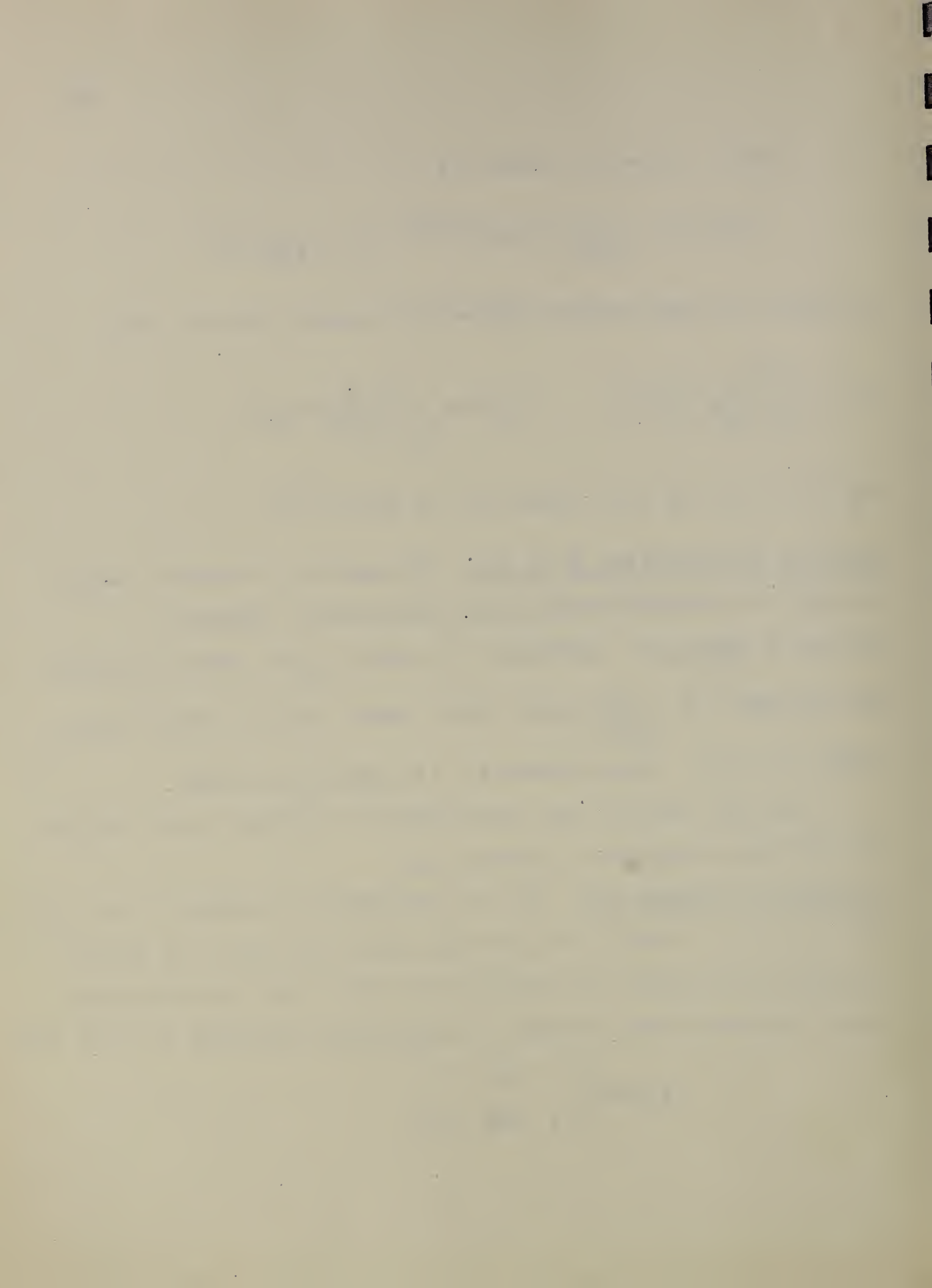
But by lemma 4.2  $\lim_{n \rightarrow \infty} \psi_n(s) = \psi(s)$  where  $\psi(s)$  is itself deter-

mined by (4.10). Thus theorem 4.1 is completely proved.

From our proof of the sufficiency of equation (4.10) follows the following corollary to theorem 4.1.

Corollary to theorem 4.1: If  $x$  is distributed according to an l.d.l. then  $x = y + z$  where  $y$  is normally distributed and  $z$  is distributed as is the limit of a sequence of finite sums of independent random variables each of which is distributed according to (4.5) with

$$\phi_1(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x \geq a \end{cases}$$



Theorem 4.2. Let  $x_t$  be a differential process of second order, then  $E(x_{t+\tau} - x_t) = \tau m$  and  $\text{Var}(x_{t+\tau} - x_t) = \tau \sigma^2$  where  $m$  and  $\sigma^2$  are constants independent of  $t$  and  $\tau$ .

Proof: Let  $\psi_\tau(z)$  be the logarithm of the c.f. of  $x_{t+\tau} - x_t$ . From (4.9) we see then that  $\psi_\tau(z) = \tau \psi_1(z)$  where  $\psi_1(z)$  is determined by (4.10). Therefore  $\psi'_\tau(0) = \tau \psi'_1(0)$  and  $\psi''_\tau(0) = \tau \psi''_1(0)$ . From this and from  $\psi'_\tau(0) = iE(x_{t+\tau} - x_t)$  and  $\psi''_\tau(0) = -\text{Var}(x_{t+\tau} - x_t)$  we see that the ~~lemma~~ <sup>theorem</sup> holds.

The estimation procedures in the case of a process given by (4.4) are very simple. The parameter to be estimated is  $\mu$ , its maximum likelihood estimate is  $x_T^*/T$ , the number of shots observed per unit of time, and the variance of this estimate is  $\mu/T$ .

If the process is given by (4.5) then the increments observed are a sample from a population with the distribution  $\phi(x)$ . If  $\phi(x)$  is given in parametric form then the proper estimation procedures are those appropriate for estimating the parameters of  $\phi(x)$  from the observed sample of increments.





## CHAPTER 5

### DIFFERENTIAL PROCESSES MODIFIED BY MECHANICAL DEVICES

#### 1. Filter effect.

In registering a stochastic process the registering device often spreads the effect over a certain period of time in such a way that an increase occurring at time  $t = 0$  will produce an effect at time  $t$  and the observed value, or the output process, is obtained from the superposition of all the effects produced from increases that occurred in preceding time periods.

If we let  $y_t$  be the output process and  $x_t$  the input process and assume that the effect is proportional to the increase we then have

$$(5.1) \quad y_t = \int_{-\infty}^{+\infty} f(t-\tau) dx_\tau$$

where  $\int_{-\infty}^{+\infty} |f(t)| dt$  and  $\int_{-\infty}^{+\infty} [f(t)]^2 dt$  and the integral (5.1) are assumed to exist.

Taking the upper limit of the integral equal to infinity instead of zero leaves the possibility open that future changes will influence the present. If the present is not affected by the future, then  $f(t)$  will be 0 for negative values of  $t$ .

In previous work we have often taken the point of view that the process  $x_t$  starts at some fixed time  $T$ . However we can also speak of the conditional distribution of  $x_t$  given  $x_\tau$  for  $\tau < t$

and thus  $\int_A^B f(t-\tau) dx_\tau$  may be formed for every  $A$  and  $B$  and we

proceed to prove



Theorem 5.1. If  $x_t$  is a differential process of second order with a weight function  $f(t)$  then the integral  $\int_{-\infty}^{+\infty} f(t-\tau) dx_\tau$  exists l.i.m.

In the following we use a simplified notation by writing  $x_i$  for  $x_{\tau_i}$  and  $\Delta_i$  for  $\tau_i - \tau_{i-1}$ .

Proof of theorem 5.1: We have

$$\begin{aligned} E\left[\int_{-\infty}^{+\infty} f(t-\tau) dx_\tau\right]^2 &= \lim_{\Delta_i \rightarrow 0} E\left\{\left[\sum_i f(t-\tau_i^*)(x_i - x_{i-1})\right]^2\right\} \\ &= \lim_{\Delta_i \rightarrow 0} E\left\{\sum_i f^2(t-\tau_i^*)(x_i - x_{i-1})^2\right. \\ &\quad \left. + \sum_{i \neq j} f(t-\tau_i^*)f(t-\tau_j^*)(x_i - x_{i-1})(x_j - x_{j-1})\right\} \end{aligned}$$

where  $\tau_{i-1} \leq \tau_i^* \leq \tau_i$ .

Since  $x_t$  is a differential process of second order we have, writing  $\sigma_{ij}$  for the covariance of  $(x_i - x_{i-1})$  and  $(x_j - x_{j-1})$

$$\sigma_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ \sigma^2(\tau_i - \tau_{i-1}) & \text{if } i = j \end{cases}$$

$$E(x_i - x_{i-1}) = m(\tau_i - \tau_{i-1}).$$

This follows from theorem 4.2.



