NATIONAL BUREAU OF STANDARDS REPORT

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CALCULATION OF CONTINUED FRACTIONS BY USE OF AN IBM 604 MULTIPLIER

By

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FOREWORD

The introduction of high-speed automatic computing machines has made it feasible to consider empirical sampling as a general method in statistical research. One of the problems encountered is the transformation of variates distributed uniformly over (0,1) to variates having other distributions. Continued fraction expansions such as the one given in this report should make it possible to carry out the transformations in cases where power series converge too slowly or do not converge at all.

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CALCULATION OF CONTINUED FRACTIONS
BY USE OF AN IBM 604 MULTIPLIER

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ABSTRACT

This paper gives an example of the use of continued fraction expansions in calculating values of functions by high-speed automatic machines. This method, applied here to the doubly-exponential distribution $F(x) = e^{-e^{-x}}$, should prove useful in a wide variety of empirical sampling problems.

I. INTRODUCTION

In mathematical and statistical calculations values of various functions are frequently required. If only low speed relay multipliers are available, the values can usually be obtained most efficiently from punch-card or punch-tape tables. However, if high speed electronic calculators are used, the time required to look up values in a table becomes prohibitive and the values of the function can be obtained more rapidly by direct calculations. It has been usual to use power series expansions for this. In this paper an example is given of the use of continued fraction expansions.

In the use of power series expansions two difficulties are encountered:

(a) the range for which the series converges is limited;

(b) the numbers entering into the calculations become very large or very small.

The latter difficulty is usually surmounted by a floating decimal point; and the first, by multiplying the argument by a suitably chosen constant and later making the necessary adjustment. In both cases extensive programming and a detailed knowledge of numbers which will arise in the calculation are required. On the other hand, the advantages of power series expansions are that programming for an iterative method of calculation is fairly simple and that only a few numbers have to be stored.

Such general statements cannot be made about continued fraction expansions. However, for some the following statements hold: the expansion converges for a large range of values of the argument; the numbers entering into the calculations remain reasonably close to unity; and the successive approximations can be calculated by an iterative method. However, more storage space is required than is needed for power series.

II. THEORY

A continued fraction expansion of a function f(x) takes the form

$$f(x) = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3} + \dots}}$$

where the a_i and b_i may be functions of x. For convenience the expansion is often written in the form

$$f(x) = b_0 + \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \frac{a_3}{b_3 +} \cdots \, .$$

The nth approximant to f(x) is given by

$$f_n(x) = b_0 + \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \cdots \frac{a_n}{b_n} \, .$$

It is shown in [1] that this approximant may be calculated as follows:

$$f_n(x) = b_0 + \rho_1 + \rho_1\rho_2 + \dots + (\rho_1\rho_2 \dots \rho_n)$$

where

$$1 + \rho_n = \frac{1}{1 + r_n(1 + \rho_{n-1})}$$

$$r_n = \frac{a_n}{b_{n-1}b_n}$$

with initial values

$$\rho_1 = \frac{a_1}{b_1}$$

$$1 + \rho_2 = \frac{1}{1 + r_2} \, .$$

III. METHOD OF CALCULATION

The value of the function can therefore be obtained by calculating successive approximations until the product $\rho_1\rho_2 \dots \rho_n$ is equal to zero to as many digits as required. This can be done by carrying out the steps in the calculation given in Column 1 of Table I. Column 2 gives the values required from storage; and Column 3, the initial values required.

TABLE I

(1) Calculation	(2) Required from Storage	(3) Initial Values
1. $r_n = \frac{a_n}{b_{n-1} , b_n}$	a_n, b_{n-1}, b_n, x	1
2. $1 + \rho_n = \frac{1}{1 + r_n(1 + \rho_{n-1})}$	$1 + \rho_{n-1}$	1
3. $(\rho_1 \ldots \rho_{n-1})\rho_n$	$\rho_1 \ldots \rho_{n-1}$	a_1/b_1
4. $f_n = f_{n-1} + \rho_1 \rho_2 \ldots \rho_n$		$b_0 + a_1/b_1$

In general an extensive memory will be required if many a_i and b_i are needed, i.e. if many iterations will be required before $\rho_1 \ldots \rho_n$ becomes zero. However, if a_n and b_n are functions of n and x only, and this function can be calculated in an iterative method, machines with relatively small memories can be used. In the following section an example is given in which the 604, with only thirty-two memory positions, was used.

