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The Dynamics of Fields of Higher Spin

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There are several difficulties that plague all existing relativistic equations of motion describing elementary fields having an intrinsic spin greater than one. While the free field equations can be shown to be explicitly covariant, the introduction of interactions gives rise to a phenomenon of noncausality. In the presence of interactions, the retarded solutions spread beyond the light cone and the influence travels faster than light. Furthermore, the solutions in certain simple potentials do not have a finite norm, violating the probabilistic requirements of quantum mechanics.

This paper develops a relativistic theory that is free of the aforementioned difficulties. This Lagrangian theory describes fields and particles with arbitrary mass and charge and having any discrete spin, integer or half integer. Apart from gauge conditions there are no subsidiary conditions.

A matrix formulation is used. The generators of the inhomogeneous Lorentz group for a field of any intrinsic spin and mass are defined in terms of Wigner operators of the group SU(2) and a metric operator. A maximal Abelian set of invariants is formed which defines two completely reducible representation bases of the inhomogeneous Lorentz group having distinct structures. A set of γ matrices, obeying a Clifford algebra, is also defined in terms of the Wigner operators and the metric operator. State vectors having different structures and Lorentz transformation properties can be related to one another by operators involving the γ matrices.

The equations of motion can be obtained from the Lagrangian by variational methods, and certain aspects of the canonical formalism can be used to quantize the fields. Invariance of the Lagrangian under infinitesimal displacements and rotations yield conservation laws and constants of the motion for pertinent physical observables. The metric of the Hilbert space of the states is uniquely defined for any spin field, assuring positive definite four momenta and charge.

The Dirac formulation for the spin one-half field and the Maxwell-Lorentz formulation for the electromagnetic field are special cases of this theory.

Key words: Causality; high spin fields; inhomogeneous Lorentz group; relativistic fields; wave equations.

I. Introduction

Elementary particles of finite mass and with spin greater than one half have assumed a role of increasing importance in physics in recent years. It is ironical, however, that no "true" dynamical theory of such particles has emerged that enjoys the privilege of existing as an entity like the Dirac theory of the spin one-half particle. There are a host of composite theories that are explicitly covariant but possess certain intrinsic difficulties in application. These theories make explicit or implicit use of the invariance properties of the inhomogeneous Lorentz group and often of the one-to-two homomorphism between the homogeneous Lorentz group and the group SL(2c). There have been proposed two classes of theories that yield first order equations of motion for the particles. The first of these is the Dirac-Fierz-Pauli type [1–3][1] involving the usual four-dimensional Dirac matrices obeying a Clifford algebra. A set of Dirac matrices is used for each constituent of the composite state vector and each set is orthogonal to all other sets. Specifically, for example, the formulation of Bargmann and Wigner [4] contains extraneous components that must be constrained or eliminated to yield the required number of degrees of freedom. The prescription for accomplishing this feat is not always unique, but depends rather on the choice of irreducible combinations that are to be considered on the basis of a preconceived model. Because of these auxiliary conditions the Bargmann-Wigner equations are somewhat intractable, and when interactions are present they become both formidable and questionable. In addition there are some difficulties in formulating a Lagrangian theory.

The requirement that all differential equations of motion result from a variational principle involving an action integral requires the introduction of auxiliary fields into the Lagrangian. Only with the requirement that these auxiliary fields vanish in the absence of interactions is an explicit Lagrangian obtained. Fierz and Pauli [3] long ago recognized the difficulties arising from inconsistencies in this formulation.

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[1] Figures in brackets indicate the literature references at the end of this paper.

Free particles of spin values 0, 1/2, and 1 described by the Bargmann-Wigner equations are equivalently described by the Klein-Gordon, Dirac and Proca equations, respectively.

Free particles having spins greater than one half are described only by a set of coupled differential first order equations [5]. The number of equations may be reduced, however, by application of conditions for a lower spin field. For example, a spin 3/2 particle may be described by a set of three coupled equations involving a three-spinor. Conditions applicable to a spin-one field allow this three-spinor to be described in terms of a vector-spinor and an antisymmetrical tensor-spinor. This formalism proposed by Rarita and Schwinger [6] allows the set of three coupled equations to be reduced to two.