Thus

$$\begin{aligned}
 (5.2) \quad E\left[\int_A^B f(t-\tau)dx_\tau\right]^2 &= \sigma^2 \int_A^B f^2(t-\tau)d\tau + m^2 \left[\int_A^B f(t-\tau)d\tau\right]^2 \\
 &= \sigma^2 \int_{t-B}^{t-A} f^2(\tau)d\tau + m^2 \left[\int_{t-A}^{t-B} f(\tau)d\tau\right]^2.
 \end{aligned}$$

Both integrals on the right of (5.2) converge to zero if  $A$  and  $B$  converge both to  $+\infty$  or to  $-\infty$ . In fact, if  $m=0$ , only the convergence of  $\int_{-\infty}^{\infty} [f(t)]^2 dt$  need be assumed; thus by lemma 1.6

$$\begin{array}{l}
 \text{l. i. m.} \\
 \begin{array}{l}
 A \rightarrow -\infty \\
 B \rightarrow +\infty
 \end{array}
 \end{array}
 \int_A^B f(t-\tau)dx_\tau \quad \text{exists.}$$

We denote the characteristic function of the increment  $x_{t+\tau} - x_t$  of a differential process by  $\phi_\tau(s)$ . From (4.9) we see that

$$(5.3) \quad \phi_\tau(s) = \exp[\tau \log \phi_1(s)]$$

We compute next the characteristic function  $\eta_t(s)$  of  $y_t$ , that is  $\eta_t(s) = E(e^{isy_t})$ . We have

$$\begin{aligned}
 (5.4) \quad \eta_t(s) &= E\left\{\exp\left[is \int_{-\infty}^{+\infty} f(t-\tau)dx_\tau\right]\right\} \\
 &= E\left\{\exp\left[is \lim_{\Delta_j \rightarrow 0} \sum_j f(t-\tau_j^*)(x_j - x_{j-1})\right]\right\}
 \end{aligned}$$



The random variable in the exponent at the right of (5.4) converges

in probability to  $y_t = \int_{-\infty}^{+\infty} f(t-\tau) dx_\tau$ . Hence its characteristic

function converges to  $\eta_t(s)$ . We thus have

$$(5.5) \quad \eta_t(s) = \lim_{\Delta_j \rightarrow 0} E\{\exp[is \sum_j f(t-\tau_j^*)(x_j - x_{j-1})]\}$$

Since the summands in the exponent are independent random variables we have

$$\eta_t(s) = \lim_{\Delta_j \rightarrow 0} \prod_j E\{\exp[is f(t-\tau_j^*)(x_j - x_{j-1})]\}$$

The characteristic function of  $x_{t+\tau} - x_t$  is  $\phi_\tau(s)$ , therefore

the characteristic function of  $f(t-\tau_j^*)(x_j - x_{j-1})$  is

$$\phi_{\Delta_j}[sf(t-\tau_j^*)] = \exp\{\Delta_j \log \phi_1[sf(t-\tau_j^*)]\} \text{ so that}$$

$$(5.6) \quad \log \eta_t(s) = \lim_{\Delta_j \rightarrow 0} \sum_j \Delta_j \log \phi_1[sf(t-\tau_j^*)] \\ = \int_{-\infty}^{+\infty} \log \phi_2[sf(t-\tau)] d\tau = \int_{-\infty}^{+\infty} \log \phi_1[sf(t)] dt .$$

The characteristic function of the joint distribution of  $y_{t_1}, \dots, y_{t_n}$ , which completely determines the output process is

formed in an analogous manner. We have





$$\begin{aligned} \eta_{t_1, \dots, t_n}(s_1, \dots, s_n) &= E\{\exp i(s_1 y_{t_1} + \dots + s_n y_{t_n})\} \\ &= E\{\exp i \int_{-\infty}^{+\infty} [s_1 f(t_1 - \tau) + \dots + s_n f(t_n - \tau)] d\mathbf{x}_\tau\} \end{aligned}$$

The argument employed in the case  $n=1$  shows that

$$\begin{aligned} (5.7) \quad \log \eta_{t_1, \dots, t_n}(s_1, \dots, s_n) \\ = \int_{-\infty}^{+\infty} \log(\phi_1[s_1 f(t_1 - \tau) + \dots + s_n f(t_n - \tau)]) d\tau \end{aligned}$$

It is seen from (5.7) that  $\eta_{t_1, \dots, t_n}(s_1, \dots, s_n)$  and thus also the joint distribution of  $y_{t_1}, \dots, y_{t_n}$  is invariant under translations in time. Hence the process is stationary. [14]

If the distribution of  $x_t$  is Gaussian (a differential process which is Gaussian is a F.R.P.) then also the distribution of  $y_t$  is Gaussian. The variance of  $y_t$  is given by  $\sigma^2 \int_{-\infty}^{+\infty} [f(t)]^2 dt$  and the covariance function is given by

$$\begin{aligned} (5.8) \quad R(t-t') &= E\left\{\left[\int_{-\infty}^{+\infty} f(t-\tau) d\mathbf{x}_\tau\right]\left[\int_{-\infty}^{+\infty} f(t'-\tau) d\mathbf{x}_\tau\right]\right\} \\ &= \sigma^2 \int_{-\infty}^{+\infty} f(t-\tau) f(t'-\tau) d\tau = \sigma^2 \int_{-\infty}^{+\infty} f(\tau) f(t-t+\tau) d\tau \end{aligned}$$

[14] A process is called stationary if the variables  $x_{t_1}, \dots, x_{t_n}$  have the same distribution as the variables  $x_{t_1+h}, \dots, x_{t_n+h}$ .



This follows immediately by writing the integrals as limits of Riemann-Stieltjes sums and by then applying theorem 5.1 and lemma 1.4. Thus the resulting process is a stationary Gaussian process with covariance function (5.8). If we put, for instance,

$$f(t) = \begin{cases} e^{-\beta t} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

then we obtain the O.U.P. of Chapter 2 with  $E(y_t^2) = \frac{\sigma^2}{2\beta}$

It may be seen from (5.7) and (5.8) that a large variety of output processes may be obtained from differential processes. If the differential process can be specified in parametric form and the function  $f(t)$  is also known at least in parametric form then (5.7) or in the most important special case (5.8) will give the output process in parametric form, so that the procedures of testing hypotheses about a finite number of parameters and of estimation in the parametric case become applicable although the difficulties of calculation may still be formidable.

In case nothing is known about either  $f(t)$  or  $\phi_p(s)$  the only way known at present by which some inferences can be obtained is by the spectral analysis described in Chapter 6.

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[14] A process is called stationary if the variables  $x_{t_1}, \dots, x_{t_n}$  have the same distribution as the variables  $x_{t_1+h}, \dots, x_{t_n+h}$ .

The first part of the document discusses the importance of maintaining accurate records of all transactions. It emphasizes that every entry should be supported by a valid receipt or invoice. This ensures transparency and allows for easy verification of the data.

Furthermore, it is noted that the records should be kept up-to-date and organized in a systematic manner. This helps in identifying trends and anomalies over time. The document also mentions that regular audits are essential to ensure the integrity of the information.

In addition, the text highlights the need for confidentiality and security of the data. Appropriate measures should be taken to protect the records from unauthorized access or loss. This includes using secure storage methods and limiting access to authorized personnel only.

Finally, the document concludes by stating that maintaining accurate and secure records is a fundamental responsibility of any organization. It is crucial for the overall success and compliance of the business.

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## 2. Counter data.

The modifying device may also operate in such a way that the modification of the input process is itself dependent on previous values of the input or output process. A frequently occurring example of this type of modification is provided by certain counter devices, which count random events. Due to the inertia of the counter device not all events will be counted. In particular we shall consider two types of such devices.

Type 1. After an event has been registered the counter remains locked during a certain time  $\tau$ .

Type 2. After an event has happened the counter remains locked during a certain time  $\tau$ .

A general and comprehensive treatment of probability problems in counter devices has been given by W. Feller (Courant Anniversary volume, 1948, pp. 105-115) and we shall here follow essentially Feller's representation. We shall assume that the input process is a Poisson process described by (4.4):

Let  $T_1, i \geq 1$  be the time interval between the  $i$ -th and the  $(i+1)$ st registration,  $T_0$  the time from the beginning, when the counter is locked, to the first registration. The  $T_i, i \geq 1$  are independently distributed all with the same distribution.

We denote the time up to the  $(k+1)$ st registration by

$$(5.9) \quad S_k = T_0 + T_1 + \dots + T_k$$

Let  $N$  be the number of registrations during time  $t$ . We clearly have



$$p_k(t) = P(N=k) = P(S_{k-1} \leq t) - P(S_k \leq t)$$

Let  $T_k$ ,  $k \geq 1$  have the distribution function  $F$ , so that  $P(T_k \leq t) = F(t)$ . We write moreover  $F_0(t)$  for the distribution function of  $T_0$ . Let  $F_k(t) = P(S_k \leq t)$  then

$$(5.10) \quad p_k(t) = F_{k-1}(t) - F_k(t).$$

Since  $S_{k+1} = S_k + T_{k+1}$  and since  $S_k$  and  $T_{k+1}$  are independent we have

$$F_{n+1}(t) = \int_0^t F_n(t-x) dF(x).$$

The characteristic function  $\phi_t(s)$  of the random variable  $N$  is thus given by

$$(5.11) \quad p_0(t) + \sum_{k=1}^{\infty} e^{isk} [F_{k-1}(t) - F_k(t)] \\ = p_0(t) + e^{is} F_0(t) + \sum_{k=1}^{\infty} e^{is(k+1)} F_k(t) - \sum_{k=1}^{\infty} e^{isk} F_k(t).$$

Thus since  $p_0(t) + F_0(t) = 1$ , we obtain

$$(5.12) \quad \phi_t(s) = 1 + (e^{is} - 1) \sum_{k=0}^{\infty} e^{isk} F_k(t).$$

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1. The first part of the book is devoted to a general introduction to the subject of the history of the world.

2. The second part of the book is devoted to a general introduction to the subject of the history of the world.

3. The third part of the book is devoted to a general introduction to the subject of the history of the world.

4. The fourth part of the book is devoted to a general introduction to the subject of the history of the world.

5. The fifth part of the book is devoted to a general introduction to the subject of the history of the world.

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7. The seventh part of the book is devoted to a general introduction to the subject of the history of the world.

