NATIONAL BUREAU OF STANDARDS REPORT

10 108

ALGORITHMS FOR CALCULATING THE TRANSIENT HEAT CONDUCTION BY THERMAL RESPONSE FACTORS FOR MULTI-LAYER STRUCTURES OF VARIOUS HEAT CONDUCTION SYSTEMS

(Theories, Computer Programs, and Sample Calculations)

NBS

U.S. DEPARTMENT OF COMMERCE

NATIONAL BUREAU OF STANDARDS

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by

T. Kusuda

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PREFACE

A paper entitled "Thermal Response Factors for Multi-layer Structures of Various Heat Conduction Systems" was published in the 1969 Transactions of the American Society of Heating, Refrigerating and Air Conditioning Engineers (Vol. 75, Part 1, pp. 246-271). A computer program mentioned in that paper alled RESPTK was used to obtain thermal response factors for walls, solid objects and semi-infinite walls of planes, cylindrical and spherical systems. In order to respond to the many requests for the computer program, the Fortran listing, the input instructions of RESPTK and various modifications to that paper have been included in this report.

Contingent upon the responses given to this report a formal NBS publication may be issued in the future as a Building Science Series paper.

ABSTRACT

The thermal response factor method for calculating transient heat conduction through multi-layer slabs is generalized to include the solutions for many other important engineering heat transfer problems. Response factor formulas for multi-layer structures of cylindrical and spherical objects (hollow as well as solid), plane and curved surface walls adjacent to an infinitely thick heat conduction medium, such as the ground, and to plane slabs, are presented in this paper. Numerical evaluation of these formulas is carried out for selected multi-layer structures and results are tabulated.

Also included in this report are Fortran listings of the response factor calculation programs and sample usages of the computer programs for evaluating heat conduction through building walls.

Key Words: Thermal response factors, multi-layer structures, transient heat conduction, cylinder and sphere

Nomenclature

Unless otherwise specified, the following symbols are used throughout this paper. Since the units attached to symbols represent the English system (still most popular among heat and air conditioning engineers in the United States), a conversion table for standard metric units is also provided at the end of this section.

A, B, C, D	Elements of overall temperature flux matrix
A_ν, B_ν, C_ν, D_ν	Elements of individual layer temperature-flux matrix
F	Heat flux, Btu hr^{-1} ft^{-2}
f	Laplace transform of heat flux
h_I, h_o	Exterior wall surface heat conductance [Btu ft^{-2} F^{-1} hr^{-1}]
I	Irradiated heat flux, [Btu ft^{-2} hr^{-1} F^{-1}]
j	Complex number notation = $\sqrt{-1}$
λ_ν	Thermal conductivity of the νth layer [Btu hr^{-1} F^{-1} ft^{-1}]
l_ν	Thickness of the νth layer, [ft.]
m	Curvature index: m = 0, plane; m = 1, cylinder; m = 2, sphere
N	A large number
n	Total number of layers to be considered
p	Laplace transform parameter (p is treated as a complex variable for the inversion integral)
Q	Heat flux, [Btu hr^{-1} ft^{-2}]

q	$= \sqrt{\frac{p}{\alpha}}$
r_ν, $r_{\nu+1}$	Radii of the bounding surfaces of the νth layer, [ft]
R_ν	Thermal resistance of the νth layer, [ft^2 F hr Btu^{-1}]
R	General response function defined in the text
T	Temperature, [F]
T_o	Initial temperature at $t = 0$, [F]
T_ν, $T_{\nu+1}$	Boundary temperatures of the νth layer, [F]
V_ν, $V_{\nu+1}$	Temperature departure of the νth layer, $(T_\nu - T_o)$ and $(T_{\nu+1} - T_o)$, respectively, [F]
$\bar{V}_\nu$, $\bar{V}_{\nu+1}$	Laplace transforms of the temperature departures, V and $V_{\nu+1}$
X_i, Y_i, and Z_i	Response factors, [Btu ft^{-2} F^{-1} hr^{-1}]
X_i, Y_i, and Z_i	Modified response factors [Btu ft^{-2} F^{-1} hr^{-1}]
α_ν	Thermal diffusivity of νth layer, [ft^2 hr^{-1}]
φ	Characteristic function of pulses
δ	Time increment, [hr]
Δ_ν	Determinant of the matrix for A_ν, B_ν, C_ν, D_ν
Γ	Determinant of the matrix of A, B, C, D
t_i	Time coordinate $t_i = t - i\delta$, [hr]
τ	Time index $t = \tau\delta$
Ω	$\Omega = \beta_k \delta$
β_k	$\beta_k = -p_k$, roots for residue evaluation $k = 1, 2, 3, \ldots.$
$\Psi(\beta_k)$	Defined in the text, eq. (26)

Subscripts

ν = layer boundaries

$\hbar$ = roots for the residue evaluation

i = response factor series in relation to time series

τ = discrete time

Unit Conversion

To convert from	Multiply by	To Obtain
Q = Btu hr^{-1} ft^{-2}	3.152481E + 00	W m^{-2}
h = Btu hr^{-1} ft^{-2} $°F^{-1}$	5.6783E + 00	W m^{-2} K^{-1}
λ = Btu, hr^{-1} ft^{-1} $°F^{-1}$	0.14423E - 02	W m^{-1} K^{-1}
c = Btu lb^{-1} F^{-1}	4.187E + 03	J Kg^{-1} $°K^{-1}$
ρ = lb ft^{-3}	1.602E + 01	Kg m^{-3}
α = ft^2 hr^{-1}	2.581E - 05	m^2 s^{-1}
ℓ = ft	3.048E - 01	m

1. Introduction

A recent advance in computer application for hour by hour building heat transfer calculations has made it possible to improve the Response Factor technique in transient heat conduction analysis. This improved response factor method permits an accurate evaluation of transient (non-steady and/or aperiodic) heat conduction through multi-layer walls and roofs, which has heretofore been extremely difficult.

It should be stated that an existing procedure commonly known in the U. S. as the Mackey and Wright[1-2] solution for evaluating the building heat transfer has been based upon the assumption that the building walls and roofs experience steady periodic temperature cycles on a diurnal basis. Their solution obtained using this assumption is inadequate for the accurate evaluation of actual hour-by-hour heat gain or loss of buildings.

Another well-known approach solving the transient heat transfer is finite difference approximations to the heat conduction equation. Although the computational procedures involved in this latter technique are less complicated than analytical procedures, extremely small grid sizes are required for finite difference time and space coordinates if computational stability is to be retained in calculating transient heat flow for a multi-layer heat flow problem.

The Response Factor Method has been treated previously by several authors[3-8]. This method basically utilizes the superposition principle in such a manner that the overall thermal response of the building structure at a selected time is the sum of the responses caused by many individual temperature pulses during preceding significant times. Thus, by simulating the transient boundary temperatures by a train of pulses, and by summing up the heat flux caused by each pulse, the total heat flux at a given time can be derived. The calculation of the thermal response of multi-layer walls and roofs due to each individual temperature pulse has in the past been simulated by the concept of a finite number of lumped-resistances-and-capacitances by equating the heat flow path to an electrical circuit analog. A significant contribution has been made recently by Mitalas and Arseneault[9], who improved considerably the accuracy of the calculation by avoiding the lumped-resistance-and-capacitance concept. Mitalas and Arseneault were able to solve the differential equation of heat conduction for the multilayer system by employing a matrix equation of Laplace transforms[10]. Although the matrix equation of the composite wall has been employed effectively in the past, previous efforts[1,2,6,7,8] have used Fourier series simulation of the boundary temperature functions. The previous difficulty of applying the matrix method for a periodic heat transfer problem was primarily due to the complexity of evaluating the inversion integrals of the Laplace transforms. Using a high-speed digital computer, Mitalas and Arsenault were able numerically to invert the Laplace transform matrix for the multi-layer heat conduction equation when a periodic or transient boundary temperature function is simulated by a train of triangular pulses.

In this paper, the method employed by Mitalas and Arseneault is extended to cover walls and roofs with cylindrical and spherical curvatures. The response factors for the cylindrical and spherical walls may also be useful in analyzing the transient heat flow through pipes, underground shelters and structures, storage tanks, and tunnels. Sample calculations for a typical brick wall were performed using a computer program to carry out the mathematical procedures outlined in this paper. Results of the calculations for plane, cylindrical and spherical walls are compared with those obtained by an exact analytical method for a steady periodic boundary temperature profile.

Also presented in this paper are formulas for evaluating heat flux in semi-infinite systems, interfacial temperatures and heat fluxes of the multi-layer transient system, and the analyses for non-linear boundary heat transfer problems.

2. Heat Conduction Through a Homogeneous Layer

The heat conduction equations for one-dimensional heat flow in a homogeneous layer of a multilayer system are first analyzed. Assume that this particular layer has thermal diffusivity α_ν, thermal conductivity λ_ν, and at time t has boundary temperatures $T_\nu(t)$ at the surface $r = r_\nu$, and $T_{\nu+1}(t)$ at the surface $r = r_{\nu+1}$. Also assume that the temperature of the layer at time $t = 0$ was constant at T_o. The differential equation and boundary conditions describing the conditions stated above are then

$$\frac{\partial^2 T}{\partial r^2} + \frac{m}{r}\frac{\partial T}{\partial r} = \frac{1}{\alpha_\nu}\frac{\partial T}{\partial t} \quad \text{for } m = 0, 1, \text{ or } 2 \tag{1}$$

$T = T_\nu$ at $r = r_\nu$ for $t > 0$

$T = T_{\nu+1}$ at $r = r_{\nu+1}$

$T = T_o$ for all r at $t = 0$

Applying the Laplace transform to the above relations, it is possible to write

$$\frac{d^2\bar{V}}{dr^2} + \frac{m}{r}\frac{d\bar{V}}{dr} = q_\nu^{\,2}\bar{V} \tag{2}$$

$\bar{V} = \bar{V}_\nu$ at $r = r_\nu$

$\bar{V} = \bar{V}_{\nu+1}$ at $r = r_{\nu+1}$

where

$$\bar{V} = \int_o^\infty (T - T_o)e^{-pt}dt \tag{3}$$

$$q_\nu = \sqrt{\frac{p}{\alpha_\nu}} \tag{4}$$

and p is the Laplace transform operator.

A general solution of these Laplace transform equations for heat conduction may be written in matrix form,

$$\begin{pmatrix} \bar{V}_\nu \\ f_\nu \end{pmatrix} = \begin{pmatrix} A_\nu & B_\nu \\ C_\nu & D_\nu \end{pmatrix} \begin{pmatrix} \bar{V}_{\nu+1} \\ f_{\nu+1} \end{pmatrix} \tag{5}$$

or by rearranging,

$$\begin{pmatrix} f_\nu \\ f_{\nu+1} \end{pmatrix} = \begin{pmatrix} \frac{D_\nu}{B_\nu} & -\frac{\Gamma_\nu}{B_\nu} \\ \frac{1}{B_\nu} & -\frac{A_\nu}{B_\nu} \end{pmatrix} \begin{pmatrix} \bar{V}_\nu \\ \bar{V}_{\nu+1} \end{pmatrix} \tag{6}$$

where f_ν and $f_{\nu+1}$ are the Laplace transforms of $-\lambda_\nu \frac{dV}{dr}$, heat flux at $r = r_\nu$, and $r_{\nu+1}$, respectively. Specific expressions for each element of the matrix in (5) for the cases of m = 0, 1, and 2 are shown in Tables 1, 2 and 3.

By using the expressions in Tables 1, 2, and 3, it can be shown that the determinant of the matrix in (5) is

$$\Gamma_\nu = \begin{vmatrix} A_\nu & B_\nu \\ C_\nu & D_\nu \end{vmatrix} = \left(\frac{r_{\nu+1}}{r_\nu}\right)^m \tag{7}$$

3. Multi-layer Heat Conduction

The solutions obtained for the single layer (the νth layer) bounded by $r = r_\nu$ and $r = r_{\nu+1}$ are valid for each of the other layers of a multi-layer slab, so that one may write for each layer:

$$\text{1st layer:} \quad \begin{pmatrix} \bar{V}_1 \\ f_1 \end{pmatrix} = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \begin{pmatrix} \bar{V}_2 \\ f_2 \end{pmatrix}$$

$$\text{2nd layer:} \quad \begin{pmatrix} \bar{V}_2 \\ f_2 \end{pmatrix} = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} \begin{pmatrix} \bar{V}_3 \\ f_3 \end{pmatrix} \tag{8}$$

$$\text{(n-1)st layer:} \quad \begin{pmatrix} \bar{V}_{n-1} \\ f_{n-1} \end{pmatrix} = \begin{pmatrix} A_{n-1} & B_{n-1} \\ C_{n-1} & D_{n-1} \end{pmatrix} \begin{pmatrix} \bar{V}_n \\ f_n \end{pmatrix}$$

This is predicted upon the assumption that there is perfect thermal contact at the interface of the layers of the multi-layer slab giving continuity of temperature and heat flux. Combining the above matrix equations give

$$\begin{pmatrix} \bar{V}_1 \\ f_1 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \bar{V}_n \\ f_n \end{pmatrix} \tag{9}$$

where

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} \cdots \begin{pmatrix} A_{n-1} & B_{n-1} \\ C_{n-1} & D_{n-1} \end{pmatrix} \tag{10}$$

If the νth layer of the multi-layer slab has a negligibly small thermal mass, (e.g., is a fully enclosed air space), the matrix elements for that layer are

$$\begin{aligned} A_\nu &= 1 \\ B_\nu &= R_\nu \\ C_\nu &= 0 \\ D_\nu &= \Gamma_\nu \end{aligned} \tag{11}$$

where R_ν is the thermal resistance of the layer.

Applying matrix algebra, the determinant of the overall matrix in (10) can be shown to be

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \Gamma = \Gamma_1 \cdot \Gamma_2 \cdot \Gamma_3 \ldots \Gamma_{n-1} = \left(\frac{r_n}{r_1}\right)^m . \tag{12}$$

From (9), the Laplace transform of the heat flux matrix relation is

$$\begin{pmatrix} f_1 \\ f_n \end{pmatrix} = \begin{pmatrix} \frac{D}{B} & -\frac{\Gamma}{B} \\ \frac{1}{B} & -\frac{A}{B} \end{pmatrix} \begin{pmatrix} \bar{V}_1 \\ \bar{V}_n \end{pmatrix} \tag{13}$$

The heat flux at each surface can be evaluated by applying the inversion theorem of the Laplace transform to equation (13).

4. Superposition Principle and the Inversion of Laplace Transforms

The inversion of (13) can be approximated easily by applying the superposition principle where the slab temperature T is represented by a linear sum of functions $V_i (i = 1, 2, ...)$ such that

$$T - T_o = \sum_{i=0}^{\infty} V_i(t_i) \qquad (14)$$

Furthermore, the boundary temperature functions T_1 and T_n at $r = r_1$ and $r = r_n$ are assumed to be represented by a series of pulse functions such that

$$V_1 = T_1 - T_o = \sum_{i=0}^{\infty} V_{1,i}\, \varphi(t_i)$$

$$V_n = T_n - T_o = \sum_{i=0}^{\infty} V_{n,i}\, \varphi(t_i)$$

In the above equation, $V_{1,i}$ and $V_{n,i}$ are pulse heights at time $t = i\delta$ for the boundary surfaces, where δ is the discrete time interval of the pulses. The pulse function $\varphi(t_i)$ is defined only for $0 < t_i < m'\delta$, where m' is the width of the pulse at the time base. The simplest pulse most commonly used is the rectangular pulse of width δ, (or $m' = 1$), such as shown in Fig. 1, and it can be described by the following pulse function

$$\begin{aligned} \varphi(t_i) &= 0 & t_i &\le 0 \\ &= 1 & 0 < t_i &\le \delta \\ &= 0 & t_i &> \delta \end{aligned}$$

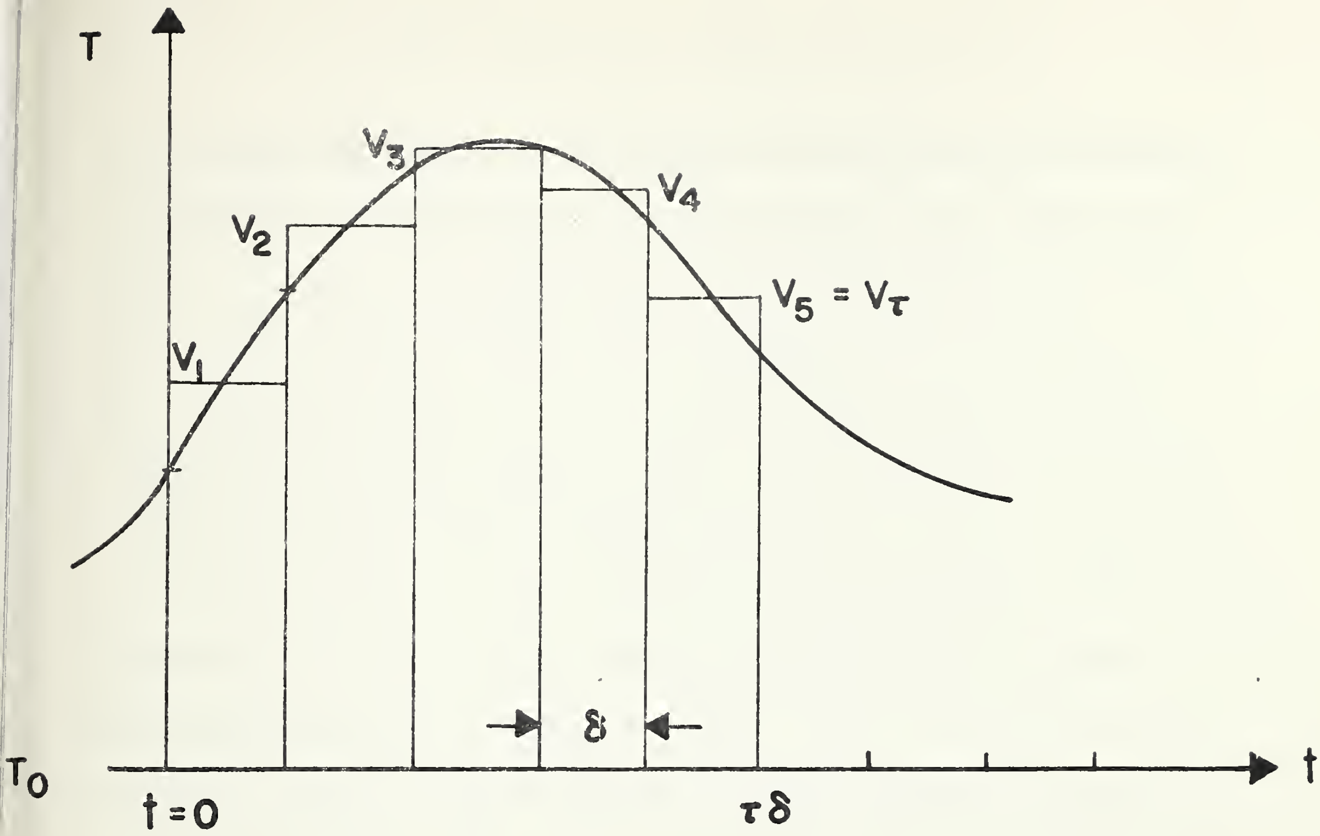

Fig. 1, Rectangular pulses

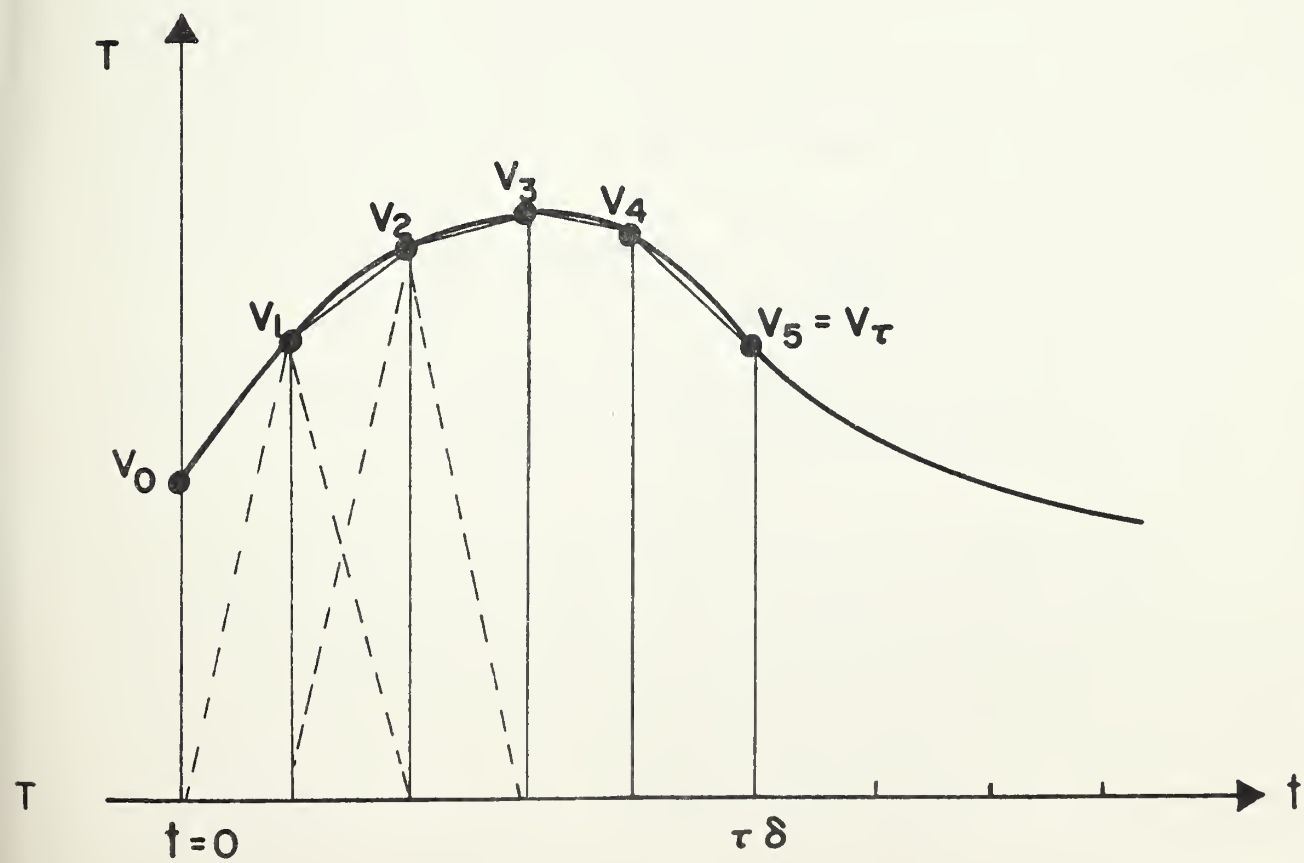

Fig. 2, Trapezoidal pulses

Although the rectangular pulse simulation of the boundary temperatures is very simple, the approximation of a complex profile by a finite number of rectangular pulses inevitably causes loss of accuracy unless the time increment δ is chosen extremely small. A considerable gain in the accuracy, however, can be restored if the boundary temperatures are simulated by trapezoidal pulses, such as shown in Fig. 2. It can be proven also that two overlapping triangular pulses (dotted line) of base width of 2δ have identical thermal response to that created by the trapezoidal pulse of width δ, which is shared by the two triangular pulses (see Fig. 2). The triangular pulse of $m' = 2$ is, however, better suited for this analysis than the trapezoidal pulse, since it represents each pulse by a single pulse instead of two. The pulse function for a triangular pulse of base 2δ is

$$\begin{aligned} \varphi(t_i) &= 0 && \text{for} \quad t_i \le 0 \\ &= t_i/\delta && \text{for } 0 < t_i \le \delta \\ &= 2 - t_i/\delta && \text{for } \delta < t_i \le 2\delta \\ &= 0 && \text{for} \quad t_i > 2\delta \end{aligned} \tag{16}$$

Substituting (14) and (16) into the original differential equation it is found that the solutions obtained for V are also valid for V_i ($i = 1, 2, \ldots. \infty$), provided that the new time coordinate t_i used for V_i is related to the original time coordinate t by

$$t_i = t - i\delta$$

The Laplace transform flux relation for $V_{1,i}$ and $V_{n,i}$, similar to equation (13) is then written as

$$\begin{pmatrix} f_{1,i} \\ f_{n,i} \end{pmatrix} = \bar{\varphi} \begin{pmatrix} \frac{D}{B} & -\frac{\Gamma}{B} \\ \frac{1}{B} & -\frac{A}{B} \end{pmatrix} \begin{pmatrix} V_{1,t-i\delta} \\ V_{n,t-i\delta} \end{pmatrix} \tag{17}$$

where $\bar{\varphi}$ is the Laplace transform of the pulse function φ, or

$$\begin{aligned} \bar{\varphi} &= \frac{1}{\delta p^2} \text{ for } 0 < t_i \le \delta \\ &= \frac{1}{\delta p^2} (1 - 2e^{-p\delta}) \text{ for } \delta < t_i \le 2\delta \\ &= \frac{1}{\delta p^2} (1 - e^{-p\delta})^2, \text{ for } 2\delta < t_i \end{aligned} \tag{18}$$

for the triangular pulse function.