A second class of theories are characterized as the Bhabba-type [7]. In these, attempts are made to avoid the auxiliary conditions by inclusion of only those irreducible composites that are necessary to maintain Lorentz covariance together with the requirement that all components of the state vector obey a second order wave equation involving the same mass. This procedure requires matrices of higher dimension than the Dirac matrices. These matrices obey a rather complicated algebra. For particles of spin 0 and 1 these matrices are of dimension five and ten, respectively, and obey the Duffin-Kemmer-Petiau algebra [8]. The Bhabba-type equations of motion may be readily obtained from a Lagrangian, however the representation of the Lagrangian is by no means unique. In fact, the Lagrangian used by Kemmer might be questionable from the point of view of having an excessive number of powers of the four-momenta needed to describe the fields. In any event, for spins greater than one there are profound difficulties in a theory with interactions. Because of the attractiveness of the possibility for the elimination of subsidiary conditions there is still lively attention [9] being given to the first order Bhabba-type theories. The problem remains to get a theory with interactions that describes a particle of unique mass and charge without subsidiary conditions *and* without extraneous components. Both classes suffer from an inherent nonuniqueness in the selection of irreducible combinations required to express the dynamics of a field of a particular spin.

Theories that lead to second order equations of motion, apart from the familiar Proca [10] and Stuekelberg [11] formalisms tor spin one fields, have received less attention. These are usually tensor formulations [12] and, like the first order theories, usually have supplementary conditions invoked to limit the number of degrees of freedom. These theories also become formidable when interactions are introduced.

Velo and Zwanziger [13] have looked into the origin of some of the difficulties. Wave propagation is usually associated with hyperbolic systems of partial differential equations. Such equations allow an initial-value problem to be posed on a class of surfaces, called "space-like" with respect to the equations, and they possess solutions with wave fronts that travel along rays at finite velocities. The rays through any point form a ray cone that is entirely determined by the coefficients of the highest derivatives. Thus for hyperbolic systems when coupling occurs only in lower derivatives, the ray cone is the same in the interacting and free case. The free Klein-Gordon and Dirac equations are familiar examples of hyperbolic systems, and so, when they are coupled through lower order derivatives, the ray cone remains the light cone.

On the other hand, for spins greater than one half, the free Lagrangian equations are not hyperbolic but constitute instead a degenerate system because they imply constraints. However, it may be shown that they are equivalent to a system of hyperbolic equations which describe the wave propagation, supplemented by constraints that are conserved in time. But, if any low or nonderivative coupling is added to the free higher-spin Lagrangian, the resulting equations do not remain equivalent to a hyperbolic system with the light cone as the ray cone supplemented by the same number of constraints. There is at present no known example of a satisfactory equation with interaction for spin greater than one. The case of spin one is marginal; some interactions appear to lead to satisfactory equations, but others are unacceptable. Similar doubts were expressed long ago by J. W. Weinberg [14]. Note that the requirements of special relativity are not automatically satisfied by equations that transform covariantly.

This paper will develop a relativistic theory of higher-spin fields employing the variational methods of classical Lagrangian field theory. The chief aim is to present an unambiguous method for constructing a dynamical description of a field having any discrete spin, integer or half integer and including zero, and having an arbitrary, but unique, mass including zero. The formalism will be similar for all spins while, at the same time, will have no supplementary conditions or inherent nonuniqueness except those resulting from internal symmetries arising from variation of the Lagrangian.

One of the main considerations in elaborating relativistic quantum mechanics for particles comes from the fact that the law of conservation of the number of particles ceases, in general, to be true. To produce a complete theory we must encompass in a single scheme dynamical states specified, not only by the quantum state, but also by the number and the nature of the elementary particles of which they are composed. This requires that the theory, to be useful, must be quantizable. If it can be a Lagrangian type theory where the usual canonical methods apply, so much the better, for then the quantization methods may be straighforward.

Another consideration in relativistic theories is concerned with positive definiteness of the probability density and the energy. In the early days of quantum mechanics, there was a belief that any effective theory describing a particle with spin should yield a first order equation of motion in such a way that the probability density should be positive definite at all times.

Viewed in hindsight, the arguments for the superiority of first order equations over second order equations interpreted as single particle theories are not convincing. Of course, at the time the Dirac equation had the obvious advantage that it described most experimental facts involving spin one-half particles. A quantized theory removes these apparent differences between first and second order equations of motion, however neither is exempt from all difficulties, or even of contradictions.

One of the characteristics of any quantized relativistic theory describing particles of spin one or greater is the appearance of a negative metric for some of the components of the Hilbert space [15]. This leads to a negative probability of a different sort than that previously discussed. Dirac suggested that, in a relativistic quantum theory, use should be made of an indefinite metric so that the negative probabilities could be eliminated in order to maintain the probabilistic interpretation of the formalism of quantum theory.