8. The eighth part of the book is devoted to a general introduction to the subject of the history of the world.



Hence

$$\begin{aligned}
 (5.13) \quad \psi_t(s) &= \frac{\phi_t(s) - 1}{(e^{1s} - 1)} = \sum_{k=0}^{\infty} e^{1sk} F_k(t) \\
 &= F_0(t) + \sum_{k=1}^{\infty} \int_0^t e^{1sk} F_{k-1}(t-x) dF(x) \\
 &= F_0(t) + e^{1s} \int_0^t \psi_{t-x}(s) dF(x).
 \end{aligned}$$

In type 1 as well as in type 2 counters the value of  $N$  is bounded

so that  $\sum_{k=0}^{\infty} \frac{V_k}{k!}$  converges, where  $V_k = V_k(t)$  is the  $k$ -th moment.

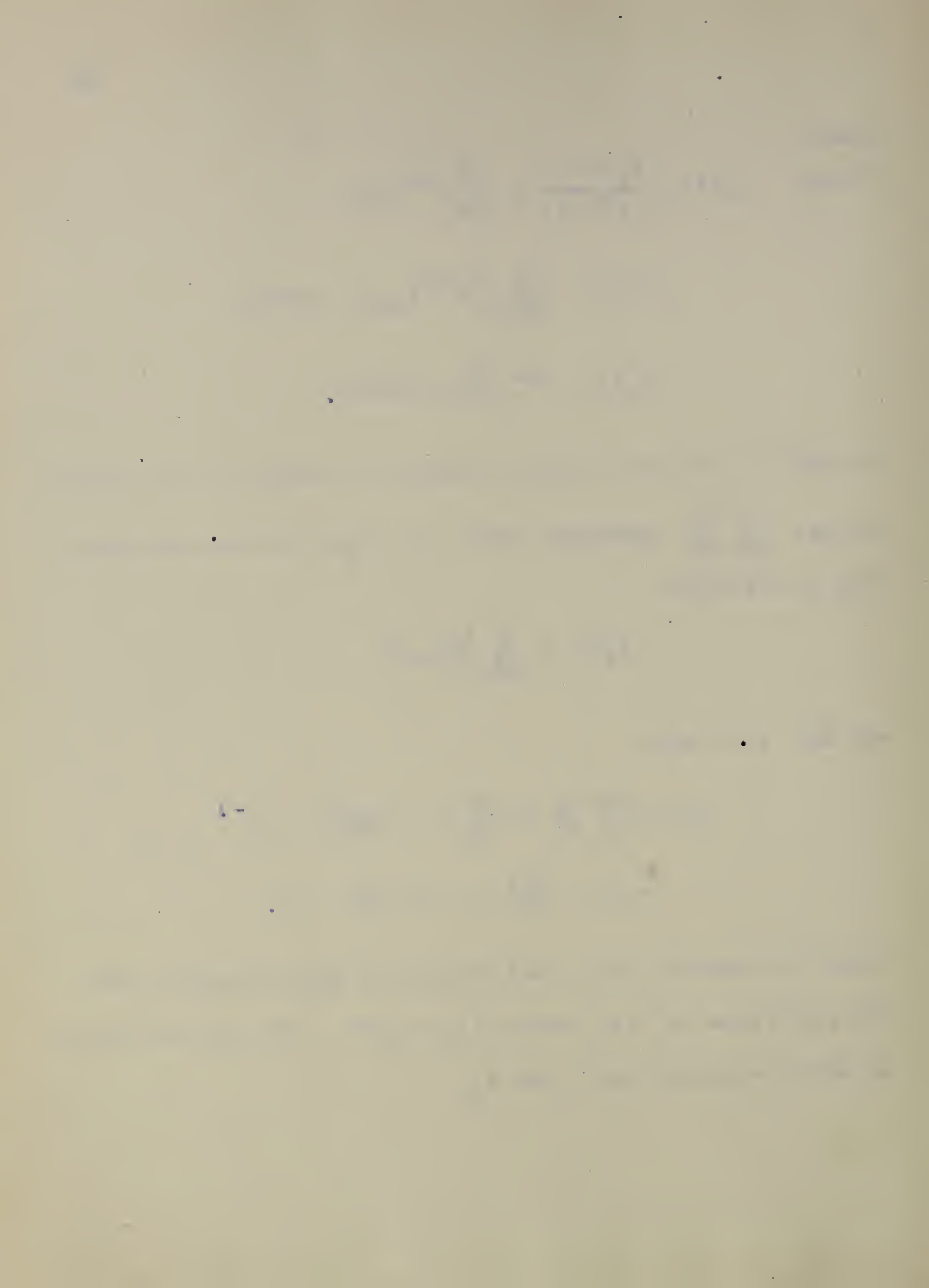
Thus we may write

$$\phi_t(s) = \sum_{k=0}^{\infty} \frac{V_k}{k!} (1s)^k$$

and for  $s < 1$  also

$$\begin{aligned}
 \psi_t(s) &= \left[ \sum_{k=1}^{\infty} \frac{V_k}{k!} (1s)^k \right] \left[ 1s + \frac{(1s)^2}{2!} + \dots \right]^{-1} \\
 &= (V_1 + \frac{1sV_2}{2!} + \dots) (1 - \frac{1s}{2} + \dots).
 \end{aligned}$$

Hence the constant term in the expansion of  $\psi_t(s)$  becomes  $V_1$  and the coefficient of  $(1s)$  becomes  $(V_2 - V_1)/2$ . From this and (5.13) we obtain equations for  $V_1$  and  $V_2$



$$(5.14) \left\{ \begin{array}{l} V_1(t) = F_0(t) + \int_0^t V_1(t-x) dF(x) = E(N) \\ V_2(t) = 2V_1(t) - F_0(t) + \int_0^t V_2(t-x) dF(x) = E(N^2) \end{array} \right.$$

We begin with the discussion of counters of type 1. From (4.2) we see that  $F_0(t) = 1 - e^{-at}$  where  $a > 0$  is the mean number of events per unit of time.

Further

$$\begin{aligned} F(t) &= 1 - e^{-a(t-\tau)} \quad \text{for } t \geq \tau, \\ F(t) &= 0 \quad \text{for } t < \tau \end{aligned}$$

since the counter remains locked during the time  $\tau$  after every registration. The first of the equations (5.14) then becomes

$$(5.15) \quad V_1(t) = \begin{cases} 1 - e^{-at} & \text{for } t \leq \tau \\ 1 - e^{-at} + a \int_{\tau}^t V_1(t-x) e^{-a(x-\tau)} dx & \text{for } t > \tau. \end{cases}$$

We compare (5.15) with the more general equation

$$A(t) = \begin{cases} H(t) & \text{for } t \leq \tau \\ H(t) + a \int_{\tau}^t A(t-x) e^{-a(x-\tau)} dx & \text{for } t > \tau. \end{cases}$$

If  $H(t) \leq 1 - e^{-at}$  then  $A(t) \leq V_1(t)$ . If  $H(t) > 1 - e^{-at}$  then  $A(t) > V_1(t)$ . This is certainly true for  $0 \leq t \leq \tau$  and can easily be shown to hold for  $t \leq (n+1)\tau$  if it holds for  $t \leq n\tau$ .



We put  $A(t) = \frac{at}{1+a\tau} + c$ . Then

$$H(t) = \begin{cases} \frac{at}{1+a\tau} + c & \text{for } t \leq \tau \\ 1 - \frac{e^{-a(t-\tau)}}{1+a\tau} + ce^{-a(t-\tau)} & \text{for } t \geq \tau \end{cases}$$

An elementary calculation shows that

$$H(t) \leq 1 - e^{-at} \quad \text{if } c = 0$$

$$H(t) > 1 - e^{-at} \quad \text{if } c = \frac{a^2\tau^2}{2(1+a\tau)}$$

Hence

$$(5.16) \quad \frac{at}{1+a\tau} \leq V_1(t) \leq \frac{at}{1+a\tau} + \frac{a^2\tau^2}{2(1+a\tau)}$$

It is also possible to obtain from (5.14) an exact expression for  $V_1(t)$ . However, this does not seem to be of great interest since (5.16) shows that  $\frac{at}{1+a\tau}$  approximates  $V_1(t)$  very closely with a bounded error which is small compared to  $V_1(t)$  unless  $a\tau$  is very large. The exact expression for  $V_1(t)$  is moreover very involved and hard to evaluate.

For the variance  $B(t)$  of  $N$  given by  $B(t) = V_2(t) - [V_1(t)]^2$ ,

Feller found the asymptotic expression

$$(5.17) \quad B(t) = \frac{at}{(1+a\tau)^3} + o(t)$$



We now put

$$(5.18) \quad \begin{cases} f(s) = \int_0^{\infty} e^{-st} dF(t) ; f_k(s) = \int_0^{\infty} e^{-st} dF_k(t) \\ \mu(s) = \int_0^{\infty} v_1(t) e^{-st} dt \end{cases}$$