The inversion of the Laplace transform can be accomplished by applying the residue theorem to the inversion integral, details of which are given elsewhere[10].

The inversion of flux equation (17) essentially involves the analysis of the following general formula

$$f_i = \bar{\varphi} \frac{R}{B} \tag{19}$$

where R represents D, Γ, 1, or A in equation (17). The inversion of (19) yields

$$F_i = \lim_{p \to 0} \frac{d}{dp} \left[\frac{p^2 \bar{\varphi} R e^{pt_i}}{B} \right] + \sum_{k=1}^{\infty} \left[\frac{\bar{\varphi} R e^{pt_i}}{\frac{dB}{dp}} \right]_{p=-\beta_k} \tag{20}$$

where β_k (k = 1, 2, ...) are the real roots of the equation

$$B(p = -\beta_k) = 0. \tag{21}$$

In order to evaluate equation (20), values of A, B, D, $\frac{dA}{dp}$, $\frac{dB}{dp}$, $\frac{dD}{dp}$ at $p = 0$ and $p = -\beta_k$, (k = 0, 1, 2, ...) are needed.

The derivatives of matrix elements A_ν, B_ν, C_ν, and D_ν are provided in tables 4, 5, and 6 to assist in the calculation of the derivatives of A, B, C, and D by the following relationship:

$$\frac{d}{dp}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \frac{dA_1}{dp} & \frac{dB_1}{dp} \\ \frac{dC_1}{dp} & \frac{dD_1}{dp} \end{pmatrix}\begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}\cdots\cdot\begin{pmatrix} A_{n-1} & B_{n-1} \\ C_{n-1} & D_{n-1} \end{pmatrix}$$

$$+ \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}\begin{pmatrix} \frac{dA_2}{dp} & \frac{dB_2}{dp} \\ \frac{dC_2}{dp} & \frac{dD_2}{dp} \end{pmatrix}\cdots\cdot\begin{pmatrix} A_{n-1} & B_{n-1} \\ C_{n-1} & D_{n-1} \end{pmatrix} \tag{22}$$

$$+ \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}\begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}\cdots\cdot\begin{pmatrix} \frac{dA_{n-1}}{dp} & \frac{dB_{n-1}}{dp} \\ \frac{dC_{n-1}}{dp} & \frac{dD_{n-1}}{dp} \end{pmatrix}$$

Applying the theory of limits, it can also be shown that

$$\lim_{p\to 0}\begin{pmatrix} A_\nu & B_\nu \\ C_\nu & D_\nu \end{pmatrix} = \begin{pmatrix} 1 & R_\nu \\ 0 & 1 \end{pmatrix} \qquad \text{for } m = 0$$

$$= \begin{pmatrix} 1 & \frac{r_{\nu+1}}{\lambda_\nu}\,\ell n\left(\frac{r_{\nu+1}}{r_\nu}\right) \\ 0 & \frac{r_{\nu+1}}{r_\nu} \end{pmatrix} \qquad \text{for } m = 1 \tag{23}$$

$$= \begin{pmatrix} 1 & R_\nu\left(\frac{r_{\nu+1}}{r_\nu}\right) \\ 0 & \left(\frac{r_{\nu+1}}{r_\nu}\right)^2 \end{pmatrix} \qquad \text{for } m = 2$$

It is extremely interesting to note that the multiplication of these successive matrices for the multi-component slab would yield

$$\lim_{p\to 0} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{U'} \\ 0 & \Gamma \end{pmatrix}$$

where U' = overall steady-state heat conductance from the surface 1 to n + 1.

The first term of the right-hand side of equation (20) can be reduced further, for the case of the triangular pulse function, to the following relation

$$\begin{aligned} \lim_{p\to 0} \frac{d}{dp}\left[\frac{p^2 \bar{\varphi} R e^{pt_i}}{B}\right] &= \frac{U'}{\delta}\left[\frac{dR}{dp} + R - \frac{R\frac{dB}{dp}}{B}\right]_{p=0}, \quad 0 < t_i \leq \delta \qquad (24) \\ &= -\frac{U'}{\delta}\left[\frac{dR}{dp} - \frac{R\frac{dB}{dp}}{B}\right]_{p=0}, \quad \delta < t_i \leq 2\delta \\ &= 0 \qquad\qquad\qquad\qquad\quad , \quad 2\delta < t_i \end{aligned}$$

Table 7 is provided for the evaluation of the limits of each of the matrix elements and their derivatives, such as R and $\frac{dR}{dp}$, as p approaches zero.

Letting $p = -\beta_{\hbar}$, then

$$q_\nu = j\sqrt{\frac{\beta_{\hbar}}{\alpha_\nu}} \qquad (25)$$

whereby the complex functions of Tables 1, 2, 3, 4, 5, and 6 and the derivatives occurring in series on the right-hand side of equation (20) can be represented as real functions as shown in Tables 8, 9, and 10.

The functions indicated in Tables 8, 9 and 10 are to be evaluated at each of the negative real roots $-\beta_{\hbar}$ ($\hbar$ = 1, 2, 3, ...) of the equation B(p) = 0, for all non-negligible terms of equation (20). The magnitude of the terms, however, decreases quite rapidly with increase in $\hbar$, particularly when t_i is large or when a particular component has a large $\frac{\ell_\nu}{\sqrt{\alpha_\nu}}$ value. For the triangular pulse function, the series of equation (20) can be evaluated by the following relations

$$\sum_{\hbar=1}^{\infty} \left[\frac{\overline{\varphi} R e^{pt_i}}{\frac{dB}{dp}}\right]_{p=-\beta_\hbar} = \sum_{\hbar=1}^{\infty} \overline{\Psi}(\beta_\hbar)\; e^{-\Omega} \qquad \text{for } t_i \le \delta$$

$$= \sum_{\hbar=1}^{\infty} \overline{\Psi}(\beta_\hbar)(1 - 2e^{\lambda})\; e^{-2\Omega} \qquad \text{for } \delta < t_i \le 2\delta \qquad (26)$$

$$= \sum_{\hbar=1}^{\infty} \overline{\Psi}(\beta_\hbar)(1 - e^{\lambda})^2\; e^{-i\Omega} \qquad \text{for } t_i = i\delta > 2\delta$$

where

$$\overline{\Psi}(\beta_\hbar) = \frac{1}{\delta\beta_\hbar^2}\left[\frac{R}{\frac{dB}{dp}}\right]_{p=-\beta_\hbar}$$

$$\Omega = \beta_\hbar \delta$$

where R may be any one of A, Γ, 1, or D of Equation (17).

By combining (24) and (26), generalized response factors X_i ($i = 0, 1, 2, \ldots \infty$) may be derived in terms of R and its derivative $\frac{dR}{dp}$ as follows:

$$X_0 = \left[\frac{R}{B}\right]_{p=0} + \left[\frac{\frac{dR}{dp}}{B\delta} - \frac{R\frac{dB}{dp}}{B^2\delta}\right]_{p=0} + \sum_{\hbar=1}^{\infty} \overline{\Psi}(\beta_\hbar)e^{-\beta_\hbar\delta} \quad (\text{for } i = 1)$$

$$X_1 = -\left[\frac{\frac{dR}{dp}}{B\delta} - \frac{R\frac{dB}{dp}}{B^2\delta}\right]_{p=0} + \sum_{\hbar=1}^{\infty} \overline{\Psi}(\beta_\hbar)(1-2e^{\beta_\hbar\delta})e^{-2\beta_\hbar\delta} \quad (\text{for } i = 2) \qquad (27)$$

$$X_i = \sum_{\hbar=1}^{\infty} \overline{\Psi}(\beta_\hbar)(1-e^{\beta_\hbar\delta})^2 \, e^{-i\beta_\hbar\delta} \qquad (\text{for } i = 3, 4, \ldots\infty)$$

Using these notations, the inversion of heat flux relation (17) may be expressed generally as

$$\begin{pmatrix} F_{1,i} \\ F_{n,i} \end{pmatrix} = \begin{pmatrix} X_i & -\Gamma Y_i \\ Y_i & -Z_i \end{pmatrix} \begin{pmatrix} V_1(t-i\delta) \\ V_n(t-i\delta) \end{pmatrix} \qquad (28)$$

where X_i, Y_i, and Z_i are response factors and correspond to X_i of equation (27), (28) and (29) when R is replaced by D, 1, and A respectively.

By denoting the time coordinate t by increments of δ, say $t = \tau\delta$, $V_1(t - i\delta)$ may be expressed simply by $V_{1,\tau-i}$. Using the subscripted temperature notation, equation (28) can be used to express the original heat conduction system as follows

$$\begin{pmatrix} F_{1,\tau} \\ F_{n,\tau} \end{pmatrix} = \sum_{i=0}^{\infty} \begin{pmatrix} X_i & -\Gamma Y_i \\ Y_i & -Z_i \end{pmatrix} \begin{pmatrix} V_{1,\tau-i} \\ V_{n,\tau-i} \end{pmatrix} \qquad (29)$$

This relation is called the convolution equation of the heat fluxes.

In equation (29), X_i, Y_i, and Z_i are called response factors. A close examination of (27), (28) and (29) reveals the following interesting facts:

1. Response factors X_i, Y_i, and Z_i tend to decrease with a common ratio $e^{-\beta_1 \delta}$ for large values of i or

$$\frac{X_{i+1}}{X_i} = \frac{Y_{i+1}}{Y_i} = \frac{Z_{i+1}}{Z_i} = e^{-\beta_1 \delta} \quad \text{if } i \geq N \tag{30}$$

   and N is a large number. For a conventional building wall, $N \approx 15$.

2. To be compatible with the steady state heat flow condition when V_1 and V_n are constant, it is necessary that

$$\left| \frac{1}{\Gamma} \sum_{i=0}^{\infty} X_i \right| = \left| \sum_{i=0}^{\infty} Y_i \right| = \left| \sum_{i=0}^{\infty} Z_i \right| = U \tag{31}$$

   where U is the overall heat transfer coefficient based upon $r = r_{n+1}$.

5. Sample Calculations

A digital computer program called RESPTK (refer to Appendix) has been developed at the National Bureau of Standards for calculating the response factors formulated in the previous sections. Sample walls with properties as shown in Fig. 3 and Table 11 were analyzed by this program for cases m = 0, 1, and 2 (for plane wall (PW), cylindrical wall (CW), and spherical wall (SW), respectively). The sample wall consists of two solid mass layers bounded by two air film layers.

Table 12 shows the residues of $\frac{D\bar{\varphi}}{B}$, $\frac{\bar{\varphi}}{B}$, and $\frac{A\bar{\varphi}}{B}$ at $p = 0$ for $0 < t \leq \delta$ and for $\delta < t \leq 2\delta$. The residue of these functions becomes zero for $t > 2\delta$.

Table 13 gives β_k, the roots of $B(p) = 0$ along the negative real axis. The response factors calculated by formula (20) for R = D, 1 and A are indicated in Table 14 as X_i, Y_i, and Z_i, respectively. Also indicated at the end of Table 14 are the common ratios from (30) attained by successive values of each of the response factors when $i \geq 14$. Each of the response factors corresponds to the value evaluated at $t = i\delta$. As seen from Table 14, the response factors for plane, cylindrical, and spherical walls are very similar to each other in this particular wall. This is due to the fact that the curved walls used in the sample calculations had an innermost radius of 5 ft. and total wall thickness of 2/3 ft., which can be very closely simulated by the plane wall heat transfer. Using the response factors the heat flux values at $r = r_1$ and $r = r_5$ were also calculated for a periodic temperature profile, results of which are shown in Table 15, 16, and 17 corresponding respectively to the plane, cylindrical and spherical walls. Although the application of the response factor calculation is not limited to the periodic heat flow problem, the periodic heat flow problem was chosen for the sample calculation because exact solutions for the periodic heat flow problem can be used to check the accuracy of the response factor method. (The response factor calculation is, in a rigorous sense, an approximate solution, where the boundary temperature profiles are approximated by a train of trapezoidal pulses). The exact solutions for the heat conduction equation under periodic boundary temperature conditions are obtained by setting in the original differential equation

$$T_1 = T_o + \sum_{i=1}^{\infty} V_{1,i}\, e^{j\omega_i t}$$

$$T_n = T_o + \sum_{i=1}^{\infty} V_{n,i}\, e^{j\omega_i t} \qquad (34)$$

$$q_{\nu,i} = \sqrt{\frac{\omega_i}{2\alpha_\nu}}\,(1+j), \quad \omega_i = \frac{i2\pi}{P} \text{ and } P = 24$$

The heat flux relations in terms of complex variables are treated in reference [10]. Another computer program called ETD 2 was developed during the course of this study to perform the complex algebra calculation for the periodic heat flow problem. The results of the exact solutions are given in Tables 15, 16, and 17.

Agreement between the exact solutions and the solutions obtained by the response factor calculations shown in Tables 15, 16, and 17 is very good. To obtain this degree of agreement, the response factors had to be calculated up to i = 72. The compilation and computation of heat flux by response factors for all three walls using a UNIVAC 1108 took 34 seconds, while for a periodic heat transfer solution by complex algebra, the time was 16 seconds. The response factor calculation involves a lengthy iterative process in searching for the roots of B(p) = 0.

The plane wall response factors treated in this paper have also been calculated by D. G. Stephenson of the National Research Council of Canada[11/]. His results agree very well with those obtained in this paper.

6. Heating and Cooling of Plates, Cylinders and Spheres

For the calculation of heating and cooling loads for buildings, it may become necessary to determine the heat storage effect of interior furnishings, partitions, floors, ceilings, etc. This heat exchange problem may also be treated by the response factor method if these materials can be represented by simple geometric shapes such as solid plates, cylinders or spheres.

The boundary conditions for these cases are $\frac{\partial T}{\partial r} = 0$ at $r = 0$ for all $t > 0$ for the cylindrical and spherical cases ($m \neq 0$). For the case of cylindrical and spherical objects, the Laplace transform heat flux relation at the outside surface of the innermost core $r = r_1$ is expressed in terms of that core's thermal properties λ_1 and α_1

$$f_1 = G'\overline{V}_1 \tag{33}$$

where

$$G' = -\lambda_1 q_1 \left[\frac{I_1(q_1 r_1)}{I_0(q_1 r_1)}\right] \text{ for } m = 1 \tag{34}$$

$$= \lambda_1 q_1 \left[\frac{1}{q_1 r_1} - \frac{\cosh(q_1 r_1)}{\sinh(q_1 r_1)}\right] \text{ for } m = 2 \tag{35}$$

Combining this relationship with the rest of the outer multi-layer system as before

$$\begin{pmatrix} f_1 \\ f_n \end{pmatrix} = \begin{pmatrix} \frac{D}{B} & -\frac{\Gamma}{B} \\ \frac{1}{B} & -\frac{A}{B} \end{pmatrix} \begin{pmatrix} \overline{V}_1 \\ \overline{V}_n \end{pmatrix} \tag{36}$$

and noting that

$$f_1 = \frac{D}{B}\overline{V}_1 - \frac{\Gamma}{B}\overline{V}_n = G'\overline{V}_1 , \tag{37}$$

$$\overline{V}_1 = \left(\frac{\frac{\Gamma}{B}\overline{V}_n}{\frac{D}{B} - G'} \right), \tag{38}$$

then
$$f_n = \frac{\overline{V}_1}{B} - \frac{A}{B}\overline{V}_n = \frac{AG' - C}{D - BG'}\overline{V}_n \tag{39}$$

The inversion of this heat flux relation is readily obtained by the residue theorem similar to equation (20). Table 18 shows specific expressions of G' dG'/dp for $m = 1$ and 2 (or for the cylinder and sphere.

For a plane shaped object heated or cooled at both surfaces in a space, the response factor representations of heat exchange between the objects and the air in space at time $t = \tau\delta$ is

$$q_\tau = \sum_{i=0}^{\infty} (X_i + Z_i - 2Y_i)\, T_{\tau-i} \tag{40}$$

where X_i, Y_i, Z_i ($i = 0, 1, 2...$) are response factors of the slab as defined in (30) including surface heat transfer coefficients, and $T_{\tau-i}$ represents space air temperature at time $(\tau - i)\delta$.

7. Semi-infinite System

In many cases the transient heat conduction characteristics of semi-infinite and composite systems are needed. Problems of heat conduction to the paved earth surface, to the basement floor and to underground pipes are good examples for the semi-infinite system.

Assume the nth layer of the previous system (equations 8, 9, and 10) to be infinitely thick, its thermal conductivity and diffusivity been λ_n and α_n. For the infinitely thick nth layer, boundary conditions can be written as follows

$$\begin{aligned} T &= T_n(t) \text{ at } r = r_n \\ T &= T_0 \text{ for all } t \text{ at } r_{n+1} \to \infty \end{aligned} \tag{41}$$

The general solution in the Laplace transform domain is

$$f_n = G\overline{V}_n \tag{42}$$

where

$$\begin{aligned} G &= \lambda_n q_n \text{ for } m = 0 \\ &= \lambda_n q_n \left[\frac{K_1(q_n r_n)}{K_0(q_n r_n)}\right], \text{for } m = 1 \\ &= \lambda_n q_n \left[1 + \frac{1}{q_n r_n}\right], \text{for } m = 2 \\ q_n &= \sqrt{\frac{p}{\alpha_n}} \end{aligned} \tag{43}$$

Combining relation (37) with (9), the Laplace transform of the heat flux equation at $r = r_1$ (or at the surface) can be written as

$$f_1 = \left(\frac{C + DG}{A + BG}\right) \overline{V}_1 \tag{44}$$

The inversion of (44) cannot be obtained by the residue theorem as in the previous cases because of a branch point[10] at p = 0. The branch point integration described in reference [10] may be performed to yield the following response factor relations for the surface heat flux over a multi-layer semi-infinite system.

$$F_\tau = \sum_{i=0}^{\infty} \bar{Z}_i \, T_{1,\tau-i}$$

$$\begin{aligned}
\text{where } \bar{Z}_0 &= \phi_1 \\
\bar{Z}_1 &= \phi_2 - 2\phi_1 \\
\bar{Z}_i &= \phi_i - 2\phi_{i-1} + \phi_{i-2} \quad \text{for } i \geq 3 \\
\phi_i &= \int_0^{i\delta} \left(\frac{-1}{2\pi i} \int_{c_1 \text{ and } c_2} \left(\frac{C + DG}{A + BG} \right) \frac{e^{pt}}{p\delta} \, dp \right) dt
\end{aligned} \qquad (45)$$

and c_1 and c_2 are paths of the line integral where p is defined by $p = re^{i\pi}$ and $p = re^{-i\pi}$ respectively for r from zero to infinity. Although the method described above is for a rigorous and generalized evaluation of response factors, an approximate solution to the problem may be obtained as described below.

For an approximate method, the response factor calculations will be performed for the surface layer region (which may or may not be of a multi-layer system) and for the semi-infinite region separately. The response factors for the former are denoted herein by X_i, Y_i and Z_i (i = 0, 1, 2, ...) and for the latter by Z_i' (i = 0, 1, ...).