A more fundamental reason for the indefinite metric may be seen in the fact that, disregarding the translations, the Lorentz group is a noncompact group, since there is no transformation that corresponds to the limiting velocity, c. Any finite representation of a non-compact group requires a space with an indefinite metric, although infinite representations with a definite metric may exist [16].

Although much of the contents of this paper may be based on arguments of group theory which could be used extensively to make the exposition more economical, much of this paper is written in terms of conventional matrix theory. The use of matrix theory in specific representations makes the physical arguments clearer and closer to the import of Dirac theory and also makes them more easily understood by people not well versed in spinor calculus and abstract algebra. In particular, we shall develop specific matrix representations of the inhomogeneous Lorentz group with the homogeneous subgroup employing the nonunitary four dimensional orthogonal group O(4) in a complex Minkowski space in which $x_\mu = (\mathbf{x}, it)$ where three of the six generators are regarded as specific representations of the three dimensional rotations about the spatial axes and the remaining three generators are specific representations of Lorentz transformations (boosts) along the three spatial axes. The use of O(4) in a Minkowski metric allows all matrix operators in spin space to be hermitian in our treatment[2].

The plan of the paper is as follows: In section 2 we establish and summarize the main features of the inhomogeneous Lorentz group. We introduce a set of invariants that supplement those customarily employed in order to obtain a set of commuting operators whose expectation values will completely characterize a physical state of a particular spin and mass.

In section 3 we first develop certain matrix representations of hermitian operators in a subspace of the Hilbert space of the physical states in terms of Wigner operators of the rotation group SU(2)

[2] Alternatively, we could have used in our formalism the group O(3,1) to describe the homogeneous Lorentz subgroup in a real space in which $x^\mu = (t, \mathbf{x})$. It should be pointed out that the apparently trivial differences in sign in the real four-dimensional Euclidian group O(4) and the group O(3,1) result in very important differences between the respective covering groups, SU(2) ⊗ SU(2) which is homomorphic to real O(4) and SL(2c) which is homomorphic to O(3,1). On the other hand, the group O(4) with the Minkowski metric has SL(2c) as its covering group, the same covering group as for O(3,1). There is no fundamental reason in special relativity, apart from personal taste, to choose O(3,1) with a real metric over O(4) with a Minkowski metric. The latter is that which is used in the original Dirac-Pauli description of spin one-half fields.

and a metric operator. Using these hermitian submatrices we develop general rules to construct a set of γ matrices obeying a Clifford algebra, and a set of spin operators $S_{\mu\nu}$ that form a six parameter Lie algebra. We establish Lorentz transformation properties of the physical states with arbitrary spin using the generators of the inhomogeneous Lorentz group.

In section 4 we establish a uniform Lagrangian formalism for classical fields of any spin and mass. From this Lagrangian formalism, the equations of motion are obtained and conservation laws established by means of variational methods and Noether's theorem.

In section 5 we obtain the plane-wave solutions for fields having several specific spin values and determine the completeness and orthogonality properties of these solutions. The Hamiltonian formalism is developed and the conditions for quantization of these fields is established in the Heisenberg picture.

In section 6 we subject the quantized fields to the discrete symmetries of space inversion, time reversal, and charge conjugation and find the transformation properties of these fields and bilinear combinations thereof. Furthermore we develop the commutation relations between the discrete symmetry operators and the generators of the inhomogeneous Lorentz group.

2. Lorentz Transformations

Of the many important invariance principles of relativistic quantum mechanics, the most fundamental is that which arises from the group of transformations denoted as the inhomogeneous Lorentz group. In our canonical approach, it is necessary to consider the transformation properties of the Hilbert space of the physical states, as well as those of the coordinates, in order to discuss the symmetries of the Langrangian and the equations of motion obtained therefrom. The properties of the inhomogeneous group and its subgroups have been established and understood for many years since the classic paper of Wigner [17]. This understanding, however, has not led to a totally satisfactory dynamical theory for particles and fields of higher spin. This section is devoted to establishing and summarizing the main facts concerning the Lorentz group, and to introduce a supplement to the customary method of construction of invariants of this group in order to classify the representations employed in this presentation.

The inhomogeneous Lorentz transformation, $\{a,\Lambda\}$, is a linear transformation of the coordinates conserving the norm of the intervals between different points of space-time. The new coordinates x_α' are obtained from the old coordinates by the relation[3]

$$x_\alpha' = \Lambda_{\alpha\beta} x_\beta + a_\alpha \tag{2.1}$$

The quantity a_α represents the translation of the space-time coordinates, x_α. The condition of invariance of the norm required that $\Lambda_{\alpha\beta}\Lambda_{\alpha\gamma} = \delta_{\beta\gamma}$, from which it follows that det $\Lambda = \pm 1$.

