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A Method for the Dynamic Determination of the Elastic, Dielectric, and Piezoelectric Constants of Quartz

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A Method for the Dynamic Determination of the Elastic, Dielectric, and Piezoelectric Constants of Quartz

Saul A. Basri

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Saul A. Basri

Several dynamic determinations have been made of the constants of quartz. Most of these determinations do not take into account the piezoelectric effect; those that do, suffer from certain deficiencies which are discussed in this paper.

Taking into account the piezoelectric effect, expressions for the frequency of longitudinal vibration of rectangular bars and thickness shear vibration of infinite plates are derived and applied to the determination of the constants of quartz. On the basis of present theoretical knowledge, it is suggested that the best procedure is to measure the frequency of vibration of 2 particular cuts for rectangular bars, and 7 cuts for plates, and to measure the capacitance at zero frequency of a rectangular bar. These 10 measurements provide the data for determining the 6 elastic, 2 dielectric, and 2 piezoelectric constants of quartz uniquely.

For accurate measurements, it will first be necessary to determine the linear coefficients of expansion of quartz to higher accuracy than is available at present.

Conventions

The following conventions will be used throughout this report:

1. The word "quartz" shall denote "α-quartz."
2. X_1, X_2, X_3 are, respectively, the coordinates along the electrical, mechanical, and optical axes of quartz.
3. $\{X_1', X_2', X_3'\}$ is a rectangular coordinate system rotated with respect to $\{X_1, X_2, X_3\}$.
4. All unprimed tensor components are referred to $\{X_1, X_2, X_3\}$ and primed tensor components to $\{X_1', X_2', X_3'\}$.
5. Latin indices such as $i, j, k, \cdots$ take the values 1 to 3, and Greek indices such as $\mu, \nu, \cdots$ take the values 1 to 6. The usual correspondences
   $11\leftrightarrow1,\ 22\leftrightarrow2,\ 33\leftrightarrow3,\ 23\leftrightarrow4,\ 31\leftrightarrow5,\ 12\leftrightarrow6$
   are adopted here.
6. All repeated indices shall be summed over unless otherwise indicated (summation convention).
7. All the constants of quartz discussed in the report are assumed to be *adiabatic* constants unless otherwise mentioned.
8. Electrostatic units will be used throughout this report.

1. General Considerations

1.1. Introduction

Alpha-quartz is characterized by 6 independent elastic constants, 2 independent dielectric constants, and 2 independent piezoelectric constants, or 10 constants in all. For the complete description of quartz, it is also necessary to know the mass density and the linear coefficients of expansion along the optical and mechanical axes.

There are essentially two methods for the measurement of the above-mentioned 10 constants of quartz, the static method and the dynamic method. The static method yields the isothermal constants, whereas the dynamic method yields the adiabatic constants. The isothermal and adiabatic constants are related through the specific heat constants. Since the specific heat constants of quartz are not accurately known, and the main interest is in the *vibrating* quartz crystal, the dynamic method is indicated.

In order to be able to make use of the dynamic method to determine the constants of quartz, it is necessary to have theoretical formulas that relate the frequency of vibration of the quartz crystal used to the constants of interest. At present, such formulas exist only for very simple shapes of crystals such as the rectangular bar and the infinite plate. There are many simplifications and assumptions made in the derivation of these formulas. Consequently, the determination of the constants of quartz with the help of these formulas is trustworthy only if the conditions under which these formulas are valid are met experimentally. For this reason, a careful analysis of the conditions of validity of these formulas is necessary before these formulas can be used.

Since the dynamic method is chosen, all of the constants used in the following will be understood to be the *adiabatic* constants.

1.2. Equations of Motion and Boundary Conditions

The equations of motion of a vibrating crystal are given by

$$\frac{\partial T'_{ij}}{\partial X'_j}=\rho\frac{\partial^2\xi'_i}{\partial t^2}, \tag{1}$$

where T'_{ij} are the stress components in an arbitrary coordinate system $\{X'_1, X'_2 X'_3\}$, ξ'_i the components of displacement along X'_i, ρ the mass density and t the time. The time variable t in (1) can be eliminated by setting

$$\xi'_i=U'_i(X'_1, X'_2, X'_3)e^{iwt}. \tag{2}$$

Substituting (2) into (1) leads to the equation

$$\frac{\partial T'_{ij}}{\partial X'_j}+\rho w^2U'_i=0. \tag{3}$$

If we write these equations explicitly, we get, with the help of convention (5),

$$\frac{\partial T'_1}{\partial X'_1}+\frac{\partial T'_6}{\partial X'_2}+\frac{\partial T'_5}{\partial X'_3}+\rho w^2U'_1=0, \tag{4a}$$

$$\frac{\partial T'_6}{\partial X'_1}+\frac{\partial T'_2}{\partial X'_2}+\frac{\partial T'_4}{\partial X'_3}+\rho w^2U'_2=0, \tag{4b}$$

$$\frac{\partial T'_5}{\partial X'_1}+\frac{\partial T'_4}{\partial X'_2}+\frac{\partial T'_3}{\partial X'_3}+\rho w^2U'_3=0, \tag{4c}$$

The stress components T'_{ij} are given by the expression

$$T'_{ij}=C'_{ijkl}\underline{S}'_{kl}-e'_{kij}E'_k, \tag{5}$$

or equivalently by

$$T'_\mu=C'_{\mu\nu}\underline{S}'_\nu-e'_{i\mu}E'_i. \tag{5'}$$

$C'_{\mu\nu}$ are the elastic stiffness constants, $e'_{i\mu}$ the stress piezoelectric constants, $\underline{S}'_\nu$ the strain components and E'_i the ith component of the electric field inside the crystal. $\underline{S}'_{ij}$ are related to U'_i by

$$\underline{S}'_{ij}=\frac{1}{2}\left(\frac{\partial U'_i}{\partial X_j}+\frac{\partial U'_j}{\partial X_i}\right). \tag{6}$$

Written explicitly, (6) becomes

$$\underline{S}'_1=\underline{S}'_{11}=\frac{\partial U'_1}{\partial X'_1}, \ \underline{S}'_2=\underline{S}'_{22}=\frac{\partial U'_2}{\partial X'_2}, \ \underline{S}'_3=\underline{S}'_{33}=\frac{\partial U'_3}{\partial X'_3},$$

$$\underline{S}'_4=2\underline{S}'_{23}=\frac{\partial U'_2}{\partial X'_3}+\frac{\partial U'_3}{\partial X'_2}, \ \underline{S}'_5=2\underline{S}'_{13}$$

$$=\frac{\partial U'_1}{\partial X'_3}+\frac{\partial U'_3}{\partial X'_1}, \ \underline{S}'_6=2\underline{S}'_{12}=\frac{\partial U'_1}{\partial X'_2}+\frac{\partial U'_2}{\partial X'_1}. \tag{7}$$

E'_i consists of two terms, namely

$$E'_i=E_i^{P'}+E_i^{A'}. \tag{8}$$

$E_i^{P'}$ is due to the piezoelectric effect of a freely vibrating crystal and $E_i^{A'}$ are the components of the applied field. The relation between $E_i^{P'}$ and $\underline{S}'_\mu$ or T'_μ will be derived in the next section.

If the surfaces of the crystal are free, the boundary conditions that must be satisfied are

$$T'_{ij}n'_j=0 \textit{ on all surfaces}, \tag{9}$$

where n'_j are the components of the unit normal to the surface under consideration, and T'_{ij} are given by (5). In the case of the rectangular crystal shown in figure 1, (9) becomes

$$T'_1=T'_6=T'_5=0 \text{ at } X'_1=\pm\frac{l_1}{2}$$

$$T'_6=T'_2=T'_4=0 \text{ at } X'_2=\pm\frac{l_2}{2}$$

$$T'_5=T'_4=T'_3=0 \text{ at } X'_3=\pm\frac{l_3}{2}. \tag{10}$$

Notice that it is important that *all* the equations (4) and *all* the conditions (10) must be satisfied, and not only some of them. This fact is too frequently forgotten in the literature, even by well-known authors.

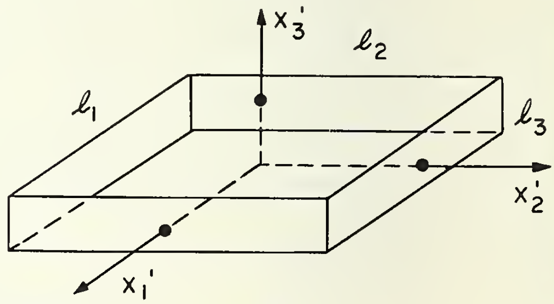


Figure 1.

1.3. Piezoelectric Effect in Thin Plates

We shall now derive the expression of E'_i in terms of $\underline{S}'_{ij}$ and T'_{ij} for a freely vibrating crystal plate whose thickness h is much smaller than the lateral dimensions.

Consider a central region R (see fig. 2) whose boundary has a distance from the edges of the crystal which is much larger than thickness h. If the crystal is isolated, the total charge inside R will be approximately zero, and the charge inside R will be expected to vary along X'_3 only. We may thus think of the charge distribution inside R to consist of plane sheets normal to X'_3, and such that to every sheet having a uniform surface charge density $+\sigma$ there is another sheet of uniform surface charge density $-\sigma$. In other words, the region R can be considered to be composed of an infinite number of parallel plate condensers.

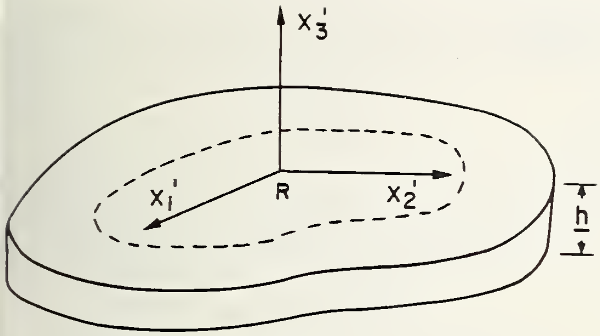


FIGURE 2.

Let us consider one of these parallel condensers with a surface charge density σ (see fig. 3). Applying Gauss' law to two pill boxes about the two plates, in a region far from the edges, we come to the conclusion that the electric displacement D is normal to the surfaces and has the magnitude $4\pi\sigma$ between the two plates, but vanishes outside them. From this it follows that the normal component of D *outside* the crystal vanishes at the surfaces of the crystal over a central region such as R.

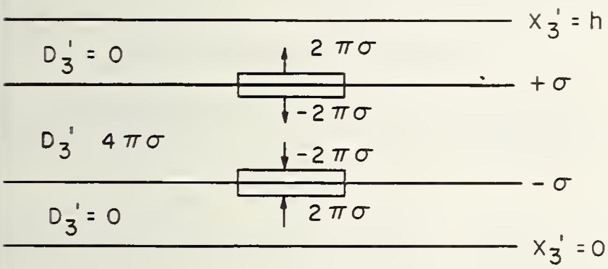


FIGURE 3.

Since the only surface charges on the crystal are polarization charges, it follows from the boundary conditions on D that the normal component of D *inside* the crystal will also vanish at the surfaces. Consequently, if we take a pill box inside the crystal whose top is adjacent to the top of the crystal and whose bottom is insde the crystal (see fig. 4), the total electrical flux through the top will be zero. Since the crystal is homogeneous, the total flux entering the sides of the pill box will be equal to the total flux leaving the sides; in other words, the total flux through the sides is zero. Furthermore, since all the charges inside the pill box are polarization charges, it follows from Gauss' law that the total flux out of the whole box must be zero. Therefore, the flux through the bottom of the pill box will be zero. From this we can conclude that the normal component of D vanishes everywhere within the region R, i.e.,

$$D_3'=0 \textit{ everywhere inside } R. \tag{1}$$

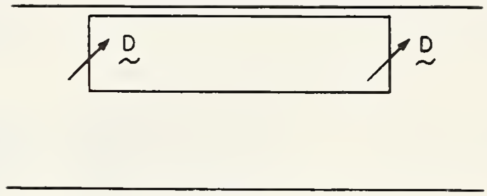


FIGURE 4.

Since the charge within R does not vary with X_1' and X_2', we conclude from symmetry that the electric field should be along X_3' everywhere within R, i.e.,

$$E_i^{P'}=E_3^{P'}\delta_{3i}, \tag{2}$$

where δ_{ij} is the Kronecker delta.

In general,

$$\begin{aligned} D_3' &= \epsilon_{3i}^{S'}E_i^{P'}+4\pi e_{3\mu}'\underline{S}_\mu' \\ &= \epsilon_{3i}^{T'}E_i^{P'}+4\pi d_{3\mu}'T_\mu', \end{aligned} \tag{3}$$

where $\epsilon_{ij}^{S'}$ and $\epsilon_{ij}^{T'}$ are, respectively, the dielectric constants at constant strain and constant stress, and $e_{i\mu}'$, $d_{i\mu}'$ are piezoelectric constants. Making use of (1) and (2), we get from (3)

$$\epsilon_{33}^{S'}E_3^{P'}+4\pi e_{3\mu}'\underline{S}_\mu'=0.$$

and

$$\epsilon_{33}^{T'}E_3^{P'}+4\pi d_{3\mu}'T_\mu'=0.$$

Therefore,

$$E_3^{P'}=-\frac{4\pi}{\epsilon_{33}^{S'}}e_{3\mu}'\underline{S}_\mu' \tag{4}$$

or

$$=-\frac{4\pi}{\epsilon_{33}^{T'}}d_{3\mu}'T_\mu'. \tag{5}$$

Either (5) or (4) can be used in (2.8) to evaluate the piezoelectric effect in a plate whose thickness is along the X_3'-direction.

Cady [1, p. 312] obtains, instead of (4), the result

$$E_3^{P'}=-\frac{4\pi}{\epsilon_{33}^{S'}}e_{3\mu}'\underline{S}_\mu'+C,$$

where C is a constant determined by conditions on the boundaries of the crystal. Cady finds $C\neq 0$, whereas our derivation shows that $C=0$. In his derivation, Cady makes use of special assumptions with regard to the charge and potential distributions inside the crystal, some of which are not justified. In contrast, the derivation of (4) or (5) is based on very general arguments which are fully stated. Our results are in agreement with those obtained by Koga et al. [5].