The surface temperature of the entire region and the interfacial temperature (at $r = r_n$) between the surface layer and the semi-infinite regions at time τ are denoted by $T_{1,\tau}$ and $T_{n,\tau}$ respectively. At the interface, the following heat transfer relations can be established:

$$F_{n,\tau} = \sum_{i=0}^{\infty} Z_i' \, T_{n,\tau-i}$$
$$= \sum_{i=0}^{\infty} Y_i T_{1,\tau-i} - \sum_{i=0}^{\infty} Z_i T_{n,\tau-i} \qquad (46)$$

Eliminating $T_{n,\tau-i}$ from the above equation, and applying it to the heat flux at the surface ($r = r_1$),

$$F_{1,\tau} = \sum_{i=0}^{\infty} \bar{Z}_i \, T_{1,\tau-i}$$

where $\bar{Z}_i$ ($i = 0,1,\ldots\infty$) are the response factors for the overall system (multi-layer surface region plus the semi-infinite region), expressed as

$$\bar{Z}_i = X_i - \frac{Y_i^2}{Z_i + Z_i'} \qquad (47)$$

Table 19 shows mathematical formulas for obtaining Z_i' for m = 0, 1, and 2. Also shown in the Appendix are system schematics and response factors of various heat conduction systems treated in this paper.

8. Interfacial Temperatures

The response factor method can also be used to calculate the temperatures at the interfaces of a multi-layer wall system when surface temperatures at the outer surfaces are prescribed.

For the multi-layer system of Fig. 3, assume that the interfacial temperature at $r = r_3$ is to be calculated. The Laplace transforms of the matrix relations for the flux and temperature for the two subsystems, one for the layers 1 and 2, and the other for the layers 3 and 4, are written as follows:

$$\begin{pmatrix} \bar{V}_3 \\ f_3 \end{pmatrix} = \begin{pmatrix} A_3 & B_3 \\ C_3 & D_3 \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{h_o} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{V}_o \\ f_o \end{pmatrix}$$

$$\begin{pmatrix} \bar{V}_I \\ f_I \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{h_I} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} \begin{pmatrix} \bar{V}_3 \\ f_3 \end{pmatrix}$$

By denoting

$$\begin{pmatrix} A^{(1)} & B^{(1)} \\ C^{(1)} & D^{(1)} \end{pmatrix} = \begin{pmatrix} A_3 & B_3 \\ C_3 & D_3 \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{h_o} \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} A(2) & B(2) \\ C(2) & D(2) \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{h_I} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix},$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A(2) & B(2) \\ C(2) & D(2) \end{pmatrix} \begin{pmatrix} A(1) & B(1) \\ C(1) & D(1) \end{pmatrix}$$

and by knowing that

$$\begin{pmatrix} f_I \\ f_o \end{pmatrix} = \begin{pmatrix} \frac{D}{B} & -\frac{1}{B} \\ \frac{1}{B} & -\frac{A}{B} \end{pmatrix} \begin{pmatrix} \bar{V}_I \\ \bar{V}_o \end{pmatrix}$$

and

$$\begin{pmatrix} \bar{V}_3 \\ f_3 \end{pmatrix} = \begin{pmatrix} A^{(1)} & B^{(1)} \\ C^{(1)} & D^{(1)} \end{pmatrix} \begin{pmatrix} \bar{V}_o \\ f_o \end{pmatrix}$$

it can be shown that

$$\bar{V}_3 = \frac{B^{(1)}}{B} \bar{V}_I + \left\{\frac{B^{(2)} A}{B}\right\} \bar{V}_o. \tag{48}$$

The inverse transform can be carried out to yield the following relation, if the triangular pulse simulations are used for V_i and V_o.

$$V_{3,\tau} = \sum_{i=0}^{\infty} (a_i V_{I,\tau-i} + b_i V_{o,\tau-i}) \tag{49}$$

where, for example, b_i can be evaluated by

$$b_i = \lim_{p\to 0} \frac{d}{dp}\left[\frac{B^{(2)}}{B\delta} \bar{\varphi} e^{ip\delta}\right] + \sum_{\hbar=1}^{\infty}\left[\frac{B^{(2)} \bar{\varphi} e^{ip\delta}}{p^2 \delta \frac{dB}{dp}}\right]_{p=p_\hbar} \tag{50}$$

when $p_\hbar$ is the $\hbar$th negative real root of $B(p) = 0$.

9. Application of Response Factors Calculation to Non-Linear Boundary Problems

In many heat conduction problems, non-linear heat transfer relations occur at boundary surfaces. Two such cases of major importance are treated in this section as illustrative examples of the response factors technique.

Case 1: Stefan-Boltzmann type radiation heat exchange at one of the surfaces:

This situation is typical of the radiation heat exchange of space craft. Assume that the surfaces receive the solar radiation I, and become heated and in turn emit long wave-length radiation, which is proportional to the fourth power of the absolute temperature. The boundary heat transfer is then

$$-\lambda\left(\frac{\partial T}{\partial r}\right)_{\text{surface}} = I(t) - \sigma\epsilon T^4 \tag{51}$$

where σ = Stefan-Boltzmann constant

ϵ = surface emittance

If the inside surface of a wall of finite thickness is kept at a constant temperature T_o, which is at the initial temperature when $t = 0$, the heat transfer relation at a time $t = \tau\delta$ is

$$Q_\tau = \sum_{i=0}^{\infty} (T_{\tau-i} - T_o)X_i = I_\tau - \sigma\epsilon T_\tau{}^4 \tag{52}$$

T_τ must be found from the following relation by iteration

$$T_\tau \; X_o + \sigma\epsilon T_\tau{}^4 = I_\tau - \sum_{i=1}^{\infty} (T_{\tau-i} - T_o) \; X_i \tag{53}$$

Substituting the solution T_τ back into (39), the heat flux at time $t = \tau\delta$ can be obtained.

Case 2: Simultaneous heat and mass transfer boundary:

Transient heat transfer through a multilayer solid wall when one surface is wet and experiencing either evaporative or condensing heat and mass transfer is treated in this section. The surface boundary condition is:

$$-\lambda\left(\frac{\partial T}{\partial r}\right)_{\text{surface}} = I + h_c(T_a - T) + K_D L(W_a - W) \tag{54}$$

where

I = Irradiated heat flux

h_c = Convective heat transfer coefficient

K_D = Mass transfer coefficient

T_a = Ambient air temperature

W_a = Ambient air humidity ratio

W = Humidity ratio of saturated air at the surface temperature, T

L = Latent heat of evaporation or condensation

Assuming again that the temperature of the other surface of the wall is kept constant at the initial system temperature at t = o, the response factor relation for the iterative procedure for finding T_T will be written as

$$X_o(T_T-T_o) - h_c(T_a-T_T) - K_D L(W_a-W_T) = I_T - \sum_{i=0}^{\infty} X_i(T_{T-i}-T_o) \tag{55}$$

As can be seen in this example, the irradiation, heat and mass transfer coefficients, air temperature and air humidity ratio can also be treated as time variables.

The value of T_τ must be found first from (55) by iteration and put back into the heat flux equation to calculate the heat transfer. Many other complex heat transfer problems can be solved in the same manner.

10. Calculation of Space Temperature

The response factors developed in this paper are not only useful for evaluating the conduction heat transfer, but also are applicable to the calculation of a space temperature.

Consider a simple room surrounded by a wall whose response factors are X_j, Y_j and Z_j ($j = 0, 1, \ldots$).

Also assume that the room air temperature at time t is established as a result of sensible heat balance among the following components:

q_G = heat generated at time τ

q_v = cooling capacity of supply air

$= 1.08 \cdot (\text{CFM}) \cdot (T_\tau - T_{s,\tau})$

where (CFM) = supply air flow rate in cu. ft per min.

T_τ = room air temperature at time τ

$T_{s,\tau}$ = supply air temperature entering the room

q_a = heat capacity of room air

$$= V_a C_a \frac{dT_\tau}{d\tau} = V_a C_a (T_\tau - T_{\tau-1})/\Delta\tau$$

where V_a = room air volume

C_a = room air specific heat

$\frac{dT_\tau}{d\tau}$ = time change of room air temperature

q_w = heat gain to the room through the wall

$$= \left[-\sum_{j=o}^{\infty} T_{\tau-j} X_j + \sum_{j=o}^{\infty} T_{o,\tau-j} Y_j \right] A_w$$

when A_w = total wall area

It is assumed that the response factors were previously calculated for the heat flow in the direction from inside to outside in time increment $\Delta\tau$

The heat balance equation is

$$q_G + q_w - q_v = q_a$$

The room air temperature T_τ is readily obtained by substituting each expression of heat flow into the above equation and factoring out T_τ as follows:

$$T_\tau = \frac{\frac{V_a C_a}{\Delta\tau} T_{\tau-1} - \left[\sum_{j=1}^{\infty} T_{\tau-j} X_j - \sum_{j=o}^{\infty} T_{o,\tau-j} Y_j \right] A_w + 1.08 \text{ (CFM) } T_{s,\tau} + q_G}{-\frac{V_a C_a}{\Delta\tau} + 1.08 \text{ (CFM)} + X_o A_w}$$

The knowledge of the past history of room air and outdoor air temperature, therefore, permits the evaluation of present room air temperature with the use of response factors. Although the example cited herein is a simple one, any degree of complexity can be added, if necessary, to make a complete heat balance of more complicated system.

11. Modified Response Factors

According to equation (29), the heat flux calculation by the convolution relation requires a large number of terms for the summation before the values of terms $X_i \, V_{\tau-i}$ becomes sufficiently small to be negligible. When the heat flux values are to be calculated successively, however, it is possible to shorten the computational efforts by making use of the modified response factors.

The modified response factor concept may be explained in conjunction with the common ratio (CR) relation of equation (30) as follows. From equation (29) the heat fluxes at $r = r_1$ for two consecutive times τ-1 and t can be expressed as

$$F_{1,\tau} = \sum_{i=o}^{\infty} (X_i \, V_{1,\tau-i} - \Gamma \, Y_i \, V_{n,\tau-i}) \tag{56}$$

$$F_{1,\tau-1} = \sum_{i=o}^{\infty} (X_i \, V_{i,\tau-1-i} - \Gamma \, Y_i \, V_{n,\tau-1-i}) \tag{57}$$

By multiplying the common ratio of the response factors (denoted here as $CR = e^{-\beta_1 \delta}$) to the both sides of equation (57) and subtracting it from equation (56),

$$F_{1,\tau} - CR \cdot F_{1,\tau-1} = (X_o V_{1,\tau} - \Gamma Y_o V_{n,\tau})$$

$$+ \sum_{i=1}^{\infty} \left\{ (X_i - X_{i-1} \cdot CR) V_{1,\tau-i} - \Gamma (Y_i - Y_{i-1} \cdot CR) V_{n,\tau-i} \right\} \tag{58}$$

By noting from equation (30) that

$$\frac{X_i}{X_{i-1}} = \frac{Y_i}{Y_{i-1}} = CR \qquad \text{for } i \geq N + 1$$

the heat flux at $r = r_1$ for time τ may be calculated by

$$F_{1,\tau} = CR \cdot F_{1,\tau-1} + \sum_{i=o}^{N} (X_i' V_{i,\tau-i} - \Gamma Y_i' V_{n,\tau-i}) \tag{59}$$

where $X_i' = X_i - X_{i-1} \cdot CR$

$Y_i' = Y_i - Y_{i-1} \cdot CR$ (60)

for $i = 1, 2, \ldots N$

and $X_o' = X_o$

$Y_o' = Y_o$

These sets of finite numbers, X_i', and Y_i' ($i = 0, 1 \ldots N$), are called the modified response factors of the first kind.

Since the value of N is usually around 15, as indicated earlier for most of the building walls and roofs, the calculation effort for $F_{1,\tau}$ can be reduced drastically by employing the modified response factors. The similar expression can be derived for the heat flux at $r = r_{n+1}$, or for $F_{n,\tau}$.

Periodic heat flow

The response factor calculation can also be modified to shorten the periodic heat flow calculations. Under a periodic boundary condition, the temperatures and heat flux must assume the following relation

$$V_{1,\tau} = V_{1,\tau-\hbar p}$$
$$V_{n,\tau} = V_{n,\tau-\hbar p}$$
$$F_{1,\tau} = F_{1,\tau-\hbar p}$$

where p is the period of the cycle and $\hbar = 0, 1, 2, \ldots \infty$.

Assuming that p is larger than N, beyond which the response factors are evaluated by the common ratio relation (30), equation (29) can be expressed as follows:

$$\begin{pmatrix} F_{1,\tau} \\ F_{n,\tau} \end{pmatrix} = \sum_{i=o}^{p-1} \begin{pmatrix} X_i & -\Gamma Y_i' \\ Y_i & -Z_i' \end{pmatrix} \begin{pmatrix} V_{1,\tau-1} \\ V_{n,\tau-1} \end{pmatrix}$$

for $o < \tau < p-1$

whereby X_i', Y_i' and Z_i are modified response factors of the second kind according to the following relationships

$$X_i' = X_i + X_p \cdot (CR)^i/(1 - CR^p)$$

$$Y_i' = Y_i + Y_p (CR)^i/(1 - CR^p)$$

$$Z_i' = Z_i + Z_p (CR)^i/(1 - CR^p)$$

12. Conclusions

General formulae for calculating thermal response factors for multi-layer structures of plane, cylindrical and spherical construction, have been developed and these formulae are listed in this report. Several applications of these response factors are also illustrated such as

a. Interface temperature of a multi-layer construction

b. Evaluation of non-linear boundary temperature problem such as radiation and evaporation

c. Evaluation of room air temperature

The computer program called RESPTK developed to obtain the response factors based upon the formulae described in this report is found in the Appendix.

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Table 1
Matrix Elements for Plane Layer

m = 0

$$A_\nu = \cosh (q_\nu \ell_\nu)$$

$$B_\nu = R_\nu \, S(q_\nu \ell_\nu)$$

$$C_\nu = \frac{q_\nu \ell_\nu}{R_\nu} \sinh (q_\nu \ell_\nu)$$

$$D_\nu = \cosh (q_\nu \ell_\nu)$$

where $\ell_\nu = r_{\nu+1} - r_\nu$

$$R_\nu = \frac{\ell_\nu}{\lambda_\nu}$$

$$S(q_\nu \ell_\nu) = \frac{\sinh (q_\nu \ell_\nu)}{q_\nu \ell_\nu}$$

Table 2

Matrix Elements for Cylindrical Layer

$m = 1$

$$A_\nu = (q_\nu r_{\nu+1})(I_{o,1} K_{1,2} + K_{o,1} I_{1,2})$$

$$B_\nu = \left(\frac{r_{\nu+1}}{\lambda_\nu}\right)\left(-I_{o,1} K_{o,2} + K_{o,1} I_{o,2}\right)$$

$$C_\nu = \lambda_\nu q_\nu^2 r_{\nu+1}(-I_{1,1} K_{1,2} + K_{1,1} I_{1,2})$$

$$D_\nu = (q_\nu r_{\nu+1})(I_{1,1} K_{o,2} + K_{1,1} I_{o,2})$$

where $I_{o,1} = I_o(q_\nu r_\nu)$

$$I_{o,2} = I_o(q_\nu r_{\nu+1})$$

$$I_{1,1} = I_1(q_\nu r_\nu)$$

$$I_{1,2} = I_1(q_\nu r_{\nu+1})$$

$$K_{o,1} = K_o(q_\nu r_\nu)$$

$$K_{o,2} = K_o(q_\nu r_{\nu+1})$$

$$K_{1,1} = K_1(q_\nu r_\nu)$$

$$K_{1,2} = K_1(q_\nu r_{\nu+1})$$

These are the modified Bessel Functions.

Table 3
Matrix Elements for Spherical Layer

m = 2

$$A_\nu = \left(\frac{r_{\nu+1}}{r_\nu}\right)\left(\cosh(q_\nu \ell_\nu) - \frac{\ell_\nu}{r_{\nu+1}} \; S(q_\nu \ell_\nu)\right)$$

$$B_\nu = R_\nu \left(\frac{r_{\nu+1}}{r_\nu}\right) S(q_\nu \ell_\nu)$$

$$C_\nu = \frac{1}{R_\nu}\left(\frac{\ell_\nu}{r_\nu}\right)^2 \left[(q_\nu^2 r_\nu r_{\nu+1} - 1) \; S(q_\nu \ell_\nu) + \cosh(q_\nu \ell_\nu)\right]$$

$$D_\nu = \left(\frac{r_{\nu+1}}{r_\nu}\right)\left(\cosh(q_\nu \ell_\nu) + \left(\frac{\ell_\nu}{r_\nu}\right) S(q_\nu \ell_\nu)\right)$$

where $\ell_\nu = r_{\nu+1} - r_\nu$

$$R_\nu = \frac{\ell_\nu}{\lambda_\nu}$$

$$S(q_\nu \ell_\nu) = \frac{\sinh(q_\nu \ell_\nu)}{q_\nu \ell_\nu}$$

Table 4

Derivatives of Matrix Elements for Plane Layer

m = 0

$$\frac{dA_\nu}{dp} = \left(\frac{\ell_\nu^2}{2\alpha_\nu}\right) S_1 (q_\nu \ell_\nu)$$

$$\frac{dB_\nu}{dp} = \left(\frac{\ell_\nu^2}{2\alpha_\nu}\right) R_\nu S_2 (q_\nu \ell_\nu)$$

$$\frac{dC_\nu}{dp} = \left(\frac{\ell_\nu^2}{2\alpha_\nu}\right) \frac{1}{R_\nu} \left[S_1 (q_\nu \ell \nu) + \cosh(q_\nu \ell_\nu) \right]$$

$$\frac{dD_\nu}{dp} = \left(\frac{\ell_\nu^2}{2\alpha_\nu}\right) S_1 (q \ell_\nu)$$

where $S_1 (q_\nu \ell_\nu) = \dfrac{\sinh(q_\nu \ell_\nu)}{q_\nu \ell_\nu}$

$$S_2 (q_\nu \ell_\nu) = \frac{\cosh(q_\nu \ell_\nu) - S_1 (q_\nu \ell_\nu)}{(q_\nu \ell_\nu)^2}$$

Table 5

Derivatives of Matrix Elements for Cylindrical Layer

m = 1

$$\frac{dA_\nu}{dp} = \left(\frac{r_\nu r_{\nu+1}}{2\alpha_\nu}\right)\left(I_{1,1}K_{1,2} - K_{1,1}I_{1,2}\right) - \left(\frac{r_\nu^2 + 1}{2\alpha_\nu}\right)\left(I_{o,1}K_{o,2} - K_{o,1}I_{o,2}\right)$$

$$\frac{dB_\nu}{dp} = -\left(\frac{r_\nu r_{\nu+1}}{2\alpha_\nu}\right)\left(\frac{I_{1,1}K_{o,2}+K_{1,1}I_{o,2}}{q_\nu \lambda_\nu}\right) + \left(\frac{r_\nu^2+1}{2\alpha_\nu}\right)\left(\frac{I_{o,1}K_{1,2} + K_{o,1}I_{1,2}}{q_\nu \lambda_\nu}\right)$$

$$\frac{dC_\nu}{dp} = -\left(\frac{r_\nu r_{\nu+1}}{2\alpha_\nu}\right)\left(q_\nu \lambda_\nu\right)\left(I_{o,1}K_{1,2} + K_{o,1}I_{1,2}\right)$$

$$+ \left(\frac{r_\nu^2+1}{2\alpha_\nu}\right)\left(q_\nu \lambda_\nu\right)\left(I_{1,1}K_{o,2} + K_{1,1}I_{o,2}\right)$$

$$\frac{dD_\nu}{dp} = \left(\frac{r_\nu r_{\nu+1}}{2\alpha_\nu}\right)\left(I_{o,1}K_{o,2} - K_{o,1}I_{o,2}\right) - \left(\frac{r_\nu^2+1}{2\alpha_\nu}\right)\left(I_{1,1}K_{1,2} - K_{1,1}I_{1,2}\right)$$

where $I_{o,1}$, $I_{o,2} \ldots K_{1,2}$ are all defined previously in Table 2.

Table 6
Derivatives of Matrix Elements for Spherical Layer

m = 2

$$\frac{dA_\nu}{dp} = \left(\frac{\ell_\nu^2}{2\alpha_\nu}\right)\left(\frac{r_{\nu+1}}{r_\nu} S_1(q_\nu \ell_\nu) - \left(\frac{\ell_\nu}{r_\nu}\right) S_2(q_\nu \ell_\nu)\right)$$

$$\frac{dB_\nu}{dp} = \left(\frac{\ell_\nu^2}{2\alpha_\nu}\right) R_\nu \left(\frac{r_{\nu+1}}{r_\nu}\right) S_2 \; (q_\nu \ell_\nu)$$

$$\frac{dC_\nu}{dp} = \left(\frac{\ell_\nu^2}{2\alpha_\nu}\right) \frac{1}{R_\nu} \left(\frac{\ell_\nu^2}{r_\nu^2}\right)\left[\left(2\frac{r_\nu r_{\nu+1}}{\ell_\nu^2} + 1\right) S_1(q_\nu^2 r_\nu r_{\nu+1} - 1) S_2(q_\nu \ell_\nu)\right]$$

$$\frac{dD_\nu}{dp} = \left(\frac{\ell_\nu^2}{2\alpha_\nu}\right)\left(\frac{r_{\nu+1}}{r_\nu} S_1(q_\nu \ell_\nu) + \left(\frac{\ell_\nu}{r_\nu}\right)\left(\frac{r_{\nu+1}}{r_\nu}\right) S_2(q_\nu \ell_\nu)\right)$$

where S_1 and S_2 have been defined in Table 4.