We have by (5.10)

$$v_1(t) = \sum_{k=1}^{\infty} k p_k(t) = \sum_{k=1}^{\infty} k [F_{k-1}(t) - F_k(t)] = \sum_{k=0}^{\infty} F_k(t) ,$$

and by induction, using the well-known multiplicative property of the Laplace transform,

$$f_k(s) = f_0(s) [f(s)]^k$$

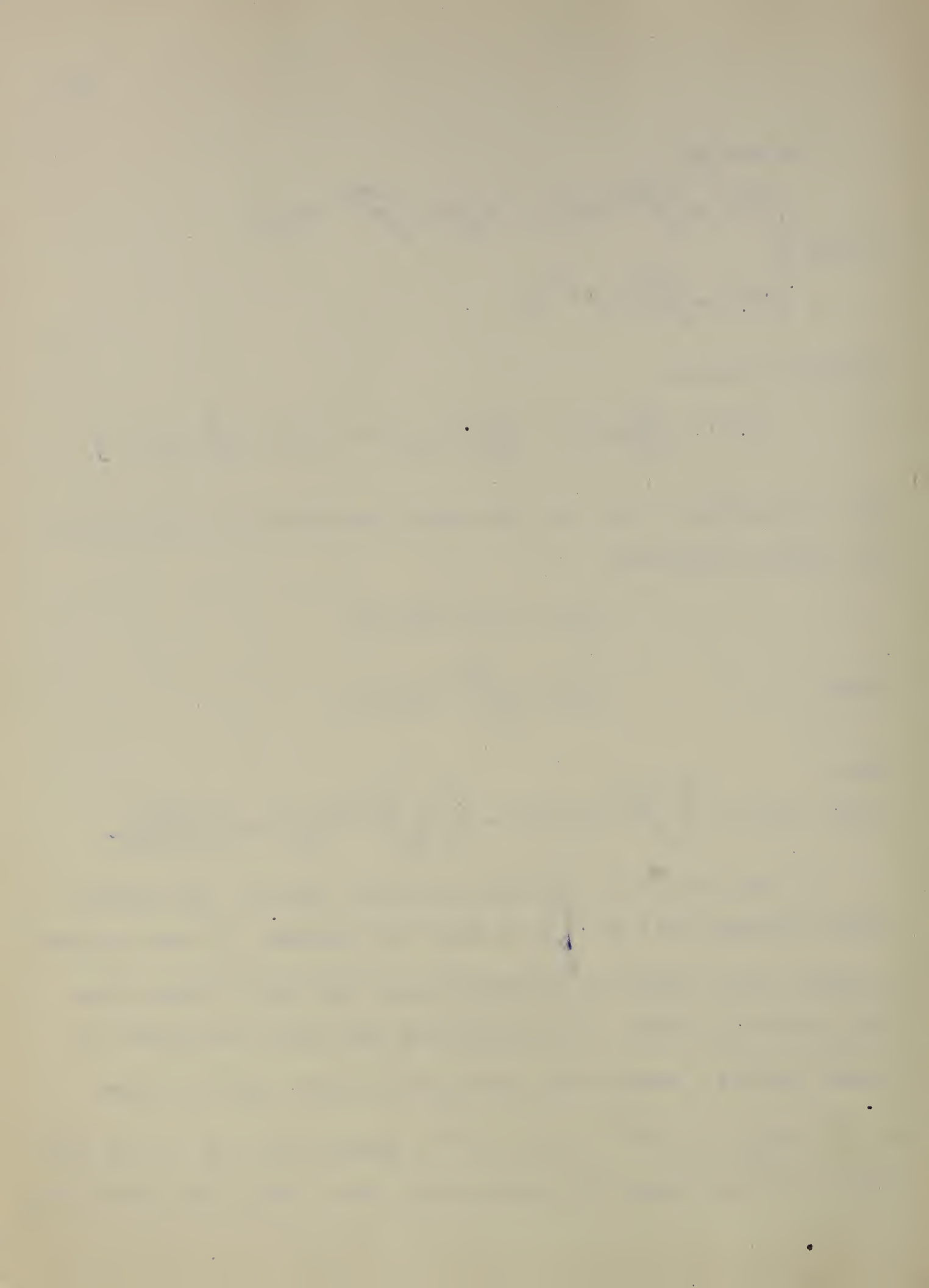
where

$$f_0(s) = \int_0^{\infty} e^{-st} dF_0(t) .$$

Thus

$$(5.19) \quad \mu(s) = \sum_0^{\infty} \int_0^{\infty} e^{-st} F_k(t) dt = \sum_0^{\infty} \frac{1}{s} \int_0^{\infty} e^{-st} dF_k(t) = \frac{1}{s} \frac{f_0(s)}{1-f(s)} .$$

We now proceed to discuss counters of type 2. The distribution function  $F(t)$  of  $T_k$  must first be obtained. To this purpose we shall first obtain the distribution of the time  $T$  during which the counter is locked. The probability that once the counter is locked exactly  $v$  events will prolong the locked time  $T$  is given by  $q^v p$ , where  $p = e^{-\lambda \tau}$ ,  $q = 1 - e^{-\lambda \tau}$ , since  $p$  by (4.2) is the probability that no event will occur during time  $\tau$  and  $q^v$  the probability





that the time intervals between  $v$  successive events will all be smaller than  $\tau$ . Let now  $T^{(1)}$  be the time elapsed between the  $(i-1)$ -st and the  $i$ -th event. The total locked time  $T$ , provided exactly  $v$  events prolong the locked time, is then given by

$$(5.20) \quad T = T^{(1)} + T^{(2)} + \dots + T^{(v)} + \tau.$$

The conditional probability  $U(t)$  that an event will occur during time  $t$  provided that  $t \leq \tau$  is then given by

$$(5.21) \quad U(t) = \frac{1}{q} (1 - e^{-at}).$$

Thus

$$(5.22) \quad \bar{u}(s) = \int_0^{\tau} e^{-st} dU(t) = \frac{1}{q} \frac{a}{a+s} [1 - e^{-(a+s)\tau}].$$

Let now  $v$  events prolong the locked time and write  $W_v(t) = P(T^{(1)} + \dots + T^{(v)} < t | v)$  for the probability that the locked time will be at most  $t + \tau$ , provided  $v$  events prolong the locked time. From (5.22) and (5.18) we see that

$$\int_0^{\tau} e^{-st} dW_v(t) = [u(s)]^v.$$

Consider now  $v$  itself as a random variable. Then  $W(t) = P(T^{(1)} + \dots + T^{(v)} < t)$  is the probability that the locked time  $T$  is at most  $t + \tau$ . We thus have



$$\int_0^{\infty} e^{-st} dW(t) = p\mathcal{L}[qu(s)]^V = \frac{p}{1-qu(s)} = p\left\{1 - \frac{a}{a+s}[1 - e^{-(a+s)\tau}]\right\}^{-1}$$

$$= \frac{(a+s)e^{-a\tau}}{s+a e^{-(a+s)\tau}} .$$

Let now  $G(t) = P(T \leq t)$  then  $G(t) = W(t-\tau)$  for  $t \geq \tau$  and  $G(t) = 0$  for  $t < \tau$ . Hence

$$(5.23) \quad \int_0^{\infty} e^{-st} dG(t) = \int_{\tau}^{\infty} e^{-st} dW(t-\tau) = e^{-s\tau} \int_0^{\infty} e^{-st} dW(t)$$

$$= \frac{(a+s)e^{-(a+s)\tau}}{s+a e^{-(a+s)\tau}} .$$

The time between two successive registrations is composed of the resolving time  $T$  and the time from the moment when the counter is free to the next event. The distribution of the latter is by (4.2)  $1 - e^{-at}$  and has the Laplace transform  $a/(a+s)$  thus

$$f(s) = \int_0^{\infty} e^{-st} dF(t) = \frac{a \exp[-(a+s)\tau]}{s+a \exp[-(a+s)\tau]}$$

while  $f_0(s) = \frac{a}{s+a}$ . Substituting this into (5.19) yields

$$(5.24) \quad \mu(s) = \frac{a[s+a e^{-(a+s)\tau}]}{s^2(a+s)} .$$



This is the Laplace transform (5.18) of the function

$$(5.25) \quad V_1(t) = \begin{cases} 1 - e^{-at} & \text{for } 0 \leq t \leq \tau \\ 1 - e^{-a\tau} + (t-\tau)ae^{-a\tau} & \text{for } t \geq \tau \end{cases}$$

Since  $V_1(t)$  is completely determined by its Laplace transform, formula (5.25) gives the expected number of registrations during time  $t$  in a counter of type 2.

A calculation similar to the one leading  <sup>$t$</sup>  to (5.25) shows that the variance  $B(t)$  of the number of counted events is given by

$$(5.26) \quad B(t) = V_2(t) - [V_1(t)]^2 \\ = ae^{-a\tau}(t-\tau)[1 - 2a\tau e^{-a\tau}] - e^{-a\tau} + (1+a\tau)^2 e^{-2a\tau}$$



The Fourier Analysis of Stochastic Processes.1. General theory.

A function  $f(t, t')$  in two variables is called monotonoid if  $f(t, t') = g(t, t') - h(t, t')$  where  $g$  and  $h$  are two functions monotonic in  $t$  and  $t'$  in the same sense. We now prove:

Theorem 6.1. Let  $x_t$  be a stochastic process with a monotonoid and continuous covariance function  $\sigma_{tt'}$  and  $E(x_t) = 0$  then

(i) For  $0 < t < T$  we have the expansion

$$(6.1) \quad x_t = \text{l.i.m.}_{m \rightarrow \infty} \sum_{n=-m}^{m} c_n \exp[2\pi i n t / T]$$

where

$$c_n = \frac{1}{T} \int_0^T x_t \exp[-2\pi i n t / T] dt$$

(ii) This limit is uniform in  $0 < \varepsilon \leq t < T - \varepsilon$

$$(iii) \quad \sigma_{c_n c_m} = \frac{1}{T^2} \int_0^T \int_0^T \sigma_{tt'} \exp\left[-2\pi i \frac{nt + mt'}{T}\right] dt dt'$$

(iv) If the process is Gaussian, then any finite set of  $c_n + \bar{c}_n$  and  $c_n - \bar{c}_n$  are jointly normally distributed.