1.4. Constants of Quartz and Their Transformations

Due to the fact that quartz has 3-fold symmetry about the X_3-axis and 2-fold symmetry about the X_1-axis, the constants of quartz are given by the following matrices:

Elastic Constants

$$\begin{array}{cccccc} C_{11} & C_{12} & C_{13} & C_{14} & 0 & 0 \\ C_{12} & C_{11} & C_{13} & -C_{14} & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ C_{14} & -C_{14} & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & C_{14} \\ 0 & 0 & 0 & 0 & C_{14} & \frac{C_{11}-C_{12}}{2} \end{array} \tag{1a}$$

$$C_{ijkl}=C_{\mu\nu} \tag{1b}$$

$$\begin{array}{cccccc} S_{11} & S_{12} & S_{13} & S_{14} & 0 & 0 \\ S_{12} & S_{11} & S_{13} & -S_{14} & 0 & 0 \\ S_{13} & S_{13} & S_{33} & 0 & 0 & 0 \\ S_{14} & -S_{14} & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{44} & 2S_{14} \\ 0 & 0 & 0 & 0 & S_{14} & 2(S_{11}-S_{12}) \end{array} \tag{2a}$$

$$S_{iijj}=S_{\mu\nu},\ S_{iijk}=\tfrac{1}{2}S_{\mu\nu}(j\neq k),\ S_{ijkl}=\tfrac{1}{4}S_{\mu\nu}(i\neq j,\ k\neq l). \tag{2b}$$

Piezoelectric Constants

$$\begin{array}{cccccc} e_{11} & -e_{11} & 0 & e_{14} & 0 & 0 \\ 0 & 0 & 0 & 0 & -e_{14} & -e_{11} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \tag{3a}$$

$$e_{ijk}=e_{i\mu}. \tag{3b}$$

$$\begin{array}{cccccc} d_{11} & -d_{11} & 0 & d_{14} & 0 & 0 \\ 0 & 0 & 0 & 0 & -d_{14} & -2d_{11} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \tag{4a}$$

$$d_{ijj}=d_{i\mu},\ d_{ijk}=\tfrac{1}{2}d_{i\mu}(j\neq k). \tag{4b}$$

Dielectric Constants

$$\begin{array}{ccc} \epsilon_{11} & 0 & 0 \\ 0 & \epsilon_{11} & 0 \\ 0 & 0 & \epsilon_{33} \end{array} \tag{5}$$

Suppose we wish to transform from the crystal axes X_i to an arbitrary rectangular coordinate system X'_i. Let

$$X'_i=\alpha^i_j X_j, \quad (6)$$

and

$$\alpha^i_j\alpha^k_l\alpha^m_n \cdots =\alpha^{ikm}_{jln} \cdots \quad (7)$$

Since the elastic, piezoelectric, and dielectric constants are the components of tensors of the 4th, 3d, and 2d rank, respectively, it follows that:

$$C'_{ijkl}=\alpha^{ijkl}_{mnpq}C_{mnpq}$$

$$=[(\alpha^{ijkl}_{1111}+{}^{ijkl}_{2222})+\tfrac{1}{2}(\alpha^{ijkl}_{1212}+\alpha^{ijkl}_{2112}+\alpha^{ijkl}_{1221}+\alpha^{ijki}_{2121})]C_{11}$$

$$+[(\alpha^{ijkl}_{1122}+\alpha^{jikl}_{2211})-\tfrac{1}{2}(\alpha^{ijkl}_{1212}+\alpha^{ijkl}_{2112}+\alpha^{ijkl}_{1221}+\alpha^{ijkl}_{2121})]C_{12}$$

$$+(\alpha^{ijkl}_{1133}+\alpha^{ijkl}_{3311}+\alpha^{ijkl}_{2233}+\alpha^{ijkl}_{3322})C_{13}+\alpha^{ijkl}_{3333}C_{33}$$

$$+(\alpha^{ijkl}_{1123}+\alpha^{ijkl}_{2311}+\alpha^{ijkl}_{1132}+\alpha^{ijkl}_{3211}+\alpha^{ijkl}_{1312}+\alpha^{ijkl}_{1321}$$

$$+\alpha^{ijkl}_{3112}+\alpha^{ijkl}_{3121}+\alpha^{ijkl}_{1213}+\alpha^{ijkl}_{1231}+\alpha^{ijkl}_{2113}+\alpha^{ijkl}_{2131}$$

$$-\alpha^{ijkl}_{2223}-\alpha^{ijkl}_{2322}-\alpha^{ijkl}_{2232}-\alpha^{ijkl}_{3222})C_{14}$$

$$+(\alpha^{ijkl}_{2323}+\alpha^{ijkl}_{3223}+\alpha^{ijkl}_{2332}+\alpha^{ijkl}_{3232}$$

$$+\alpha^{ijkl}_{1313}+\alpha^{ijkl}_{3113}+\alpha^{ijkl}_{1331}+\alpha^{ijkl}_{3131})C_{44}. \quad (8)$$

The transformation of S'_{ijkl} is the same as (8) except for the replacements,

$$C_{11}\to S_{11},\ C_{12}\to S_{12},\ C_{13}\to C_{13},\ C_{33}\to S_{33}$$

$$C_{14}\to\tfrac{1}{2}S_{14},\ C_{44}\to\tfrac{1}{4}S_{44}. \quad (9)$$

$$e'_{ijk}=\alpha^{ijk}_{lmn}e_{lmn}$$

$$=(\alpha^{ijk}_{111}-\alpha^{ijk}_{122}-\alpha^{ijk}_{212}-\alpha^{ijk}_{221})\,e_{11}$$

$$+(\alpha^{ijk}_{123}+\alpha^{ijk}_{132}-\alpha^{ijk}_{213}-\alpha^{ijk}_{231})e_{11} \quad (10)$$

The transformation of d'_{ijk} is the same as (10) except for the replacements,

$$e_{11}\to d_{11}, e_{14}\to\tfrac{1}{2}d_{14}. \quad (11)$$

$$\epsilon'_{ij}=\alpha^{ij}_{kl}\epsilon_{kl}=(\alpha^{ij}_{11}+\alpha^{ij}_{22})\epsilon_{11}+\alpha^{ij}_{33}\epsilon_{33}. \quad (12)$$

2. Longitudinal Vibration of Rectangular Bars

2.1. Zeroth Order Solution

Consider a quartz rectangular bar such as that shown in figure 1. It will be assumed that

$$l_1\gg l_2\gg l_3. \quad (1)$$

The reason for taking l_1 and $l_2\gg l_3$ is to satisfy the conditions for the validity of the expressions (sec. 1.3, eq. 4 or 5), which will make it possible to evaluate the piezoelectric effect. The assumption $l_1\gg l_2$ is made in order to minimize coupling with other modes of vibartion and thus makes it possible to simplify the problem and get a sufficiently reliable solution. The effect on the frequency of small but appreciable values of (l_2/l_1) will be given in the following section.

We are interested here in a solution of the form

$$U'_1=A\cos k\,X'_1,$$

which satisfies the equation

$$\partial^2U'_1/\partial X'^2_1=-\text{ (constant) }U'_1.$$

Since

$$\underline{S}'_1=\partial U'_1/\partial X'_1,$$

this equation can be written as

$$\partial\underline{S}'_1/\partial X'_1=-(\text{constant})U'_1. \quad (2)$$

In order to relate (2) to the equations of motion (sec. 1.2, eq 4) it is necessary to find the relation between $\underline{S}'_\mu$ and T'_μ. This, we shall do now.

In general,

$$\underline{S}'_\mu=S'_{\mu\nu}T'_\nu+d'_{i\mu}E'_i, \quad (3)$$

where $S'_{\mu\nu}$ are the elastic compliance coefficients, $d'_{i\mu}$ the strain piezoelectric constants, and E'_i is given by (sec. 1.2; eq 8). For a freely vibrating crystal,

$$E^{A\prime}_i=0, \text{ and } E'_i=E^{P\prime}_i.$$

Because of (1), we can use (sec. 1.3 eq 2 and 5) and get

$$E'_i=-(4\pi/\epsilon^{T\prime}_{33})d'_{3\nu}T'_\nu\delta_{3i}.$$

Substituting this expression into (3), we find

$$\underline{S}'_\mu=S'_{\mu\nu}T'_\nu-(4\pi/\epsilon^{T\prime}_{33})d'_{3\mu}d'_{3\nu}T'_\nu.$$

If we let

$$\tilde{S}'_{\mu\nu}\equiv S'_{\mu\nu}-(4\pi/\epsilon^{T\prime}_{33})d'_{3\mu}d'_{3\nu}, \quad (4)$$

we can write

$$\underline{S}'_\mu=\tilde{S}'_{\mu\nu}T'_\nu, \quad (5)$$

which is the desired expression. In particular, we have

$$\underline{S}'_1=\tilde{S}'_{1\mu}T'_\mu.$$

Substituting this expression into (2), gives

$$\tilde{\underline{S}}'_{1\mu}\partial T'_\mu/\partial X'_1=-(\text{constant})U'_1.$$

In order to make this equation agree with (sec. 1.2, eq 4a), it is necessary to assume that

$$\frac{\partial T_1'}{\partial X_1'} \gg \frac{\partial T_6'}{\partial X_2'} + \frac{\partial T_5'}{\partial X_3'}, \tag{6}$$

and also

$$T_1' \gg T_\mu' \text{ for all } \mu \neq 1. \tag{7}$$

The assumption (7) is quite frequently made in the literature. Some authors erroneously justify (7) as follows: The boundary conditions at $X_2 = \pm l_2/2$ and $X_3 = \pm l_3/2$ are given by (sec. 1.2, eq 10). Since l_2 and l_3 are taken to be small, the stresses $T_\mu'(\mu \neq 1)$ cannot differ appreciably from zero in the interior and therefore can be assumed to be zero everywhere. If this argument is correct then one may conclude, in the case of a crystal plate whose thickness is in the X_2 direction, that $T_6' = T_2' = T_4' \simeq 0$ everywhere. This is in conflict with the well established fact that T_6' is the principle stress in the thickness-shear vibration of such a plate. The main justification of (6) and (7), would be whether it is possible with their help, to satisfy all of the equations of motion (sec. 1.2, eq 4) and all of the boundary conditions (sec. 1.2, eq 10), at least approximately.

As far as the boundary conditions are concerned, all of the conditions imposed on $T_\mu'(\mu \neq 1)$ will be at least approximately satisfied because of (7). The remaining boundary condition

$$T_1' = 0 \text{ at } X_1' = \pm l_1/2, \tag{8}$$

can be exactly satisfied, as will be shown later.

Equation (sec. 1.2, eq 4a) becomes, with the help of (6),

$$\frac{\partial T_1'}{\partial X_1'} + \rho w^2 U_1' = 0. \tag{9}$$

Furthermore, (5) and (7) yield

$$\underline{S}_\mu' \simeq \tilde{S}_{\mu 1}' T_1'. \tag{10}$$

and in particular

$$\underline{S}_1' \simeq \tilde{S}_{11}' T_1',$$

or

$$T_1' \simeq \frac{1}{\tilde{S}_{11}'} \underline{S}_1' = \frac{1}{\tilde{S}_{11}'} \frac{\partial U_1'}{\partial X_1'}. \tag{11}$$

Substituting (11) into (9) gives the equation

$$\frac{\partial^2 U_1'}{\partial X_1'^2} + \rho w^2 \tilde{S}_{11}' U_1' = 0. \tag{12}$$

The solution of (12) is of the form

$$U_1' = A \cos kX_1' + B \sin kX_1', \tag{13}$$

where

$$k = w(\rho \tilde{S}_{11}')^{\frac{1}{2}}. \tag{14}$$

Making use of (11) the boundary condition (8) becomes

$$\frac{dU_1'}{dX_1'} = 0 \text{ at } X_1' = \pm l_1/2.$$

Imposing this condition on (13) yields

$$-A \sin k\frac{l_1}{2} + B \cos k\frac{l_1}{2} = 0, \tag{15a}$$

$$+A \sin k\frac{l_1}{2} + B \cos k\frac{l_1}{2} = 0. \tag{15b}$$

The only way to get nonzero values for A and B is to set

$$\begin{vmatrix} -\sin k\frac{l_1}{2} & \cos k\frac{l_1}{2} \\ +\sin k\frac{l_1}{2} & \cos k\frac{l_1}{2} \end{vmatrix} = -2 \sin k\frac{l_1}{2} \cos k\frac{l_1}{2} = -\sin kl_1 = 0,$$

which implies

$$kl_1 = n\pi$$

or

$$k = n\pi/l_1,\ n = 1, 2, 3, \ldots \tag{16}$$

We can evaluate B from either (15a) or (15b). Using (15a), we get

$$B = A \sin k\frac{l_1}{2} \Big/ \cos k\frac{l_1}{2}.$$

Substituting this value of B into (13) we get

$$U_1' = \left(A/\cos k\frac{l_1}{2}\right)\left(\cos kX_1' \cos k\frac{l_1}{2} + \sin kX_1' \sin k\frac{l_1}{2}\right).$$

If we set

$$A/\cos k\frac{l_1}{2} \equiv a = \text{constant},$$

we can write U_1' in the form

$$U_1' = a \cos k\left(X_1' - \frac{l_1}{2}\right). \tag{17}$$

The frequency of vibration, f_n, can be obtained from (14) and (16).

$$f_n = \frac{w_n}{2\pi} = \frac{k_n}{2\pi}(\rho \tilde{S}_{11}')^{-\frac{1}{2}} = \frac{n\pi}{2\pi l_1}(\rho \tilde{S}_{11}')^{-\frac{1}{2}}.$$

Thus

$$f_n=\frac{n}{2l_1}(\rho\tilde{S}'_{11})^{-\frac{1}{2}}, \tag{18}$$

where we have from (4),

$$\tilde{S}'_{11}=S'_{11}-(4\pi/\epsilon^{T'}_{33})(d'_{31})^2. \tag{19}$$

The solution obtained here is only a zeroth order solution. For this reason, from now on we shall denote U'_1 in (17) by $U^{0\prime}_1$ and f_n in (18) by f^0_n. The 1st order solution and the two remaining equations (sec. 1.2, eq b and c) will be discussed in the next section.

Cady [1, p. 317] states that the piezoelectric correction is zero for the longitudinal vibration of a quartz bar. The above result does not seem to bear out this conclusion. The origin of this disagreement was discussed at the end of section 1.3.

