

NATIONAL BUREAU OF STANDARDS REPORT

1162

BIBLIOGRAPHY
ON
BOUNDS FOR CHARACTERISTIC ROOTS
OF FINITE MATRICES

September 1951

Olga Taussky



U. S. DEPARTMENT OF COMMERCE
NATIONAL BUREAU OF STANDARDS



THE NATIONAL BUREAU OF STANDARDS

The scope of activities of the National Bureau of Standards is suggested in the following listing of the divisions and sections engaged in technical work. In general, each section is engaged in specialized research, development, and engineering in the field indicated by its title. A brief description of the activities, and of the resultant reports and publications, appears on the inside of the back cover of this report.

1. **ELECTRICITY.** Resistance Measurements. Inductance and Capacitance. Electrical Instruments. Magnetic Measurements. Electrochemistry.
2. **OPTICS AND METROLOGY.** Photometry and Colorimetry. Optical Instruments. Photographic Technology. Length. Gage.
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5. **CHEMISTRY.** Organic Coatings. Surface Chemistry. Organic Chemistry. Analytical Chemistry. Inorganic Chemistry. Electrodeposition. Gas Chemistry. Physical Chemistry. Thermochemistry. Spectrochemistry. Pure Substances.
6. **MECHANICS.** Sound. Mechanical Instruments. Aerodynamics. Engineering Mechanics. Hydraulics. Mass. Capacity, Density, and Fluid Meters.
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8. **METALLURGY.** Thermal Metallurgy. Chemical Metallurgy. Mechanical Metallurgy. Corrosion.
9. **MINERAL PRODUCTS.** Porcelain and Pottery. Glass. Refractories. Enameled Metals. Building Stone. Concreting Materials. Constitution and Microstructure. Chemistry of Mineral Products.
10. **BUILDING TECHNOLOGY.** Structural Engineering. Fire Protection. Heating and Air Conditioning. Exterior and Interior Coverings. Codes and Specifications.
11. **APPLIED MATHEMATICS.** Numerical Analysis. Computation. Statistical Engineering. Machine Development.
12. **ELECTRONICS.** Engineering Electronics. Electron Tubes. Electronic Computers.
13. **ORDNANCE DEVELOPMENT.** Mechanical Research and Development. Electromechanical Fuzes. Technical Services. Missile Fuzing Research. Missile Fuzing Development. Projectile Fuzes. Ordnance Components. Ordnance Tests. Ordnance Research.
14. **RADIO PROPAGATION.** Upper Atmosphere Research. Ionospheric Research. Regular Propagation Services. Frequency Utilization Research. Tropospheric Propagation Research. High Frequency Standards. Microwave Standards.
15. **MISSILE DEVELOPMENT.** Missile Engineering. Missile Dynamics. Missile Intelligence. Missile Instrumentation. Technical Services.

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ington 25, D. C.

This is a revised edition of part of an earlier bibliography prepared in 1949 by A. Ostrowski, O. Taussky Todd, J. Todd which included sections on Inversion of Matrices and Computation of characteristic Roots. These have not been revised in view of the fact that more complete bibliographies are in preparation. In particular we refer to the contribution by G. E. Forsythe to the Proceedings of the Symposium on Simultaneous Linear Equations and the Determination of Eigenvalues. (National Bureau of Standards Semi-centennial 1951)

Bounds for characteristic roots of matrices are of importance from various aspects. For instance in some methods of computing the characteristic roots as approximate value is required: a reasonable value may be found if the bounds are known. On the other hand, it is frequently necessary, e.g. in stability problems, to know whether the roots of a matrix are inside the unit circle or whether they have positive or negative real or imaginary parts. A number of results in this connection has been obtained. Some of these results give bounds for the absolute values of the roots, others deal with regions in the complex plane inside which the roots must lie. In some cases the regions or bounds are invariant under unitary transformations. An important classical result which comes under the subject "Bounds for characteristic roots of matrices" is Frobenius theorem: a matrix with positive elements has a real and positive number as dominant characteristic root.

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Addenda:

- A. Brauer, Bounds for characteristic roots of matrices, to appear in Proceedings of the Symposium on Simultaneous Linear Equations and the determination of Eigen values.
- A. Ostrowski, Bounds for the greatest latent root of a positive matrix, manuscript.
- H. Wielandt, Inclusion theorems for eigen values, to appear in Proceedings of the Symposium on Simultaneous Linear Equations and the Determination of Eigen values

THE NATIONAL BUREAU OF STANDARDS

Functions and Activities

The National Bureau of Standards is the principal agency of the Federal Government for fundamental and applied research in physics, mathematics, chemistry, and engineering. Its activities range from the determination of physical constants and properties of materials, the development and maintenance of the national standards of measurement in the physical sciences, and the development of methods and instruments of measurement, to the development of special devices for the military and civilian agencies of the Government. The work includes basic and applied research, development, engineering, instrumentation, testing, evaluation, calibration services, and various scientific and technical advisory services. A major portion of the NBS work is performed for other government agencies, particularly the Department of Defense and the Atomic Energy Commission. The functions of the National Bureau of Standards are set forth in the Act of Congress, March 3, 1901, as amended by Congress in Public Law 619, 1950. The scope of activities is suggested in the listing of divisions and sections on the inside of the front cover.

Reports and Publications

The results of the Bureau's work take the form of either actual equipment and devices or published papers and reports. Reports are issued to the sponsoring agency of a particular project or program. Published papers appear either in the Bureau's own series of publications or in the journals of professional and scientific societies. The Bureau itself publishes three monthly periodicals, available from the Government Printing Office: the Journal of Research, which presents complete papers reporting technical investigations; the Technical News Bulletin, which presents summary and preliminary reports on work in progress; and Basic Radio Propagation Predictions, which provides data for determining the best frequencies to use for radio communications throughout the world. There are also five series of nonperiodical publications: the Applied Mathematics Series, Circulars, Handbooks, Building Materials and Structures Reports, and Miscellaneous Publications.

Information on the Bureau's publications can be found in NBS Circular 460, Publications of the National Bureau of Standards (75 cents). Information on calibration services and fees can be found in NBS Circular 483, Testing by the National Bureau of Standards (25 cents). Both are available from the Government Printing Office. Inquiries regarding the Bureau's reports and publications should be addressed to the Office of Scientific Publications, National Bureau of Standards, Washington 25, D. C.