To simplify the proof we put  $\tau = \frac{2\pi t}{T}$ ,  $y_\tau = x_t$ . Then  $\tau$

goes from 0 to  $2\pi$  as  $t$  goes from 0 to  $T$  and we have to prove the formula

$$(6.2) \quad y_\tau = \text{l.i.m.}_{m \rightarrow \infty} \sum_{-m}^m c_n e^{in\tau}, \quad c_n = \frac{1}{2\pi} \int_0^{2\pi} y_\tau e^{-in\tau} d\tau.$$

1. James M. Smith



We thus have to prove

$$\lim_{m \rightarrow \infty} E[y_{\tau}^{(m)} - y_{\tau}]^2 = 0$$

uniformly in every interval  $\varepsilon \leq \tau \leq 2\pi - \varepsilon$  where

$$y_{\tau}^{(m)} = \frac{1}{\sqrt{m}} \sum_{-m}^{+m} c_n e^{in\tau}.$$

We have

$$(6.3) \quad y_{\tau}^{(m)} = \frac{1}{2\pi} \int_0^{2\pi} y_{\tau'} \left[ \sum_{-m}^{+m} e^{in(\tau - \tau')} \right] d\tau'$$

Now

$$\begin{aligned} \sum_{n=-m}^m e^{ina} &= e^{-ima} \frac{1 - e^{i(2m+1)a}}{1 - e^{ia}} = \frac{e^{-ima} - e^{-i(m+1)a}}{1 - e^{ia}} \\ &= \frac{e^{-ima} - e^{i(m+1)a} - e^{-i(m+1)a} + e^{ima}}{2(1 - \cos a)} \\ &= \frac{\cos ma - \cos(m+1)a}{1 - \cos a} = \frac{\sin \frac{2m+1}{2} a}{\sin \frac{a}{2}} \end{aligned}$$

Putting  $\tau' = \tau + h$  we thus have

$$(6.4) \quad y_{\tau}^{(m)} = \frac{1}{2\pi} \int_{-\tau}^{2\pi - \tau} y_{\tau+h} \frac{\sin \frac{2m+1}{2} h}{\sin \frac{h}{2}} dh$$

and



$$(6.5) \quad E[y_{\tau}^{(m)} - y_{\tau}]^2 = \frac{1}{4\pi^2} \int_{-\tau}^{2\pi-\tau} \int_{-\tau}^{2\pi-\tau} \sigma_{\tau+h, \tau+k} \frac{\sin \frac{2m+1}{2}h}{\sin \frac{h}{2}} \cdot \frac{\sin \frac{2m+1}{2}k}{\sin \frac{k}{2}} dh dk$$

$$- 2 \frac{1}{2\pi} \int_{-\tau}^{2\pi-\tau} \sigma_{\tau, \tau+h} \frac{\sin \frac{2m+1}{2}h}{\sin \frac{h}{2}} dh + \sigma_{\tau\tau} \quad .$$

By well-known theorems on the Dirichlet integral [15] we have uniformly in  $\varepsilon \leq \tau \leq 2\pi - \varepsilon$  ( $\varepsilon > 0$ ),

$$(6.6) \quad \lim_{m \rightarrow \infty} \frac{1}{4\pi^2} \int_{-\tau}^{2\pi-\tau} \int_{-\tau}^{2\pi-\tau} \sigma_{\tau+h, \tau+k} \frac{\sin \frac{2m+1}{2}h}{\sin \frac{h}{2}} \cdot \frac{\sin \frac{2m+1}{2}k}{\sin \frac{k}{2}} dh dk$$

$$= \lim_{m \rightarrow \infty} \frac{1}{2\pi} \int_{-\tau}^{2\pi-\tau} \sigma_{\tau, \tau+h} \frac{\sin \frac{2m+1}{2}h}{\sin \frac{h}{2}} dh = \sigma_{\tau\tau} \quad .$$

From (6.5) and (6.6) it follows that

$$(6.7) \quad \lim_{m \rightarrow \infty} E[y_{\tau}^{(m)} - y_{\tau}]^2 = 0 \quad \text{or} \quad \text{l.i.m.}_{m \rightarrow \infty} y_{\tau}^{(m)} = y_{\tau}$$

uniformly in  $\varepsilon \leq \tau \leq 2\pi - \varepsilon$  for every  $\varepsilon > 0$ .

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[15] For the double Dirichlet integral see Hobson: "The theory of functions of a real variable and the theory of Fourier series", vol. II, pp. 705-9.



This completes the proof of the first two statements of theorem 6.1; (iii) is easily obtained by an elementary computation while (v) follows from the representation of the Fourier coefficients as limits of Riemann sums.

## 2. Trigonometric expansion of the F. R. P.

As an example we shall represent the F.R.P. by a trigonometric series with random coefficients. The covariance function of the F.R.P. is  $c \min(t, t')$  [see formula (2.4)], this is a monotonic and non-decreasing function of  $t$  and  $t'$  so that theorem 6.1 is applicable.

In this case  $c_n$  (that is the real and imaginary part of  $a_n$ ) is normally distributed with mean zero and we have

$$(6.8) \quad E(y_\tau y_{\tau'}) = E(x_\tau x_{\tau'}) = c \min(t, t') = \frac{c\Gamma}{2\pi} \min(\tau, \tau') = c' \min(\tau, \tau')$$

Thus

$$\begin{aligned} E(c_n c_m) &= c' \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \min(\tau, \tau') \exp[-in\tau - im\tau'] d\tau d\tau' \\ &= \frac{c'}{4\pi^2} \left\{ \int_0^{2\pi} \left[ \int_0^\tau \tau' \exp(-in\tau - im\tau') d\tau' \right] d\tau + \int_0^{2\pi} \left[ \int_\tau^{2\pi} \tau \exp(-in\tau - im\tau') d\tau' \right] d\tau \right\} \end{aligned}$$

For  $n=m=0$  we obtain

$$(6.9.1) \quad E(c_0^2) = \frac{c'}{4\pi^2} \left\{ \int_0^{2\pi} \frac{\tau^2}{2} d\tau + \int_0^{2\pi} \tau(2\pi-\tau) d\tau \right\} = \frac{2\pi c'}{3} = \frac{c\Gamma}{3}$$

For  $n \neq 0$  we have



$$(6.9.2) \quad E(c_n c_m) = \frac{c'}{4\pi^2} \int_0^{2\pi} e^{-int} \left( \frac{e^{-im\tau}}{m^2} - \frac{1}{m^2} - \frac{\tau}{im} \right) d\tau .$$

This gives

$$(6.9.3) \quad E(c_0 c_m) = \frac{c'}{4\pi^2} \left[ -\frac{2\pi^2}{im} - \frac{2\pi}{m^2} \right] = \frac{-cT}{4\pi m i} - \frac{cT}{4\pi^2 m^2} \quad \text{for } m \neq 0 ,$$

$$(6.9.4) \quad E(c_m c_{-m}) = \frac{cT}{2\pi^2 m^2} \quad \text{for } m \neq 0 .$$

For  $n \neq -m$ ,  $n \neq 0$ ,  $m \neq 0$  we obtain from (6.9.2)

$$(6.9.5) \quad E(c_n c_m) = \frac{-cT}{4\pi^2 mn} .$$

If we write

$$x_t = a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{2\pi n t}{T} + b_n \sin \frac{2\pi n t}{T} \right) \quad \text{l.i.m.}$$

we have

$$a_n = c_n + c_{-n}, \quad b_n = i(c_n - c_{-n}) \quad \text{for } n > 0, \quad a_0 = c_0$$

and from this and the formulae (6.9.1) - (6.9.5) we find





$$(6.10) \left\{ \begin{array}{l} E(a_0^2) = cT/5 \\ E(a_0 a_n) = -cT/2\pi^2 n^2 \\ E(a_m a_n) = 0 \quad \text{for } m \neq n, m \neq 0, n \neq 0 \\ E(a_n^2) = cT/2\pi^2 n^2 \\ E(a_n b_m) = 0 \quad \text{for } n \neq 0 \\ E(a_0 b_m) = -cT/2\pi m \\ E(b_n^2) = 3cT/2\pi^2 n^2 \\ E(b_n b_m) = cT/\pi^2 mn \quad \text{for } m \neq n. \end{array} \right.$$

We shall now estimate  $E[x_t - x_t^{(m)}]^2$  for a fixed  $m$  where

$$x_t^{(m)} = a_0 + \sum_{n=1}^m a_n \cos 2\pi \frac{nt}{T} + \sum_{n=1}^m b_n \sin 2\pi \frac{nt}{T}.$$

We have, using (6.10)

$$\begin{aligned} E[x_t - x_t^{(m)}]^2 &= \sum_{n=m+1}^{\infty} E(a_n^2) \cos^2 2\pi \frac{nt}{T} + \sum_{n=m+1}^{\infty} E(b_n^2) \sin^2 2\pi \frac{nt}{T} \\ &\quad + \sum_{\substack{n=m+1 \\ n \neq k}}^{\infty} \sum_{\substack{k=m+1 \\ k \neq n}}^{\infty} E(b_n b_k) \sin 2\pi \frac{nt}{T} \sin 2\pi \frac{kt}{T} \\ &= \frac{cT}{\pi^2} \left\{ \frac{1}{2} \sum_{n=m+1}^{\infty} \frac{1}{n^2} + \left[ \sum_{n=m+1}^{\infty} \frac{\sin \frac{2\pi n}{T}}{n} \right]^2 \right\}. \end{aligned}$$



Expanding the function  $\pi - \alpha$  into a Fourier series we get

$$\pi - \alpha = 2 \sum_{n=1}^{\infty} \frac{\sin n\alpha}{n}$$

hence

$$\sum_{n=m+1}^{\infty} \frac{\sin n\alpha}{n} = \frac{1}{2}(\pi - \alpha) - \sum_{n=1}^m \frac{\sin n\alpha}{n} = f(\alpha).$$

Differentiating this equation we have

$$f'(\alpha) = -\frac{1}{2} - \sum_{n=1}^m \cos n\alpha = -\frac{1}{2} \frac{\sin(m+\frac{1}{2})\alpha}{\sin \frac{\alpha}{2}},$$

and therefore

$$\sum_{n=m+1}^{\infty} \frac{\sin n\alpha}{n} = \frac{\pi}{2} - \frac{1}{2} \int_0^{\alpha} \frac{\sin(m+\frac{1}{2})t}{\sin \frac{t}{2}} dt = \frac{\pi}{2} - \int_0^{\frac{\alpha}{2}} \frac{\sin(2m+1)v}{\sin v} dv.$$

Hence

$$\mathbb{E}[x_t - x_t^{(m)}]^2 = \frac{cT}{\pi^2} \left[ \frac{1}{2} \sum_{m+1}^{\infty} \frac{1}{n^2} + \left( \frac{\pi}{2} - \int_0^{\frac{\pi t}{T}} \frac{\sin(2m+1)v}{\sin v} dv \right)^2 \right].$$

Thus for values of  $t$  not too close to 0 or  $T$ ,  $x_t^{(m)}$  is a good approximation to  $x_t$ .