2.2. First Order Solution

We now have to find a solution which is consistent with (sec. 2.1, eqs 6 and 7) and satisfies equations (sec. 1.2, eq b and c) to terms of the order of (l_2/l_1) or less. We have seen in section 1 that one of the consequences of (sec. 2.1, eq 7) is (sec. 2.1 eq 10), namely

$$\underline{S}'_\mu\simeq\tilde{S}'_{\mu1}T'_1,$$

and in particular,

$$\underline{S}'_1\simeq\tilde{S}'_{11}T'_1.$$

Eliminating T'_1 between these two expressions gives

$$\underline{S}'_\mu\simeq(\tilde{S}'_{\mu1}/\tilde{S}'_{11})S'_1.$$

For convenience, we shall let

$$s_\mu\equiv\tilde{S}'_{\mu1}/\tilde{S}'_{11}, \tag{1}$$

and get

$$\underline{S}'_\mu\simeq s_\mu\underline{S}'_1\simeq s_\mu\underline{S}^{0\prime}_1. \tag{2}$$

We shall now make use of (2) and the assumption

$$U'_1=U^{0\prime}_1+f_1(X'_1,X'_2,X'_3) \tag{3}$$

in order to calculate the first order solution. It should be noticed that

$$U^{0\prime}_1=a\cos k_0\left(X'_1-\frac{l_1}{2}\right), \tag{4a}$$

$$\underline{S}^{0\prime}_1=\frac{dU^{0\prime}_1}{dX'_1}=-ak_0\sin k_0\left(X'_1-\frac{l_1}{2}\right), \tag{4b}$$

$$\frac{d\underline{S}^{0\prime}_1}{dX'_1}=\frac{d^2U^{0\prime}_1}{dX'^2_1}=-k_0^2U^{0\prime}_1. \tag{4c}$$

$$\underline{S}'_2=\frac{\partial U'_2}{\partial X'_2}\simeq s_2\underline{S}^{0\prime}_1\rightarrow U'_2=s_2X'_2\underline{S}^{0\prime}_1+f_2(X'_1,X'_3). \tag{5a}$$

$$\underline{S}'_3=\frac{\partial U'_3}{\partial X'_3}\simeq s_3\underline{S}^{0\prime}_1\rightarrow U'_3=s_3X'_3\underline{S}^{0\prime}_1+f_3(X'_1,X'_2). \tag{5b}$$

$$\underline{S}'_4=\frac{\partial U'_2}{\partial X'_3}+\frac{\partial U'_3}{\partial X'_2}=\frac{\partial f_2}{\partial X'_3}+\frac{\partial f_3}{\partial X'_2}\simeq s_4\underline{S}^{0\prime}_1. \tag{5c}$$

$$\underline{S}'_5=\frac{\partial U'_1}{\partial X'_3}+\frac{\partial U'_3}{\partial X'_1}=\frac{\partial f_1}{\partial X'_3}+s_3X'_3(-k_0^2U^{0\prime}_1)+\frac{\partial f_3}{\partial X'_1}\simeq s_5\underline{S}^{0\prime}_1. \tag{5d}$$

$$\underline{S}'_6=\frac{\partial U'_1}{\partial X'_2}+\frac{\partial U'_2}{\partial X'_1}=\frac{\partial f_1}{\partial X'_2}+s_2X'_2(-k_0^2U^{0\prime}_1)+\frac{\partial f_2}{\partial X'_1}\simeq s_6\underline{S}^{0\prime}_1. \tag{5e}$$

From (5d) and (5e) we get, respectively,

$$f_1=s_5X'_3\underline{S}^{0\prime}_1+\tfrac{1}{2}s_3X'^2_3k_0^2U^{0\prime}_1-\frac{\partial f_3}{\partial X'_1}X'_3+g(X'_1,X'_2),$$

$$f_1=s_6X'_2\underline{S}^{0\prime}_1+\tfrac{1}{2}s_2X'^2_2k_0^2U^{0\prime}_1-\frac{\partial f_2}{\partial X'_1}X'_2+h(X'_1,X'_3).$$

Comparing these two expressions for f_1, we get

$$g(X'_1,X'_2)=s_6X'_2\underline{S}^{0\prime}_1+\tfrac{1}{2}s_2X'^2_2k_0^2U^{0\prime}_1,$$

$$h(X'_1,X'_3)=s_5X'_3\underline{S}^{0\prime}_1+\tfrac{1}{2}s_3X'^2_3k_0^2U^{0\prime}_1,$$

$$X'_3\partial f_3/\partial X'_1=X'_2\partial f_2/\partial X'_1.$$

The last equation yields

$$f_2=f(X'_1)X'_3, f_3=f(X'_1)X'_2.$$

Substituting these expressions into (5c), we find

$$2f(X'_1)=s_4\underline{S}^{0\prime}_1 \text{ or } f(X'_1)=\tfrac{1}{2}s_4\underline{S}^{0\prime}_1.$$

Gathering all of the above results, we get

$$U'_1\simeq U^{0\prime}_1+(s_6X'_2+s_5X'_3)\underline{S}^{0\prime}_1+\tfrac{1}{2}(s_2X'^2_2+s_3X'^2_3+s_4X'_2X'_3)k_0^2U^{0\prime}_1 \tag{6a}$$

$$U'_2\simeq(s_2X'_2+\tfrac{1}{2}s_4X'_3)\underline{S}^{0\prime}_1 \tag{6b}$$

$$U'_3\simeq(s_3X'_3+\tfrac{1}{2}s_4X'_2)\underline{S}^{0\prime}_1. \tag{6c}$$

It remains to find the first order correction to the frequency and satisfy (sec. 1.2, eq b and c). For this purpose, it is necessary to calculate T'_μ. Making use of (2) and (3), we get

$$\underline{S}'_1=\frac{\partial U'_1}{\partial X'_1}=\underline{S}^{0\prime}_1+\partial f_1/\partial X'_1, \underline{S}'_\mu\simeq s_\mu\underline{S}^{0\prime}_1(\mu\neq1). \tag{7}$$

From (sec. 1.2, eq 5) and (sec. 1.3, eq 4) we have

$$T'_\mu = C'_{\mu\nu}\underline{S}'_\nu - e'_{i\mu}E'_i$$

$$= C'_{\mu\nu}\underline{S}'_\nu - e'_{3\mu}E'_3$$

$$= C'_{\mu\nu}\underline{S}'_\nu + \frac{4\pi}{\epsilon^{S'}_{33}} e'_{3\mu}e'_{3\nu}\underline{S}'_\nu.$$

Thus if we let

$$\tilde{C}'_{\mu\nu} = C'_{\mu\nu} + \frac{4\pi}{\epsilon^{S'}_{33}} e'_{3\mu}e'_{3\nu}, \tag{8}$$

we can write

$$T'_\mu = \tilde{C}'_{\mu\nu}S'_\nu. \tag{9}$$

Making use of (sec. 2.1, eq 5), we get

$$T'_\mu = \tilde{C}'_{\mu\nu}\underline{S}'_\nu = \tilde{C}'_{\mu\nu}\tilde{S}'_{\nu\lambda}T'_\lambda = \delta_{\mu\lambda}T'_\lambda,$$

which implies

$$\tilde{C}'_{\mu\nu}\tilde{S}'_{\nu\lambda} = \delta_{\mu\lambda}. \tag{10}$$

Substituting the values of $\underline{S}'_\mu$ given in (7) into (9), and making use of (3), (6a), (10), and (1), we get

$$T'_1 = \frac{1}{\tilde{S}'_{11}}\underline{S}^{0\prime}_1 + \tilde{C}'_{11}k_0^2[-(s_6X'_2+s_5X'_3)U^{0\prime}_1 + \tfrac{1}{2}(s_2X'^2_2+s_3X'^2_3+s_4X'_2X'_3)\underline{S}^{0\prime}_1], \tag{11a}$$

and for $\mu\neq 1$

$$T'_\mu = \tilde{C}'_{\mu 1}k_0^2[-(s_6X'_2+s_5X'_3)U^{0\prime}_1 + \tfrac{1}{2}(s_2X'^2_2+s_3X'^2_2+s_4X'_2X'_3)\underline{S}^{0\prime}_1]. \tag{11b}$$

If we now substitute (11) into (sec. 1.2, eq 4) and make use of (1), (sec. 2.1, eq 14), and assume that

$$\xi_i = U'_i(X'_1, X'_2, X'_3)e^{i\omega t}, \tag{12}$$

where U'_i are given by (6) and w is the corrected frequency, we get

$$[1+\tfrac{1}{2}k_0^2\tilde{C}'_{11}(\tilde{S}'_{12}X'^2_2+\tilde{S}'_{13}X'^2_3+\tilde{S}_{14}X'_2X'_3) + (\tilde{C}'_{15}\tilde{S}'_{15}+\tilde{C}'_{16}\tilde{S}'_{16})]U^{0\prime}_1 + [\tilde{C}'_{11}(\tilde{S}'_{16}X'_2+\tilde{S}'_{15}X'_3)$$
$$-\tfrac{1}{2}\tilde{C}'_{16}(2\tilde{S}'_{12}X'_2+\tilde{S}'_{14}X'_3) - \tfrac{1}{2}\tilde{C}'_{15}(\tilde{S}'_{14}X'_2+2\tilde{S}'_{13}X'_3)]\underline{S}^{0\prime}_1$$
$$=\left(\frac{w}{w_0}\right)^2\{[1+\tfrac{1}{2}\frac{k_0^2}{\tilde{S}'_{11}}(\tilde{S}'_{12}X'^2_2+\tilde{S}'_{13}X'^2_3 + \tilde{S}'_{14}X'_2X'_3)]U^{0\prime}_1 + \frac{1}{\tilde{S}'_{11}}(\tilde{S}'_{16}X'_2+\tilde{S}'_{15}X'_3)\underline{S}^{0\prime}_1\}, \tag{13a}$$

$$[\tfrac{1}{2}k_0^2\tilde{C}'_{16}(\tilde{S}'_{12}X'^2_2+\tilde{S}'_{13}X'^2_3+\tilde{S}'_{14}X'_2X'_3) + (\tilde{C}'_{12}\tilde{S}'_{16} + \tilde{C}'_{14}\tilde{S}'_{15})]U^{0\prime}_1 + [\tilde{C}'_{16}(\tilde{S}'_{16}X'_2+\tilde{S}'_{15}X'_3) - \tfrac{1}{2}\tilde{C}'_{12}(2\tilde{S}'_{12}X'_2$$
$$+\tilde{S}'_{14}X'_3) - \tfrac{1}{2}\tilde{C}'_{14}(\tilde{S}'_{14}X'_2+2\tilde{S}'_{13}X'_3)]\underline{S}^{0\prime}_1$$
$$=\left(\frac{w}{w_0}\right)^2\frac{1}{\tilde{S}'_{11}}(\tilde{S}'_{12}X'_2+\tfrac{1}{2}\tilde{S}'_{14}X'_3)\underline{S}^{0\prime}_1, \tag{13b}$$

$$[\tfrac{1}{2}k_0^2\tilde{C}'_{15}(\tilde{S}'_{12}X'^2_2+\tilde{S}'_{13}X'^2_3+\tilde{S}'_{14}X'_2X'_3) + (\tilde{C}'_{14}\tilde{S}'_{16} + \tilde{C}'_{13}\tilde{S}'_{15})]U^{0\prime}_1 + [\tilde{C}'_{15}(\tilde{S}'_{16}X'_2+\tilde{S}'_{15}X'_3)$$
$$-\tfrac{1}{2}\tilde{C}'_{14}(2\tilde{S}'_{12}X'_2+\tilde{S}'_{14}X'_3) - \tfrac{1}{2}\tilde{C}'_{14}(\tilde{S}'_{14}X'_2+2\tilde{S}'_{13}X'_3)]\underline{S}^{0\prime}_1$$
$$=\left(\frac{w}{w_0}\right)\frac{1}{\tilde{S}'_{11}}(\tfrac{1}{2}\tilde{S}'_{14}X'_2+\tilde{S}^1_{13}X'_3)\underline{S}^{0\prime}_1. \tag{13c}$$

From (4) we have

$$\int_{-l_1/2}^{+l_1/2}(U^{0\prime}_1)^2dX'_1 = a^2l_1/2,$$

$$\int_{-l_1/2}^{l_1/2}(\underline{S}^{0\prime}_1)^2dX'_1 = k_0^2a^2l_1/2,$$

$$\int_{-l_1/2}^{l_1/2}U^{0\prime}_1\underline{S}^{0\prime}_1dX'_1 = 0.$$

Multiplying equations (13) by

$$\int_{-l_1/2}^{l_1/2}U^{0\prime}_1dX'_1\int_{-l_2/2}^{l_2/2}dX'_2\int_{-l_3/2}^{l_3/2}dX'_3$$

and making use of the above integrals, we get

$$1+(\tilde{C}'_{15}\tilde{S}'_{15}+\tilde{C}'_{16}\tilde{S}'_{16}) + \frac{k_0^2}{24}\tilde{C}'_{11}(\tilde{S}'_{12}l_2^2+\tilde{S}'_{13}l_3^2)$$
$$=\left(\frac{w}{w_0}\right)^2\left[1+\frac{k_0^2}{24}\frac{1}{\tilde{S}'_{11}}(\tilde{S}'_{12}l_2^2+\tilde{S}'_{13}l_3^2)\right], \tag{14a}$$

$$(\tilde{C}'_{12}\tilde{S}'_{16}+\tilde{C}'_{14}\tilde{S}'_{15}) + \frac{k_0^2}{24}\tilde{C}'_{16}(\tilde{S}'_{12}l_2^2+\tilde{S}'_{13}l_3^2) = 0, \tag{14b}$$

$$(\tilde{C}'_{14}\tilde{S}'_{16}+\tilde{C}'_{13}\tilde{S}'_{15}) + \frac{k_0^2}{24}\tilde{C}'_{15}(\tilde{S}'_{12}l_2^2+\tilde{S}'_{13}l_3^2) = 0. \tag{14c}$$

According to (sec. 2.1, eq 16)

$$k_0^2 = n^2\pi^2/l_1^2.$$

Substituting this value into (14) and setting

$$\Delta \equiv n^2\frac{\pi^2}{24}\left[\frac{\tilde{S}'_{12}}{\tilde{S}'_{11}}\left(\frac{l_2}{l_1}\right)^2 + \frac{\tilde{S}'_{13}}{\tilde{S}'_{11}}\left(\frac{l_3}{l_1}\right)^2\right], \tag{15}$$

we get from (14a)

$$\left(\frac{w}{w_0}\right)^2 = \frac{1+(\tilde{C}'_{15}\tilde{S}'_{15}+\tilde{C}'_{16}\tilde{S}'_{16})+\tilde{C}'_{11}\tilde{S}'_{11}\Delta}{1+\Delta} = \left(\frac{f}{f_0}\right)^2. \tag{16}$$

Equations (14b) and (14c) can be solved for $\tilde{S}'_{15}$ and $\tilde{S}'_{16}$, and yield

$$\tilde{S}'_{15}=\frac{\tilde{C}'_{12}\tilde{C}'_{15}-\tilde{C}'_{14}\tilde{C}'_{16}}{\tilde{C}'^{2}_{14}-\tilde{C}'_{12}\tilde{C}'_{13}}\,\tilde{S}'_{11}\Delta, \tag{17a}$$

$$\tilde{S}'_{16}=\frac{\tilde{C}'_{13}\tilde{C}'_{16}-\tilde{C}'_{14}\tilde{C}'_{15}}{\tilde{C}'_{14}-\tilde{C}'_{12}\tilde{C}'_{13}}\,\tilde{S}'_{11}\Delta, \tag{17b}$$

which show that S'_{15} and S'_{16} are of the order of, Δ or $(l_2/l_1)^2$. In practice, since all the constants of quartz are not known to much better than about 1 percent, it is not feasible to try to find the orientations at which (17) is satisfied. If $(l_2/l_1)^2 \lesssim 0.01$, then (16) should provide a satisfactory approximation at orientations such that

$$\tilde{S}'_{15}\simeq 0 \text{ and } \tilde{S}'_{16}\simeq 0. \tag{18}$$

From this we see that *the assumptions* (sec. 2.1, eq 6 and 7) *are only valid at the orientations where* (18) *holds,* contrary to the usual belief that they are good for any orientation. The terms in (16) containing S'_{15} and S'_{16}, will take care of the fact that these coefficients may not be exactly zero. The solutions (16) will be reliable provided orientations are used for which (18) is fufilled.

