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TABLE OF THE FIRST MOMENT OF RANKED EXTREMES

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FOREWORD

The tabulation of the first moment of ranked extremes will considerably simplify the use of certain methods of analyzing extreme-value data. The present table was computed by the Computation Laboratory of the National Applied Mathematics Laboratories, National Bureau of Standards, at the suggestion of Bradford F. Kimball, who developed the necessary formulas.

This table is one item of a project aimed at the improved application of the theory of extreme values to the analysis of gust loads of airplanes. This program of the Statistical Engineering Laboratory (Section 3 of the National Applied Mathematics Laboratories, Division 11, NBS) is supported by the National Advisory Committee for Aeronautics.

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TABLE OF THE FIRST MOMENT OF RANKED EXTREMES

This table gives the expected values of order statistics from the distribution of largest values Prob $\{Y \leq y\} = \exp(-e^{-y})$. The entries were computed from the formula given by B. F. Kimball*

$$E(y_r) = C + \sum_{t=0}^{r} (-1)^t \binom{n}{t} \Delta^t \ln(n-t) \quad ,$$

where $y_0 \geq y_1 \geq \ldots y_r \geq \ldots\ldots\ldots \geq y_{n-1}$ denote the n order statistics in a sample of n values and C is Euler's constant (.57722...). The expected values of y_r, $E(y_r)$, are given in the table for

$$r = 0(1)n-1 \text{ or } 25, \text{ whichever is smaller}$$

$$n = 1(1)10(5)60(10)100.$$

The table has two important uses. One, described by Kimball (ibid), is in connection with a method of graphical analysis of an ordered sample of extremes.

A second use relates to determination of the "best" linear unbiased estimator T of the mode u of the distribution of largest values, Prob $\{X \leq x\} = \exp(-e^{-y})$, where $y = \alpha(x - u)$, regardless of whether or not the parameter α is known. If

*Kimball, B. F.: Assignment of Frequencies to a Completely Ordered Set of Sample Data, Trans. Amer. Geophysical Union, vol. 27, no. 6, 1946, pp. 843-846. See also discussion of this article in vol. 28, no. 6, 1947, pp. 951-953.

$T = \sum_{i=0}^{n-1} \tau_i x_i$, where the x_i are n ordered sample values (order statistics) $x_0 \geq x_1 \geq \cdots \geq x_{n-1}$ from this distribution, then one set of conditions on the unknowns τ_i is given by the vanishing of the bias of T, that is, $E(T) - u = 0$ identically in u. From the above linear relation between x and y, we have $E(y_i) = \alpha[E(x_i) - u]$, whence

$$E(T) - u = \Sigma\, \tau_i E(x_i) - u = \Sigma \tau_i [u + \frac{1}{\alpha} E(y_i)] - u,$$

whose vanishing requires that the τ_i satisfy the two conditions $\Sigma\tau_i = 1$, $\Sigma\tau_i E(y_i) = 0$. In order to determine the "best" or most efficient linear estimator T (i.e., the one with smallest variance), the τ_i must be determined by further conditions so that the variance of T, $\sigma^2(T)$, is a minimum. This will require the computation of a table of variances and covariances of ranked extremes, $\sigma(y_i y_j)$.

$$E(y_r) = C + \sum_{t=0}^{r} (-1)^t \binom{n}{t} \Delta^t \ln(n-t), \text{ where } C = \text{Euler's}$$

Constant = 0.5772157

r	n=1	n=2	n=3	r
0	0.57721 566	+1.2703 628	1.6758 280	0
1		-0.1159 315	+0.4594 326	1
2			-0.4036 136	2

r	n=4	n=5	n=6	r
0	1.9635 100	2.1866 536	2.3689 751	0
1	0.8127 317	1.0709 358	1.2750 458	1
2	+0.1060 835	**+0.4255 506**	0.6627 159	2
3	-0.5735 126	**-0.1068 945**	+0.1383 853	3
4		**-0.6901 671**	-0.2545 345	4
5			-0.7772 937	5

r	n=7	n=8	n=9	r
0	2.5231 258	2.6566 572	2.7744 402	0
1	1.4440 711	1.5884 061	1.7143 929	1
2	0.8524 826	1.0110 660	1.1474 521	2
3	0.4096 935	0.5881 770	0.7382 939	3
4	+0.0224 042	+0.2312 101	0.4005 308	4
5	-0.3653 099	-0.1028 793	+0.0957 534	5
6	-0.8459 576	-0.4527 868	-0.2021 957	6
7		-0.9021 249	-0.5243 843	7
8			-0.9493 425	8