Table 7
Limits of Derivative Matrices

m = 0

$$\frac{dA_\nu}{dp} = \frac{\ell_\nu^2}{2\alpha_\nu}$$

$$\frac{dB_\nu}{dp} = \frac{\ell_\nu^2}{6\alpha_\nu} R_\nu$$

$$\frac{dC_\nu}{dp} = \frac{1}{R_\nu}\frac{\ell_\nu^2}{\alpha_\nu}$$

$$\frac{dD_\nu}{dp} = \frac{\ell_\nu^2}{2\alpha_\nu}$$

m = 1

$$\frac{dA_\nu}{dp} = \left(\frac{r_{\nu+1}^2}{2\alpha_\nu}\right)\left[\frac{1}{2}\left[\left(\frac{r_\nu}{r_{\nu+1}}\right)^2 - 1\right] + \ell_n \frac{r_{\nu+1}}{r_\nu}\right]$$

$$\frac{dB_\nu}{dp} = \left(\frac{r_{\nu+1}^2}{4\alpha_\nu}\right)\left(\frac{r_{\nu+1}}{\lambda_\nu}\right)\left[\left[1 + \left(\frac{r_\nu}{r_{\nu+1}}\right)^2\right] \ell_n\left(\frac{r_{\nu+1}}{r_\nu}\right) - \left(1 - \left(\frac{r_\nu}{r_{\nu+1}}\right)^2\right)\right]$$

$$\frac{dC_\nu}{dp} = \left(\frac{\lambda_\nu}{r_\nu}\right)\left(\frac{r_{\nu+1}^2}{2\alpha}\right)\left[1 - \left(\frac{r_\nu}{r_{\nu+1}}\right)^2\right]$$

$$\frac{dD_\nu}{dp} = \left(\frac{r_{\nu+1}^2}{2\alpha_\nu}\right)\left[\frac{1}{2}\left(\frac{r_{\nu+1}^2 - r_\nu^2}{r_\nu r_{\nu+1}}\right) - \left(\frac{r_\nu}{r_{\nu+1}}\right) \ell_n \left(\frac{r_{\nu+1}}{r_\nu}\right)\right]$$

When $(r_{\nu+1} - r_\nu)/r_\nu$ is sufficiently small, these derivatives can be approximated as follows:

$$\frac{dA_\nu}{dp} = \left(\frac{\ell_\nu^2}{2\alpha}\right)\left[1 - \frac{1}{2}\left(\frac{\ell_\nu}{r_\nu}\right)^2\right]$$

$$\frac{dB_\nu}{dp} = \left(\frac{\ell_\nu^2}{2\alpha}\right)\left[\frac{1}{3} + \frac{5}{12}\left(\frac{\ell}{r_\nu}\right) + \frac{1}{4}\left(\frac{\ell}{r_\nu}\right)^2 + \frac{1}{6}\left(\frac{\ell}{r_\nu}\right)^3\right] R_\nu$$

$$\frac{dC_\nu}{dp} = \left(\frac{\ell_\nu^3}{2\alpha}\right)\left[2 + \frac{\ell}{r_\nu}\right]\frac{1}{R_\nu}$$

$$\frac{dD_\nu}{d} = \left(\frac{\ell_\nu^2}{2\alpha}\right)\left[1 + \frac{3}{2}\left(\frac{\ell_\nu}{r_\nu}\right)\right]$$

Table 7 (con't)

m = 2

$$\frac{dA_\nu}{dp} = \left(\frac{\ell\nu^2}{6\alpha_\nu}\right)\left(\frac{2r_{\nu+1}}{r_\nu} + 1\right)$$

$$\frac{dB_\nu}{dp} = R_\nu \left(\frac{r_{\nu+1}}{r_\nu}\right)\left(\frac{\ell\nu^2}{6\alpha_\nu}\right)$$

$$\frac{dC_\nu}{dp} = \frac{1}{R_\nu}\left(\frac{\ell\nu^2}{2\alpha_\nu}\right)\left[\frac{2r_{\nu+1}}{r_\nu} + \frac{2}{3}\left(\frac{\ell\nu}{r_\nu}\right)^2\right]$$

$$\frac{dD_\nu}{dp} = \left(\frac{\ell\nu^2}{6\alpha_\nu}\right)\left(\frac{r_{\nu+1}}{r_\nu}\right)\left(\frac{r_{\nu+1}}{r_\nu} + 2\right)$$

When $(r_{\nu+1} - r_\nu)/r_\nu$ is sufficiently small, these derivatives can be approximated as follows

$$\frac{dA_\nu}{dp} = \frac{\ell\nu^2}{2\alpha_\nu}\left(1 + \frac{2}{3}\frac{\ell\nu}{r_\nu}\right)$$

$$\frac{dB_\nu}{dp} = \frac{\ell\nu^2}{6\alpha_\nu} R_\nu \left(1 + \frac{\ell_\nu}{r_\nu}\right)$$

$$\frac{dC_\nu}{dp} = \frac{1}{R_\nu}\left(\frac{\ell\nu^2}{\alpha_\nu}\right)\left[1 + \frac{\ell\nu}{r_\nu} + \frac{1}{3}\left(\frac{\ell\nu}{r_\nu}\right)^2\right]$$

$$\frac{dD_\nu}{dp} = \frac{\ell\nu^2}{2\alpha_\nu}\left(1 + \frac{1}{3}\frac{\ell\nu}{r_\nu}\right)\left(1 + \frac{\ell\nu}{r_\nu}\right)$$

Table 8

Matrix Elements for $p = -\beta_{\hbar}$ for Plane Layer

$m = 0$

$$A_{\nu} = \cos E$$

$$B_{\nu} = R_{\nu}\, S_1(E)$$

$$C_{\nu} = -\frac{E}{R_{\nu}} \sin E$$

$$D_{\nu} = \cos E$$

$$\frac{dA_{\nu}}{dp} = \gamma S_1(E)$$

$$\frac{dB_{\nu}}{dp} = \gamma R_{\nu} S_2(E)$$

$$\frac{dC_{\nu}}{dp} = \frac{\gamma}{R_{\nu}} (S_1(E) + \cos(E))$$

$$\frac{dD_{\nu}}{dp} = \gamma S_1(E)$$

where $E = \sqrt{\frac{\beta_{\hbar}}{\alpha} \ell^2}$

$$S_1(E) = \frac{\sin E}{E}$$

$$S_2(E) = \frac{S_1(E) - \cos E}{E^2}$$

$$\gamma = \frac{\ell^2 \nu}{2\alpha_{\nu}}$$

Table 9

Matrix Elements for $p = -\beta_{\hbar}$ for Cylindrical Layer

m = 1

$$A_\nu = -\frac{\pi}{2} E_2 (J_{01}Y_{12} - Y_{01}J_{12})$$

$$B_\nu = \frac{\pi}{2}\left(\frac{r_{\nu+1}}{\lambda_\nu}\right)\left(J_{01}Y_{02} - Y_{01}J_{02}\right)$$

$$C_\nu = \frac{\pi}{2}\left(\frac{\lambda_\nu}{r_{\nu+1}}\right)\left(-J_{11}Y_{12} + Y_{11}J_{12}\right)E_2^2$$

$$D_\nu = \frac{\pi}{2} E_2 (J_{11}Y_{02} - Y_{11}J_{02})$$

$$\frac{dA_\nu}{dp} = \frac{\pi}{4\alpha}\left\{-r_{\nu+1}r_\nu(J_{11}Y_{12} - Y_{11}J_{12}) + r_{\nu+1}^2(J_{01}Y_{02} - Y_{01}J_{02})\right\}$$

$$\frac{dB_\nu}{dp} = \frac{\pi}{4\alpha}\left(\frac{r_{\nu+1}}{\lambda_\nu}\right)\frac{1}{E_2}\left\{(r_{\nu+1}r_\nu)(J_{11}Y_{02}-Y_{11}J_{02}) + (r_{\nu+1})^2(J_{01}Y_{12}-Y_{01}J_{12})\right\}$$

$$\frac{dC_\nu}{dp} = \frac{\pi}{4\alpha}\left(\frac{r_{\nu+1}}{\lambda_\nu}\right)E_2\left\{(r_{\nu+1}r_\nu)(J_{01}Y_{12}-Y_{01}J_{12}) + (r_{\nu+1})^2(J_{11}Y_{02}-Y_{11}J_{02})\right\}$$

$$\frac{dD_\nu}{dp} = \frac{\pi}{4\alpha}\left\{(r_{\nu+1}r_\nu)(-J_{01}Y_{02} + Y_{01}J_{02}) - (r_{\nu+1})^2(-J_{11}Y_{12} + Y_{11}J_{12})\right\}$$

where $E_2 = \sqrt{\frac{\beta_{\hbar}}{\alpha_\nu}}\, r_{\nu+1}$, $E_1 = \sqrt{\frac{\beta_{\hbar}}{\alpha_\nu}}\, r_\nu$

$J_{01} = J_o(E_1)$, $J_{11} = J_1(E_1)$

$J_{02} = J_o(E_2)$, $J_{12} = J_1(E_2)$

$K_{01} = K_o(E_1)$, $K_{11} = K_1(E_1)$

$K_{02} = K_o(E_2)$, $K_{12} = K_1(E_2)$

Table 10

Matrix Elements for $p = -\beta_{k}$ for Spherical Layer

$m = 2$

$$A_\nu = \left(\frac{r_{\nu+1}}{r_\nu}\right)\left(\cos E - \frac{\ell_\nu}{r_{\nu+1}} S_1(E)\right)$$

$$B_\nu = R_\nu\left(\frac{r_{\nu+1}}{r_\nu}\right)S_1(E)$$

$$C_\nu = \frac{1}{R_\nu}\left(\frac{\ell_\nu}{r_\nu}\right)^2\left[\cosh E - (E_1 E_2 + 1)S_1(E)\right]$$

$$D_\nu = \left(\frac{r_{\nu+1}}{r_\nu}\right)\left(\cos E + \frac{\ell_\nu}{r_\nu} S_1(E)\right)$$

$$\frac{dA_\nu}{dp} = \gamma\left\{\frac{r_{\nu+1}}{r_\nu} S_1(E) - \frac{\ell_\nu}{r_\nu} S_2(E)\right\}$$

$$\frac{dB_\nu}{dp} = \gamma R_\nu\left(\frac{r_{\nu+1}}{r_\nu}\right)S_2(E)$$

$$\frac{dC_\nu}{dp} = \gamma\left(\frac{1}{R_\nu}\right)\left(\frac{\ell_\nu}{r_\nu}\right)^2\left(\left(\frac{r_\nu r_{\nu+1}}{\ell_\nu^2} + 1\right)S_1(E) - (E_1 E_2 + 1)S_2(E)\right)$$

$$\frac{dD_\nu}{dp} = \gamma\left(\frac{r_{\nu+1}}{r_\nu} S_1(E) + \left(\frac{\ell_\nu}{r_\nu}\right)\left(\frac{r_{\nu+1}}{r_\nu}\right)S_2(E)\right)$$

where $E = \sqrt{\frac{\beta_{k}}{\alpha_\nu}}\,\ell_\nu$

$$E_1 = \sqrt{\frac{\beta_{k}}{\alpha_\nu}}\,r_\nu, \quad E_2 = \sqrt{\frac{\beta_{k}}{\alpha_\nu}}\,r_{\nu+1}$$

$$S_1(E) = \frac{\sin E}{E}$$

$$S_2(E) = \frac{S_1(E) - \cos E}{E^2}$$

$$\gamma = \frac{\ell_\nu^2}{2\alpha_\nu}$$

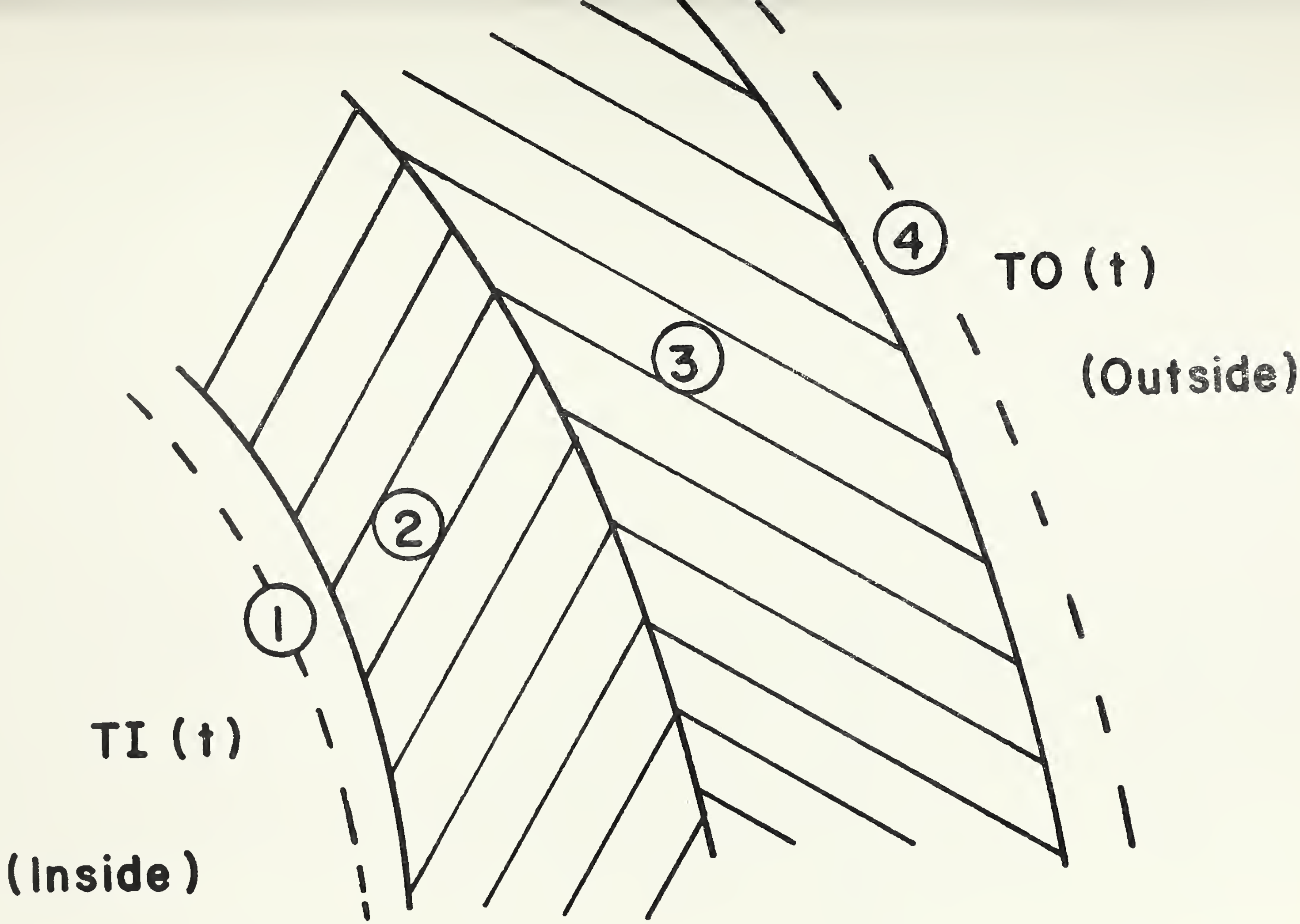

Fig. 3. Sample multi-layer walls of Table 11

Table 11

layer ν	description	ℓ_ν (ft)	λ_ν (Btu/hr.)	α_ν (ft^2/hr.)	r_ν (ft)
1	Inside air film $h_I = 1.20$	0	---	---	5.000
2	Common brick 4"	0.333	0.42	0.019	5.000
3	Face brick 4"	0.333	0.77	0.028	5.333
4	Outside air film $h_o = 3.0$	0	---	---	5.666

Table 12
Residues at p = 0 for

$0 < t \le \delta$		$D\bar{\varphi}/B$	$\bar{\varphi}/B$	$A\,\bar{\varphi}/B$
	PW	2.73117	-2.94849	7.88866
	CW	2.6297	-2.6874	7.90340
	SW	2.52953	-2.44114	7.89881
$\delta < t \le 2\delta$				
	PW	-2.31309	3.36656	-7.47058
	CW	-2.19279	3.07306	-7.51776
	SW	-2.07374	2.79607	-7.54387

PW = plane wall m = 0

CW = cylindrical wall m = 1

SW = spherical wall m = 2

Table 13

β_k: Roots of B(p) = 0

k	PW	CW	SW
1	.17452	.17701	.17980
2	.84430	.84634	.84866
3	2.56859	2.57005	2.57188
4	4.85967	4.86146	4.86360
5	8.85960	8.86093	8.86265
6	12.84988	12.85127	12.85303
7	19.15047	19.15200	19.15398
8	25.00846	25.0095	25.01083
9	33.33174	33.33359	33.33583
10	41.45064	41.45137	41.45249

PW = plane wall

CW = cylindrical wall

SW = spherical wall

Table 14
Response Factors, Btu ft^{-2}, F^{-1}, hr^{-1}

i		X_i	Y_i	Z_i
0	PW	.91949	0.00013	1.9834
	CW	.92162	0.00014	1.97607
	SW	.92365	0.00011	1.96864
1	PW	-.16678	0.00812	-.51260
	CW	-.16392	0.00759	-.52127
	SW	-.16099	0.00713	-.52993
2	PW	-0.07950	0.03112	-.23226
	CW	-0.07744	0.02916	-.23749
	SW	-0.07540	0.02726	-.24268
3	PW	-0.05150	0.04482	-.15634
	CW	-0.04987	0.04185	-.15997
	SW	-0.04826	0.03903	-.16353
4	PW	-0.03715	0.04658	-.11690
	CW	-0.03580	0.04340	-.11954
	SW	-0.03447	0.04038	-.12207
5	PW	-0.02861	0.04304	-0.09216
	CW	-0.02746	0.04000	-0.09410
	SW	-0.02632	0.03712	-0.09592
6	PW	-0.02292	0.03784	-0.07482
	CW	-0.02192	0.03508	-0.07625
	SW	-0.02094	0.03247	-0.07756
7	PW	-0.01877	0.03250	-0.66173
	CW	-0.01790	0.03006	-0.06277
	SW	-0.01704	0.02775	-0.06369
8	PW	-0.01556	0.02761	-0.05137
	CW	-0.01480	0.02548	-0.05212
	SW	-0.01404	0.02344	-0.05274
9	PW	-0.01298	0.02333	-0.04294
	CW	-0.01231	0.02147	-0.04346
	SW	-0.01165	0.01970	-0.04386
10	PW	-0.01086	0.01965	-0.03598
	CW	-0.01028	0.01804	-0.03632
	SW	-0.00970	0.01651	-0.03655

Table 14 (con't)

i		X_i	Y_i	Z_i
11	PW	-0.00911	0.01653	-0.03018
	CW	-0.00860	0.01513	-0.03039
	SW	-0.00809	0.01381	-0.03050
12	PW	-0.00764	0.01389	-0.02533
	CW	-0.00719	0.01269	-0.02544
	SW	-0.00675	0.01155	-0.02547
13	PW	-0.00642	0.01167	-0.02126
	CW	-0.00602	0.01063	-0.02131
	SW	-0.00564	0.00965	-0.02127
14	PW	-0.00539	0.00980	-0.01786
	CW	-0.00505	0.00891	-0.01785
	SW	-0.00471	0.00807	-0.01777
CR	PW	0.8398		
	CW	0.8378		
	SW	0.8358		

PW = plane wall $m = 0$

CW = cylindrical wall $m = 1$

SW = spherical wall $m = 2$

CR = common ratio where for $i \geq 15$

$$\frac{X_{i+1}}{X_i} = \frac{Y_{i+1}}{Y_i} = \frac{Z_{i+1}}{Z_i} = CR$$

Table 15
Plane Wall Model

t	TI_t	$T\phi_t$	QI_t Exact Solution	QI_t Response Factor Solution	$Q\phi_t$ Exact Solution	$Q\phi_t$ Response Factor Solution
24	75	77	-17.15	-17.15	31.04	30.58
23	75	79	-19.20	-19.18	32.92	33.77
22	75	81	-20.90	-20.83	39.47	38.71
21	75	83	-21.48	-21.33	44.79	47.35
20	75	85	-20.00	-19.91	69.48	70.30
19	75	87	-17.10	-17.10	-20.15	-24.28
18	75	138	-13.72	-13.76	-72.88	-78.01
17	75	162	-10.45	-10.53	-101.46	-103.27
16	75	168	-7.82	-7.91	-111.45	-111.14
15	75	163	-6.06	-6.13	-98.60	-99.37
14	75	148	-5.05	-5.07	-76.34	-74.90
13	75	128	-4.04	-4.36	-32.97	-32.06
12	75	104	-3.86	-3.89	-28.51	-28.87
11	75	100	-3.65	-3.68	-23.32	-23.04
10	75	95	-3.73	-3.75	-16.17	-16.18
9	75	90	-4.13	-4.15	-10.75	-10.70
8	75	86	-4.86	-4.87	-4.76	-4.73
7	75	82	-5.82	-5.83	7.48	8.04
6	75	76	-6.94	-6.95	14.46	14.34
5	75	74	-8.21	-8.22	17.11	17.42
4	75	74	-9.65	-9.66	19.08	18.77
3	75	75	-11.29	-11.30	19.94	20.27
2	75	76	-13.11	-13.12	25.02	24.84
1	75	76	-15.07	-15.08	27.90	28.39

TI: Inside temperature, (F)

Tϕ: Outside temperature, (F)

QI: Inside heat flux, (Btu hr^{-1} ft^{-2})

Qϕ: Outside heat flux, (Btu hr^{-1} ft^{-2})

$$QI_t = \sum_{j=o}^{\infty} X_j \cdot TI_{t-j} - \sum_{j=o}^{\infty} Y_j \cdot T\phi_{t-j}$$

$$Q\phi_t = \sum_{j=o}^{\infty} Y_j \cdot TI_{t-j} - \sum_{j=o}^{\infty} Z_j \cdot T\phi_{t-j}$$

Cylindrical Wall Table 16

t	TI_t	$T\phi_t$	QI_t		$Q\phi_t$	
			Exact Solution	Response Factor Solution	Exact Solution	Response Factor Solution
24	75	77	-17.94	-17.95	31.68	31.23
23	75	79	-20.13	-20.12	33.76	34.62
22	75	81	-22.00	-21.88	40.37	39.80
21	75	83	-22.60	-22.44	46.17	48.76
20	75	85	-21.06	-20.96	71.40	72.21
19	75	87	-18.01	-18.01	-17.72	-21.91
18	75	138	-14.44	-14.48	-75.48	-75.61
17	75	162	-10.99	-11.07	-99.28	-101.12
16	75	168	-8.21	-8.30	-109.69	-109.39
15	75	163	-6.33	-6.41	-97.30	-98.08
14	75	148	-5.25	-5.29	-75.50	-74.04
13	75	128	-4.50	-4.53	-32.40	-31.48
12	75	104	-3.99	-4.02	-28.06	-28.43
11	75	100	-3.76	-3.78	-23.02	-22.73
10	75	95	-3.82	-3.85	-15.97	-15.98
9	75	90	-4.24	-4.26	-10.64	-10.59
8	75	86	-5.00	-5.01	-4.73	-4.70
7	75	82	-6.00	-6.01	7.47	8.04
6	75	76	-7.18	-7.18	14.49	14.36
5	75	74	-8.50	-8.51	17.18	17.49
4	75	74	-10.02	-10.03	19.23	18.92
3	75	75	-11.75	-11.75	20.17	20.50
2	75	76	-13.67	-13.68	25.36	25.18
1	75	76	-15.75	-15.76	28.37	28.86

TI: Inside temperature, (F)

TΦ: Outside temperature, (F)

QI: Inside heat flux (Btu hr^{-1} ft^{-2})