It should be mentioned that the type of calculation used by Davies [6] to estimate the correction to the frequency due to the other modes of vibration is wrong, since it is assumed in this calculation that *only* the kinetic energy changes, due to these extra modes. An examination of the derivation here would show that the change in potential energy is equally important and cannot be neglected. In fact, all the terms in (16) containing $C'_{\mu\nu}$ are due to the change in potential energy.

2.3. Optimum Orientations of Bars

We have seen in the previous section that the only orientations for which (sec. 2.2, eq 16) is reliable, are those for which (sec. 2.2, eq 18) is satisfied, namely

$$\tilde{S}'_{15}\simeq 0 \text{ and } \tilde{S}'_{16}\simeq 0.$$

From (sec. 2.1, eq 4) and the approximately known values of the constants of quartz, it can be verified that

$$[(4\pi/\epsilon^{T'}_{33})d'_{3\mu}d'_{3\nu}]/S'_{\mu\nu}<.01.$$

We are thus justified in replacing the above conditions by the simpler ones

$$S'_{15}\simeq 0 \text{ and } S'_{16}\simeq 0. \tag{1}$$

We shall now find out the orientations at which (1) is satisfied and for which the longitudinal vibration can be excited piezoelectrically.

Consider the bar shown in figure 5. If we let

$$X'_i=\alpha^i_j X_j,$$

then we have from figure 5

$$(\alpha^i_j)=\begin{bmatrix} S_\theta C_\phi & S_\theta S_\phi & C_\theta \\ -C_\theta C_\phi & -C_\theta S_\phi & S_\theta \\ S_\phi & -C_\phi & 0 \end{bmatrix} \tag{2}$$

where:

$$S_\theta=\sin\theta,\ C_\theta=\cos\theta,$$

$$S_\phi=\sin\phi,\ C_\phi=\cos\phi.$$

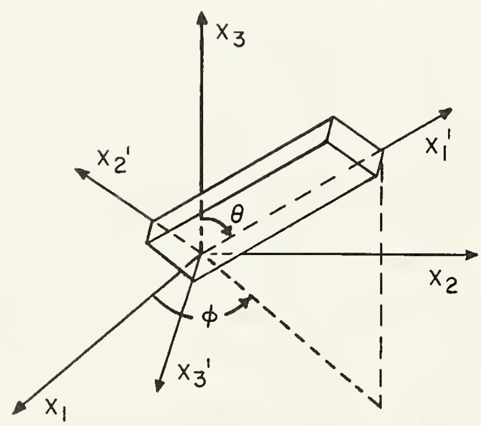
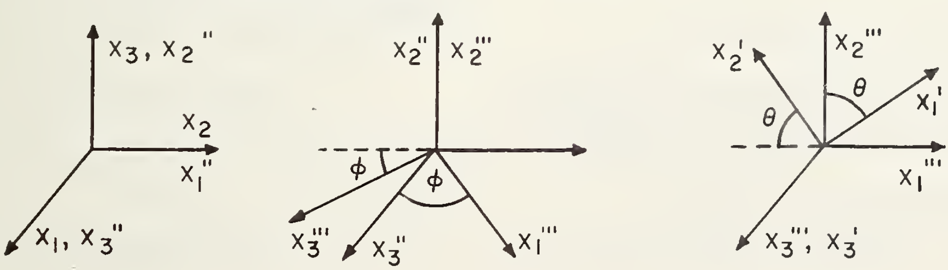


FIGURE 5.

The matrix (2) can be obtained as the product of three rotation matrices representing the three successive rotations shown in the figures below.



Thus

$$(\alpha_j^i)=\begin{bmatrix} S_\theta & C_\theta & 0 \\ -C_\theta & S_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} S_\phi & 0 & C_\phi \\ 0 & 1 & 0 \\ -C_\phi & 0 & S_\phi \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

In the primed coordinate system we have, using (sec. 1.4. eq 8, 9) and (2),

$$\begin{aligned}\tfrac{1}{2}S'_{15}=&S_{11}(S_\theta^3C_\phi^3S_\phi-S_\theta^3S_\phi^3C_\phi)+S_{12}(-S_\theta^3C_\phi^3S_\phi\\ &+S_\theta^3S_\phi^3C_\phi)+S_{13}(C_\theta^2S_\theta C_\phi S_\phi-C_\theta^2S_\theta S_\phi C_\phi)\\ &+\tfrac{1}{2}S_{14}(9S_\theta^2S_\phi^2C_\theta C_\phi-3S_\theta^2C_\phi^3C_\theta)+\tfrac{1}{4}S_{44}\\ &(-2S_\theta S_\phi C_\theta^2C_\phi+2S_\theta C_\phi C_\theta^2S_\phi)+\tfrac{1}{4}S_{66}\\ &(-2S_\theta^3C_\phi^3S_\phi+2S_\theta^3C_\phi S_\phi^3)\\ =&S_{11}S_\theta^3C_\phi S_\phi(C_\phi^2-S_\phi^2)-S_{12}S_\theta^3C_\phi S_\phi(C_\phi^2-S_\phi^2)\\ &+\tfrac{3}{2}S_{14}S_\theta^2C_\theta C_\phi(4S_\phi^2-1)-\tfrac{1}{2}S_{66}S_\theta^3C_\phi S_\phi(C_\phi^2-S_\phi^2)\\ =&(S_{11}-S_{12}-\tfrac{1}{2}S_{66})S_\theta^3C_\phi S_\phi(C_\phi^2-S_\phi^2)\\ &+\tfrac{3}{2}S_{14}S_\theta^2C_\theta C_\phi(4S_\phi^2-1).\end{aligned}$$

For quartz,

$$S_{66}=2(S_{11}-S_{12}).$$

Thus

$$S'_{15}=-3S_{14}\sin^2\theta\cos\theta\cos 3\phi, \qquad (3)$$

from which we conclude that

$$\begin{aligned}S'_{15}=0 \text{ when } &\theta=0, 90^\circ \text{ for all values of } \phi,\\ \text{and } &\phi=30, 90^\circ \text{ for all values of } \theta. \qquad (4)\end{aligned}$$

Similarly, it can be shown that

$$\begin{aligned}S'_{16}=&2S'^{1112}=2\alpha_i^1\alpha_j^1\alpha_k^1\alpha_l^2S^{ijkl}\\ =&\sin\theta\{\cos\theta\,[2(S_{33}\cos^2\theta-S_{11}\sin^2\theta)-\\ &(S_{44}+2S_{13})\cos 2\theta]-S_{14}\sin 3\theta\sin 3\phi\}. \qquad (5)\end{aligned}$$

Probably the most reliable values of $S_{\mu\nu}$, are those given by Koga et al. [5] (with a correction in sign). In units of 10^{-13} cm²/dyne, these values are:

$$\begin{aligned}&S_{11}=12.8,\ S_{33}=9.7,\ S_{44}=20.0,\ S_{12}=-1.8,\\ &S_{13}=-1.2,\quad S_{14}=-4.5,\quad S_{66}=29.2. \qquad (6)\end{aligned}$$

Substituting these values into (5), we get

$$\begin{aligned}S'_{16}\simeq&\sin\theta\,[\cos\theta(19.4\cos^2\theta-25.6\sin^2\theta\\ &-17.8\cos 2\theta)+4.5\sin 3\theta\sin 3\phi],\end{aligned}$$

from which we conclude that

$$\begin{aligned}S'_{16}\simeq 0 \qquad \text{when } &\theta=0 \text{ for all } \phi,\\ &\theta=90 \text{ for } \phi=0, 60,\\ &\phi=90 \text{ for } \theta=70, \pi-49,\\ &\phi=30 \text{ for } \theta=\pi-70, 49. \qquad (7)\end{aligned}$$

No other values of ϕ are listed in (7) since, according to (4), $S'_{15}=0$ only at $\phi=30°$ and 90°.

Examination of figure 6 will show that $\phi=30°$, $\theta=\pi-70°$ is the same orientation as $\phi=90°+120°$, $\theta=+70°$. Since quartz has trigonal symmetry about X_3, it follows that all the properties of quartz at $\phi=30°$, $\theta=\pi-70°$, are the same as those at $\phi=90°$, $\theta=70°$. In the same way, it can be shown that the properties of quartz at $\phi=30°$, $\theta=49°$, are the same as they are at $\phi=90°$, $\theta=\pi-49°$. Consequently, the orientations in (7) at $\phi=30°$, give no more information than those at $\phi=90°$, and need not be considered further.

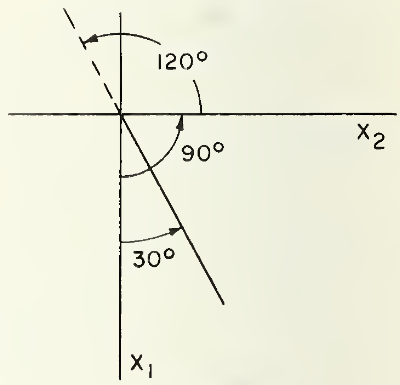


FIGURE 6.

It remains now to find out at which of the orientations listed in (7) it is possible to excite longitudinal vibrations piezoelectrically. Since T'_{11} is the stress mainly responsible in exciting the longitudinal vibration, the driving electric field is taken along the thickness direction of the bar, i.e., along the X'_3 direction, we have from (sec. 1.2, eq 5),

$$T'_{11}=-e'_{311}E'_3.$$

Consequently, it is possible to excite the longitudinal vibration piezoelectrically for all orientations at which $e'_{311}\neq 0$.

Making use of (2), we have, with the help of (sec. 1.4, eq 10),

$$e'_{31}=\sin\theta\,(\sin 3\phi\, e_{11}+2\cos\theta\, e_{14}).$$

Making use of the values of e_{11} and e_{14} by Koga et al. [5] (with correction in sign) we get in cgs units

$$e'_{31}\simeq\sin\theta\,(5.25\sin\theta\sin 3\phi+2.44\cos\theta)\times 10^4.$$

From this, we conclude that

$$e'_{31}=0 \text{ when } \theta=0^\circ \text{ for all } \phi, \quad \theta=90^\circ \text{ for } \phi=0,\ 60^\circ, \quad \theta=25^\circ \text{ for } \phi=90^\circ. \tag{8}$$

Comparing (4), (7), and (8), we see that the orientation at which $S'_{15}\simeq 0$, $S'_{16}\simeq 0$, and longitudinal vibrations can be excited piezoelectrically, are:

$$\phi=90^\circ,\ \theta=70^\circ,\ \pi-49^\circ. \tag{9}$$

At $\phi=90^\circ$, there is the additional advantage that $S'_{14}=0$ for all θ.

2.4. Constants of Quartz at $\phi=90^\circ$

We have seen in the previous section that all the orientations that can be used, occur at $\phi=90^\circ$ (see figure 5). We shall, therefore, write down all the constants of quartz occurring in (sec. 2.2, eq 16) for these orientations.

Making use of (sec. 2.3, eq 2) and letting

$$\sin n\,\theta=S_n,\ \cos n\,\theta=C_n,$$

we get

$$(\alpha^i_j)=\begin{bmatrix} 0 & S & C \\ 0 & -C & S \\ 1 & 0 & 0 \end{bmatrix} \text{ for } \phi=90^\circ. \tag{1}$$

Substituting the values of α^i_j given by (1) into (sec. 1.4, eq 8 to 12), we get

$$S'_{11}=S^4S_{11}+C^4S_{33}+S^2C^2(2S_{13}+S_{44})-2S^3CS_{14}.$$

$$S'_{12}=S^2C^2(S_{11}+S_{33}-S_{44})+(S^4+C^4)S_{13}-SC\,C_2S_{14}.$$

$$S'_{13}=S^2S_{12}+C^2S_{13}+SCS_{14}.$$

$$S'_{15}=0.$$

$$S'_{16}=2S[S^2C(-S_{11}+S_{13}+\tfrac{1}{2}S_{44}) + C^3(S_{33}-S_{13}-\tfrac{1}{2}S_{44})+\tfrac{1}{2}S_3S_{14}].$$

$$C'_{11}=S^4C_{11}+C^4C_{33}+S_2^2(\tfrac{1}{2}C_{13}+C_{44})-4S^3CC_{14}.$$

$$C'_{15}=0.$$

$$C'_{16}=2S[S^2C(-C_{11}+C_{13}+2C_{44}) + C^3(C_{33}-C_{13}-2C_{44})+S_3C_{14}].$$

$$d'_{31}=-S^2d_{11}+SCd_{14}.$$

$$d'_{32}=-C^2d_{11}-SCd_{14}.$$

$$d'_{33}=d_{11}.$$

$$d'_{35}=0.$$

$$d'_{36}=S_2d_{11}-C_2d_{14}.$$

$$e'_{31}=-S^2e_{11}+S_2e_{14}.$$

$$e'_{35}=0.$$

$$e'_{36}=\tfrac{1}{2}S_2e_{11}-C_2e_{14}.$$

$$\epsilon'^S_{33}=\epsilon^S_{11}.$$

$$\epsilon'^T_{33}=\epsilon^T_{11}.$$

$$\tilde{S}'_{1\mu}=S'_{1\mu}-(4\pi/\epsilon'^T_{33})d'_{31}d'_{3\mu}$$

$$\tilde{C}'_{1\mu}=C'_{1\mu}-(4\pi/\epsilon'^S_{33})e'_{31}e'_{3\mu}.$$

It was stated at the outset that there are only 6 independent elastic constants, 2 piezoelectric constants, and 2 dielectric constants. The equations that relate $C_{\mu\nu}$ and $S_{\mu\nu}$ are given by [2, p. 207]:

$$\left.\begin{aligned} &2C_{11}=\frac{S_{33}}{\alpha}+\frac{S_{44}}{\beta},\ 2C_{12}=\frac{S_{33}}{\alpha}-\frac{S_{44}}{\beta},\ C_{13}=-\frac{S_{13}}{\alpha}, \\ &C_{14}=-\frac{S_{14}}{\beta},\ C_{33}=\frac{S_{11}+S_{12}}{\alpha},\ C_{44}=\frac{S_{11}-S_{12}}{\beta}, \\ &C_{66}=\frac{C_{11}-C_{12}}{2}=\frac{S_{44}}{2\beta}, \\ &\text{where} \\ &\alpha=S_{33}(S_{11}+S_{12})-2S_{13}^2, \\ &\beta=S_{44}(S_{11}-S_{12})-2S_{14}^2. \end{aligned}\right\} \tag{2}$$

$$\left.\begin{aligned} &2S_{11}=\frac{C_{33}}{\alpha'}+\frac{C_{44}}{\beta'},\ 2S_{12}=\frac{C_{33}}{\alpha'}-\frac{C_{44}}{\beta'},\ S_{13}=-\frac{C_{13}}{\alpha'}, \\ &S_{14}=-\frac{C_{14}}{\beta'},\ S_{33}=\frac{C_{11}+C_{12}}{\alpha'},\ S_{44}=\frac{C_{11}-C_{12}}{\beta'}, \\ &S_{66}=2(S_{11}-S_{12})=2\frac{C_{44}}{\beta'}, \\ &\text{where} \\ &\alpha'=C_{33}(C_{11}+C_{12})-2C_{13}^2, \\ &\beta'=C_{44}(C_{11}-C_{12})-2C_{14}^2. \end{aligned}\right\} \tag{3}$$