Another and perhaps more useful formula may be obtained by deriving the following expansion

$$x_t - \frac{t}{T} x_T = \sum_{n=1}^{\infty} \left\{ a_n \left[ \cos \frac{2\pi n t}{T} - 1 \right] + b_n \sin \frac{2\pi n t}{T} \right\}.$$



In this expansion the  $a_n$  and  $b_n$  are independently and normally distributed variables with mean zero and variances  $\frac{cT}{2\pi^2 n^2}$ , the  $a_n$  and  $b_n$  are also independent of  $x_T$ . The right side converges, moreover, uniformly in the mean to the left side. The proof can be obtained by first applying theorem 6.1 to the stochastic process  $x_t - \frac{t}{T} x_T$  and determining the Fourier coefficients and their variances. It is then seen that the Fourier expansion thus obtained converges also l.i.m. for  $t=T$  and thus  $a_0 = - \sum_{n=1}^{\infty} a_n$ . The proof is rather laborious but elementary and is therefore omitted.

Thus writing  $x_T = a_0$  we have

$$(6.11) \quad x_t = a_0 \frac{t}{T} + \sum_{n=1}^{\infty} \left\{ a_n \left[ \cos \frac{2\pi n t}{T} - 1 \right] + b_n \sin \frac{2\pi n t}{T} \right\}$$

where

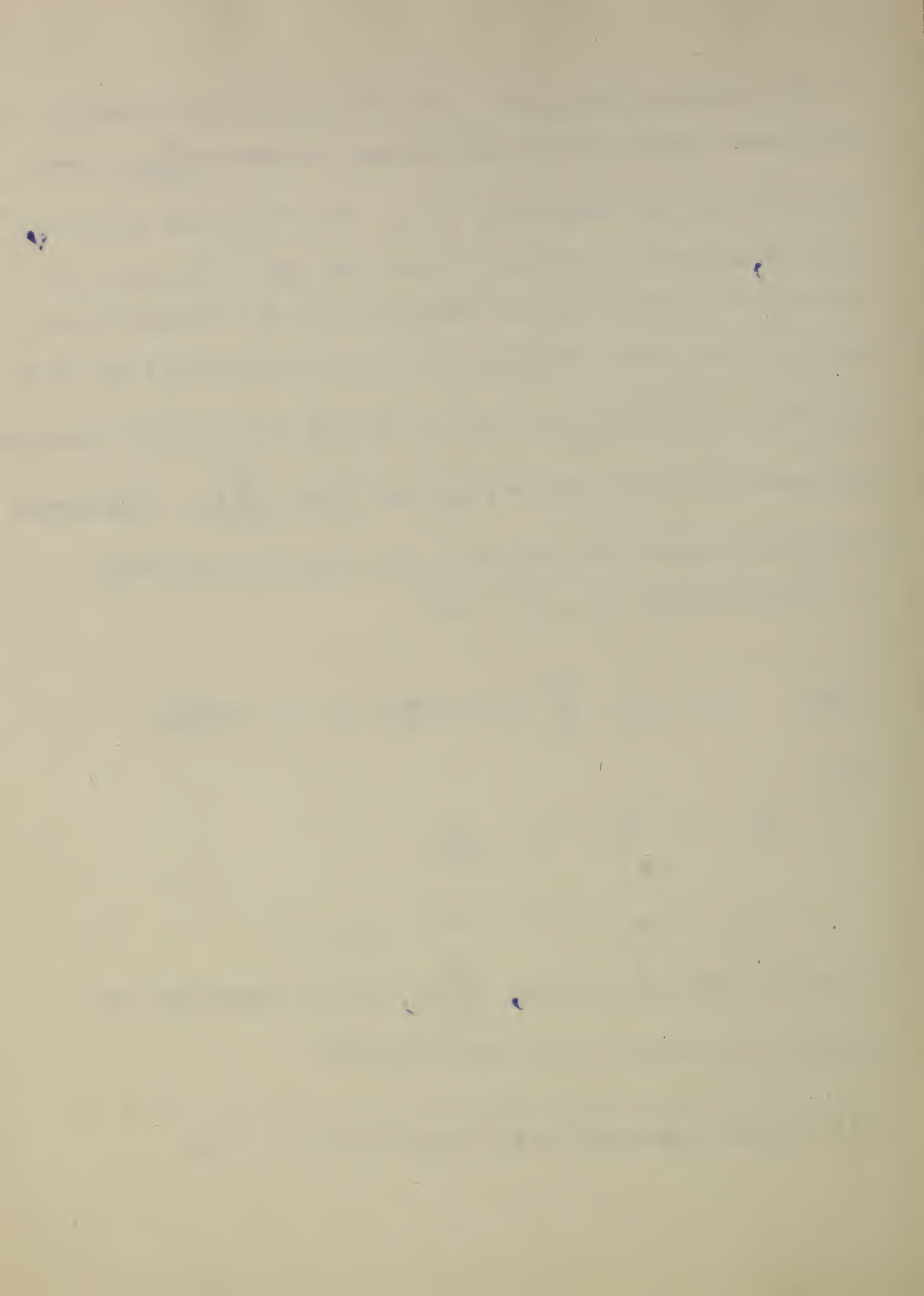
$$\sigma_{a_0}^2 = cT, \quad \sigma_{a_n}^2 = \sigma_{b_n}^2 = \frac{cT}{2\pi^2 n^2},$$

$$\sigma_{a_i a_j} = \sigma_{b_i b_j} = 0 \quad \text{for } i \neq j, \quad \sigma_{a_i b_j} = 0$$

Except for the constant term,  $- \sum_{n=1}^{\infty} a_n$ , this is essentially the expansion discovered by Paley and Wiener. [16]

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[16] Fourier transforms in the complex domain, p. 147.



### 3. Stationary processes.

We now return to the general theory and consider stationary processes.

We shall further assume that the covariance  $E(x_t x_{t'}) = \sigma_{tt'}$  exists. We then have  $\sigma_{tt'} = R(t - t')$  where  $R(\tau)$  is an even function of  $\tau$ .

We shall also consider a slightly more general class of processes, called quasistationary processes. A process  $x_t$  is said to be quasistationary if  $E[x_t]$  is independent of  $t$  and if its covariance function exists and is given by  $\sigma_{tt'} = R(t - t')$  where  $R(\tau)$  is an even function of  $\tau$ .

We assume now that  $R(\tau)$  is continuous at the point  $\tau = 0$  and show that  $R(\tau)$  is then continuous everywhere. If  $R(\tau)$  is continuous at  $\tau = 0$  then we have  $\lim_{\tau \rightarrow 0} E(x_{t+\tau} - x_t)^2 = 0$ . From the definition of  $R(\tau)$  we see that

$$\lim_{h \rightarrow 0} [R(\tau+h) - R(\tau)] = \lim_{h \rightarrow 0} E[(x_{\tau+h} - x_\tau)x_0] = 0$$

since

$$|E[(x_{\tau+h} - x_\tau)x_0]| \leq \sqrt{E(x_{\tau+h} - x_\tau)^2 E(x_0^2)}.$$





We next introduce the following definition: A function  $f(t)$  is said to be positive definite if

(a)  $f(t)$  is continuous and bounded on the real axis;

(b)  $f(t)$  is Hermitian, that is  $f(-t) = \overline{f(t)}$ ;

(c) for any positive integer  $m$  and any real numbers

$z_1, z_2, \dots, z_m$  and any complex numbers  $u_1, u_2,$

$\dots, u_m$  we have

$$\sum_{h=1}^m \sum_{k=1}^m f(z_h - z_k) u_h \bar{u}_k \geq 0 .$$

From the preceding it is clear that  $R(t)$  satisfies conditions (a) and (b) since  $R(t)$  is real and even. We have only to prove that

$$S = \sum_{h=1}^m \sum_{k=1}^m R(t_h - t_k) u_h \bar{u}_k \geq 0 .$$

We have

$$S = \sum_{h=1}^m \sum_{k=1}^m u_h \bar{u}_k E(x_{\tau+t_h} x_{\tau+t_k}) = E \left\{ \left( \sum_{h=1}^m u_h x_{\tau+t_h} \right) \left( \sum_{k=1}^m \bar{u}_k x_{\tau+t_k} \right) \right\}$$

$$= E \left\{ \left| \sum_{h=1}^m u_h x_{\tau+t_h} \right|^2 \right\} \geq 0 . \quad \text{Therefore } R(t) \text{ is a positive definite}$$

function.



According to a theorem of S. Bochner<sup>[17]</sup> every positive definite function  $f(t)$  may be represented in the form

$$f(t) = \int_{-\infty}^{+\infty} e^{it\alpha} dV(\alpha)$$

where  $V(\alpha)$  is a bounded non-decreasing function.

Thus we have

$$(6.15) \quad R(t) = \int_{-\infty}^{+\infty} e^{it\omega} dg(\omega)$$

where  $g(\omega)$  is a bounded and non-decreasing function. We may take  $g(-\infty) = 0$ . Then  $g(\infty) = R(0)$  and  $g(\alpha)/R(0)$  could therefore be defined as a distribution function. It will however simplify our formulae if we determine  $g(\alpha)$  so that

$$g(\alpha) = \frac{g(\alpha+) + g(\alpha-)}{2} .$$

Since  $R(t) = R(-t)$ , we have

$$\begin{aligned} R(t) &= \int_{-\infty}^{+\infty} e^{it\omega} dg(\omega) = \int_{-\infty}^{+\infty} e^{-it\omega} dg(\omega) = - \int_{-\infty}^{+\infty} e^{it\omega} dg(-\omega) \\ &= \int_{-\infty}^{+\infty} e^{it\omega} d[g(\infty) - g(-\omega)] , \end{aligned}$$

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[17] S. Bochner, Vorlesungen über Fouriersche Integrale, p. 76, Satz 23.