r	n=10	n=15	n=20	r
0	2.8798 008	3.2852 659	3.5729 479	0
1	1.8261 956	2.2503 728	2.5470 821	1
2	1.2671 822	1.7132 872	2.0200 359	2
3	0.8680 818	1.3403 534	1.6583 531	3
4	0.5436 122	1.0478 353	1.3785 557	4
5	+0.2574 495	0.8013 873	1.1471 396	5
6	-0.0120 439	0.5851 754	0.9472 338	6
7	-0.2836 893	0.3872 821	0.7690 671	7
8	-0.5845 581	0.2010 161	0.6063 833	8
9	-0.9898 741	+0.0206 140	0.4548 347	9
10		-0.1594 579	0.3111 607	10
11		-0.3458 052	0.1727 124	11
12		-0.5485 327	+0.0371 321	12
13		-0.7883 747	-0.0979 135	13
14		-1.1326 841	-0.2350 333	14
15			-0.3775 607	15
16			-0.5304 220	16
17			-0.7022 321	17
18			-0.9119 529	18
19			-1.2232 104	19

$S(y_r) = C + \sum_{t=0} (-1)^t \binom{n}{t} \Delta^t \ln(n-t)$, where C = Euler's Constant = 0.5772157

r	n=25	n=30	n=35	r
0	3.7960 915	3.9784 130	4.1325 637	0
1	2.7735 416	2.9613 665	3.1179 999	1
2	2.2542 557	2.4438 174	2.6030 713	2
3	1.8988 407	2.0923 991	2.2544 241	3
4	1.6258 959	1.8237 416	1.9887 029	4
5	1.4020 114	1.6044 708	1.7725 500	5
6	1.2104 327	1.4178 745	1.5892 723	6
7	1.0415 346	1.2543 779	1.4293 167	7
8	0.8892 472	1.1079 713	1.2866 988	8
9	0.7494 644	0.9746 216	1.1574 147	9
10	0.6192 453	0.8514 772	1.0386 471	10
11	0.4963 746	0.7364 333	0.9283 319	11
12	0.3791 010	0.6278 773	0.8249 044	12
13	0.2659 691	0.5245 305	0.7271 437	13
14	0.1557 008	0.4253 445	0.6340 712	14
15	+0.0471 025	0.3294 297	0.5448 828	15
16	-0.0610 213	0.2360 031	0.4589 015	16
17	-0.1699 655	0.1443 471	0.3755 427	17
18	-0.2812 545	+0.0537 740	0.2942 883	18
19	-0.3968 504	-0.0364 087	0.2146 659	19
20	-0.5195 330	-0.1269 375	0.1362 310	20
21	-0.6536 981	-0.2186 413	+0.0585 513	21
22	-0.8073 485	-0.3125 115	-0.0188 098	22
23	-0.9984 861	-0.4098 117	-0.0963 110	23
24	-1.2882 598	-0.5122 664	-0.1744 547	24
25		-0.6224 196	-0.2538 157	25

r	n=40	n=45	n=50	r
0	4.2660 951	4.3838 782	4.4892 387	0
1	3.2533 828	3.3725 996	3.4791 033	1
2	2.7403 936	2.8611 039	2.9687 929	2
3	2.3937 809	2.5160 483	2.6249 676	3
4	2.1301 969	2.2540 897	2.3642 868	4
5	1.9162 926	2.0418 841	2.1534 099	5
6	1.7353 848	1.8627 537	1.9756 620	6
7	1.5779 311	1.7071 623	1.8215 109	7
8	1.4379 596	1.5691 446	1.6849 953	8
9	1.3114 804	1.4447 183	1.5621 370	9
10	1.1956 921	1.3310 904	1.4501 482	10
11	1.0885 487	1.2262 243	1.3469 974	11
12	0.9885 067	1.1285 871	1.2511 576	12
13	0.8943 693	1.0369 939	1.1614 507	13
14	0.8051 863	0.9505 080	1.0769 470	14
15	0.7201 865	0.8683 738	0.9968 991	15
16	0.6387 319	0.7899 709	0.9206 958	16
17	0.5602 845	0.7147 813	0.8478 294	17
18	0.4843 816	0 6423 661	0.7778 727	18
19	0.4106 182	0.5723 476	0.7104 609	19
20	0.3386 325	0.5043 959	0.6452 792	20