The equations that relate $d_{i\mu}$, $e_{i\mu}$, ϵ^S_{ij}, and ϵ^T_{ij} are given by [2, p. 452]:

$$e_{11}=(C_{11}-C_{12})d_{11}+C_{14}d_{14},\ e_{14}=2C_{14}d_{11}+C_{44}d_{14}. \tag{4}$$

$$\frac{\epsilon^S_{11}}{4\pi}=\frac{\epsilon^T_{11}}{4\pi}-(2e_{11}d_{11}+e_{14}d_{14}),\ \epsilon^S_{33}=\epsilon^T_{33}. \tag{5}$$

Since $S'_{15}=0$, $d'_{35}=0$, $C'_{15}=0$, and $e'_{35}=0$, it follows that $\tilde{S}'_{15}=0$ and $\tilde{C}'_{15}=0$ for $\phi=90^\circ$ at any value of θ. Consequently, the expression (sec. 2.2, eq 16) for the frequency becomes

$$(1+\Delta)\left(\frac{f}{f_0}\right)^2=1+\tilde{C}'_{16}\tilde{S}'_{16}+\tilde{C}'_{11}\tilde{S}'_{11}\Delta.$$

Making use of (sec. 2.1, eq 18) and (sec. 2.2, eq 15), i.e.

$$f_{0n}=\frac{n}{2l_1}(\rho\tilde{S}'_{11})^{-\frac{1}{2}} \tag{6}$$

and

$$\tilde{S}'_{11}\Delta=\frac{n^2}{24}\left(\frac{\pi}{l_1}\right)^2(l_2^2\tilde{S}'_{12}+l_3^2\tilde{S}'_{13}),$$

and letting

$$\gamma=(2l_1f)^2\rho \tag{7}$$

we get, for $n=1$, the equation

$$\gamma\tilde{S}'_{11}+(\gamma-\tilde{C}'_{11})\frac{1}{24}\left(\frac{\pi}{l_1}\right)^2(l_2^2\tilde{S}'_{12}+l_3^2\tilde{S}'_{13})-\tilde{C}'_{16}\tilde{S}'_{16}=1. \tag{8}$$

Equation (8) yields 2 independent equations, one for each of the 2 orientations given in (sec. 2.3, eq 9). Therefore, at least 8 more equations are necessary to determine all of the 10 constants.

2.5. Concluding Remarks

In order to be able to use (sec. 2.4, eq 16) it is necessary to use a rectangular bar for which (see fig. 1)

$$l_2\ll l_1 \text{ and } l_3\ll l_2.$$

The bar must be fully plated on the surfaces $X'_3=\pm l_3/2$. The lowest mode of vibration ($n=1$) should be used, since the correction factor Δ in (sec. 2.2, eq 15) depends on n^2.

The mass density ρ occurring in (4.7) can be obtained from the data in [4, p. 288]. An equation of the form

$$\rho=A+B\tau+C\tau^2+D\tau^3$$

can be used to obtain an exact fit of this data, and the result valid for the range $-200°\ C\leq\tau\leq+100°$ C, is

$$\rho=2.6510-10^{-5}\tau[9.167+10^{-2}\tau-0.167(10^{-2}\tau)^2] \tag{1}$$

where τ is the temperature in degrees centigrade and ρ is in gm/cm^3.

The dimensions l_1, l_2, l_3 of the bar should be measured accurately before plating at some temperature τ_0, say room temperature. The dimensions at any other temperature can be calculated from the curves given in [4, p. 384] for the expansivity

$$a(\tau)=\frac{l(\tau)-l(0°\text{C})}{l(0°C)}. \tag{2}$$

A fit to 2 or 3 significant figures (the same accuracy to which the curves can be read) can be obtained with an equation of the form

$$a=(A+B\tau+C\tau^2)\tau.$$

The result of such fitting valid for the range $-200°\ C\leq\tau\leq+100°\ C$, is given by

$$10^3a_1(\tau)=10^3a_2(\tau)$$

$$=[1.282+0.117(10^{-2}\tau)+0.028(10^{-2}\tau)^2+0.023(10^{-3}\tau)^3](10^{-2}\tau). \tag{3a}$$

$$10^3a_3(\tau)=[0.687+0.087(10^{-2}\tau)+0.003(10^{-2}\tau)^2+0.003(10^{-2}\tau)^3](10^{-2}\tau), \tag{3b}$$

where $a_i(\tau)$ is the expansivity associated with the X_i direction.

From (1) we have

$$l(\tau)=l(0)[1+a(\tau)],$$

and

$$l(\tau_0)=l(0)[1+a(\tau_0)].$$

Thus

$$\frac{l(\tau)}{l(\tau_0)}=\frac{1+a(\tau)}{1+a(\tau_0)}\simeq[1+a(\tau)][1-a(\tau_0)]\simeq 1-a(\tau_0)+a(\tau),$$

from which we obtain

$$l(\tau)=l(\tau_0)[1-a(\tau_0)+a(\tau)]. \tag{4}$$

Let $\mathbf{e}_i$, $\mathbf{e}'_i$ be the unit vectors along X_i and X'_i, respectively. We can expand the l_i dimension along the crystal axes as follows (not using the summation convention):

$$\mathbf{l}_i(\tau_0)=l_i(\tau_0)\mathbf{e}'_i=l_i(\tau_0)\sum_{1}^{3}{}_j\alpha_i^{-1j}\mathbf{e}_j=\Sigma_j l_i^j(\tau_0)\mathbf{e}_j. \tag{5a}$$

where

$$l_i^j(\tau_0)=l_i(\tau_0)\alpha_i^{-1j} \tag{5b}$$

and α_j^{-1i} is the inverse of α_j^i. Since (α_j^i) is orthogonal, $\alpha_j^{-1i}=\alpha_j^i$, and we get from (4.8)

$$(\alpha_j^{-1i})=\begin{bmatrix}0 & 0 & 1\\ S & -C & 0\\ C & S & 0\end{bmatrix}\text{ for }\phi=90°. \tag{6}$$

According to (4) we have with the help of (5b)

$$l_i^j(\tau)=l_i^j(\tau_0)[1-a_j(\tau_0)+a_j(\tau)]$$

$$=l_i(\tau_0)[1-a_j(\tau_0)+a_j(\tau)]\alpha_i^{-1j}.$$

Consequently,

$$\mathbf{l}_i(\tau)=\Sigma_j l_i^j(\tau)\mathbf{e}_j$$

$$=l_i(\tau_0)\Sigma_j[1-a_j(\tau_0)+a_j(\tau)]\alpha_i^{-1j}\mathbf{e}_j \tag{7}$$

which shows that $l_i(\tau)$ is not parallel to $l_i(\tau_0)$. i.e., the orientation of the bar changes slightly with temperature. Strictly speaking, a correction should be made for this, but this correction is $\lesssim 0.1$ percent, as can be seen from (3), and is thus of the same order of magnitude as the terms we have been neglecting.

From (7) we have

$$|\mathbf{l}_i(\tau)|^2 = \mathbf{l}_i(\tau)\cdot\mathbf{l}_i(\tau)$$
$$= l_i(\tau_0)^2 \sum_1^3{}_j [1-a_j(\tau_0)+a_j(\tau)]^2(\alpha_i^{-1j})^2,$$

or

$$l_i(\tau) = l_i(\tau_0)\left\{\sum_1^3{}_j [1-a_j(\tau_0)+a_j(\tau)]^2(\alpha_i^{-1j})^2\right\}^{\frac{1}{2}}, \quad (8)$$

which is the desired expression for l_i at any temperature τ.

The frequency will be shifted due to plating, and the frequency for zero plating can best be obtained by measuring the frequency as a function of plating thickness and extrapolating to zero thickness.

3. Thickness Vibration of Quartz Plates

3.1. Thickness Vibration of an Infinite Plate

We saw in the last chapter that we have to find at least 8 more trustworthy formulas relating the frequency of vibration to the constants of quartz. The thickness vibration of infinite plates has been studied extensively and thus offers a possibility for obtaining a few more reliable formulas, if the conditions of validity of the formulas can be met experimentally. We shall see that this is possible.

Much of the theory of thickness vibration of infinite plates can be found in [1]. However, for ease of reference, and in order to have a theory in the form we have been developing, we shall present this theory in this section.

Consider a quartz plate of lateral dimensions much larger than the thickness h. It is customary to choose X_2' along the thickness; and we shall follow this custom here.

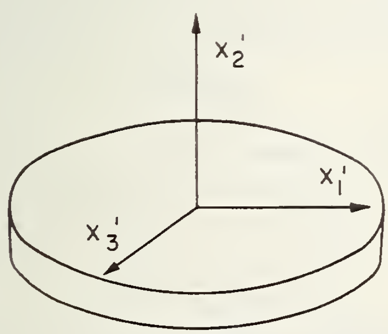


Figure 7.

The equations of motion are given by (sec. 1.2, eq 3), i.e.,

$$\frac{\partial T_{ij}'}{\partial X_j'} + \rho w^2 U_i = 0. \quad (1)$$

The expression for T_{ij}' can be obtained from (sec. 2.2, eq 8 and 9) by replacing the index '3' by '2'. The result is

$$T_{ij}' = \tilde{C}_{ijkl}' S_{kl}'. \quad (2)$$

where

$$\tilde{C}_{ijkl}' = C_{ijkl}' + \frac{4\pi}{\epsilon_{22}^{S'}} e_{2ij}' e_{2kl}'. \quad (3)$$

Making use of (sec. 1.2, eq 6) and the symmetry properties of $\tilde{C}_{ijkl}'$, we can write (2) in the form

$$T_{ij}' = \tilde{C}_{ijkl}' \frac{1}{2}\left(\frac{\partial U_k'}{\partial X_l'} + \frac{\partial U_l'}{\partial X_k'}\right) = \tilde{C}_{ijkl}' \frac{\partial U_k'}{\partial X_l'}. \quad (4)$$

Substituting (4) into (1), we get

$$\tilde{C}_{ijkl}' \frac{\partial^2 U_k'}{\partial X_j' \partial X_l'} + \rho w^2 U_i' = 0. \quad (5)$$

If the lateral dimensions of the plate are sufficiently large compared to the thickness, the edge effects can be neglected, and we can assume that

$$U_k' = U_k'(X_2').$$

Thus, (5) becomes

$$\tilde{C}_{i2j2}' \frac{\partial^2 U_j'}{\partial X_2'^2} + \rho w^2 U_i' = 0. \quad (6)$$

Equation (6) constitutes 3 coupled equations of the second order. In order to solve these equations, we use the usual method of transforming to 'normal' coordinates in which the equations are decoupled. For this purpose it is necessary to rewrite (6) with due attention to contravarient and covarient components, and get

$$\tilde{C}'^{i2j2} \frac{\partial^2 U_j'}{\partial X_2'^2} + pw^2 U'^i = 0. \quad (6')$$

If β_j^i are the coefficients of an orthogonal transformation, then we have

$$U_j' = \beta_j^m \psi_m,\ U'^i = \beta_n^{-1i} \psi^n, \quad (7)$$

and

$$\tilde{C}'^{i2j2} = \beta_k^{-1i} \beta_l^{-1j} \gamma^{kl}. \quad (8)$$

We shall assume that β^i_j is so chosen, that

$$\gamma^{kl}\begin{Bmatrix} 0 \text{ if } k\neq l \\ \gamma_k \text{ if } k=l \end{Bmatrix}. \tag{9}$$

Substituting these expressions into (6′), and discontinuing the summation convention for the time being, we get

$$\Sigma_{jkl}\beta_k^{-1i}\beta_k^{-1j}\gamma_k\beta_j^l\frac{\partial^2\psi_l}{\partial X_2'^2}=\Sigma_{km}\beta_k^{-1i}\delta_k^l\gamma_k\frac{\partial^2\psi_l}{\partial X_2'^2}$$

$$=\Sigma_k\beta_k^{-1i}\gamma_k\frac{\partial^2\psi_k}{\partial X_2'^2}=-\rho w^2\Sigma\beta_k^{-1i}\psi^k.$$

Multiplying both sides of last equality by β_i^j and summing over i, we get

$$\Sigma_{ik}\beta_i^j\beta_k^{-1i}\gamma_k\frac{\partial^2\psi_k}{\partial X_2'^2}=\Sigma_k\,\delta_k^j\gamma_k\frac{\partial^2\psi_k}{\partial X_2'^2}=\frac{\partial^2\psi_j}{\partial X_2'^2}\gamma_j$$

$$=-\rho w^2\Sigma_{ik}\beta_i^j\beta_k^{-1i}\psi^k=-\rho w^2\Sigma_k\,\delta_k^l\psi^k=-\rho w^2\psi^j.$$

Thus

$$\gamma_i\frac{\partial^2\psi_i}{\partial X_2'^2}+\rho w^2\,\psi_i=0, \tag{10}$$

which are the desired decoupled equations. The solution of (10) can be readily written as

$$\psi_i=A_i\sin k_iX_2'+B_i\cos k_i\,X_2', \tag{11}$$

where

$$k_i=\sqrt{\rho/\gamma_i}\,w. \tag{12}$$

The problem now is to calculate β_j^i, γ_i and k_i. Multiplying both sides of (8) by β_j^l, summing over j, and making use of (9), we get

$$\Sigma_j\tilde{C}'^{\,i2j2}\beta_j^l=\Sigma_{jk}\beta_k^{-1i}\beta_k^{-1j}\beta_j^l\gamma_k=\Sigma_k\beta_k^{-1i}\delta_k^l\gamma_k=\beta_l^{-1i}\gamma_l.$$