91

93

and since the function  $g(\omega)$  is unique if  $g(-\infty) = 0$  and

$$g(\omega) = \frac{g(\omega+) + g(\omega-)}{2}, \text{ we must have } g(\omega) = g(\infty) - g(-\omega)$$

and for  $\omega = 0$ ,  $g(\infty) = 2g(0)$  and  $g(\omega) - g(0) = g(0) - g(-\omega)$ .

It is further well known that

$$g(\omega) - g(0) = \lim_{T \rightarrow \infty} \frac{1}{2\pi i} \int_{-T}^T R(t) \frac{e^{it\omega} - 1}{t} dt .$$

We may also write

$$\begin{aligned} R(t) &= \frac{R(t) + R(-t)}{2} = \int_{-\infty}^{+\infty} \frac{e^{it\omega} + e^{-it\omega}}{2} dg(\omega) \\ &= \int_{-\infty}^{\infty} \cos t\omega dg(\omega) = \int_0^{\infty} \cos t\omega d[g(\omega) - g(-\omega)] \\ &= \int_0^{\infty} \cos t\omega dF(\omega) , \end{aligned}$$

where

$$(5.16) \quad \begin{cases} F(\omega) = g(\omega) - g(-\omega) = \frac{F(\omega+) + F(\omega-)}{2} \\ F(\infty) = g(\infty) = R(0) \\ F(0) = 0 . \end{cases}$$



Further

$$F(\omega) = g(\omega) - g(-\omega) = \lim_{T \rightarrow \infty} \frac{1}{2\pi i} \int_{-T}^{+T} R(t) \frac{e^{it\omega} - e^{-it\omega}}{t} dt$$

so that

$$F(\omega) = \frac{2}{\pi} \int_0^{\infty} R(t) \frac{\sin t\omega}{t} dt .$$

It may also be remarked that to every positive definite function  $R(\tau)$  we may construct a Gaussian process with  $R(\tau)$  as covariance function. This can be done by defining the distribution of  $x_{t_1}, \dots, x_{t_n}$  to be a multivariate Gaussian distribution with covariance matrix  $\| R(t_i - t_j) \|$ . Since  $R(t)$  is positive definite such a distribution always exists. It is then easy to verify that the family of distribution functions so defined satisfies the consistency conditions of chapter 1. Combining this with the result of Bochner we obtain

Theorem 6.2. The function  $R(t)$  is the covariance function of a quasistationary process if and only if it is the Fourier transform of a bounded non-decreasing function.

#### 4. The mean ergodic theorem.

We shall conclude this chapter with a proof of the mean ergodic theorem.

Theorem 6.3. (Mean ergodic theorem): [18] Let  $x_t$  be a quasistationary process with continuous covariance function  $R(t)$  and mean value zero.

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[18] This theorem is due to J. v. Neumann; Proc. Nat. Acad. Sci., vol. 18(1932), pp. 70-82.





Then

$$(6.17) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \text{i.o.m.} \int_0^T e^{i\lambda t} x_t dt = a_\lambda$$

where  $a_\lambda$  is a random variable with variance  $g(\lambda+) - g(\lambda-)$  and mean zero and where  $E(a_\lambda a_\mu) = 0$  for  $\lambda \neq -\mu$ . The function  $g(\lambda)$  is defined by (6.15).

We first prove the following lemma

Lemma 6.1: For any  $0 \leq t \leq T$ ,  $T \geq 1$  and for every  $\varepsilon$

$$\left| \frac{1}{T} \int_0^T e^{i\lambda(t-\tau)} R(t-\tau) d\tau - \left[ g\left(\lambda + \frac{\varepsilon}{T}\right) - g\left(\lambda - \frac{\varepsilon}{T}\right) \right] \right|$$

$$\leq [g(\lambda+\varepsilon) - g(\lambda+) + g(-\lambda+\varepsilon) - g(-\lambda+)]$$

$$+ \frac{\varepsilon}{2} [g(\lambda+\varepsilon) - g(\lambda-\varepsilon)] + \frac{4}{\varepsilon T} [g(\infty) - g(-\infty)]$$

Proof: We put

$$\frac{1}{T} \int_0^T e^{i\lambda(t-\tau)} R(t-\tau) d\tau = \frac{1}{T} \int_0^T \int_{-\infty}^{\infty} \exp[i\lambda(t-\tau) + i\omega|t-\tau|] dg(\omega) d\tau$$

$$= I_1 + I_2 + I_3 + J_1 + J_2 + J_3$$

where



$$I_1 = \frac{1}{T} \int_0^t e^{i\lambda(t-\tau)} \int_{|\omega+\lambda| \geq \epsilon} e^{i\omega(t-\tau)} dg(\omega) d\tau; \quad J_1 = \frac{1}{T} \int_t^T e^{i\lambda(t-\tau)} \int_{|\omega-\lambda| \geq \epsilon} e^{-i\omega(t-\tau)} dg(\omega) d\tau;$$

$$I_2 = \frac{1}{T} \int_0^t e^{i\lambda(t-\tau)} \int_{\frac{\epsilon}{T} \leq |\omega+\lambda| \leq \epsilon} e^{i\omega(t-\tau)} dg(\omega) d\tau; \quad J_2 = \frac{1}{T} \int_t^T e^{i\lambda(t-\tau)} \int_{\frac{\epsilon}{T} \leq |\omega-\lambda| \leq \epsilon} e^{-i\omega(t-\tau)} dg(\omega) d\tau;$$

$$I_3 = \frac{1}{T} \int_0^t e^{i\lambda(t-\tau)} \int_{-\lambda-\frac{\epsilon}{T}}^{-\lambda+\frac{\epsilon}{T}} e^{i\omega(t-\tau)} dg(\omega) d\tau; \quad J_3 = \frac{1}{T} \int_t^T e^{i\lambda(t-\tau)} \int_{\lambda-\frac{\epsilon}{T}}^{\lambda+\frac{\epsilon}{T}} e^{-i\omega(t-\tau)} dg(\omega) d\tau.$$

These integrals converge absolutely. Hence we may interchange the order of integration whenever necessary. In this manner we obtain

$$|I_1| = \left| \frac{1}{T} \int_{|\omega+\lambda| \geq \epsilon} dg(\omega) \int_0^t e^{i(\omega+\lambda)(t-\tau)} d\tau \right| = \left| \frac{1}{T} \int_{|\omega+\lambda| \geq \epsilon} \frac{e^{i(\omega+\lambda)t} - 1}{i(\omega+\lambda)} dg(\omega) \right|,$$

so that

$$(6.18) \quad |I_1| \leq \frac{2}{\epsilon T} [g(\infty) - g(-\infty)]$$

Similarly

$$(6.18a) \quad |J_1| \leq \frac{2}{\epsilon T} [g(\infty) - g(-\infty)]$$

We have, for  $x$  real,

$$(*) \quad |e^{ix} - 1| \leq |x|;$$



using this inequality we obtain from

$$|I_2| = \left| \frac{1}{T} \int_{\frac{\varepsilon}{T} \leq |\omega + \lambda| \leq \varepsilon} \int_0^t \exp[i(\lambda + \omega)(t - \tau)] d\tau dg(\omega) \right|,$$

$$(6.19) \quad |I_2| \leq \frac{t}{T} [g(\lambda + \varepsilon) - g(\lambda) + g(-\lambda + \varepsilon) - g(-\lambda)]$$

and similarly

$$(6.19a) \quad |J_2| \leq \frac{T-t}{T} [g(\lambda + \varepsilon) - g(\lambda) + g(-\lambda + \varepsilon) - g(-\lambda)].$$

Since

$$\frac{t}{T} [g(\lambda + \frac{\varepsilon}{T}) - g(\lambda - \frac{\varepsilon}{T})] = \frac{1}{T} \int_{-\lambda - \frac{\varepsilon}{T}}^{-\lambda + \frac{\varepsilon}{T}} \int_0^t dg(\omega) d\tau$$

we have, using (\*)

$$|I_3 - \frac{t}{T} [g(\lambda + \frac{\varepsilon}{T}) - g(\lambda - \frac{\varepsilon}{T})]| = \left| \frac{1}{T} \int_{-\lambda - \frac{\varepsilon}{T}}^{-\lambda + \frac{\varepsilon}{T}} \int_0^t [e^{i(\lambda + \omega)(t - \tau)} - 1] dg(\omega) d\tau \right|$$

$$\leq \frac{1}{T} \int_{-\lambda - \frac{\varepsilon}{T}}^{-\lambda + \frac{\varepsilon}{T}} \int_0^t |\lambda + \omega| (t - \tau) dg(\omega) d\tau$$

$$\leq \frac{\varepsilon t^2}{2T^2} \int_{-\lambda - \frac{\varepsilon}{T}}^{-\lambda + \frac{\varepsilon}{T}} dg(\omega) \leq \frac{\varepsilon t}{2T} [g(\lambda + \frac{\varepsilon}{T}) - g(\lambda - \frac{\varepsilon}{T})]$$



Since  $T \geq 1$  and since  $g(x)$  is non-decreasing it is seen easily that

$$(6.20) \quad \left| I_3 - \frac{t}{T} [g(\lambda + \frac{\epsilon}{T}) - g(\lambda - \frac{\epsilon}{T})] \right| \leq \frac{\epsilon t}{2T} [g(\lambda + \epsilon) - g(\lambda - \epsilon)] .$$

Similarly we obtain

$$(6.20a) \quad \left| J_3 - \frac{T-t}{T} [g(\lambda + \frac{\epsilon}{T}) - g(\lambda - \frac{\epsilon}{T})] \right| \leq \frac{\epsilon(T-t)}{2T} [g(\lambda + \epsilon) - g(\lambda - \epsilon)] .$$

Lemma 6.1 then follows easily from (6.18), (6.18a), (6.19), (6.19a), (6.20), (6.20a).