$E(\gamma_r) = C + \sum_{i=0}^{r} (-1)^i \binom{r}{i} \Delta^i \ell n(n-r)$, where C = Euler's Constant = 0.577 2157

r	n=40	n=45	n=50	r
21	0.2630 942	0.4[illegible]82 186	0.5820 524	21
22	0.1986 951	0.3[illegible]35 525	0.5205 370	22
23	0.1301 405	0.3[illegible]01 567	0.4605 148	23
24	+0.0621 408	0.2[illegible]78 069	0.4017 883	24
25	-0.0055 967	0.1[illegible]62 904	0.3441 759	25

r	n=55	n=60	n=70	r
0	4.5845 489	4.6[illegible]15 602	4.8257 109	0
1	3.5753 462	3.6[illegible]31 331	3.8184 993	1
2	3.0659 996	3.1[illegible]45 855	3.3111 996	2
3	2.7231 708	2.8[illegible]25 802	2.9704 763	3
4	2.4635 211	2.5[illegible]37 800	2.7129 954	4
5	2.2537 117	2.3[illegible]48 473	2.5054 151	5
6	2.0770 701	2.1[illegible]91 110	2.3310 720	6
7	1.9240 664	2.0[illegible]70 428	2.1804 376	7
8	1.7887 415	1.8[illegible]26 854	2.0475 [illegible]	8
9	1.6671 204	1.7[illegible]20 653	1.9284 576	9
10	1.5564 180	1.6[illegible]23 995	1.8203 598	10
11	1.4546 060	1.5[illegible]16 619	1.7212 594	11
12	1.3601 612	1.4[illegible]83 314	1.6295 780	12
13	1.2719 091	1.3[illegible]12 360	1.5442 062	13
14	1.1889 245	1.2[illegible]94 532	1.4642 047	14
15	1.1104 647	1.2[illegible]22 431	1.3888 367	15
16	1.0359 236	1.1[illegible]90 029	1.3175 026	16
17	0.9647 993	1.0[illegible]92 342	1.2497 078	17
18	0.8966 708	1.0[illegible]25 197	1.1850 392	18
19	0.8311 807	0.9[illegible]85 062	1.1231 476	19
20	0.7680 221	0.8[illegible]68 913	1.0637 356	20
21	0.7069 285	0.8[illegible]74 137	1.0065 469	21
22	0.6476 664	0.7[illegible]98 456	0.9513 592	22
23	0.5900 293	0.7[illegible]39 865	0.8979 731	23
24	0.5338 321	0.6[illegible]96 584	0.8462 324	24
25	0.4789 079	0.5[illegible]67 022	0.7959 700	25

r	n=80	n=90	n=100	r
0	4.9592 423	5.0[illegible]70 253	5.1825 859	0
1	3.9529 397	4.0[illegible]14 283	4.1773 523	1
2	3.4465 700	3.5[illegible]57 784	3.6722 760	2
3	3.1067 984	3.2[illegible]67 412	3.3338 229	3
4	2.8502 897	2.9[illegible]09 823	3.0786 589	4
5	2.6437 091	2.7[illegible]51 672	2.8734 497	5
6	2.4703 881	2.5926 280	2.7015 279	6
7	2.3208 011	2.[illegible]438 397	2.5533 687	7
8	2.1889 938	2.[illegible]128 486	2.4230 189	8
9	2.0709 961	2.[illegible]956 850	2.3065 092	9
10	1.9640 281	2.0[illegible]95 700	2.2010 610	10

$R(g_r) = C + \sum_{t=0} (-1)^t \binom{r}{t} \Delta^t \ln(n-t)$, where C = Euler's Constant = 0.5772157

r	n=80	n=90	n=100	r
11	1.8660 674	1.9924 818	2.1046 529	11
12	1.7755 969	1.9029 038	2.0157 689	12
13	1.6914 486	1.8196 690	1.9332 423	13
14	1.6127 044	1.7418 600	1.8561 563	14
15	1.5386 292	1.6687 427	1.7837 772	15
16	1.4686 252	1.5997 200	1.7155 086	16
17	1.4021 995	1.5343 001	1.6508 589	17
18	1.3389 408	1.4720 725	1.5894 186	18
19	1.2785 021	1.4126 915	1.5308 422	19
20	1.2205 878	1.3558 625	1.4748 362	20
21	1.1649 442	1.3013 329	1.4211 483	21
22	1.1113 514	1.2488 842	1.3695 608	22
23	1.0596 175	1.1985 258	1.3198 839	23
24	1.0095 742	1.1494 908	1.2719 514	24
25	0.9610 726	1.1022 318	1.2256 168	25

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