Since β_i^l is orthogonal, $\beta_l^{-1i}=\beta_i^l$, and we get

$$\Sigma_j\tilde{C}'^{\,i2j2}\beta_j^k=\beta_i^k\gamma_k=\Sigma_j\delta_i^j\beta_j^k\gamma_k,$$

or

$$\Sigma_j(\tilde{C}'^{\,i2j2}-\delta_i^j\gamma_k)\beta_j^k=0. \tag{13}$$

Equation (13) represents 3 homogeneous linear equations, and thus has a nonzero solution only if

$$\det\,(\tilde{C}'^{\,i2j2}-\delta_i^j\gamma_k)=0, \tag{14}$$

which is the well known *secular* equation. Expanding (14) and making use of convention (5), we get

$$\begin{vmatrix} \tilde{C}'_{66}-\gamma_k & \tilde{C}'_{26} & \tilde{C}'_{46} \\ \tilde{C}'_{26} & \tilde{C}'_{22}-\gamma_k & \tilde{C}'_{24} \\ \tilde{C}'_{46} & \tilde{C}'_{24} & \tilde{C}'_{44}-\gamma_k \end{vmatrix}=0. \tag{14′}$$

Which is a cubic equation in γ_k and has the three real roots γ_1, γ_2, and γ_3. For each of the roots γ_k, two of equations (13) and the normalization condition

$$\sum_1^3{}_i\,(\beta_i^k)^2=1 \tag{15}$$

can be solved uniquely for β_1^k, β_2^k, β_3^k. Thus with each value of γ_k there is associated a frequency of vibration given by (12) and a direction of vibration given by

$$\mathbf{e}_k''=\Sigma_i\beta_k^{-1i}\mathbf{e}_i'=\Sigma_i\beta_i^k\mathbf{e}_i'. \tag{16}$$

The only thing remaining to complete the solution is to solve for k_i, using the boundary conditions on the surfaces $X_2'=\pm h/2$, given by the second equation of sec. 1.2, eq 10. Making use of (4) and (7), the boundary conditions take the form

$$T'^{2i}=\Sigma_{jk}\tilde{C}'^{\,2ijk}\frac{\partial U_j'}{\partial X_k'}=\Sigma_j\tilde{C}'^{\,2ij2}\frac{\partial U_j'}{\partial X_2'}$$

$$=\Sigma_{jk}\tilde{C}^{2ij2}\beta_j^k\frac{\partial\psi_k}{\partial X_2'}=0\text{ at }X=\pm\frac{h}{2}.$$

Since these constitute homogeneous linear equations, and $\det(\Sigma_j\tilde{C}'^{\,2ij2}\beta_j^k)\neq0$, it follows that

$$\frac{\partial\psi_k}{\partial X_2'}=0\text{ at }X_2'=\pm\frac{h}{2}. \tag{17}$$

Because the crystal is driven by electrodes of opposite polarity, only that part of the solution (11) which is antisymmetric in X_2' is of interest, and thus

$$\psi_i=A_i\sin k_iX_2'. \tag{18}$$

Applying (17) to (18), we get

$$\cos k_ih/2=0,$$

from which it follows that

$$k_{in}\frac{h}{2}=(2n+1)\frac{\pi}{2},\ n=0,1,2,\ldots,$$

or

$$k_{in}=n\,(\pi/h),\ n=1,3,5,\ldots \tag{19}$$

Substituting (19) into (12), we get finally for the frequency of vibration

$$f_{in}=\frac{w_{in}}{2\pi}=\frac{1}{2\pi}\sqrt{\frac{\gamma_i}{\rho}}\,k_{in}=\frac{n}{2h}\sqrt{\frac{\gamma_i}{\rho}}, \tag{20a}$$

where

$$i=1,2,3,\ n=1,3,5,\ldots \tag{20b}$$

In order to find out whether any of the normal modes described by (18) and (20) can be excited piezoelectrically, we have to calculate the piezoelectric coefficients pertaining to these modes.

Making use of (1) and (7) we get

$$\Sigma_j \frac{\partial T'_{ij}}{\partial X'_j} = \frac{\partial T'_{i2}}{\partial X'_2} = -\rho w^2 U'^i = -\rho w^2 \Sigma_k \beta_k^{-1i} \psi^k$$

from which we get

$$\Sigma_i \beta_i^l \frac{\partial T'_{i2}}{\partial X'_2} = -\rho w^2 \Sigma_{ik} \beta_i^l \beta_k^{-1i} \psi^k$$

$$= -\rho w^2 \Sigma_k \delta_k^l \psi^k = -\rho w^2 \psi^l.$$

Thus the effective stress components that excite the mode ψ^l are not T'_{i2} but $\Sigma_i \beta_i^l T'_{i2}$. Making use of (sec. 1.2, eq 5) and recalling that the applied field is along the X'_2 direction, we get

$$T'_{i2} = -e'_{ki2} E'_k = -e'_{2i2} E \qquad (21)$$

and

$$\Sigma_i \beta_i^l T'_{i2} = -\Sigma_i \beta_i^l e'_{2i2} E = -e'_l E,$$

where E is the magnitude of the applied field, and

$$e'_l = \beta_i^l e'_{2i2} \qquad (22)$$

is the desired piezoelectric coefficient. Making use of convention (5), we can write (22) in the form

$$e'_l = \beta_1^l e'_{26} + \beta_2^l e'_{22} + \beta_3^l e'_{24}. \qquad (23)$$

3.2. Rotated Y-Cut plates

The secular equation (sec. 3.1, eq 14′) is very complicated in general, and it is desirable to find special cases in which (sec. 3.1, eq 14′) gets simplified. A well known special case in which this occurs is the case of the so-called "rotated Y-cut plates" or "Y′-cut plates" shown in figure 8. For such plates, we have from figure 7.

$$X'_i = \alpha_j^i X_j$$

where

$$\alpha_j^i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C & -S \\ 0 & S & C \end{bmatrix}$$

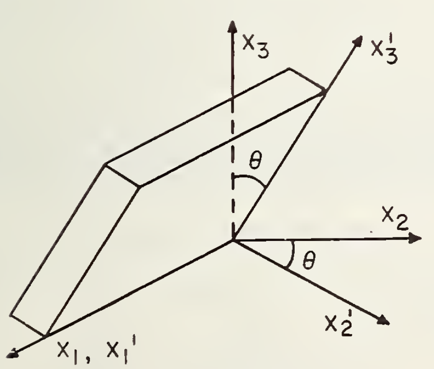


Figure 8.

and

$$S = \sin\theta, \; C = \cos\theta.$$

Furthermore, according to [3, p. 247, 248],

$$C'_{\mu 5} = C'_{\mu 6} = 0 \text{ for } \mu = 1,2,3,4 \qquad (1a)$$

$$e'_{21} = e'_{22} = e'_{23} = e'_{24} = 0 \qquad (1b)$$

It thus follows from (1) and (sec. 3.1, eq 3) that

$$\tilde{C}'_{26} = 0, \; \tilde{C}'_{46} = 0, \; \tilde{C}'_{22} = C'_{22}, \; \tilde{C}'_{24} = C'_{24}, \; \tilde{C}'_{44} = C'_{44}$$

and (sec 3.1, eq 14′) becomes

$$\begin{bmatrix} \tilde{C}'_{66} - \gamma_k & 0 & 0 \\ 0 & C'_{22} - \gamma_k & C'_{24} \\ 0 & C'_{24} & C'_{44} - \gamma_k \end{bmatrix} = 0.$$

Expanding this secular equation, we get

$$(\tilde{C}'_{66} - \gamma_k)[\gamma_k^2 - (C'_{22} + C'_{44})\gamma_k + (C'_{22} C'_{44} - C'_{24})^2] = 0$$

whose roots are

$$\gamma_1 = \tilde{C}'_{66} = C'_{66} + (4\pi/\epsilon'^S_{22})(e'_{26})^2 \qquad (2a)$$

and

$$\gamma_{2,3} = \left(\frac{C'_{22} + C'_{44}}{2}\right) \pm \left[\left(\frac{C'_{22} - C'_{24}}{2}\right)^2 + C'^2_{24}\right]^{\frac{1}{2}}. \qquad (2b)$$

Equations (sec 3.1, eq 13 and 15) can be written as

$$(\tilde{C}'_{66} - \gamma_k)\, \beta_1^k = 0 \qquad (3a)$$

$$(C'_{22} - \gamma_k)\beta_2^k + C'_{24}\beta_3^k = 0 \qquad (3b)$$

$$C'_{24}\, \beta_2^k + (C'_{44} - \gamma_k)\beta_3^k = 0 \qquad (3c)$$

$$(\beta_1^k)^2 + (\beta_2^k)^2 + (\beta_3^k)^2 = 1. \qquad (3d)$$

When $k=1$, we notice that

$$\begin{vmatrix} (C'_{22} - \gamma_1) & C'_{24} \\ C'_{24} & (C'_{44} - \gamma_1) \end{vmatrix} \neq 0$$

and thus

$$\beta_1^1 = 1, \; \beta_2^1 = \beta_3^1 = 0. \qquad (4a)$$

On the other hand when $k=2$ and [3, $\tilde{C}'_{66} \neq \gamma_{2,3}$ and thus $\beta_1^{2,3} = 0$. We then get from (3b, c, d), after a little algebra.

$$\beta_1^2 = 0, \; \beta_2^2 = \frac{C'_{24}}{[C'^2_{24} + (C'_{22} - \gamma_2)^2]^{\frac{1}{2}}}, \; \beta_3^2 = \frac{C'_{22} - \gamma_2}{[C'^2_{24} + C'_{22} - \gamma_2)^2]^{\frac{1}{2}}}, \qquad (4b)$$

and

$$\beta_1^3=0,\ \beta_2^3=\frac{C'_{24}}{[C'^2_{24}+(C'_{22}-\gamma_3)^2]^{\frac{1}{2}}},\ \beta_3^3=\frac{C'_{22}-\gamma_3}{[C'^2_{24}+(C'_{22}-\gamma_3)^2]^{\frac{1}{2}}}. \tag{4c}$$

Since the direction of vibration is given by (sec. 3.1, eq 16), we see from (4) that

$$\mathbf{e}''_1=\beta_1^1\mathbf{e}'_1,\ \mathbf{e}'_k=\beta_2^k\mathbf{e}'_2+\beta_3^k\mathbf{e}'_3(\text{for } k=2,3),$$

and thus ψ_1 represents thickness-shear vibration in the $X'_1X'_2$-plane and ψ_2, ψ_3 represent vibrations in mutually perpendicular directions in the $X'_2X'_3$-plane.

Making use of (1b) the piezoelectric excitation coefficient given in (sec. 3.1, eq 23) becomes

$$e'_k=\beta_1^k e'_{26}.$$

It thus follows from (4) that

$$e'_1=e'_{26},\ e'_2=e'_3=0. \tag{5}$$

Consequently ψ_2 and ψ_3 cannot be excited piezoelectrically, i.e., the only mode which can be excited piezoelectrically is the thickness-shear mode

$$\psi_{1n}=A\sin\frac{n\pi}{h}X'_2,(n=1,3,5,\ldots) \tag{6}$$

which has a frequency of vibration

$$f_n=\frac{n}{2h}\sqrt{\frac{\tilde{C}'_{66}}{\rho}}, \tag{7}$$

where we have made use of (sec. 3.1, eq 18 to 20), and (2a).

From [3, p. 247 and 248] we have

$$C'_{66}=S^2C_{44}+C^2C_{66}-2SC\,C_{14}, \tag{8a}$$

$$e'_{26}=C(Se_{14}-Ce_{11}), \tag{8b}$$

$$\epsilon'^S_{22}=C^2\epsilon^S_{11}+S^2\epsilon^S_{33}. \tag{8c}$$

We thus get from (8), (7) and (2a)

$$\tilde{C}'_{66}=S^2C_{44}+C^2C_{66}-2SC\,C_{14}$$
$$+4\pi[C^2(Se_{14}-Ce_{11})^2/(C^2\epsilon^S_{11}+S^2\epsilon^S_{33})]$$
$$=\rho(2hf_n/n)^2. \tag{9}$$

From (9) we see that by cutting Y′-cut plates at 7 different orientations, we can evaluate all of the dielectric constants, all of the piezoelectric coefficients and three of the elastic constants.

The question now is what are the orientations that give minimum coupling with other modes of vibration. Let us first find out what stresses can be excited piezoelectrically. From (sec. 1.2, eq 5) we have

$$T'_{ij}=-e'_{kij}E'_k=-e'_{2ij}E,$$

or

$$T'_\mu=-e'_{2\mu}E.$$

Making use of (1b), we see that T'_1, T'_2, T'_3, and T'_4 cannot be excited piezoelectrically, and we only need to worry about T'_5 and T'_6. Furthermore, from (sec. 3.1, eq 2) and (1) we get

$$T'_5=\tilde{C}'_{55}S'_5+\tilde{C}'_{56}S'_6, \tag{10a}$$

$$T'_6=\tilde{C}'_{56}S'_5+\tilde{C}'_{66}S'_6. \tag{10b}$$

The only strain component which is involved in the thickness-shear mode in the $X'_1\,X'_2$-plane, is S'_6. It thus follows from (10) that the interfering mode is determined by S'_5, which is coupled to S'_6 by means of $\tilde{C}'_{56}$. We can thus reduce coupling to other modes to a minimum by choosing such orientations as to make

$$\tilde{C}'_{56}=C'_{56}+(4\pi/\epsilon'^S_{22})e'_{25}e'_{26}=0.$$

Since the second term is less than 1 percent of C'_{56} and the constants of quartz are not known to better than 1 percent, we can neglect the second term in the preceeding equation and get [3, p. 247].

$$C'_{56}=(C^2-S^2)C_{14}-SC(C_{44}-C_{66})\simeq 0. \tag{11}$$

This is exactly the condition that leads to the *AC* and *BC* cuts [1, p. 454] . Making use of the values of $C_{\mu\nu}$ given by Koga et al, [5] (after a correction in sign),

$$C_{14}=18.06\times 10^{10}\text{ dynes/cm}^2,\ C_{44}-C_{66}$$
$$=18.39\times 10^{10}\text{ dynes/cm}^2,$$

we find that (11) is satisfied for

$$\theta=+31.5°,-58.5°.$$

However, since these values depend upon temperature, we shall use, for convenience, the angles

$$\theta=+30°,-60°. \tag{12}$$

It will be shown in (4.2) that this choice will make it much easier to obtain a solution for the elastic constants in terms of the frequencies of vibration.