Corollary 1 to lemma 6.1.

$$(6.21) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T e^{\lambda(t-\tau)} R(t-\tau) d\tau = g(\lambda+) - g(\lambda-) .$$

Corollary 2 to lemma 6.1.

$$(6.22) \quad \lim_{\substack{T \rightarrow \infty \\ T' \rightarrow \infty}} \frac{1}{TT'} \int_0^T \int_0^{T'} e^{\lambda(t-t')} R(t-t') dt dt' = g(\lambda+) - g(\lambda-) .$$

*Proof.* We may always write the double integral so that  $T' > T$  so that lemma 6.1 is applicable and corollary 2 follows easily since  $\epsilon$  is arbitrary.





In the proof of the mean ergodic theorem we shall operate with complex random variables. If  $z = x + iy$  is a complex random variable with mean zero we shall define

$$(6.23) \quad \sigma_z^2 = E(z\bar{z}) = \sigma_x^2 + \sigma_y^2$$

where  $\bar{z} = x - iy$  is the complex conjugate to  $z$ . A sequence  $\{z_n\} = \{x_n + iy_n\}$  of complex random variables converges if both  $\{x_n\}$  and  $\{y_n\}$  converge. From lemma 1.6 it follows that  $\{z_n\}$  converges l.i.m. if and only if  $E[(z_n - z_m)(\bar{z}_n - \bar{z}_m)]$  is arbitrarily small for sufficiently large  $n, m$ .

To show that  $X_T = \frac{1}{T} \int_0^T x_t e^{i\lambda t} dt$  converges in the mean we consider

$$\begin{aligned} L_{TT'} &= E[(X_T - X_{T'})(\bar{X}_T - \bar{X}_{T'})] \\ &= \frac{1}{T^2} \int_0^T \int_0^T e^{i\lambda(t-t')} R(t-t') dt dt' + \frac{1}{T'^2} \int_0^{T'} \int_0^{T'} e^{i\lambda(t-t')} R(t-t') dt dt' \\ &\quad - \frac{2}{TT'} \int_0^T \int_0^{T'} e^{i\lambda(t-t')} R(t-t') dt dt'. \end{aligned}$$



All three integrals converge to the same limit by (6.22). Thus

$\lim_{T \rightarrow \infty} \text{l.i.m. } X_T = a_\lambda$  exists. Moreover, by lemma 1.7 and (6.22)

$$\sigma_{a_\lambda}^2 = \lim_{T \rightarrow \infty} \sigma_{X_T}^2 = g(\lambda^+) - g(\lambda^-) .$$

For  $\lambda \neq -\mu$  we further have

$$\begin{aligned} E(a_\lambda a_\mu) &= \lim_{T \rightarrow \infty} \frac{1}{T^2} \int_0^T \int_0^T \exp(i\lambda t + i\mu t') R(t-t') dt dt' \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T e^{i(\lambda+\mu)t} \frac{1}{T} \int_0^T e^{-i\mu(t-t')} R(t-t') dt dt' \end{aligned}$$

The second integral converges by corollary 1 of lemma 6.1 to  $g(\mu^+) - g(\mu^-)$  uniformly in  $t$ . Thus

$$E(a_\lambda a_\mu) = \lim_{T \rightarrow \infty} \left\{ \frac{1}{T} \int_0^T e^{i(\lambda+\mu)t} [g(\mu^+) - g(\mu^-)] dt + \eta(T) \right\}$$

where  $\lim_{T \rightarrow \infty} \eta(T) = 0$ . It easily follows that  $E(a_\lambda a_\mu) = 0$ .

Theorem 6.4. Let  $x_t$  be any quasistationary process with ~~correlation~~ <sup>covariance</sup>

function  $R(\tau)$  and let  $g(\omega)$  be defined by (6.15). Further let  $\lambda_1,$

$\lambda_2, \dots$  be the discontinuities of  $g(\omega)$  and

$$a_\lambda = \lim_{T \rightarrow \infty} \text{l.i.m. } \frac{1}{T} \int_0^T x_t e^{i\lambda t} dt$$



$$x_t = \sum_{j=1}^n a_j \lambda_j e^{-i\lambda_j t} + y_t$$

where  $y_t$  is a quasistationary process such that

$$\text{l.i.m.}_{T \rightarrow \infty} \frac{1}{T} \int_0^T y_t e^{i\mu t} dt = 0$$

for all real numbers  $\mu$ .

Proof: The sum  $z_t = \sum_{j=1}^n a_j \lambda_j e^{i\lambda_j t}$  converges in the mean

since

$$\sum_{j=m}^n \sigma_{a_j}^2 = \sum_{j=m}^n [\sigma(\lambda_{j+}) - \sigma(\lambda_{j-})]$$

hence

$$\sum_{j=1}^n [\sigma(\lambda_{j+}) - \sigma(\lambda_{j-})] \leq \sigma(\infty) - \sigma(-\infty) \text{ for all } n.$$

Moreover

$$\text{l.i.m.}_{T \rightarrow \infty} \frac{1}{T} \int_0^T z_t e^{i\mu t} dt = \begin{cases} a_{\lambda_1} & \text{for } \mu = \lambda_1 (i=1, 2, \dots) \\ 0 & \text{otherwise} \end{cases}$$

which proves theorem 5.4.



BOCHNER, S., 110.

Brownian motion, 32, 34, 47.

Characteristic function; definition, 75; of increment of differential process, 87; of number of registrations, 92; logarithm of, 76.

Conditional probability, 28.

Confidence region for FOURIER coefficients, 63.

Convergence in probability, 2.

Convergence in the mean, 11.

Convergence of distribution functions; at points of continuity, 4; HELLY-BRAY theorem, 78; HELLY'S theorem, 5; LEVY'S continuity theorem, 76.

Counters, 91 ff..

Covariance, 10.

Covariance function, 10.

CRAMÉR, H., 11, 77, 78.

Derivative of stochastic process, 14.

Differentiable l.i.m., 15.

Differential processes, 34; of second order, 84; modified, 85.

Distribution functions, 1 ff.; of number of registrations, 92; of registration times, 92, 96.

DOOB, J. L., (IV), 53.

Estimation of parameters, 55 ff..

Expectation, 10.

FELLER, W., (IV), 91, 95.

Filter effects, 85 ff..





- FOURIER analysis of stochastic processes, 100 ff..
- FOURIER transform, 77.
- Frictional effects, 47.
- Fundamental Random Process (F.R.P.), 32 ff.; covariance function, 89; definition, 37; estimation of parameters, 55 ff.; integral, 46; properties, 37 ff.; trigonometric expansion, 103 ff.; with mean value function, 57.
- Gaussian process, 46, 89, 100, 112.
- Generalized differential process, 72 ff..
- Generalized POISSON distribution, 75.
- Generalized second derivative, 15.
- HAHN, H., 34.
- HELLY-BRAY theorem, 78.
- HELLY'S theorem, 5.
- Hermitian function, 109.
- HOBSON, E. W., 102.
- Independent stochastic processes, 27.
- Infinitely divisible law (i.d.l.), 76 ff..
- Input process, 85.
- Integrable l.i.m., 24.
- Integration of stochastic processes, 17; of one process with respect to another, 27.
- KHINTCHINE, A., 36.
- KOLMOGOROFF, A., 28.
- LAPLACE transform, 96.



- LÉVY, P., (IV), 18, 76, 81.
- LÉVY'S continuity theorem, 76.
- Limit in the mean (l.i.m.), 11.
- LINDBERBERG condition, 35-37.
- Locked time, 97.
- LOÈVE, M., (IV), 14, 76.
- LUKACS, E., (IV).
- MARKOFF processes, 32, 34, 55.
- Mathematical expectation, 10.
- Maximum of process on interval, 18.
- Mean ergodic theorem, 112 ff..
- Mesh, 8.
- Minimum of process on interval, 18.
- Modified differential processes, 85 ff..
- Modulus; of mesh, 8; of subdivision, 17.
- Monotonid, 100.
- MORANDA, P., (IV)
- Netting, 8.
- ORNSTEIN, L. S., 49.
- ORNSTEIN-UHLENBECK process (O.U.P.), 47 ff.; covariance function, 51; estimation of parameters, 64, integral of, 51.
- Output process, 85.
- PALEY, R. E. A. C., 107.
- plim, 2.
- POISSON distribution, 74, 75.



- POISSON process, 91.
- Positive definite function, 109.
- Probability limit of a sequence of random variables, 2.
- Quasistationary processes, 108 ff..
- Random variables, 1 ff..; complex valued, 117; convergence of sequence, 2; non-negative, 11; probability limit of sequence, 2; sequences of, 2.
- Registration of stochastic processes, 85 ff..
- RUBIN, H., (III)
- SAVAGE, L., (IV)
- Shot, 73.
- SLUTSKY, E., 3.
- Spectral analysis, 100 ff..
- Stationary processes, 49, 89, 90, 108.
- Stochastic differential equation, 53.
- Stochastic processes; continuous, 10; differentiable, 14; derivative of, 14; generalized second derivative, 15; integration of, 17; maximum of, 18; minimum of, 18; of a continuous parameter, 10; of second order, 15; registration of, 85; stationary, 49; strongly continuous, 22; uniformly continuous, 28.
- Strong continuity, 18.
- UHLENBECK, G. E., 49.
- VON NEUMANN, J., 112.
- WIENER, N., 107.
- WIDDER, D. V., 5.



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