IV. EXAMPLE

In the study of extreme values the distribution function

$$y = F(x) = e^{-e^{-x}}$$

appears. In sampling experiments it is necessary to calculate the inverse function

$$x = -\log_e [-\log_e(+y)]$$

for given values of y. Since $0 \leq y \leq 1$, the power series expansion could be used to calculate

$$z = -\log_e(+y) \quad .$$

Since z can take any value between zero and infinity, the power series is not sufficient to calculate, efficiently,

$$x = -\log_e z \quad .$$

However, the continued fraction expansion

$$\log_e z = \frac{(z-1)}{1} + \frac{(z-1)}{2} + \frac{(z-1)}{3} + \frac{4(z-1)}{4} + \frac{4(z-1)}{5} + \dots$$

converges for all $0 < z$.[1] Furthermore, the a_n and b_n are such that

$$r = \frac{\left|\frac{n}{2}\right|^2 (z-1)}{n(n-1)}$$

where $\left|\frac{n}{2}\right|$ denotes the greatest integer in $\frac{n}{2}$.

The storage assignment in the 604 was as follows:

Storage	Factor	Decimal
FS-1	n	xxx.
FS-2	$z-1$	xx.xxx
FS-3		
FS-4	Temporary	
MQ		
Counter		
GS-1 - GS-3	f_{n-1}	xx.xxxx
GS-2	$1 + \rho_{n-1}$	x.xxxx
GS-4	$\rho_1 \dots \rho_n$	x.xxxx

In this case z was less than 10. Even with this restriction the calculations can be carried out to only four decimal places because of the lack of storage positions Therefore one can not expect the results to be accurate to four decimals. Table II gives the intermediate calculations for z = 10. The only value punched on the card is the final one, i.e. 2.3024 .

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[1] The expansion for $\log_e x$ and numerous others are given in [2] and [3].

Table III gives some results for the range $.1 \leqq z \leqq 10$. The accuracy decreases as $z \rightarrow \infty$ because $1 + \rho_n \rightarrow 0$ and there are fewer than 4 significant figures. In order to get greater accuracy a larger memory would be needed.

This work was done on a 604 with 60 program steps. Each iteration was obtained by impulsing the program repeat, suppression of the program repeat being based on a balance test of GS-4. The machine completed approximately 1100 cards per hour.

TABLE II

CALCULATION OF $\log_e 10$

n	r_n	$1 + \rho_n$	ρ_n	$\rho_1 \rho_2 \dots \rho_n$	f_n
1	9.0000	1.0000	9.0000	9.0000	9.0000
2	4.5000	0.1818	-0.8182	-7.3638	1.6362
3	1.5000	0.7857	-0.2143	1.5781	3.2143
4	3.0000	0.2979	-0.7021	-1.1080	2.1063
5	1.8000	0.6510	-0.3490	0.3867	2.4930
6	2.7000	0.3626	-0.6374	-0.2465	2.2465
7	1.9286	0.5885	-0.4115	0.1014	2.3479
8	2.5714	0.3979	-0.6021	-0.0611	2.2868
9	2.0000	0.5569	-0.4431	0.0271	2.3139
10	2.5000	0.4180	-0.5820	-0.0158	2.2981
11	2.0455	0.5391	-0.4609	0.0073	2.3054
12	2.4545	0.4304	-0.5696	-0.0042	2.3012
13	2.0769	0.5280	-0.4720	0.0020	2.3032
14	2.4231	0.4387	-0.5613	-0.0011	2.3021
15	2.1000	0.5205	-0.4795	0.0005	2.3026
16	2.4000	0.4446	-0.5554	-0.0003	2.3023
17	2.1176	0.5134	-0.4866	0.0001	2.3024

TABLE III

ACCURACY OF VALUES CALCULATED BY THE 604 MULTIPLIER

x	$\log_e x$	calculated $\log_e x$
.1	-2.30259	-2.3024
.2	-1.60944	-1.6091
.3	-1.20397	-1.2045
.4	-0.91629	-0.9164
.5	-0.69315	-0.6931
.6	-0.51083	-0.5110
.7	-0.35667	-0.3567
.8	-0.22314	-0.2231
.9	-0.10536	-0.1054
1.0	0.0	0.0
2.0	0.69315	0.6932
3.0	1.09861	1.0985
4.0	1.38629	1.3863
5.0	1.60944	1.6095
6.0	1.79176	1.7914
7.0	1.94591	1.9460
8.0	2.07944	2.0794
9.0	2.19722	2.1971
10.0	2.30258	2.3024

REFERENCES

[1] Ralph E. Lane, "Interpolation by Means of Continued Fractions," Proceedings of the Fraternal Actuarial Association, No. 19 (1944-46).

[2] O. Perron, Die Lehre von der Kettenbruchen, Leipzig, 1913.

[3] H. S. Wall, The Analytic Theory of Continued Fractions, Van Nostrand, New York, 1948.

THE NATIONAL BUREAU OF STANDARDS

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