QΦ: Outside heat flux (Btu hr^{-1} ft^{-2})

$$QI_t = \sum X_j \, TI_{t-j} - \Gamma \sum Y_j \, T\phi_{t-j}$$

$$QO_t = \sum Y_j \, TI_{t-j} - \Gamma \sum Z_j \, T\phi_{t-j}$$

$$\Gamma = \frac{5.666}{5.000}$$

Spherical Wall Table 17

Time (hr)	TI	TØ	QI Exact Solution	QI Response Factor Solution	QØ Exact Solution	QØ Response Factor Solution
24	75	77	-18.74	-18.75	32.21	31.80
23	75	79	-21.07	-21.05	34.52	35.39
22	75	81	-23.03	-22.95	41.60	40.82
21	75	83	-23.74	-23.57	47.49	50.12
20	75	85	-22.15	-22.04	73.26	74.06
19	75	87	-18.94	-18.93	-15.32	-19.57
18	75	138	-15.18	-15.22	-73.10	-73.24
17	75	162	-11.54	-11.62	-97.13	-98.98
16	75	168	-8.60	-8.69	-107.96	-107.65
15	75	163	-6.60	-6.69	-96.03	-96.81
14	75	148	-5.46	-5.50	-74.68	-73.20
13	75	128	-4.66	-4.69	-31.86	-30.92
12	75	104	-4.11	-4.13	-27.64	-28.02
11	75	100	-3.85	-3.88	-22.74	-22.45
10	75	95	-3.91	-3.93	-15.80	-15.82
9	75	90	-4.32	-4.35	-10.57	-10.52
8	75	86	-5.11	-5.12	-4.74	-4.71
7	75	82	-6.16	-6.16	7.42	7.99
6	75	76	-7.39	-7.39	14.46	14.33
5	75	74	-8.78	-8.78	17.20	17.51
4	75	74	-10.37	-10.38	19.32	19.01
3	75	75	-12.18	-12.19	20.33	20.67
2	75	76	-14.21	-14.22	25.62	25.45
1	75	76	-16.41	-16.42	28.77	29.27

TI: Inside temperature, (F)

TØ: Outside temperature, (F)

QI: Inside heat flux, (Btu hr^{-1} ft^{-2})

QØ: Outside heat flux, (Btu hr^{-1} ft^{-2})

$$QI_t = \sum_{j=o}^{\infty} X_j \cdot TI_{t-j} - \Gamma \sum_{j=o}^{\infty} Y_j \cdot T\phi_{t-j}$$

$$Q\phi_t = \sum_{j=o}^{\infty} Y_j \cdot TI_{t-j} - \sum_{j=o}^{\infty} Z_j \cdot T\phi_{t-j}$$

$$\Gamma = \left(\frac{5.666}{5.000}\right)^2$$

Table 18

Formulas for Cylinder and Sphere

m = 1.

$$G' = \left(\frac{\lambda_1}{r_1}\right) E_1 \frac{J_1(E_1)}{J_o(E_1)} \qquad \text{at } p = i\beta$$

$$\frac{dG'}{dp} = -\left(\frac{\lambda_1}{r_1}\right)\left(\frac{r_1^2}{2\alpha}\right)\left[\frac{J_1(E)}{EJ_o(E)} + \frac{J_o^2(E) - \frac{J_o(E)J_1(E)}{E}}{J_o^2(E)}\right]$$

m = 2

$$G' = \left(\frac{\lambda_1}{r_1}\right)\left(1 - \frac{\cos(E)}{S_1(E)}\right)$$

$$\frac{dG'}{dp} = -\left(\frac{\lambda_1}{r_1}\right)\left(\frac{r_1^2}{2\alpha}\right)\left[1 - \frac{\cos(E)\cdot S_2(E)}{S_1^2(E)}\right]$$

where definitions of E, $S_1(E)$ and $S_2(E)$ are identical to those used in the previous tables. As E_1 approaches to zero G' approaches zero for both m = 1 and 2, $\frac{dG'}{dp}$ becomes $-\left(\frac{r_1^2}{2\alpha}\right)\left(\frac{\lambda_1}{r_1}\right)$ for m = 1 and $-\left(\frac{r_1^2}{3\alpha}\right)\left(\frac{\lambda_1}{r_1}\right)$ for m = 2.

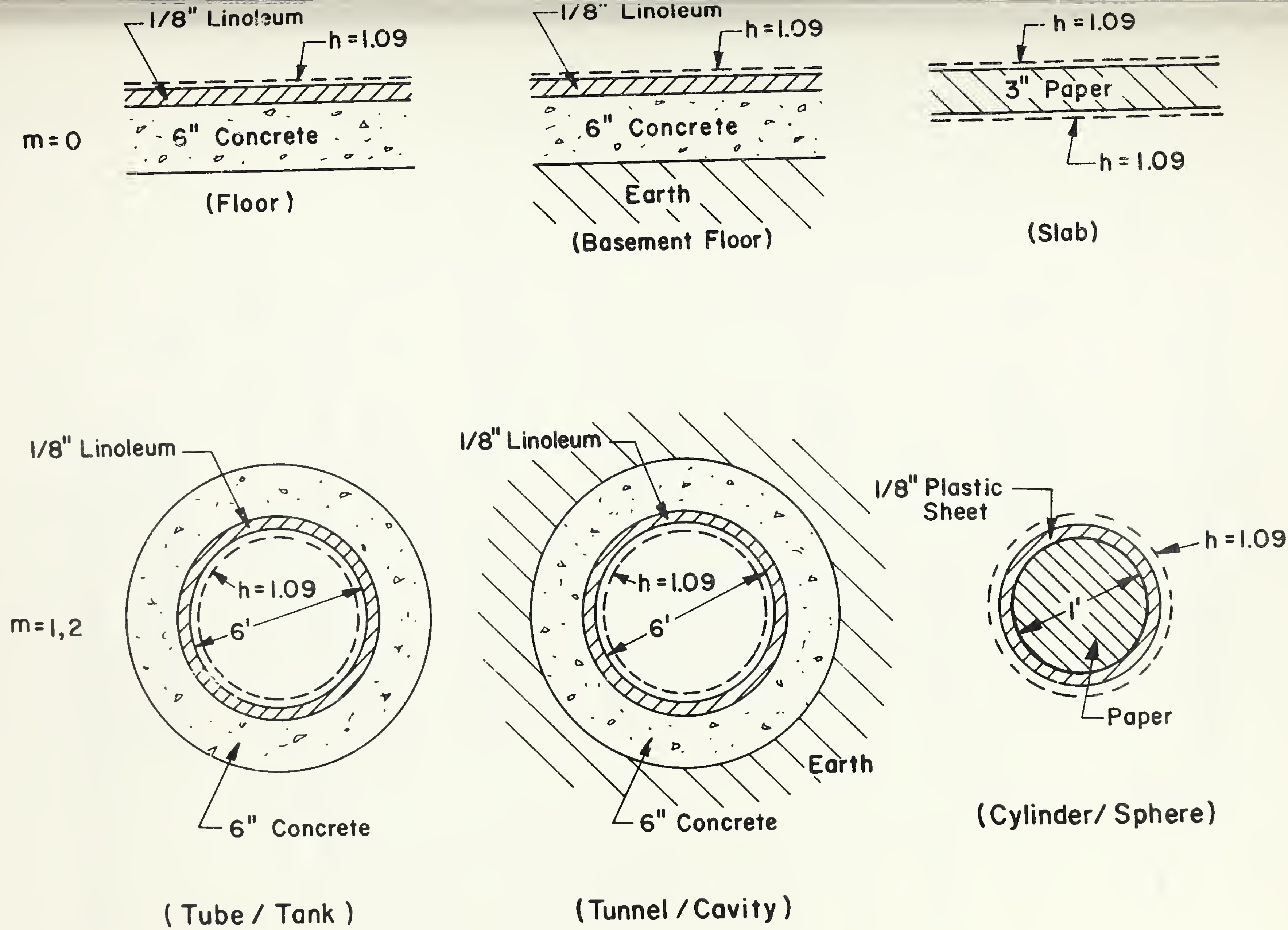

Fig.4 Sample structures For Appendix

Table 19

Response Factors for Semi-infinite Region

Common Symbols

$$L = \frac{\lambda_n}{r_n} \text{ and } \mu = \frac{r_n^2}{\alpha_n \delta}$$

$$\phi_i = \left(\frac{2}{\pi}\right)^2 \int_0^\infty \frac{1 - e^{-\beta^2 i/\mu}}{\beta^3 \{Y_o^2(\beta) + J_o^2(\beta)\}} d\beta$$

m = 0

$$\bar{Z}_1 = 2L \sqrt{\frac{\mu}{\pi}}$$

$$\bar{Z}_2 = \bar{Z}_1 (\sqrt{2} - 2)$$

$$\bar{Z}_i = \bar{Z}_1 (\sqrt{i} - 2\sqrt{i-1} + \sqrt{i-2}), \ i \geq 3$$

m = 1

$$\bar{Z}_1 = L\mu (\phi_1)$$

$$\bar{Z}_2 = L\mu (\phi_2 - 2\phi_1)$$

$$\bar{Z}_i = L\mu (\phi_i - 2\phi_{i-1} + \phi_{i-2}) \text{ for } i \geq 3$$

m = 2

$$\bar{Z}_1 = 2L \sqrt{\frac{\mu}{\pi}} \left(1 + \frac{\sqrt{\pi}}{2} \frac{1}{\sqrt{\mu}}\right)$$

$$\bar{Z}_2 = 2L \sqrt{\frac{\mu}{\pi}} (\sqrt{2} - 2)$$

$$\bar{Z}_i = 2L \sqrt{\frac{\mu}{\pi}} (\sqrt{i} - 2\sqrt{i-1} + \sqrt{i-2}) \text{ for } i > 3$$

These relationships show a very interesting fact such that Z_i for $i = 2, 3, \ldots \infty$ are identical for the cases where m = 0, 1 and 2. Moreover, from the cases of m = 0 and 1, it should follow that

$$(\phi_i - 2\phi_{i-1} + \phi_{i-2}) = \frac{2}{\sqrt{\mu\pi}} (\sqrt{i} - 2\sqrt{i-1} + \sqrt{i-2})$$

or

$$\left(\frac{2}{\pi}\right)^2 \int_0^\infty \frac{1 - e^{-\beta^2 i/\mu} \, d\beta}{\beta^3 \{Y_o^2(\beta) + J_o^2(\beta)\}} = 2\sqrt{\frac{i}{\mu\pi}}$$

which seems to be a remarkable relationship.

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APPENDIX

Response factor formulas developed in the main text of this paper were used in the computer program called RESPTK. The Fortran listings of RESPTK and other necessary subroutines to calculate the thermal response factors of various multi-layer heat conduction systems such as depicted in Figure 4 are attached herewith. The main program to perform the input/output operation for RESPTK is called RESP. Sample input and output for RESP obtained for the systems described in Figure 4 are also attached. Certain portions of the computer program are written in Fortran V (Univac 1108) and certain modifications to the program will be necessary for use with a compiler that does not recognize statements made for the Univac Fortran V compiler.

Input Requirement of the Computer Program RESPTK

RESPTK (K, L, R, G, AG, KG, X, Y, Z, NL, DT, NR, CR, U, IM, IS, F)

Input:

K = Thermal conductivity (BTU/hr, ft, F) of each layer given in the order for minimum radius to the larger radii (Fig. 2). For the plane wall, it should be given from inside surface layer to the outer layers. For the layer with no thermal mass, such as surface boundary layer, conductance values should be used.

L = Thickness of each layer (ft) given in the order for minimum radius to the larger radii (Fig. 2). This could be zero for some layers, i.e. surface boundary layer.

R = Radius (ft) of each layer boundary given in the order of minimum value to the larger values (Fig. 2). For plane wall model, any arbitrary value being same for all the layer, should be given. Note that the number of R is NL + 1.

G = Thermal diffusivity (ft^2/hr) of each layer given in the order for the minimum radius to the larger radii (Fig. 2). For the layers with no thermal mass, such as surface boundary layers and air space, G should be zero for this program.

AG = Thermal diffusivity (ft^2/hr) of solid core or semi-infinite layer (Fig. 1), given only when IS = 1, or 2.

KG = Thermal conductivity (Btu/hr, ft, °F) of solid core or of semi-infinite region (Fig. 1), given only when IS = 1 or 2.

X, Y, Z = Thermal response factors (Btu/hr, ft^2) generated by this program for a wall of finite thickness.

NL = Number of layers to be considered for the heat conduction system. Surface boundary layers and air spaces should be treated as separate layers with G = 0 (Fig. 2). When IS = 1 or 2, the solid core or the semi-infinite region should not be counted as a layer.

DT = Time increament (hr) for which the calculation of heat flux is desired. For hourly calculation DT = 1.

NR = Number of X, Y, and Z generated by the program. NR is the output of this program such that the values of X, Y, Z can be calculated by a common ratio CR as follows:

$$\frac{X(J+1)}{X(J)} = \frac{Y(J+1)}{Y(J)} = \frac{Z(J+1)}{Z(J)} = CR$$

when $J \geq NR$.

CR = Common ratio described above.

U = Overall thermal conductance obtained by the reciprocal of total thermal resistance of the heat conduction system under consideration, Btu/hr, ft^2, °F.

IM = Curvature index (Fig. 1)

if IM = 0 plane system

= 1 cylindrical system

= 2 spherical system

IS = Heat conduction system index (Fig. 1)

if IS = 0 finite wall

= 1 semi-infinite region attached

= 2 solid core attached

F = Response factors for the system with solid core or semi-infinite region.

Calculation of heat flux

(A) Referring to Fig. 2, the heat flux QI(N) and QO(N) can be evaluated as follows, where N is the time index such that time = DT*N.

$$QI(N) = \sum_{J=1}^{M} X(J)*TI(N-J) - GM* \sum_{J=1}^{M} Y(J)*TO(N-J)$$

$$QO(N) = \sum_{J=1}^{M} Y(J)*TI(N-J) - \sum_{J=1}^{M} Z(J)*TO(N-J)$$

where QI and QO, and TI and TO are heat fluxes (Btu/hr, ft^2) and temperatures (F) at surfaces where the radii are minimum and maximum, or at the inside and outside surfaces. The values of QI and QO are positive when heat is flowing from TI side to TO side or from inside to outside.

In above equation for QI(N),

M = maximum number of response factors to be used, value of which will be determined by the significance of X(M)*TI(N-M). Usually M $\lesssim$ 72 (when M > NR, X(J), Y(J) and Z(J) should be calculated by the common ratio CR such as described earlier) and

$$GM = \frac{R(NL + 1)}{R(L)} ** IM$$, which is unity for the plane wall problem.

B. For calculating heat conduction for the system with the semi-infinite region (when IS = 1),

$$QI(N) = \sum_{J=1}^{M} F(J)*TI(N-J)$$

TI(N-J) is the temperature at the surface where the radius is minimum (inner surface) at time (N-J)*DT. The value of QI(N) is positive when heat is flowing in the direction from the minimum radius (inside surface) to the larger radii (to outer layer and to the semi-infinite region).

(C) For calculating the heat conduction for solid core system (IS = Z),

$$QO(N) = -\sum_{J=1}^{M} F(J)\ TO(N-J) \quad , \text{ Btu/hr, ft}^2$$

TO(N-J) is the temperature of the surface where the radius is maximum (outside surface). The heat flux QO(N) is positive as it is defined in the above equation when heat is flowing in the direction from the smaller radius to the larger radii.

Bessel function

The calculation for IM = 1 (cylindrical system) requires a double precision Bessel function subroutine in the following forms:

$$J_0(X) = DBEJ(X, 0)$$

$$J_1(X) = DBEJ(X, 1)$$

$$Y_0(X) = DBEY(X, 0)$$

$$Y_1(X) = DBEY(X, 1)$$

```
WRN ASG  A=0491
WN FOR RESP,RESP
C      THIS PROGRAM IS DEVELOPED BY T.KUSUDA OF THE NATIONAL BURFAU OF
C      STANDARDS FOR CALCULTING THE THERMAL RESPONSE FACTORS FOR
C      COMPOSITE WALLS,FLOORS,ROOFS,BASEMENT WALLS BASEMENT FLOORS
C      AND INTERNAL FURNISHINGS OF SIMPLE SHAPES
C      RESPONSE FACTORS ARE USED IN THE FOLLOWING MANNER
C      X,Y,Z ARE RESPONSE FACTORS
C     QI=X*TI-Y*TO*GMA    INSIDE WHERE R IS MINIMUM
C      QO=Y*TI-Z*TO    OUTSIDE WHERE R IS MAXIMUM
C      TI    INSIDE TEMPERARURE WHERE R IS MINIMUM
C      TO    OUTSIDE TEMPERATURE WHERE R IS MAXIMUM
C      K   THERMAL CONDUCTIVITY
C      G   THERMAL DIFFUSIVITY
C      L   THICKNESS
C    IM=0 OR BLANK  PLANE WALL
C    IM=1  CYLINDRICAL WALL
C    IM=2 SPHERICAL WALL
C    IN=0    FINITE THICK WALL
C    IN=1    SEMI-FINITE WALL
C    IN=2    SOLID OBJECT
C    IF RESPONSE FACTORS OF THE SOLID CYLINDER OR SPHERE OF HOMOGENEOUS
C    PROPETY ARE DESIRED, TREAT THE PROBLEM OF MULTILAYER BUT WITH THE
C    IDENTICAL PROPERTIES FOR ALL THE LAYERS EXCEPT THE RADIUS
C     IF IHEAT=0  NO TEMPERATURE DATA THUS NO HEAT CALCULATION
C     IF IHEAT=1  PERIODIC BOUNDATRY CONDITIONS
  400 FORMAT(2H0   )
      REAL K(10),G(10),L(10),R(11),KG
      DIMENSION X(200),Y(200),Z(200),TI(1000),TO(1000),C(10),D(10),RES(1
     10),RMK(10,4),RMKG(4),F(200)
    1 FORMAT(10I7)
    2 FORMAT(10F7.0)
  100 FORMAT(10H1         )
  101 FORMAT(77H0  LAYER      L(I)         K(I)          (I)        C(I)     RES(I
     1)    DESCRIPTION                                                       )
  102 FORMAT(77H    NO
     2     OF LAYERS                                                         )
  103 FORMAT(1I6,1F11.3,1F10.3,1F10.2,1F10.3,1F8.2,2X,4A6)
  104 FORMAT(58H0                                          THERMAL CONDUCTANCE
     30=1F7.3)
  105 FORMAT(49H0                                       TIME INCREMENT DT=1F3.0 )
  106 FORMAT(50H0                                            RESPONSE FACTORS     )
  107 FORMAT(120H0                    J                         X                 Y
     1           Z                                                               )
  108 FORMAT(1I17,1F23.4,2F15.4)
  112 FORMAT(4A6)
  117 FORMAT(44H0                                 COMMON RATIO  CR=1F7.5)
      READ(5,1) IHEAT
      IF(IHEAT.NE.0) CALL TDATA(TO,TI,NP,IHEAT)
  700 READ(5,2) DELTAT
  500 READ(5,1) NLAYR,NTEST,IM,IN
      IF(NLAYR.GT.10) GO TO 600
      NNLAYR=NLAYR+1
      IF(NLAYR.EQ.0) GO TO 300
      DO 200 I=1,NLAYR
  200 READ(5,2) L(I),K(I),D(I),C(I),RES(I)
      IF(IN.EQ.2.AND.IM.EQ.0) GO TO 301
C    READ K,RHO, AND C OF GROUND IF IN=1
  300 IF(IN.NE.0) READ(5,2)KG,DG,CG
C     AG  THERMAL DIFFUSIVITY  OF EARTH
      IF(IN.NE.0) AG=KG/CG/DG
      IF(NLAYR.EQ.0) GO TO 501
```

```
      IF(IM.EQ.0) GO TO 301
      READ(5,2)(R(I),I=1,NNLAYR)
      GO TO 302
  301 R(1)=10.
      DO 303 I=2,NNLAYR
  303 R(I)=R(I-1)+L(I)
  302 IF(IN.EQ.2.AND.IM.NE.0) READ(5,112)(RMKG(J),J=1,4)
      DO 113 I=1,NLAYR
  113 READ(5,112)(RMK(I,J),J=1,4)
      IF(IN.EQ.1) READ(5,112) (RMKG(J),J=1,4)
      DO 109 I=1,NLAYR
      IF(L(I)) 110,111,110
  111 G(I)=0.
      K(I)=1./RES(I)
      GO TO 109
  110 G(I)=K(I)/C(I)/D(I)
  109 CONTINUE
  501 GMA=(R(NNLAYR)/R(1))**IM
      WRITE(6,207)
  207 FORMAT(2H1      )
      CALL RESPIN(K,L,R,G,AG,KG,X,Y,Z,NLAYR,DELTAT,NRT,CR,UT,IM,IN,F)
      WRITE(6,100)
      IF(IM.EQ.0) WRITE(6,701)
  701 FORMAT(50H0  PLANE WALL SYSTEM                                    )
      IF(IM.EQ.1) WRITE(6,702)
  702 FORMAT(50H0  CYLINDRICAL  SYSTEM                                  )
      IF(IM.EQ.2) WRITE(6,703)
  703 FORMAT(50H0  SPHERICAL  SYSTEM                                    )
      WRITE(6,101)
      WRITE(6,102)
      WRITE(6,400)
       IF(NLAYR.EQ.0) GO TO 502
      IF(IN.EQ.2.AND.IM.NE.0) WRITE(6,120) KG,DG,CG,(RMKG(J),J=1,4)
      DO 202 I=1,NLAYR
      IF(L(I)) 202,203,202
  203 K(I)=0.
  202 WRITE(6,103) I,L(I),K(I),D(I),C(I),RES(I),(RMK(I,J),J=1,4)
      IF(IN.EQ.1) WRITE(6,120) KG,DG,CG,(RMKG(J),J=1,4)
  120 FORMAT(1F27.3,1F10.2,1F10.3,10X,4A6)
  502 WRITE(6,105) DELTAT
      WRITE(6,104)UT
      WRITE(6,106)
      WRITE(6,400)
      IF(IN.NE.0) GO TO 1535
      WRITE(6,107)
      DO 114 N=1,NRT
      JN=N-1
  114 WRITE(6,108)JN,X(N),Y(N),Z(N)
      GO TO 504
 1535 WRITE(6,555)
  555 FORMAT(50H0                    J                   F              )
      IF(IN.EQ.1) GO TO 505
      IF(IN.EQ.2.AND.IM.EQ.0) GO TO 505
      DO 506 N=1,NRT
      JN=N-1
      X(N)=-X(N)
  506 WRITE(6,508) JN,X(N)
      GO TO 504
  505 DO 509 N=1,NRT
      JN=N-1
  509 WRITE(6,508) JN,F(N)
  508 FORMAT(1I24,1F21.5)
```

```
504 WRITE(6,400)
    WRITE(6,400)
    WRITE(6,117) CR
    IF(NTEST.EQ.0) GO TO 500
    CALL HEAT(X,Y,Z,TI,TO,DELTAT,NP,NRT,GMA,CR)
    GO TO 500
600 STOP
    END
```

```
@FOR SAMPLE,SAMPLE
C     THIS PROGRAM ILLUSTRATES THE USE OF RESPTK AND HEATX
C       DEVELOPED BY THE NATIONAL BUREAU OF STANDARDS
C       TI      INSIDE TEMPERATURE WHERE R IS THE SMALLEST
C       TO       OUTSIDE TEMPERATURE WHERE R IS THE LARGEST
C       K       THERMAL CONDUCTIVITY, K IS THE THRMAL  CONDUCTANCE OF THE
C       LAYER IF THERE IS NO THERMAL MASS SUCH AS AIR SPACE
C       L       THICKNESS OF THE  LAYER  L=0 IF NO THERMAL MASS
C       R       RADIUS OF THE LAYER   USE ARBITRARY VALUE FOR THE PLANE WALL
C       G       THERMAL DIFFUSIVITY OF THE LAYER    G=0 FOR  NO THERMAL MASS
      DIMENSION K(10),L(10),R(10),G(10),X(100),Y(100),Z(100)
      DIMENSION WV(200),SOL(200),TCT(200),F(200),DB(200),TZ(200),TO(200)
      DIMENSION TX(48),TY(48),HTW(11)
      DIMENSION T1(200),T2(200),Q2(200)
    2 FORMAT(10I7)
    5 FORMAT(10F7.0)
  104 FORMAT(50H0 WALL COMPOSITION STATED FROM INSIDE SURFACE          )
  105 FORMAT(120H0     LAYER           K                            L
     1      R                    G                                  )
  110 FORMAT(50H0 PLANE WALL MODEL                                    )
  111 FORMAT(50H0 CYLINDRICAL WALL MODEL                              )
  112 FORMAT(50H0 SPHERICAL WALL MODEL                                )
      REAL K,L
      READ(5,2) NDAY,NIMAX
      IF(NDAY.EQ.0) GO TO 100
C     TZ  ROOM AIR TEMPERATURE
C     DB    OUTDOOR AIR TEMPERATURE
C     WV       WIND VELOCITY IN KNOTS
C     SOL       SOLAR RADIATION
C     TCT     TOTAL CLOUD AMOUNT
      READ(5,1) (TZ(JT),JT=1,24)
      READ(5,1) (DB(JT),JT=1,24)
      READ(5,1)(WV(JT),JT=1,24)
      READ(5,1)(SOL(JT),JT=1,24)
      READ(5,1)(TCT(JT),JT=1,24)
    1 FORMAT(12F6.0)
      DO 103 II=2,NDAY
      DO 103 JK=1,24
      JT=24*(II-1)+JK
      DB(JT)=DB(JK)
      TO(JT)=DB(JK)
      WV(JT)=WV(JK)
      TCT(JT)=TCT(JK)
      SOL(JT)=SOL(JK)
  103 TZ(JT)=TZ(JK)
  100 READ(5,2) NL,IM
      READ(5,5) (K(I),I=1,NL)
      READ(5,5) (L(I),I=1,NL)
      NLL=NL+1
      READ(5,5) (R(I),I=1,NLL)
      READ(5,5)(G(I),I=1,NL)
      READ(5,5) DT
      IF(IM.EQ.0) WRITE(6,110)
      IF(IM.EQ.1) WRITE(6,111)
      IF (IM.EQ.2) WRITE(6,112)
      WRITE(6,104)
      WRITE(6,105)
      DO 4 J=1,NLL
    4 WRITE(6,3) K(J),L(J),R(J),G(J)
    3 FORMAT(6F20.6)
      AG=0
      KG=0
```

```
      IN=0
      GX=R(NLL)/R(1)
      IF(IM.EQ.0) GMA=1.
      IF(IM.EQ.1) GMA=GX
      IF(IM.EQ.2) GMA=GX**2
      CALL RESPTR(K,L,R,G,AG,KG,X,Y,Z,NL,DT,NR,CR,U,IM,IN,F)
      HTW(8)=0.7
      HTW(9)=0.9
      HTW(10)=1.
      HTW(1)=DT
      HTW(2)=NR
      HTW(3)=48
      HTW(4)=GMA
      HTW(5)=CR
      WRITE(6,13)
   13 FORMAT(120H1       DB(KT)        TZ(KT)      SOL(KT)       TCT(KT)        W
     1V(KT)            FOC           TOT       HEATWT                           )
      Q1P=0.
      Q2P=0.
      DO 11 KT=49,NTMAX
      CALL FO(WV(KT),1,FOC,FOT,1)
      HTW(6)=FOC
      HTW(7)=SOL(KT)
      ODB=DB(KT)
      HTW(11)=TCT(KT)
C  IN THIS PROGRAM THE PRESENT TIME IS DT*48 TH HOUR
      CALL REVI(TO,TX,48,KT)
      CALL REVI(TZ,TY,48,KT)
      CALL HEATX(X,Y,Z,TX,TY,ODB,TOT,HEATWT,HTW,Q1P,Q2P)
      WRITE(6,12) DB(KT),TZ(KT),SOL(KT),TCT(KT),WV(KT),FOC,TOT,HEATWT
   12 FORMAT(10F12.2)
      T1(KT-48)=TOT
      T2(KT-48)=TZ(KT)
      Q2(KT-48)=HEATWT
   11 TO(KT)=TOT
      WRITE(7)T1
      WRITE(7)T2
      WRITE(7)Q2
      REWIND 7
      END
```

```
RUN FOR A,A
      SUBROUTINE RESPTK(K,L,R,G,AG,KG,X,Y,Z,NL,DT,NR,CR,U,IM,IS,F)
      DIMENSION K(10),L(10),R(10),G(10),X(100),Y(100),Z(100),AP(10),BP(1
     10),CP(10),DP(10),A(10),B(10),C(10),D(10),ZR1(3),ZR2(3),RB(3),RAP(3
     2),ROOT(100),RA(3,100),ZRK(3,100),RX(100),RY(100),AZ(100)
     3,F(100)
      REAL K,L,KG
      PI=3.1415927
      IF(IS.EQ.1.AND.IM.EQ.0) WRITE(6,604)
  604 FORMAT(50H0   SEMI-INFINITE   PLANE WALL                        )
      IF(IS.EQ.1.AND.IM.EQ.1) WRITE(6,605)
  605 FORMAT(50H0   SEMI-INFINITE   CYLINDER                          )
      IF(IS.EQ.1.AND.IM.EQ.2) WRITE(6,606)
  606 FORMAT(50H0   SEMI-INFINITE   SPHERE                            )
      IF(IS.EQ.2.AND.IM.EQ.0) WRITE(6,607)
  607 FORMAT(50H0   SOLID SLAB                                        )
      IF(IS.EQ.2.AND.IM.EQ.1)  WRITE(6,602)
  602 FORMAT(50H    SOLID----CYLINDER                                 )
      IF(IS.EQ.2.AND.IM.EQ.2) WRITE(6,603)
  603 FORMAT(50H     SOLID------ SPHERE                               )
      M3=3
      IF(IS.EQ.2.AND.IM.NE.0) M3=1
      IF(IS.NE.1) GO TO 613
  608 ZL=KG/R(NL+1)
      DY=R(NL+1)**2/AG/DT
      CALL GPF(DY,ZL,IM,AZ )
      IF(IS.EQ.1.AND.NL.EQ.0)  GO TO 901
  613 CALL ABCD2(0.,K,L,R,G,AX,BX,CX,DX,IM,NL)
      RB(1)=DX
      RB(2)=1.
      RB(3)=AX
      U=1./BX
      DO 1 I=1,NL
      PX=0
      CALL ABCDP2(PX,K(I),L(I),R(I),G(I),AP(I),BP(I),CP(I),DP(I),IM)
    1 CALL  ABCD2(PX,K(I),L(I),R(I),G(I),A(I),B(I),C(I),D(I),IM,1)
      IF(NL.LT.2) GO TO 502
      CALL DERVT(A,B,C,D,AP,BP,CP,DP,APP,BPP,CPP,DPP,NL)
      GO TO 503
  502 APP=AP(1)
      BPP=BP(1)
      CPP=CP(1)
      DPP=DP(1)
  503 IF(IS.NE.2) GO TO 501
      IF(IM.EQ.0) GO TO 501
      CALL SOLID(0.,R(1),KG,AG,IM,HF,HFP)
      ZR1(1)=(-CPP+HFP*AX)/DX/DT
      ZR2(1)=-ZR1(1)
      WRITE(6,1300)
 1300 FORMAT(120H                       CPP                        HFP
    1     AX                   DX                                     )
 1400 FORMAT(4F20.5)
      WRITE(6,1400) CPP,HFP,AX,DX
      GO TO 1212
  501 RAP(1)=DPP
      RAP(2)=0.
      RAP(3)=APP
      DO 2 I=1,3
      C1=RAP(I)/BX/DT
      C2 = RB(1)*BPP/BX/BX/DT
      ZR2(I)=-C1+C2
    2 ZR1(I)=-ZR2(I)+RB(I)/BX
```

```
1212 WRITE(6,64)
  64 FORMAT(50H0  RESIDUES AT P=0                                        )
     WRITE(6,100) (ZR1(I),I =1,M3)
     WRITE(6,100) (ZR2(I),I=1,M3)
 100 FORMAT(3F20.6)
C    ROOTS OF B(P)=0.
 212 NMAX=40
     IF(IS.EQ.2.AND.IM.NE.0) NMAX=100
     PX=0.001
     DPO=0.1/DT
     IF(IS.EQ.2.AND.IM.NE.0) DPO=3.1416*3.1416*AG/R(1)/R(1)*0.25
     DLX=0.0001
     IF(IS.EQ.2.AND.IM.NE.0)DLX=DPO/1000
     N=0
     WRITE(6,63)
  63 FORMAT(50H0  ROOTS OF B(P)=0                                         )
  11 DL=DPO
     CALL ABCD2(PX,K,L,R,G,AX,BX,CX,DX,IM,NL)
     IF(IS.EQ.2.AND.IM.NE.0) CALL SOLDX(PX,R(1),KG,AG,IM,BX,DX,TEST1)
  15 PXP=PX+DL
     CALL ABCD2(PXP,K,L,R,G,AXP,BXP,CXP,DXP,IM,NL)
     IF(IS.NE.2) GO TO 213
     IF(IM.EQ.0) GO TO 213
     CALL SOLDX(PXP,R(1),KG,AG,IM,BXP,DXP,TEST2)
     IF(TEST1*TEST2) 112,113,114
 114 PX=PXP
     TEST1=TEST2
     GO TO 15
 112 IF(DL-DLX) 130,130,117
 117 DL=DL/2.
     GO TO 15
 113 IF(TEST1) 118,119,118
 119 RXX=PX
     GO TO 31
 118 RXX=PXP
     GO TO 31
 130 AB=ABS(TEST1/TEST2)
     RXX=(PX+AB*PXP)/(1+AB)
     GO TO 31
 213 IF(BX*BXP) 12,13,14
  14 PX=PXP
     BX=BXP
     GO TO 15
  12 IF(DL-DLX)30,30,17
  17 DL=DL/2.
     GO TO 15
  13 IF(BX) 18,19,18
  19 RXX=PX
     GO TO 31
  18 RXX=PXP
     GO TO 31
  30 AB=ABS(BX/BXP)
     RXX=(PX+AB*PXP)/(1.+AB)
  31 N=N+1
     ROOT(N)=RXX
     IF(N.GT.1) DPO=ROOT(N)-ROOT(N-1)
     NRT=N
     WRITE(6,41) N,ROOT(N)
  41 FORMAT(I10,1F20.6)
     PX=RXX+DLX
     TESTMX=40
     TESTX=RXX*DT
```

```
      IF(TESTX-TESTMX)42,42,43
   42 IF(N.LT.NMAX) GO TO 11
   43 WRITE(6,65)
   65 FORMAT(50H0  RESUDUES AT  P=ROOT(N)                                   )
      DO 600 JJ=1,NRT
      PX=ROOT(JJ)
      DO 51 J=1,NL
      CALL ABCD2(PX,K(J),L(J),R(J),G(J),A(J),B(J),C(J),D(J),IM,1)
   51 CALL ABCDP2(PX,K(J),L(J),R(J),G(J),AP(J),BP(J),CP(J),DP(J),IM)
      CALL ABCD2(PX,K,L,R,G,AX,BX,CX,DX,IM,NL)
      IF(NL.LT.2) GO TO 504
      CALL DERV1(A,B,C,D,AP,BP,CP,DP,APP,BPP,CPP,DPP,NL)
      GO TO 505
  504 APP=AP(1)
      BPP=BP(1)
      CPP=CP(1)
      DPP=DP(1)
  505 IF(IS.NE.2) GO TO 214
      IF(IM.EQ.0) GO TO 214
      CALL SOLID(PX,R(1),KG,AG,IM,HF,HFP)
      IF(HF) 401,400,401
  401 PYS     =(HF*AX-CX)/PX/PX/(DPP-HFP*BX-HF*BPP)/DT
      GO TO 402
  400 PYS=0.
  402 RA(1,JJ)=PYS
      GO TO 601
  214 PY=BPP*PX*PX*DT
      RA(1,JJ)=DX/PY
      RA(2,JJ)=1./PY
      RA(3,JJ)=AX/PY
  601 PZ=PX*DT
   52 RX(JJ)=EXP(-PZ)
    5 RY(JJ)=(1.-EXP(PZ))**2
  600 WRITE(6,54) ROOT(JJ),(RA(M,JJ),M=1,M3)
   54 FORMAT(4F20.6)
      DO 154 JJ=1,NRT
      DO 154 M=1,M3
      ZR1(M)=RA(M,JJ)*RX(JJ)+ZR1(M)
  154 ZR2(M)=RA(M,JJ)*RX(JJ)*RX(JJ)*(1-2/RX(JJ))+ZR2(M)
      II=1
      III=2
   80 FORMAT(50H0  RESPONSE FACTORS OF  FINITE SLAB                         )
   81 FORMAT(120H0          J                    X(J)                  Y(J)
     1            Z(J)                                                      )
  701 FORMAT(120H1  RESPONSE FACTORS FOR SOLID CYLINDRICAL OBJECTS
     1                                                                      )
  702 FORMAT(120H1    RESPONSE FACTORS FOR SOLID  SPHERICAL OBJECTS
     1                                                                      )
      IF(IS.EQ.2.AND.IM.EQ.1) WRITE(6,701)
      IF(IS.EQ.2.AND.IM.EQ.2) WRITE(6,702)
      IF(IS.EQ.0) WRITE(6,80)
      WRITE(6,81)
      IF(ZR1(2).LT.0) ZR1(2)=0.
      WRITE(6,55) II,(ZR1(M),M=1,M3)
      WRITE(6,55) III,(ZR2(M),M=1,M3)
      DO 67 M=1,M3
      ZRK(M,1)=ZR1(M)
   67 ZRK(M,2)=ZR2(M)
   55 FORMAT(I10,3F20.6)
      NT=100
      DO 58 N=3,NT
      NR=N
```

```
      DO 61 M=1,M3
   61 ZRK(M,N)=0.
      DO 57 M=1,M3
      DO 57 JJ=1,NRT
      PZ=(RX(JJ))**N
   57 ZRK(M,N)=ZRK(M,N)+PZ*RY(JJ)*RA(M,JJ)
      WRITE(6,55) N,(ZRK(M,N),M=1,M3)
      IF(N.LT.5) GO TO 58
      TEST1=ZRK(1,N)/ZRK(1,N-1)
      TEST2=ZRK(1,N-1)/ZRK(1,N-2)
      TEST3=ABS(TEST1-TEST2)
      IF(TEST3-0.00001) 59,59,58
   58 CONTINUE
   59 DO 60  N=1,NR
      X(N)=ZRK(1,N)
      Y(N)=ZRK(2,N)
   60 Z(N)=ZRK(3,N)
      CR=TEST2
      WRITE(6,62) CR
   62 FORMAT(10H0       CR=1F10.6)
      IF(IS.EQ.2.AND.IM.EQ.0) GO TO 800
      IF(IS.NE.1) GO TO 900
  901 IF(NL.EQ.0) GO TO 905
      GF=2*KG/SQRT(DI*AG*PI)
      IF(NR.LT.50) GO TO 610
      DO 204 J=50,NR
      ZJ=J
  204 AZ(J)=GF*(SQRT(ZJ)-2.*SQRT(ZJ-1.)+SQRT(ZJ-2.))
      NRR=NR
      GO TO 300
  610 DO 301 J=NR,50
      Z(J+1)=Z(J)*CR
      X(J+1)=X(J)*CR
  301 Y(J+1)=Y(J)*CR
      NRR=50
  300 DO 205 J=1,NRR
  205 F(J)=X(J)-Y(J)*Y(J)/(Z(J)+AZ(J))
      NR=NRR
      GO TO 906
  905 DO 904 J=1,NR
  904 F(J)=AZ(J)
  906 WRITE(6,207)
  207 FORMAT(50H0           J              F                          )
      CR1=1.
      DO 208 J=1,50
      CR=F(J+1)/F(J)
      TESTCR=ABS(CR-CR1)
      IF(TESTCR-0.00001) 611,611,612
  612 CR1=CR
      JJ=J-1
  208 WRITE(6,209) JJ,F(J)
  209 FORMAT(1I10,1F20.5)
  611 NR=J
      CR=CR1
      GO TO 900
  800 WRITE(6,207)
      DO 210 J=1,NR
      F(J)=2*Y(J)-(X(J)+Z(J))
      JJ=J-1
  210 WRITE(6,209) JJ,F(J)
  900 RETURN
      END
```

WN FOR B,B

```
      SUBROUTINE DERVT(A,B,C,D,AP,BP,CP,DP,APP,BPP,CPP,DPP,N)
      DIMENSION A(N),B(N),C(N),D(N),AP(N),BP(N),CP(N),DP(N),AT(10),BT(1
     1),CT(10),DT(10),ATT(10),BTT(10),CTT(10),DTT(10)
      DO 1 I=1,N
      DO 2 J=1,N
      IF(I.EQ.J) GO TO 3
      AT(J)=A(J)
      BT(J)=B(J)
      CT(J)=C(J)
      DT(J)=D(J)
      GO TO 2
    3 AT(J)=AP(J)
      BT(J)=BP(J)
      CT(J)=CP(J)
      DT(J)=DP(J)
    2 CONTINUE
    1 CALL MULT(AT,BT,CT,DT,ATT(I),BTT(I),CTT(I),DTT(I),N)
      APP=ATT(1)
      BPP=BTT(1)
      CPP=CTT(1)
      DPP=DTT(1)
      DO 4 I=2,N
      APP=APP+ATT(I)
      BPP=BPP+BTT(I)
      CPP=CPP+CTT(I)
    4 DPP=DPP+DTT(I)
      RETURN
      END
```

```
WN FOR C,C
      SUBROUTINE ABCD2(Z,K,L,R,G,A,B,C,D,IM,NL)
      DIMENSION AX(10),BX(10),CX(10),DX(10),R(10),G(10)
      DOUBLE PRECISION   DBEJ,DBEY,ZQ1,ZQ2
      REAL K(10),L(10),J01,J02,J11,J12
      PI=3.1415927
      PP=PI*0.5
      IF(NL.LT.2) R(2)=R(1)+L(1)
      DO 4 I=1,NL
      IF(G(I)) 103,103,102
  102 IF(Z) 1,1,101
  101 ZQ=SQRT(Z/G(I))
      ZQ1=ZQ*R(I)
      ZQ2=ZQ*R(I+1)
      ZQL=ZQ*L(I)
      IF(IM.NE.1) GO TO 3
      J01=DBEJ(ZQ1,0)
      J11=DBEJ(ZQ1,1)
      J02=DBEJ(ZQ2,0)
      J12=DBEJ(ZQ2,1)
      Y01=DBEY(ZQ1,0)
      Y11=DBEY(ZQ1,1)
      Y02=DBEY(ZQ2,0)
      Y12=DBEY(ZQ2,1)
      AX(I)=-PP*ZQ2*(J01*Y12-Y01*J12)
      BX(I)=PP*R(I+1)/K(I)*(-Y01*J02+J01*Y02)
      CX(I)=K(I)/R(I+1)*(-J11*Y12+Y11*J12)*PP*ZQ2*ZQ2
      DX(I)=PP*ZQ2*(J11*Y02-Y11*J02)
      GO TO 4
    3 C0=SIN(ZQL)
      C1=COS(ZQL)
      S1=C0/ZQL
      S2=(S1-C1)/ZQL/ZQL
      IF(IM.EQ.2) GO TO 5
      AX(I)=C1
      BX(I)=L(I)/K(I)*S1
      CX(I)=-ZQL*K(I)/L(I)*C0
      DX(I)=C1
      GO TO 4
    5 GM=R(I+1)/R(I)
      AX(I)=GM*(C1-L(I)/R(I+1)*S1)
      BX(I)=L(I)/K(I)*GM*S1
      CX(I)=L(I)*L(I)/R(I)/R(I)*K(I)/L(I)*(-(ZQ1*ZQ2+1)*S1+C1)
      DX(I)=GM*(C1+L(I)/R(I)*S1)
      GO TO 4
    1 AX(I)=1.
      CX(I)=0.
      DX(I)=(R(I+1)/R(I))**IM
      IF(IM.EQ.0) BX(I)=L(I)/K(I)
      IF(IM.EQ.1) BX(I)=R(I+1)/K(I)*LOG(R(I+1)/R(I))
      IF(IM.EQ.2) BX(I)=L(I)/K(I)*(R(I+1)/R(I))
      GO TO 4
  103 AX(I)=1.
      BX(I)=1/K(I)
      CX(I)=0.
      DX(I)=(R(I+1)/R(I))**IM
    4 CONTINUE
      A=AX(1)
      B=BX(1)
      C=CX(1)
      D=DX(1)
      IF(NL.LT.2) GO TO 6
```

```
      CALL MULT(AX,BX,CX,DX,A,B,C,D,NL)
    6 RETURN
      END
```

```
      FOR D,D
      SUBROUTINE ABCDP2(Z,K,L,R,G,AP,BP,CP,DP,IM)
      DOUBLE PRECISION Z01,Z02,DBEJ,DBEY
      REAL K,L,J01,J02,J11,J12
      PI=3.1415927
      PP=PI/4./G
      IF(G) 103,103,104
  104 IF(Z) 101,101,105
  105 Z0=SQRT(Z/G)
      Z0L=Z0*L
      Z01=Z0*R
      Z02=Z01+Z0L
      IF(IM.NE.1) GO TO 3
      X=R*(R+L)
      Y=(R+L)**2
      Z1=(R+L)/R
      J01=DBEJ(Z01,0)
      J02=DBEJ(Z02,0)
      J11=DBEJ(Z01,1)
      J12=DBEJ(Z02,1)
      Y01=DBEY(Z01,0)
      Y02=DBEY(Z02,0)
      Y11=DBEY(Z01,1)
      Y12=DBEY(Z02,1)
      AP=(-X*(J11*Y12-Y11*J12)+Y*(J01*Y02-Y01*J02))*PP
      BP=(X*(J11*Y02-Y11*J02)*Z1/Z02+Y*(J01*Y12-Y01*J12)*Z1/Z02)*PP
      CP=PP*Z02/Z1*(X*(J01*Y12-Y01*J12)+Y*(J11*Y02-Y11*J02))
      DP=(X*(-J01*Y02+Y01*J02)-Y*(-J11*Y12+Y11*J12))*PP
      GO TO 4
    3 X=L*L*0.5/G
      R1=R+L
      RES=L/R
      C0=SIN(Z0L)
      C1=COS(Z0L)
      S1=C0/Z0L
      S2=(S1-C1)/Z0L/Z0L
      IF(IM.EQ.0) GO TO 5
      AP=X*(R1*S1/R-L*S2/R)
      BP=RES*X*R1*S2/R
      ZP1=Z01
      ZP2=Z02
      CP=X*(L/R)**2/RES*((2.*R*R1/L/L+1)*S1-(ZP1*ZP2+1.)*S2)
      DP=X*(R1/R*S1+(L/R)*(R1/R)*S2)
      GO TO 4
    5 AP=X*S1
      BP=X*RES*S2
      CP=X*(S1+C1)/RES
      DP=X*S1
      GO TO 4
  103 AP=0.
      BP=0.
      CP=0.
      DP=0.
      GO TO 4
  101 IF(IM.NE.0) GO TO 6
      X=L*L*0.5/G
      AP=X
      BP=X*L/K/3
      CP=K/L*X*2.
      DP=X
      GO TO 4
    6 IF(IM.NE.1) GO TO 7
```

```
      R1=R+L
      AP=(0.50*R*R-R1*R1)+R1*R1*LOG(R1/R))*0.5/G
      BP=R1/4/G/R*((R1*R1+R*R)*LOG(R1/R)-(R1*R1-R*R))
      CP=K/R*0.5/G*(R1*R1-R*R)
      DP=0.5/G*(0.5*(R1*R1-R*R)*R1/R-R*R1*LOG(R1/R))
      GO TO 4
    7 X=L*L*0.5/G
      R1=R+L
      AP=X/3.*(2*R1/R+1.)
      BP=L/R*R1/R*X/3.
      CP=K/L*X*L/R*L/R*(2.*R*R1/L/L+0.66667)
      DP=X/3.*R1/R*(R1/R+2)
    4 RETURN
      END
```