3.3. Rotated X-Cut Plates

For the rotated *X*-cut plates shown in figure 9, we have

$$(\alpha^i_j)=\begin{bmatrix} S & 0 & C \\ C & 0 & -S \\ 0 & 1 & 0 \end{bmatrix}, \tag{1}$$

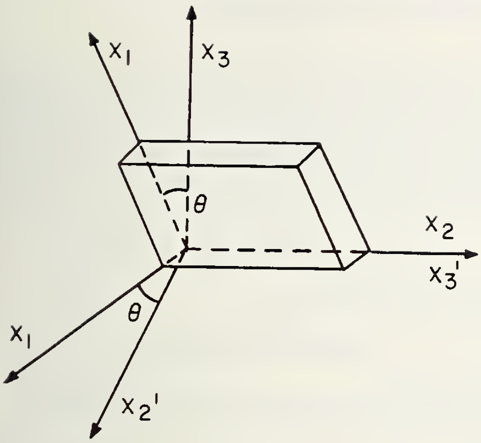


FIGURE 9.

where $S=\sin\theta$ and $C=\cos\theta$, as before. Making use of (1), we obtain, for the constants occurring in (sec. 3.1, eq 14)

$$C'_{22}=C^4C_{11}+2C^2S^2C_{13}+S^4C_{33}+4C^2S^2C_{44},$$

$$C'_{44}=S^2C_{44}+C^2C_{66},$$

$$C'_{66}=C^2S^2C_{11}-2S^2C^2C_{13}+C^2S^2C_{33}+(C^2-S^2)^2C_{44},$$

$$C'_{24}=-3C^2SC_{14},$$

$$C'_{26}=SC^3C_{11}+(S^3C-C^3S)C_{13}-CS^3C_{33}+2(CS^3-C^3S)C_{44},$$

$$C'_{46}=C(C^2-2S^2)C_{14},$$

$$e'_{22}=C^3e_{11},$$

$$e'_{24}=-CSe_{14},$$

$$e'_{26}=C^2Se_{11},$$

$$\epsilon^{S'}_{22}=C^2\epsilon^S_{11}+S^2\epsilon^S_{33}.$$

As in the previous section,

$$\tilde{C}'_{\mu\nu}=C'_{\mu\nu}+\frac{4\pi}{\epsilon^{S'}_{22}}e'_{2\mu}e'_{2\nu}. \qquad (2)$$

From these expressions, it can be seen that, in general, none of the $C_{\mu\nu}$'s are zero; which implies that the equation for γ_k is cubic. Thus, the formula for the frequency will be very complicated, and it becomes difficult to solve for the constants of quartz.

At $\theta=90°$ ($C=0$, $S=1$), the situation becomes simpler; but $e'_{22}=e'_{24}=e'_{26}=0$ and, according to (sec. 3.1, eq 23), thickness vibration cannot be excited piezoelectrically.

The X-cut plate $\theta=0, S=0, C=1$) is more promising, and, in fact, is the only simple case among the X′-cut plates which can be excited piezoelectrically. If we set $S=0$, $C=1$ in the above equations, we get for an X-cut.

$$C'_{22}=C_{11},\quad C'_{44}=C_{66},\quad C'_{66}=C_{44}$$

$$C'_{24}=0,\quad C'_{26}=0,\quad C'_{46}=C_{14}$$

$$e'_{22}=e_{11},\quad e'_{24}=0,\quad e'_{26}=0$$

$$\epsilon'^S_{22}=\epsilon^S_{11}.$$

Consequently, (sec. 3.1, eq 14′) becomes

$$\begin{vmatrix} C_{44}-\gamma_k & 0 & C_{14} \\ 0 & \tilde{C}'_{22}-\gamma_k & 0 \\ C_{14} & 0 & C_{66}-\gamma_k \end{vmatrix}=0, \qquad (3)$$

where

$$\tilde{C}'_{22}=C_{11}-\frac{4\pi}{\epsilon^S_{11}}e^2_{11},$$

and (sec 3.1, eq 23) becomes

$$e'_k=\beta^k_2 e_{11}. \qquad (4)$$

Solving (3) for γ_k, we get:

$$\gamma_2=\rho(2hf_n/n)^2=\tilde{C}'_{22}=C_{11}-\frac{4\pi}{\epsilon^S_{11}}e^2_{11}, \qquad (5)$$

$$\gamma_{1,3}=\left(\frac{C_{44}+C_{66}}{2}\right)\pm\left[\left(\frac{C_{44}-C_{66}}{2}\right)^2+C^2_{14}\right]^{\frac{1}{2}}.$$

In exactly the same way as in (sec. 3.2, eq 3 and 4), it can be shown that:

$$\beta^2_1=\beta^2_3=0,\ \beta^2_2=1;\ \beta^1_2=\beta^3_2=0.$$

Therefore, it follows from (4) that only the mode for $k=2$ can be excited piezoelectrically. The frequency of this mode is given by (5) and the direction of vibration by:

$$\mathbf{e}''_2=\Sigma_i\beta^2_i\mathbf{e}'_i=\mathbf{e}'_2.$$

Consequently this mode represents *thickness extensional* vibration.

4. Determination of the Constants of Quartz

4.1. Program

In the last two chapters, we found 4 different crystal cuts for which we have reliable formulas relating the frequency of vibration to the 10 elastic, dielectric, and piezoelectric constants of quartz. They were rectangular bars cut at (see sec. 2.3, eq 9, and fig. 8)

$$\phi=90°,\ \theta=70°,\ 131°, \tag{1}$$

and Y′-cut plates oriented at (see 3.2, eq 12, and fig. 8)

$$\theta=+30,\ -60°. \tag{2}$$

Solutions for other modes of vibration, such as flexure and face shear, are not as reliable as the ones we have obtained so far, and will therefore not be considered.

Since 4 out of 6 of the stress components (T_1', T_2', T_3', T_4') are zero for Y′-cut plates, and the expression for the frequency of vibration of these plates is relatively simple (see sec 3.2, eq 9) it is advisable to make as much use of Y′-cut plates as is possible.

From (sec. 3.2, eq 9), i.e.,

$$\tilde{C}_{66}'=S^2C_{44}+C^2C_{66}-2SC\,C_{14}+4\pi C^2\frac{(Se_{14}-Ce_{11})^2}{C^2\epsilon_{11}^S+S^2\epsilon_{33}^S}=\rho(2hf_n/n)^2, \tag{3}$$

we see that 7 constants are involved, namely; C_{44}, C_{66}, C_{14}, ϵ_{11}^S, ϵ_{33}^S, e_{11}, and e_{14}. Thus, all the information that can be obtained from such plates, should be possible to obtain from 7 different orientations. We already have 2 orientations in (2), and we need to choose 5 more. For simplicity and convenience, we shall take the following values of θ:

$$0°,\ 45°,\ 60°, -30°, -45°. \tag{4}$$

However, because the dielectric constants occur in the denominator of the last term of (3), we shall see in the following section that this will make it possible to determine all the constants in (3) except ϵ_{11}^S.

The elastic constant C_{11} can be determined from the formula for the extensional-thickness vibration of an X-cut plate (sec. 3.3, eq 5), namely

$$\tilde{C}_{22}'=C_{11}-\frac{4\pi}{\epsilon_{11}^S}e_{11}^2=\rho(2hf_n/n)^2, \tag{5}$$

in conjunction with the information obtained from (3).

In order to determine ϵ_{11}^S, a method used by Mason [2, p. 65] is probably as good as any. This method will be described in section 3.

The only constants that remain to be determined are C_{13} and C_{33}. These two constants can be obtained from the formula for the longitudinal vibration of a rectangular bar given by (sec. 2.4, eq 7 and 8), as will be shown in section 4.

4.2. C_{44}, C_{66}, C_{14}, and C_{11}

For convenience, let

$$\epsilon_{11}^S/4\pi=\epsilon_1,\ \epsilon_{33}^S/4\pi=\epsilon_3. \tag{1}$$

Evaluating (sec. 4.1, eq 3) at the angles in (sec. 4.1, eq 1 and 4), we get

$$A_0=\tilde{C}_{66}'(0)=C_{66}+\frac{e_{11}^2}{\epsilon_1} \tag{2}$$

$$\tilde{C}_{66}'(45)=\tfrac{1}{2}(C_{44}+C_{66}-2C_{14})+\tfrac{1}{2}\frac{(e_{14}-e_{11})^2}{\epsilon_1+\epsilon_3}$$

$$\tilde{C}_{66}'(-45)=\tfrac{1}{2}(C_{44}+C_{66}+2C_{14})+\tfrac{1}{2}\frac{(e_{14}+e_{11})^2}{\epsilon_1+\epsilon_3}$$

$$\tilde{C}_{66}'(30)=\tfrac{1}{4}(C_{44}+3C_{66}-2\sqrt{3}C_{14})+3/4\frac{(e_{14}-\sqrt{3}e_{11})^2}{3\epsilon_1+\epsilon_3}$$

$$\tilde{C}_{66}'(-30)=\tfrac{1}{4}(C_{44}+3C_{36}+2\sqrt{3}C_{14})+3/4\frac{(e_{14}+\sqrt{3}e_{11})^2}{3\epsilon_1+\epsilon_3}$$

$$\tilde{C}_{66}'(60)=\tfrac{1}{4}(3C_{44}+C_{66}-2\sqrt{3}C_{14})+\tfrac{1}{4}\frac{(\sqrt{3}e_{14}-e_{11})^2}{\epsilon_1+3\epsilon_3}$$

$$\tilde{C}_{66}'(-60)=\tfrac{1}{4}(3C_{44}+C_{66}+2\sqrt{3}\,C_{14})+\tfrac{1}{4}\frac{(\sqrt{3}\,e_{14}+e_{11})^2}{\epsilon_1+3\epsilon_3}.$$

From these equations, we have

$$A_1=\tfrac{1}{2}[\tilde{C}_{66}'(-45)-\tilde{C}_{66}'(45)]=C_{14}+\frac{e_{11}e_{14}}{\epsilon_1+\epsilon_3} \tag{3}$$

$$A_2=\frac{1}{\sqrt{3}}[\tilde{C}_{66}'(-30)-\tilde{C}_{66}'(30)]=C_{14}+\frac{3e_{11}e_{14}}{3\epsilon_1+\epsilon_3} \tag{4}$$

$$A_3=\frac{1}{\sqrt{3}}[\tilde{C}_{66}'(-60)-\tilde{C}_{66}'(60)]=C_{14}+\frac{e_{11}e_{14}}{\epsilon_1+3\epsilon_3} \tag{5}$$

$$B_1=[\tilde{C}_{66}'(-45)+\tilde{C}_{66}'(45)]=C_{44}+C_{66}+\frac{e_{11}^2+e_{14}^2}{\epsilon_1+\epsilon_2} \tag{6}$$

$$B_2=2[\tilde{C}_{66}'(-30)+\tilde{C}_{66}'(30)]=C_{44}+3C_{66}+3\,\frac{3e_{11}^2+e_{14}^2}{3\epsilon_1+\epsilon_3} \tag{7}$$

$$B_3=2[\tilde{C}_{66}'(-60)+\tilde{C}_{66}'(60)]=3C_{44}+C_{66}+\frac{e_{11}^2+3e_{14}^2}{\epsilon_1+3\epsilon_3}. \tag{8}$$

Equations (3 to 5) can be rewritten as

$$A_1\epsilon_1+A_1\epsilon_3=\epsilon_1C_{14}+\epsilon_3C_{14}+e_{11}e_{14}, \qquad (9)$$

$$3A_2\epsilon_1+A_2\epsilon_3=3\epsilon_1C_{14}+\epsilon_3C_{14}+3e_{11}e_{14}, \qquad (10)$$

$$A_3\epsilon_1+3A_3\epsilon_3=\epsilon_1C_{14}+3\epsilon_3C_{14}+e_{11}e_{14}. \qquad (11)$$

Multiplying (9) by 3, and subtracting it from (10), we get

$$3(A_2-A_1)\epsilon_1+(A_2-3A_1)\epsilon_3=-2\epsilon_3C_{14}. \qquad (12)$$

Moreover, if we subtract (9) from (11), we obtain

$$(A_3-A_1)\epsilon_1+(3A_3-A_1)\epsilon_3=+2\epsilon_3C_{14}. \qquad (13)$$

Adding (12) and (13), we find the following relation between ϵ_1 and ϵ_3:

$$(3A_2+A_3-4A_1)\epsilon_1+(A_2+3A_3-4A_1)\epsilon_3=0,$$

or

$$\epsilon_3=-\alpha\epsilon_1, \qquad (14)$$

where

$$\alpha=\frac{4A_1-3A_2-A_3}{4A_1-A_2-3A_3}. \qquad (15)$$

We can now obtain C_{14} by substituting the value of ϵ_3 given by (14) into (13). Thus

$$C_{14}=\tfrac{1}{2}(3A_3-A_1)+\frac{1}{2\alpha}(A_1-A_3). \qquad (16)$$

From (2), we have

$$\frac{e_{11}^2}{\epsilon_1}=A_0-C_{66}. \qquad (17)$$

Furthermore, solving (3) for $e_{11}e_{14}$, we get with the help of (14),

$$\frac{e_{11}e_{14}}{\epsilon_1}=(1-\alpha)(A_1-C_{14}). \qquad (18)$$

Squaring both sides of (18), and making use of (17), we obtain

$$\frac{e_{14}^2}{\epsilon_1}=\frac{(1-\alpha)^2(A_1-C_{14})^2}{A_0-C_{66}}. \qquad (19)$$

Consequently, if ϵ_1 is determined, ϵ_3, e_{11}, and e_{14} can be calculated from (14), (17), and (18), respectively. We shall show how $\epsilon_1=\epsilon_{11}^S/4\pi$ can be determined in the following section.