WN FOR E.E

```
      SUBROUTINE MULT(A,B,C,D,AT,BT,CT,DT,N)
      DIMENSION A(N),B(N),C(N),D(N)
      AIT=A(1)
      BIT=B(1)
      CIT=C(1)
      DIT=D(1)
      IF(N.LT.2) GO TO 3
      DO 1 J=2,N
      AT=AIT*A(J)+BIT*C(J)
      BT=AIT*B(J)+BIT*D(J)
      CT=CIT*A(J)+DIT*C(J)
      DT=CIT*B(J)+DIT*D(J)
      AIT=AT
      BIT=BT
      CIT=CT
    1 DIT=DT
      GO TO 4
    3 AT=AIT
      BT=BIT
      CT=CIT
      DT=DIT
    4 RETURN
      END
```

```
RUN FOR F,F
      SUBROUTINE SOLID(Z,R1,KG,AG,IM,HF,HFP)
      REAL KG,J01,J11
      DOUBLE PRECISION DBEJ,Z0D
      Z0=SQRT(Z/AG)
      Z01=Z0*R1
      Z0D=Z01
      ZA=R1*R1/AG
      CON=KG/R1
      IF(Z) 2,1,2
    2 IF(IM.NE.1) GO TO 100
      J01=DBEJ(Z0D,0)
      TX=ABS(J01)
      IF(TX-0.00001) 4,4,5
    5 J11=DBEJ(Z0D,1)
      HF=CON*Z01*J11/J01
      HF1=J11/J01/Z01
      HF2=(J01*J01+J11*J11-J01*J11/Z01)/J01/J01
      HFP=-CON*0.5*ZA*(HF1+HF2)
      GO TO 300
  100 C=COS(Z01)
      S=SIN(Z01)/Z01
      TX=ABS(SIN(Z01))
      IF(TX-0.00001) 4,4,3
    3 HF=-CON*(C/S-1)
      HFP=-CON*0.5*ZA*(1+C*(C-S)/S/S/Z01/Z01)
      GO TO 300
    1 HF=0.
      IF(IM.EQ.2) HFP=-CON*ZA/3.
      IF(IM.EQ.1) HFP=-0.5*CON*ZA
      GO TO 300
    4 HF=0.
      HFP=0.
  300 RETURN
      END
```

```
RUN FOR G,G
      SUBROUTINE SOLDX(Z,R1,KG,AG,IM,B,D,TEST)
      REAL KG,J01,J11
      DOUBLE PRECISION DBEJ,ZGD
      ZG=SQRT(Z/AG)
      ZG1=ZG*R1
      ZGD=ZG1
      CON=KG/R1
      IF(IM.NE.1) GO TO 100
      J01=DBEJ(ZGD,0)
      J11=DBEJ(ZGD,1)
      TEST=D*J01-B*J11*CON*ZG1
      GO TO 200
  100 TEST=D*SIN(ZG1)-B*CON*(SIN(ZG1)-ZG1*COS(ZG1))
  200 RETURN
      END
```

```
WN FOR J,J
      SUBROUTINE GPF(U,ZL,IM,Z)
      DIMENSION Z(100),ZT(5000),ZS(5000)
      DOUBLE PRECISION DBEJ,DBEY,ZG
      PI=3.1415927
      SQTPI=SQRT(PI)
      PI2=2./PI
      EB=0.001
      DB=0.1
  100 FORMAT(50H0  RESPONSE FACTORS FOR  SEMI-INFINITE BED               )
      WRITE(6,100)
      WRITE(6,101)
  101 FORMAT(50H0          K      Z(K)                                   )
      Z(1)=2*ZL*SQRT(U)/SQTPI
      ZZ=Z(1)
      Z(2)=Z(1)*(SQRT(2.)-2.)
      DO 2 K=3,50
      ZK=K
    2 Z(K)=Z(1)*(SQRT(ZK)-2.*SQRT(ZK-1)+SQRT(ZK-2.))
      IF(IM.EQ.0) GO TO 70
      IF(IM.EQ.1) GO TO 1
      Z(1)=Z(1)+ZL
      GO TO 70
    1 X=PI2  *LOG(0.5*EB )+0.36746691
      SUN=PI*0.5*(ATAN(X)+0.5*PI)
      IX=0
      B=EB-DB
      DO 17 L=1,5000
      B=B+DB
    8 ZG=B
      IF(IX.EQ.10) GO TO 30
      ZJ0=DBEJ(ZG,0)
      ZY0=DBEY(ZG,0)
      TESTX=ZJ0*ZJ0+ZY0*ZY0
      TESTY=PI2/B
      TESTZ=ABS(TESTX-TESTY)
      IF(TESTZ-0.00001) 30,30,31
   31 ZZ=B*B*B*TESTX
      GO TO 32
   30 ZZ=B*B*PI2
      IX=10
   32 ZT(L)=1./ZZ
      LT=L
      TEST=ABS(ZT(L))*10
      IF(TEST-0.0001) 11,11,17
   17 CONTINUE
   11 LTY=LT/2
      LTX=LTY*2-1
      BMAX=EB+(LTX-1)*DB
      BB=1./BMAX
      ZU=1./U
      SUT=SUN*ZU
      B=EB-DB
      DO 28 L=1,LT
      B=B+DB
      ZB=B*B*ZU
    6 ZP=EXP(-ZB)
   28 ZS(L)=(1.-ZP)*ZT(L)
      CALL SIMS(ZS,DB ,SUM,LTX)
      GK=(SUM+SUT)*PI2  +BB
      GG=GK*PI2
      Z(1)=GG*ZL*U
```

```
70 DO 15 K=1,50
15 WRITE(6,16) K,Z(K)
16 FORMAT(1I10,3F10.5)
   RETURN
   END
```

```
RUN FOR H,H
      SUBROUTINE TDATA(TO,TI,NP,IHT)
C     IF IHT=1  READ DAILY CYCLE
C     IF IHT=2  READ WEEKLY DATA
      DIMENSION TO(1000),TI(1000),DB(200),DP(200),SOL(200)
  115 FORMAT(12F5.1)
    2 FORMAT(12F6.0)
      IF(IHT.EQ.1) GO TO 121
      DO 116 ND=1,7
      N1=(ND-1)*24+1
      NF=N1+23
      READ(5,115)(DB(N),N=N1,NF)
      READ(5,115)(DP(N),N=N1,NF)
  116 READ(5,115)(SOL(N),N=N1,NF)
      DO 119 N=1,168
      TO(N)=DB(N)+0.5*(SOL(N)*3.67-7)
  119 TI(N)=75.
      GO TO 123
  121 READ(5,2) (DB(J),J=1,24)
      DO 122 ND=1,7
      DO 122 N=1,24
      N1=(ND-1)*24+N
      TO(N1)=DB(N)
  122 TI(N1)=75.
  123 WRITE(6,204)
  204 FORMAT(2H1 ,4X,4HTIME,6X,2HDB,7X,3HSUN,8X,2HTO,8X,2HDP,8X,2HTI)
      DO 205 NN=1,168
      N=168-NN+1
      TIME=NN
  205 WRITE(6,206) TIME,DB(N),SOL(N),TO(N),DP(N),TI(N)
  206 FORMAT(6F10.1)
      NP=168
      RETURN
      END
```

```
@LT FOR HEATX,HEATX
      SUBROUTINE HEATX(X,Y,Z,TO,TZ,DB,TOT,HEATWT,HTW,Q1P,Q2P)
C     HEAT FLUX CALCULATION WHEN  THE OUTSIDE SURFACE UNDERGOES
C     COMBINATION OF  RADIATIVE AND CONVECTIVE HEAT TRANSFER
      DIMENSION X(1),Y(1),Z(1),TO(1),TZ(1),HTW(1)
      REAL KR
C     X(J),Y(J),Z(J),J=1,NR)))) RESPONSE FAC
C     TO(I),TZ(I),I=1,NT   OUTSIDE SURFACE AND SPACE TEMPERATURES
C     DB  OUTSIDE AIR TEMPERATURE
C     HTW(1)=DT  TIME INCREMENT FOR RESPONSE FACTORS
C     HTW(2)=NR  NUMBER OF RESPONSE FACTORS TO BE SUPPLIED
C     HTW(3)=NT  NUMBER OF TEMPERATURE DATA TO BE SUPPLIED
C     HTW(4)=GMA  RADIUS RATIO FOR CYLINDER AND SPHERE
C     HTW(5)=CR  COMMON RATIO FOR THE RESPONSE FACTOR CALCULATION FOR
C     J .GT.NR
C     HTW(6)=FOC  OUTSIDE SURFACE HEAT TRANSFER COEFFICIENT
C     CONVECTIVE PORTION
C     HTW(7)=IT  SOLAR AND SKY RADIATION
C     HTW(8)=A  SOLAR RADIATION  ABSORPTION FACTOR OF THE OUTSIDE SURFACE
C     HTW(9)=F  EMMITTANCE OF THE OUTSIDE SURFACE
C     HTW(10)=COSWT  COSINE OF WALL TILT ANGLE
C     HTW(11)TCT  TOTAL CLOUD AMOUNT
C     TOT  OUTSIDE SURFACE TEMPERATURE CALCULATED=TO(1)
C     HEATWT  HEAT GAIN TO THE SPACE
      KR=0.1714E-8
      DXMIN=0.5
      DX=40.
      DT=HTW(1)
      NR=IFIX(HTW(2))
      NT=IFIX(HTW(3))
      GMA=HTW(4)
      CR=HTW(5)
      COSWT=HTW(10)
      EAT=KR*(460.+DB)**4
      IF(COSWT)1,1,2
    2 EAT=EAT-2*COSWT*(10.-HTW(11))
    1 TX=0.
      TXX=TX+DX
      TO(1)=TX
      JK=1
    3 ET=KR*(460.+TO(1))**4
      CALL HEAT(X,Y,Z,TZ,TO,NT,NR,GMA,CR,Q1,Q2,Q1P,Q2P)
      H1=HTW(8)*HTW(7)
      H2=HTW(9)*(EAT-ET)
      H3=HTW(6)*(DB-TO(1))
      B=H1+H2+H3+Q2
      GO TO(4,5  ,12),JK
    4 B1=B
      TO(1)=TXX
      JK=2
      GO TO 3
    5 B2=B
      IF(B1*B2) 8,7,6
    8 TX=TXX
      B1=B2
   18 TXX=TX+DX
      TO(1)=TXX
      JK=2
      GO TO 3
    6 IF(DX-DXMIN) 14,14,13
   13 DX=DX*0.5
      GO TO 18
```

```
14 C=ABS(B2/B1)
   TO(1)=(TXX+C*TX)/(1.+C)
   JK=3
   GO TO 3
 7 IF(B1) 10,11,10
11 TO(1)=TX
   GO TO 12
10 TO(1)=TXX
12 TOT=TO(1)
   HEATWT=Q1
   RETURN
   END
```

```
WIT FOR FO,FO
C        THIS SUBROUTINE CALCULATES OUTSIDE SURFACE HEAT TRANSFER
C       COEFFICIENTS,FOT AND FOC
C       FOT.... RADIATION PLUS CONVECTION
C       FOC....CONVECTION
C       V.....WIND VWLOCITY IN KNOTS
        SUBROUTINE FO(V,IS,FOC,FOT,IWD)
        DIMENSION A(6)/0.,0.001,0.,-0.002,0.,-0.00125/,B(6)/.464,0.320,0.
       1550,0.515,0.244,0.262/,C(6)/2.04,2.20,1.90,1.45,1.80,1.45/
        VP=V*1.155
        FOT=A(IS)*VP*VP+B(IS)*VP+C(IS)
C       IWD=1    IF THE SURFACE IS WINDWARD OR PARALLEL TO THE WIND
C       IWD=0  IF THE SURFACE IS LEEWARD
        IF(IWD.EQ.0) GO TO 1
        IF(VP-7.0) 1,1,2
      2 FOC=0.23*VP+1.02
        GO TO 3
      1 FOC=2.63
      3 RETURN
        END
```

```
W11 FOR I,1
      SUBROUTINE HEAT(X,Y,Z,TI,TO,NT,NR,GMA,CR,Q1,Q2,Q1P,Q2P)
C     RESPONSE FACTOR CALCULATION OF HEAT FLUXES
C     PRIOR TO THE APPLICATION OF THIS ROUTINE THE TEMPERATURE DATA
C     SHOULD HAVE BEEN REVERSED BY SUBROUTINE REVT
C     TI    INSIDE TEMPERATURE WHERE R IS MINIMUM
C     TO    OUTSIDE TEMPERATURE WHERE R IS MAXIMUM
      DIMENSION X(1),Y(1),Z(1),TI(1),TO(1),XX(30),YY(30),ZZ(30)
      DO 2000 J=2,NR
      XX(J)=X(J)-CR*X(J-1)
      YY(J)=Y(J)-CR*Y(J-1)
 2000 ZZ(J)=Z(J)-CR*Z(J-1)
 1000 SUMX=0.
      SUMY=0.
      SUMYY=0.
      SUMZ=0.
      NT=NR
      DO 4 J=1,NT
      SUMX=SUMX+TI(J)*XX(J)
      SUMY=SUMY+TO(J)*YY(J)
      SUMYY=SUMYY+TI(J)*YY(J)
    4 SUMZ=SUMZ+TO(J)*ZZ(J)
      Q1=SUMX-SUMY*GMA     +Q1P
      Q2=SUMYY-SUMZ   +Q2P
      Q1P=Q1
      Q2P=Q2
      RETURN
      END
```

```
WIT FOR REVT,REVT
      SUBROUTINE REVT(T,RT,N,NT)
      DIMENSION T(1),RT(1)
C     THIS ROUTINE REVERSE THE ORDER OF TEMPERATURE SEQUENCE
C     FOR  N TEMPERATURE POINTS
C     STARTING FROM PRESENT TIME  NT AND ENDING UP WITH  TIME NT-N+1
      DO 1 J=1,N
    1 RT(J)=T(NT-J+1)
      RETURN
      END
```

```
WN FOR BIKE,BIKE
C     THE BESSEL FUNCTION SUBROUTINES WERE DEVELOPED BY B.A. PEAVY
C     OF THE NATIONAL BUREAU OF STANDARDS
      FUNCTION DBEJ(X,M)
      COMMON /CTCHF/P,Q,R,S,U,V,A,B,C,D,E,F,G,H,PI
      DOUBLE PRECISION A(16),B(16),C(16),D(16),E(12),F(12),G(12),H(12),
     1P(18),Q(18),R(18),S(18),U(46),V(46),PI
      DOUBLE PRECISION T(46),AA,AB,AC,AD,X,DBEJ
      J=2
  100 AA=X/8.D0
      AC=X
      IF (AA.GT.1.D0) GO TO 6
      T(1)=1.D0
      T(2)=2.D0*AA**2-1.D0
      AB=4.D0*AA**2-2.D0
      DO 1 N=3,18
    1 T(N)=AB*T(N-1)-T(N-2)
      IF (J.EQ.0) GO TO 20
      IF (J.EQ.2) GO TO 30
      IF (DABS(X).LT.1.D-8) AC=1.D-8
      IF (J.EQ.3) GO TO 40
      AB=LOG(AC/8.D0)
      IF (M.EQ.1) GO TO 3
      AC=(Q(1)-AB*P(1))/2.D0
      DO 2 N=2,18
    2 AC=AC+T(N)*(Q(N)-AB*P(N))
      GO TO 5
    3 AC=(S(1)-AB*R(1))/2.D0
      DO 4 N=2,18
    4 AC=AC+T(N)*(S(N)-R(N)*AB)
      AC=1.D0/X-AA*AC
    5 DBEJ=AC
      RETURN
    6 AA= 1.D0/AA
      IF (J.EQ.1) AA=-AA
      L=12
      IF (J.LT.2) L=46
      T(1)=1.D0
      T(2)=AA
      IF (J.GE.2) T(2)=2.D0*AA**2-1.D0
      AB=2.D0*AA
      IF (J.GE.2) AB=2.D0*AA*AB-2.D0
      DO 7 N=3,L
    7 T(N)=AB*T(N-1)-T(N-2)
      IF (J.GT.1) GO TO 50
      AA=1.D0
      IF (X.LT.700.D0) AA=EXP(-X)
      AB=1.D0/SQRT(X)
      IF (J.EQ.0) GO TO 24
      AB=AB*PI*AA
      IF (M.EQ.1) GO TO 9
      AC=U(1)/2.D0
      DO 8 N=2,46
    8  AC=AC+T(N)*U(N)
      AC=AC*AB
      GO TO 5
    9 AC=V(1)/2.D0
      DO 10 N=2,46
   10 AC=AC+T(N)*V(N)
      AC=AC* AB
      GO TO 5
   20 IF (M.EQ.1) GO TO 22
```

```
      AC=P(1)/2.D0
      DO 21 N=2,16
   21 AC=AC+T(N)*P(N)
      GO TO 5
   22 AC=R(1)/2.D0
      DO 23 N=2,16
   23 AC=AC+T(N)*R(N)
      AC=AC*AA
      GO TO 5
   24 AB=AB/AA
      IF (M.EQ.1) GO TO 26
      AC=U(1)/2.D0
      DO 25 N=2,40
   25 AC=AC+T(N)*U(N)
   28 AC=AB*AC
      GO TO 5
   26 AC=V(1)/2.D0
      DO 27 N=2,38
   27 AC=AC+T(N)*V(N)
      GO TO 28
   30 IF (M.EQ.1) GO TO 32
      AC=A(1)/2.D0
      DO 31 N=2,16
   31 AC=AC+T(N)*A(N)
      GO TO 5
   32 AC=C(1)/2.D0
      DO 33 N=2,16
   33 AC=AC+T(N)*C(N)
      AC=AC*AA
      GO TO 5
   40 AB=2.D0*LOG(AC)/PI
      IF (M.EQ.1) GO TO 42
      AC=(B(1)+AB*A(1))/2.D0
      DO 41 N=2,16
   41 AC=AC+T(N)*(B(N)+AB*A(N))
      GO TO 5
   42 AC=(D(1)+AB*C(1))/2.D0
      DO 43 N=2,16
   43 AC=AC+T(N)*(D(N)+AB*C(N))
      AC=AC*AA-2.D0/(PI*X)
      GO TO 5
   50 AD=SQRT(2.D0/(PI*X))
      IF (M.EQ.1) GO TO 60
      AB=E(1)/2.D0
      AC=F(1)/2.D0
      DO 51 N=2,12
      AB=AB+T(N)*E(N)
   51 AC=AC+T(N)*F(N)
      AC=AC*AA
      AA=X-PI/4.D0
   55 IF (J.EQ.3) GO TO 52
      AC=AD*(AB*COS(AA)-AC*SIN(AA))
      GO TO 5
   52 AC=AD*(AC*COS(AA)+AB*SIN(AA))
      GO TO 5
   60 AB=G(1)/2.D0
      AC=H(1)/2.D0
      DO 61 N=2,12
      AB=AB+T(N)*G(N)
   61 AC=AC+T(N)*H(N)
      AC=AC*AA
      AA=X-.75D0*PI
```

```
GO TO 55
ENTRY DBE1(X,M)
J=0
GO TO 100
ENTRY DBEK(X,M)
J=1
GO TO 100
ENTRY DBEY(X,M)
J=3
GO TO 100
END
```

```
FOR BITH,BITH
   BLOCK DATA
   COMMON /CICHE/P,Q,R,S,U,V,A,B,C,D,E,F,G,H,PI
   DOUBLE PRECISION A(16),B(16),C(16),D(16),E(12),F(12),G(12),H(12),
  1P(18),Q(16),R(18),S(18),U(46),V(46),PI,T,W,X,Y,Z
 CHEBYSHEV COEFFICIENTS FOR BESSEL FUNCTIONS
   DATA A/.315455942949780239D0,-.8723442352852221D-2,.26517861320333
  1681D0,-.370094993872649779D0,.158067102332097261D0,-.0348937694114
  2088885D0,.4619180069467604D-2,-.460626166206275D-3,.32460328821D-4,
  3-.1761946907762D-5,.76081635924D-7,-.267925353D-8,.78486963D-10,-.
  41943835D-11,.41253D-13,-.759D-15/,B/-.060292226406569883D0,-.27447
  54305529745265D0,.179034314077182663D0,.261567346255046637D0,-.1773
  6020127811435820D0,.047196689595763367D0,-.7287962479552079D-2,.7531
  7135932577 74D-3,-.5632079141057D-4,.3206532537655D-5,-.14407233274D
  8-6,.5248794787D-8,-.158375525D-9,.4026331D-11,-.87473D-13,.1643D-1
  94/,C/1.29671754121052964D0,-1.19180116054121687D0,1.28799409885767
  1762D0,-.661443934134543253D0,.177709117239728283D0,-.0291755248061
  254208D0,.3240270182683857D-2,-.260444389348581D-3,.15887019239932D
  3-4,-.76175678054D-6,.29497070073D-7,-.942421298D-9,.25281237D-10,-
  4.57774D-12,.11386D-13,.1955D-15/,D/.040608211771868508D0,-.1286973
  564381350D0,-.76729636288664594D0,.675615780772187667D0,-.226624991
  6556754924D0,.042319180355336904D0,-.5131641161061085D-2,.440478629
  78671D-3,-.28304640149515D-4,.1416624364492D-5,-.56884400399D-7,.18
  8754703 24D-8,-.51721215D-10,.1211433D-11,-.24409D-13,.428D-15/,PI/3
  9.1415926535897932D0/
   DATA E/1.998920698695037531D0,-.536522040813212D-3,.3075184787519D
  1-5,-.51705945376D-7,.1630646464D-8,-.76640914D-10,.5168262D-11,-.4
  2304586D-12,.43266D-13,-.5069D-14,.6748D-15,-.1D-15/,F/-.0311117092 1
  30674D0,.06385199426116D-4,-.741449841106D-6,.17972457248D-7,-.7271
  491594D-9,.42201219D-10,-.3206747D-11,.3006145D-12,-.333633D-13,.42
  5552D-14,-.6099D-15,.966D-16/,G/2.0018060817200274D0,.898989633085
  6941D-3,-.3987284300489D-5,.61776339606D-7,-.1871890749D-8,.8816898
  77D-10,-.5704864D-11,.469919D-12,-.46842D-13,.5453D-14,-.722D-15,.1
  807D-15/,H/.09355557413907065D0,-.96277235491571D-4,.913861525796D-
  96,-.20959761384D-7,.82291933D-9,-.46863637D-10,.3515219D-11,-.326
  1432D-12,.35968D-13,-.4561D-14,.6508D-15,-.1027D-15/
   DATA    U/.79833170337771806D0,.627824030273932236D-2,.22510873571
  1159949D-3,.1527877872300049D-4,.157817791105719D-5,.22708300408408
  2D-6,.4342656294922D-7,.1045317003796D-7,.287935462925D-8,.78080927
  3512D-9,.15141541538D-9,-.1954135293D-10,-.436057528D-10,-.28962217
  45D-10,-.1222743919D-10,-.245354238D-11,.110880533D-11,.139814476D-
  511,.70425609D-12,.12051141D-12,-.11025346D-12,-.10945044D-12,-.417
  68293D-13,.48683D-14,.160627D-13,.898852D-14,.65964D-15,-.253132D-1
  74,-.183222D-14,-.29012D-15,.45641D-15,.3855D-15,.7251D-16,-.9677D-
  816,-.8537D-16,-.1473D-16,.2382D-16,.1975D-16,.223D-17,-.653D-17,-.
  9466D-17,-.2D-19,.19D-17,.108D-17,-.19D-18,-.56D-18/
   DATA    V/.797142770484402007D0,-.167627251400034881D-1,-.372851820
  110091957D-3,-.2119852311849735D-4,-.200394040362657D-5,-.272962045
  293394D-6,-.5026542819771D-7,-.1180972280086D-7,-.321152515316D-8,-
  3.87078483344D-9,-.17395449009D-9,.1644931968D-10,.4513375613D-10,.
  43066929043D-10,.1317959611D-10,.278710776D-11,-.107958557D-11,-.14
  5544311D-11,-.75060504D-12,-.13840416D-12,.11008746D-12,.11415063D-
  612,.4496993D-13,-.410384D-14,-.1649841D-13,-.950147D-14,-.8545D-15
  7,.25694D-14,.191914D-14,.33188D-15,-.45975D-15,-.40183D-15,-.813D-
  816,.9725D-16,.8882D-16,.1661D-16,-.2399D-16,-.2056D-16,-.264D-17,.
  9661D-17,.467D-17,.11D-18,-.193D-17,-.113D-17,.17D-18,.58D-18/
   DATA    P/255.466879624362167D0,190.494320172742844D0,82.4890327440
  1240996D0,22.2748192424622309D0,4.01167376017934853D0,.509493365439
  2982871D0,.477187487961741352D-1,.341633176601234095D-2,.1924693596
  38811366D-3,.873831549662236D-5,.32609105057896D-6,.1016972672769D-
  47,.26882612695D-9,.60968928D-11,.11989083D-12,.206305D-14,.3132D-1
  56,.42D-18/
```

```
 DATA    Q/-21.05766017740244020D0,-4.56343358644839501D0,8.005368868
1700334770D0,5.26363286687392001D0,1.51155676029227910D0,.2590844324
2349001970D0,.30080722420511874SD-1,.25363081680861990D-2,.16270837
5904302330D-3,.82160259399306600D-5,.33519525563133D-6,.1128121138760-
47,.31858797963D-9,.76575743SD-11,.158554130-12,.2857520-14,.45230-
516,.63D-18/
 DATA    R/259.89023786047729200,181.31261604057026500,69.3959176337
1344475D0,16.33455055252206020D0,2.57145990634775491D0,.2878555118O4
267205100,.23993079147840551 8D-1,.15430190156272191 4D-2,.78756785/5
3416515D-4,.32641381223098600-5,.11194628456389D-6,.32276165202 3D-8,
4.79290559290D-10,.16789726 2D-11,.309529 6D-13,.5012D-15,.718D-17,.9D
5-19/
 DATA    S/-26.68809548086266 78D0,-1.83923922428619943D0,9.36161 7831
13953886800,4.6663870266628418500,1.10146199300485220D0,.16107430165
26147824D0,.16300049289616417 6D-1,.121705699451574089D-2,.70010 6278
3547575 3D-4,.52025106919350 5D-5,.11936797074664D-6,.3696783278 36D-8
4,.96659752D-10,.21625531 9D-11,.41872790-13,.7086D-15,.1057D-16,.14
5D-18/
 COMMON //ZRZY/T(20),W(20),X(20),Y(20),Z(18)
 DATA T/2.40482555769577277D0,5.52007811028631065D0,8.65372791291 10
11221 7D0,11.79153443901428160D0,14.93091770848778 59D0,18.0710639679 1
20922 5D0,21.21163662987925 89D0,24.35247153074930 27D0,27.49347913204
30254 8D0,30.63460646843197 5D0,33.77582021357356 87D0,36.91709835366 4
40439D0,40.05842576462823 9D0,43.19979171317673 0D0,46.34118837166181
54D0,49.48260989739781 7D0,52.62405184111499 6D0,55.76551075501997900
6,58.90698392608094 2D0,62.04846919022716 9D0/
 DATA W/3.83170597020751230D0,7.01558666981561 87D0,10.1734681350 6272
12D0,13.32369193631422 3D0,16.47063005087763 28D0,19.61585851046824 20
20,22.76008438059277 19D0,25.90367208761838 3D0,29.04682853491685 5D0,
332.18967991097440400,35.33230755008386 5D0,38.47476623477161 5D0,41.
46170942128144 51D0,44.75931899765282 1D0,47.90146088718544 7D0,51.043
53518357150 9D0,54.18555364106132 0D0,57.32752543790101 D0,60.46945 78
64534 7491D0,63.61135669848123 2D0/
 DATA X/.51914749728946 6776D0,-.34026480655836 812D0,.27145229992 6361
19D0,-.23245983136472476D0,.20654643307799 6D0,-.18772880304043941 00
2,.17326589422922985 0D0,-.16170155068924998D0,.15218121377059451D0,-
3 .14416597768637 319D0,.13729694340850291D0,-.13132462666866801D0,.
41260694971272734 3D0,-.12139862477175019D0,.11721119889066592D0,-.1
51342919261642952D0,.10999114304627713D0,-.10684788825471745D0,.103
6959572669371 8D0,-.10129349893401842D0/,Y/-.40275939570255 393D0,.3
7001157525261325 4D0,-.24970487705784317D0,.21835940724787294D0,-.19
86465371466657 16D0,.18006337534431553D0,-.16718460047381802D0,.1567
9249862528522 1D0,-.14801110997277 5D0,.14060579818398220D0,-.1342112
14631000065D0,.12861662207207D0,-.12366790076983 7D0,.11924981201069
2005D0,-.11527369412016875D0,.11167049665920975D0,-.10838534894368 0
356D0,.10537405539524985D0,-.10260056710340887D0,.10003514681162D0/
4,/1.57079632679489 7D0,62.D0,15116.D0,12554474.D0,4716.D0,390241 6.
5D0,56.D0,90664.D0,7381280.D0,24.D0,1956.4D0,6.D0,4.D0,2466720.
6D0,2D0,3.D0,5.D0,7.D0/
 END
```

```
WN XQT SAMPLE
       7      168
    75.     75.     75.     75.     75.     75.     75.     75.     75.     75.     75.     75.
    76.     77.     78.     79.     83.     81.     82.     81.     80.     79.     78.     77.
    76.     76.     75.     74.     74.     74.     75.     77.     80.     83.     87.     90.
    93.     94.     95.     94.     93.     91.     87.     85.     83.     81.     79.     77.
     0.      0.      2.      2.     10.     15.     15.     10.      3.      2.      0.      0.
    10.     10.     30.     20.     10.      5.      4.      2.      1.      0.      0.      0.
     0.      0.      0.      0.      0.      3.     16.     29.     41.     42.    343.    350.
   350.    348.     47.     44.     41.     29.     20.      8.      0.      0.      0.      0.
     0.      0.      0.      2.      2.      2.      0.      0.      0.      2.      2.      2.
     1.      1.      1.      0.      0.      0.      1.      1.      1.      1.      1.      1.
      3       0
   1.86    0.42    0.77
 0.      0.3333  0.3333
    5.0     5.0   5.333   5.666
     0.   0.015   0.028
    1.0
```

```
@XQT RESP      INPUT DATA FORMAT FOR RESP
      0
     1.
      3        0        0        0
     0.       0.       0.       0.     0.92
   0.01     0.12      80.      0.3       0.
    0.5     1.00     140.     0.20     0.00
INSIDE SURFACE
1/8-IN LIN/LEUM
6-IN CONCRETE
      3        0        0        1
     0.       0.       0.       0.     0.92
   0.01     0.12      80.      0.3       0.
    0.5     1.00     140.     0.20     0.00
    0.7     120.     0.25
INSIDE SURFACE
1/8-IN LIN/LEUM
6-IN CONCRETE
GROUND
      3        0        0        2
     0.       0.       0.       0.     0.92
   0.25     0.12      80.      0.3       0.
     0.       0.       0.       0.     0.92
 OUTSIDE SURFACE
3-IN PAPER
 OUTSIDE SURFACE
      3        0        1        0
     0.       0.       0.       0.     0.92
   0.01     0.13      80.      0.3       0.
    0.5     1.00     140.      0.2     0.00
     3.       3.     3.01     3.51     3.51
INSIDE SURFACE( 3FT R)
1/8-IN LINOLEUM
6-IN CONCRETE
      3        0        1        1
     0.       0.       0.       0.     0.92
   0.01     0.13      80.      0.3       0.
    0.5     1.00     140.      0.2     0.00
    0.7     120.     0.25
     3.       3.     3.01     3.51     3.51
INNER SURFACE(3FT R)
1/8-IN LINOLEUM
6-IN CONCRETE
GROUND
      2        0        1        2
   0.01     0.13      80.      0.3       0.
     0.       0.       0.       0.     0.92
   0.25      50.     0.32
    0.5     0.51     0.51
PAPER(6 IN R)
1/8-IN LINOLEUM
SURFACE
      3        0        2        0
     0.       0.       0.       0.     0.92
   0.01     0.13      80.      0.3       0.
    0.5     1.00     140.      0.2     0.00
     3.       3.     3.01     3.51     3.51
INNER SURFAC(3-FT R)
1/8-IN LINOLEUM
6-IN CONCRETE
      3        0        2        1
     0.       0.       0.       0.     0.92
```

```
   0.01    0.13    80.     0.3     0.
    0.5    1.00   140.     0.2    0.00
    0.7    120.   0.25
     3.      3.   3.01    3.51    3.51
 INSIDE SURFACE( 3FT R)           6
 1/8-IN LINOLEUM
 6-IN CONCRETE
 GROUND
      2       0      2       2
   0.01    0.13    80.     0.3     0.
     0.      0.     0.      0.    0.92
   0.25     50.   0.32
    0.5    0.51   0.51
 6-IN RAD PAPER
 1/8-IN LINOLEUM
 SURFACE
     99
@ EOF
@ FIN
```

PLANE WALL

LAYER NO	L(I)	K(I)	(I)	C(I)	RES(I)	DESCRIPTION OF LAYERS
1	.000	.000	.00	.000	.92	INSIDE SURFACE
2	.010	.120	80.00	.300	.00	1/8-IN LIN/LEUM
3	.500	1.000	140.00	.200	.00	6-IN CONCRETE

TIME INCREMENT DT= 1.

THERMAL CONDUCTANCE U= .665

RESPONSE FACTORS

J	X	Y	Z
0	.8809	.0276	5.9704
1	-.0911	.1801	-3.5343
2	-.0470	.1684	-.6904
3	-.0291	.1081	-.4057
4	-.0182	.0678	-.2527
5	-.0114	.0424	-.1580
6	-.0071	.0265	-.0989
7	-.0044	.0166	-.0618

COMMON RATIO CR= .62549

LAYER NO	L(I)	K(I)	(I)	C(I)	RES(I)	DESCRIPTION OF LAYERS
1	.000	.000	.00	.000	.92	INSIDE SURFACE
2	.010	.120	80.00	.300	.00	1/8-IN LINOLEUM
3	.500	1.000	140.00	.200	.00	6-IN CONCRETE
		.700	120.00	.230		GROUND

TIME INCREMENT DT= 1.

THERMAL CONDUCTANCE U= .665

RESPONSE FACTORS

J	F
0	.88083
1	-.08610
2	-.02270
3	-.01117
4	-.00700
5	-.00472
6	-.00329
7	-.00232
8	-.00163
9	-.00114
10	-.00079
11	-.00053
12	-.00036
13	-.00023
14	-.00015
15	-.00010
16	-.00006
17	-.00004
18	-.00002
19	-.00002
20	-.00001
21	-.00001
22	-.00000
23	-.00000
24	-.00000
25	-.00000
26	-.00000
27	-.00000
28	-.00000
29	-.00000
30	-.00000
31	-.00000
32	-.00000

COMMON RATIO CR= .62552

PLANE

LAYER NO	L(I)	K(I)	(I)	C(I)	RES(I)	DESCRIPTION OF LAYERS
1	.000	.000	.00	.000	.92	OUTSIDE SURFACE
2	.250	.120	60.00	.300	.00	3-IN PAPER
3	.000	.000	.00	.000	.92	OUTSIDE SURFACE

TIME INCREMENT DT= 1.

THERMAL CONDUCTANCE U= .255

RESPONSE FACTORS

J	F
0	+1.44446
1	- .40390
2	- .23878
3	- .18292
4	- .14116
5	- .10896
6	- .08411
7	- .06492
8	- .05012
9	- .03868
10	- .02986
11	- .02305
12	- .01779
13	- .01373

COMMON RATIO CR= .77190

CYLINDRICAL WALL

LAYER NO	L(I)	K(I)	(I)	C(I)	RES(I)	DESCRIPTION OF LAYERS
1	.000	.000	.00	.000	.92	INSIDE SURFACE
2	.010	.130	80.00	.300	.00	1/8-IN LINOLEUM
3	.500	1.000	140.00	.200	.00	6-IN CONCRETE

TIME INCREMENT DT= 1.

THERMAL CONDUCTANCE U= .586

RESPONSE FACTORS

J	X	Y	Z
0	.8881	.0254	5.8264
1	-.0880	.1638	-3.5342
2	-.0443	.1507	-.6843
3	-.0269	.0950	-.3953
4	-.0165	.0584	-.2418
5	-.0101	.0359	-.1484
6	-.0062	.0220	-.0911
7	-.0038	.0135	-.0560

COMMON RATIO CR= .61405

CYLINDRICAL WALL

LAYER NO	L(I)	K(I)	(I)	C(I)	RES(I)	DESCRIPTION OF LAYERS
1	.000	.000	.00	.000	.92	INNER SURFACE(3FT
2	.010	.130	80.00	.300	.00	1/8-IN LINOLEUM
3	.500	1.000	140.00	.200	.00	6-IN CONCRETE
		.700	120.00	.230		GROUND

TIME INCREMENT DT= 1.

THERMAL CONDUCTANCE U= .586

RESPONSE FACTORS

J	F
0	.88803
1	-.08387
2	-.02481
3	-.01290
4	-.00799
5	-.00521
6	-.00348
7	-.00234
8	-.00158
9	-.00106
10	-.00070
11	-.00046
12	-.00029
13	-.00019
14	-.00012
15	-.00007
16	-.00005
17	-.00003
18	-.00002
19	-.00001
20	-.00001
21	-.00000
22	-.00000
23	-.00000
24	-.00000
25	-.00000
26	-.00000
27	-.00000
28	-.00000
29	-.00000
30	-.00000
31	-.00000

COMMON RATIO CR= .61407

SPHERICAL WALL

LAYER NO	L(I)	K(I)	(I)	C(I)	RES(I)	DESCRIPTION OF LAYERS
1	.000	.000	.00	.000	.92	INNER SURFAC(3-FT
2	.010	.130	80.00	.300	.00	1/8-IN LINOLEUM
3	.500	1.000	140.00	.200	.00	6-IN CONCRETE

TIME INCREMENT DT= 1.

THERMAL CONDUCTANCE U= .514

RESPONSE FACTORS

J	X	Y	Z
0	.8905	.0232	5.6855
1	-.0842	.1480	-3.5316
2	-.0414	.1338	-.6766
3	-.0246	.0828	-.3840
4	-.0148	.0500	-.2305
5	-.0089	.0301	-.1388
6	-.0054	.0181	-.0837
7	-.0032	.0109	-.0504

COMMON RATIO CR= .60252

CYLINDRICAL

LAYER NO	L(I)	K(I)	(I)	C(I)	RES(I)	DESCRIPTION OF LAYERS
		.250	56.00	.320		PAPER(6 IN R)
1	.010	.130	80.00	.300	.00	1/8-IN LINOLEUM
2	.000	.000	.00	.000	.92	SURFACE

TIME INCREMENT DT= 1.

THERMAL CONDUCTANCE U= 1.002

RESPONSE FACTORS

J	F
0	.73468
1	-.20036
2	-.10006
3	-.07064
4	-.05563
5	-.04593
6	-.03869
7	-.03284
8	-.02797
9	-.02385
10	-.02035
11	-.01736
12	-.01481
13	-.01264
14	-.01079

COMMON RATIO CR= .85332

SPHERICAL WALL

LAYER NO	L(I)	K(I)	(I)	C(I)	RES(I)	DESCRIPTION OF LAYERS
1	.000	.000	.00	.000	.92	INSIDE SURFACE(3I
2	.010	.130	80.00	.300	.00	1/8-IN LINOLEUM
3	.500	1.000	140.00	.200	.00	6-IN CONCRETE
		.700	120.00	.230		GROUND

TIME INCREMENT DT= 1.

THERMAL CONDUCTANCE U= .514

RESPONSE FACTORS

J	F
0	.89045
1	-.08081
2	-.02585
3	-.01379
4	-.00841
5	-.00533
6	-.00344
7	-.00223
8	-.00145
9	-.00093
10	-.00060
11	-.00038
12	-.00024
13	-.00015
14	-.00009
15	-.00005
16	-.00003
17	-.00002
18	-.00001
19	-.00001
20	-.00000
21	-.00000
22	-.00000
23	-.00000
24	-.00000
25	-.00000
26	-.00000
27	-.00000
28	-.00000
29	-.00000

COMMON RATIO CR= .60254

SPHERICAL

LAYER NO	L(I)	K(I)	(I)	C(I)	RES(I)	DESCRIPTION OF LAYERS
		.250	50.00	.320		6-IN RAD PAPER
1	.010	.130	80.00	.300	.00	1/8-IN LINOLEUM
2	.000	.000	.00	.000	.92	SURFACE

TIME INCREMENT DT= 1.

THERMAL CONDUCTANCE U= 1.002

RESPONSE FACTORS

J	F
0	.71441
1	-.23048
2	-.12067
3	-.08440
4	-.06333
5	-.04860
6	-.03756
7	-.02909
8	-.02254
9	-.01747
10	-.01355
11	-.01050
12	-.00814

COMMON RATIO CR= .77522

DEPARTMENT OF COMMERCE
NATIONAL BUREAU OF STANDARDS