If we substitute the value of ϵ_3 given by (14) into (6) to (8), and then the value of e_{11}^2/ϵ_1 given by (17) into these equations, we get after a little algebra,

$$B_1(1-\alpha)-A_0=(1-\alpha)C_{44}-\alpha C_{66}+\frac{e_{14}^2}{\epsilon_1}, \qquad (20)$$

$$B_2\left(1-\frac{\alpha}{3}\right)-3A_0=\left(1-\frac{\alpha}{3}\right)C_{44}-\alpha C_{66}+\frac{e_{14}^2}{\epsilon_1}, \qquad (21)$$

$$B_3\left(\frac{1}{3}-\alpha\right)-\frac{1}{3}A_0=(1-3\alpha)C_{44}-\alpha C_{66}+\frac{e_{14}^2}{\epsilon_1}. \qquad (22)$$

C_{44} can now be obtained, by subtracting (20) from (21) and then solving for C_{44}. Thus

$$C_{44}=\frac{1}{2\alpha}[B_2(3-\alpha)-3B_1(1-\alpha)-6A_0]. \qquad (23)$$

Substituting this value of C_{44}, and the value of e_{14}^2/ϵ_1, given by (19) into (20), we get after a little algebraic manipulation,

$$\alpha C_{66}^2+[(1-\alpha)(B_1-C_{44})-(1+\alpha)A_0]C_{66}$$
$$+[(1-\alpha)^2(A_1-C_{14})^2-(1-\alpha)A_0(B_1-C_{44})+A_0^2]=0.$$

The solution to this quadratic equation can be written in the form

$$C_{66}=\frac{1}{2\alpha}[(1+\alpha)A_0-(1-\alpha)(B_1-C_{44})$$
$$\pm(1-\alpha)\sqrt{(B_1-C_{44}-A_0)^2-4\alpha(A_1-C_{14})^2}]. \qquad (24)$$

Only one of the signs in (24) will give the correct answer. C_{11} can be obtained by eliminating e_{11}^2/ϵ_1 from (sec. 4.1, eq 5) and (2). Thus

$$C_{11}=\tilde{C}_{22}'-A_0-C_{66}. \qquad (25)$$

From the values of C_{14}, C_{44}, C_{66}, and C_{11} obtained in this section, we can calculate the following constants (see sec. 2.4, eq 2 and 3):

$$C_{12}=C_{11}-2C_{66}, \qquad (26)$$

$$S_{14}=-C_{14}/\beta', \quad S_{44}=2C_{66}/\beta', \quad S_{66}=2C_{44}/\beta' \qquad (27)$$

where

$$\beta'/2=C_{66}C_{44}-C_{14}^2. \qquad (28)$$

4.3. Dielectric and Piezoelectric Constants

Since the plated rectangular bar can be considered as a parallel plate condenser, it should be possible to determine the dielectric constant of the bar by measuring the capacitance between the two electrodes at *zero frequency*. This method was already used by Mason [2, p. 65]; but he employed a roundabout method of obtaining the relation between the capacitance C and the dielectric constant. We shall give here a more straight-forward derivation of the same result.

As usual, the capacitance C is defined by

$$C=\frac{Q}{V}=\frac{l_1l_2\sigma}{l_3E}, \qquad (1)$$

where Q is the total electric charge on one electrode, V the potential difference between the two electrodes, σ the surface *real* charge density on the electrodes, and E is the *static* electric field *inside* the crystal. The problem now is to evaluate σ/E.

Since the situation in figure 3 is applicable here, we have for the normal component of the electric displacement inside the crystal,

$$D'_3=4\pi\sigma, \text{ or } \sigma=D'_3/4\pi, \tag{2}$$

where

$$D'_3=\epsilon^{T'}_{3i}E'_i+4\pi d'_{3\mu}T'_\mu \tag{3a}$$

$$=\epsilon^{S'}_{3i}E'_i+4\pi e'_{3\mu}S'_\mu. \tag{3b}$$

In our case,

$$E'_i=\delta_{i3}E,\ T'_\mu=\delta_{\mu1}T'_1,\ \epsilon^{T'}_{33}=\epsilon^{T}_{11}.$$

Thus,

$$\sigma=\frac{D'_3}{4\pi}=\frac{\epsilon^T_{11}}{4\pi}E+d'_{31}T'_1. \tag{4}$$

Since T'_1 is the only important stress,

$$\underline{S}'_1=\tilde{S}'_{1\mu}T'_\mu+d'_{i1}E'_i=\tilde{S}'_{11}T'_1+d'_{31}E,$$

or

$$T'_1=\frac{1}{\tilde{S}'_{11}}(\underline{S}'_1-d'_{31}E). \tag{5}$$

The equation of motion is given by

$$\frac{dT'_1}{dX'_1}=\frac{1}{\tilde{S}'_{11}}\frac{d\underline{S}'_1}{dX'_1}=\frac{1}{\tilde{S}'_{11}}\frac{d^2U'_1}{dX'^2_1}=-\rho w^2U'_1.$$

The solution to this equation is

$$U'_1=A\cos kX'_1+B\sin kX'_1.$$

where

$$k=(\rho\tilde{S}'_{11})^{\frac{1}{2}}w,$$

as was shown in (sec. 2.1, eq 13 and 14). However, in this case, the boundary conditions are

$$T'_1=\frac{1}{\tilde{S}'_{11}}(\underline{S}'_1-d'_{31}E)=0 \text{ at } X'_1=\pm l_1/2,$$

or

$$\underline{S}'_1=\frac{dU'_1}{dX'_1}=k(-A\sin kX'_1+B\cos kX'_1)$$

$$=d'_{31}E \text{ at } X'_1=\pm\frac{l_1}{2},$$

instead of $\underline{S}'_1=0$, as before. The solution obtained here is the *particular integral*, whereas the solution (sec. 2.1, eq 17) was the *complementary function*. Writing down the boundary conditions explicitly, we have

$$k\left(-A\sin k\frac{l_1}{2}+B\cos k\frac{l_1}{2}\right)=d'_{31}E,$$

$$k\left(A\sin k\frac{l_1}{2}+B\cos k\frac{l_1}{2}\right)=d'_{31}E.$$

Thus

$$A=0,\ kB=d'_{31}E/\cos k\frac{l_1}{2},$$

and

$$\underline{S}'_1=kB\cos kX'_1=d'_{31}E\cos kX'_1/\cos k\frac{l_1}{2}.$$

At $w=0$, $k=0$, $\underline{S}'_1=d'_{31}E$ and, according to (5), $T'_1=0$. Thus, we have from (4),

$$\sigma=\frac{\epsilon^T_{11}}{4\pi}E.$$

Substituting this value of σ into (1), we get

$$C=\frac{l_1l_2}{l_3}\frac{\epsilon^T_{11}}{4\pi},$$

or

$$\frac{\epsilon^T_{11}}{4\pi}=\frac{l_3}{l_1l_2}C, \tag{6}$$

in agreement with the result obtained by Mason [2, p. 65]. The same result would have been obtained, had we used (3b) instead of (3a).

The relation between ϵ^T_{11} and ϵ^S_{11} is given by (sec. 2.4, eq 5) to be

$$\frac{\epsilon^S_{11}}{4\pi}=\frac{\epsilon^T_{11}}{4\pi}-(2e_{11}d_{11}+e_{14}d_{14}). \tag{7}$$

Making use of the relations

$$d_{i\mu}=e_{i\nu}S_{\nu\mu},$$

and the symmetry properties of quartz, we get

$$d_{11}=e_{11}(S_{11}-S_{12})+e_{14}S_{14}=\tfrac{1}{2}e_{11}S_{66}+e_{14}S_{14}, \tag{8a}$$

$$d_{14}=e_{11}(S_{14}-S_{24})+e_{14}S_{44}=2e_{11}S_{14}+e_{14}S_{44}. \tag{8b}$$

We thus have, with the help of (sec. 4.2, eq 17, 18, and 19)

$$2e_{11}d_{11}+e_{14}d_{14}=S_{66}e^2_{11}+4S_{14}e_{11}e_{14}$$
$$+S_{44}e^2_{14}=\beta\epsilon^S_{11}/4\pi, \tag{9}$$

where

$$\beta=S_{66}(A_0-C_{66})+4S_{14}(1-\alpha)(A_1-C_{14})$$
$$+S_{44}\frac{(1-\alpha)^2(A_1-C_{14})^2}{A_0-C_{66}}. \tag{10}$$

From (7), (9), and (6), we have finally,

$$\epsilon_1=\frac{\epsilon^S_{11}}{4\pi}=\frac{l_3}{l_1l_2}\frac{C}{1+\beta}. \tag{11}$$

Now that ϵ^S_{11} and ϵ^T_{11} can be calculated from (11) and (6), $\epsilon^S_{33}=\epsilon^T_{33}$ can be obtained from (Sec. 4.2, eq 14). Furthermore, e_{11}, e_{14} can be calculated from (sec. 4.2, eq 17 and 19) and d_{11}, d_{14} from (8). Thus, all the dielectric and piezoelectric constants can be determined.

4.4. C_{13} and C_{33}

The only remaining independent constants to be determined are C_{13} and C_{33}. As was already pointed out in section 1, these constants can be obtained from (sec. 2.4, eq 8), evaluated at the two angles in (sec. 4.1, eq 1).

We notice from (sec. 2.4) that we can write

$$\tilde{S}'_{1\mu}=\frac{A_{1\mu}C_{13}+B_{1\mu}C_{33}+C_{1\mu}}{(C_{11}+C_{12})\,C_{33}-2C_{13}^2}+D_{1\mu}, \tag{1}$$

and

$$\tilde{C}'_{1\mu}=E_{1\mu}C_{13}+F_{1\mu}C_{33}+G_{1\mu}, \tag{2}$$

where all the constants $A_{1\mu}$ . . ., $G_{1\mu}$ are known. In view of (1) and (2), (sec 2.4, eq 8) has the following form:

$$\mathrm{A_1C_{13}+A_2C_{33}+A_3C_{13}^2+A_4C_{13}C_{33}+A_5C_{33}^2} \\ +A_6C_{13}^2C_{33}+A_7C_{13}^3=A_8. \tag{3}$$

It is difficult to see how to solve simultaneously two equations of the form (3) for the two unknowns C_{13} and C_{33}. For this reason, we shall use the following iteration method for solving (Sec. 2.4, eq 8): First of all, we use the best known values of C_{13}, C_{33} only in (2), i.e., in the correction terms containing $\tilde{C}'_{11}$ and $\tilde{C}'_{16}$. This will make (sec. 2.4, eq 8) take the simple form

$$b_1C_{13}+b_2C_{33}+b_3C_{13}^2=b_4. \tag{4}$$

Two equations of this form can be easily solved for C_{13} and C_{33} by eliminating C_{33} first. The solution obtained in this way, can then be used in (2) and the process repeated. This method is expected to lead quickly to the correct values for C_{13} and C_{33}, since the terms containing $\tilde{C}'_{11}$ and $\tilde{C}'_{16}$ in (sec. 2.4, eq 8) are about 1 percent of the other terms, and the error made by using the best known values of C_{13} and C_{33} in (2) is probably no greater than 1 percent. Thus the error in the first solution obtained will be about 0.01 percent.

Probably the best values available for C_{13} and C_{33}, are those given by Koga et al. [5], namely

$$C_{13}=1.193\times10^{11}\ \text{dynes/cm}^2, \tag{5}$$

$$C_{33}=10.59\times10^{11}\ \text{dynes/cm}^2.$$

After C_{13} and C_{33} are evaluated, S_{11}, S_{12}, S_{13}, S_{33} can be evaluated from (sec. 2.4, eq 3), i.e.,

$$2S_{11}=\frac{C_{33}}{\alpha'}+\frac{C_{44}}{\beta'}, \tag{6}$$

$$S_{12}=S_{11}-\tfrac{1}{2}S_{66}, \tag{7}$$

$$S_{13}=-C_{13}/\alpha', \tag{8}$$

$$S_{33}=(C_{11}+C_1^2)/\alpha', \tag{9}$$

where

$$\alpha'=(C_{11}+C_{12})C_{33}-2C_{13},\ \beta'=2(C_{44}C_{66}-C_{14}). \tag{10}$$

With this, all the elastic, dielectric, and piezoelectric constants of quartz can be determined. It should be recalled that the elastic constants are the adiabatic ones at constant electric field.

4.5. Experimental Conditions

The following conditions must be fulfilled in using Y′-cut plates: (1) The plates must be flat on both sides. (2) The ratio of lateral dimensions to thickness should be as large as possible (perhaps about 20:1); (3) The plates must be excited at a harmonic high enough so that f_n/n is a constant. One should measure f_n/n experimentally, and find out the value of n at which this occurs. (4) The thickness of the plates before plating must be measured accurately. (5) The electrodes should cover the full surfaces of the plates. (6) The frequency for zero plating must be determined by extrapolation of the curve of frequency versus plating thickness. (7) Both circular and square plates should be tried. Theoretically, if the harmonic is high enough, the shape of the plate should not make any difference.

The conditions for longitudinal vibration or rectangular bars are as follows: (1) $l_1/l_2>10$ and $l_2/l_3>10$ (see fig. 10). The bars should be fully plated at the top and bottom surfaces. (3) The fundamental frequency of vibration must be used. (4) The recommended values of θ, 70° and 131° are only rough, and a little experimentation with the value of θ about these values, will determine the best cut for minimum coupling with other modes of vibration. (5) All dimensions should be measured accurately before plating. (6) As condition 6 above.

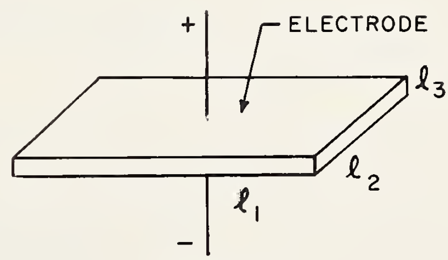


FIGURE 10.

It is understood that all the crystals are made to vibrate at the same temperature τ. The dimensions at τ can be calculated from the measured dimensions at room temperature τ_0 by means of (sec. 2.5, eq 8).

If the variation of the constants of quartz with temperature is desired, then all the constants must be determined at different temperatures, and a plot for the temperature variation of each constant can be obtained. The temperature coefficients can then be calculated from the definition,

$$T(C_{\mu\nu})=\frac{1}{C_{\mu\nu}}\frac{\partial C_{\mu\nu}}{\partial\tau}. \qquad (1)$$

4.6. Recent Determinations of the Constants of Quartz

Several authors have tried to determine dynamically the constants of quartz, e.g., Anatasoff and Hart [7] and Mason [8] and other authors mentioned in Cady's book [1]. Most of these authors have not taken the piezoelectric effect into account and have used expressions for the frequency of vibration which were not too reliable for the experimental conditions they had.

Recently, Koga et al., [5] have evaluated the constants of quartz taking the piezoelectric effect into account. However, they used the value 4.50 for the dielectric constant in all their formulas; whereas it is clear from the formulas of section 4.2 that the dielectric constant $\epsilon_{22}^{S'}$ is not the same for all the plates, but depends upon the angle of cut. Furthermore, Koga et al., only determined the constants of quartz in the neighborhood of 20° C; and there is much need for the evaluation of these constants at low temperatures, where the aging characteristics of quartz improve considerably.

The particular choice of crystal cuts recommended in this report should give a closer estimate of the actual values of the constants of quartz than those used by other authors; at least for the cutsin (sec. 4.1, eq 1 and 2). It thus seems that the determination of the constants of quartz by the method proposed here, in the temperature range −200° C to +100° C will be a valuable contribution to our knowledge of quartz.

It should be mentioned here that the linear coefficients of expansion of quartz, given by (sec. 2.5, eq 3), are only known to three significant figures, and there does not seem to be any data on the subject beyond that given in Sosman's book [4] since 1927. Thus, for highly accurate determinations of the constants of quartz, it seems that more accurate data on the linear coefficients of expansion of quartz is necessary.

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