The set of transformations in which the translations are omitted ($a_\alpha = 0$) is denoted as the full homogeneous Lorentz group, which in turn can be divided into four subsets:

(1) The subset with det $\Lambda = +1$ and $\Lambda_{44} \geq 1$ is called the group of proper homogeneous Lorentz transformations. It is a six-parameter continuous group, containing the identity operator.
(2) The subset with det $\Lambda = -1$ and $\Lambda_{44} \geq 1$ is called space inversion.
(3) The subset with det $\Lambda = -1$ and $\Lambda_{44} \leq -1$ is called time inversion.
(4) The subset with det $\Lambda = 1$ and $\Lambda_{44} \leq -1$ is called space-time inversion.

The latter three subsets are disjoint and not continuously connected. Those subsets having $\Lambda_{44} \geq 1$ may be classified as orthochronous transformations, transforming a time-like vector into a time-like vector.

[3] We employ the following notation: All boldface letters, **A**, **φ**, **p**, **x**, **J**, etc., denote three vectors. The fourth components of the coordinates and momenta, $x_4 = it$ and $p_4 = iE$, are pure imaginaries. All Greek subscripts α, β, μ, ν, $\cdots$ vary from 1 to 4 and all Roman subscripts i, j, k vary from 1 to 3, except when specifically indicated otherwise. Repeated indices are to be summed over. Scalar products of four vectors are written as $p \cdot x \equiv p_\mu x_\mu$. The operator $\partial / \partial x_\mu$ is often written as ∂_μ. The D'Alembertian operator $\Box = \partial_\mu \partial_\mu$. Units are chosen so that $\hbar$ and c are equal to unity.

To each a and Λ there corresponds an operator $L(a,\Lambda)$ which acts on the Hilbert space of the physical states. A sequence of Lorentz transformations is again a Lorentz transformation according to the law,

$$L(a',\Lambda')L(a,\Lambda) = L(a' + \Lambda'a,\Lambda'\Lambda) \tag{2.2}$$

A physical state of a free particle of spin, s, and mass, m, may be characterized by a four momentum, $p = (\mathbf{p},p_4)$, and the component of spin along $\mathbf{p}$, the latter quantity is designated the helicity, λ. For an irreducible representation there are at least $2s + 1$ linearly independent basis states in a complete orthogonal set, one for each value of $\lambda = s, s - 1, \cdots, -s$ when the mass is nonzero. If the mass is zero there are only two linearly independent states corresponding to $\lambda = \pm s$. We denote the state vector describing a physical state by $| p,s,\lambda\rangle$.

The Hilbert space of the physical states is endowed with a metric such that the norm is determined by the Lorentz scalar product of the state vector and its adjoint. The adjoint state vector is denoted as $\langle p,s,\lambda | \beta\eta$. The hermitian unitary operator β assures that the scalar product is independent of the Lorentz frame. The operator η introduces an indefinite metric into the Hilbert space so that the norm is positive definite for all state vectors. In an unquantized theory, the η operates only upon the state vectors and has eigenvalues plus or minus unity.

The covariant norm of this Hilbert space is given by

$$\langle p's\lambda' | \beta\eta | ps\lambda\rangle = \delta(\mathbf{p} - \mathbf{p}')\delta_{\lambda\lambda}' \tag{2.3}$$

Any operator, involving the space-time symmetry of the system, that preserves the norm of the Hilbert space need not be unitary but must obey the relation

$$\mathcal{O}^{-1} = \beta\mathcal{O}^{\dagger}\beta \tag{2.4}$$

Of course, all observables will correspond to unitary operators, $\mathcal{U}$, that do not depend upon the metric of Hilbert space so that they obey the relation

$$[\mathcal{U},\beta] = 0 \tag{2.5}$$

Any representation of the group is specified by the infinitesimal transformation. A translation in the x_μ direction is generated by P_μ. The unitary operator which represents a infinitesimal translation a_μ in the x_μ direction is

$$L(a,1) = 1 - ia_\mu P_\mu \tag{2.6}$$

Any finite transformation of the proper Lorentz group may be considered as the product of successive infinitesimal transformations. The finite translation by the amount a_μ in the direction of x_μ is obtained by exponentiation of (2.6), thus

$$L(a,1) = e^{-ia_\mu P_\mu} \tag{2.7}$$

Clearly the P_μ are the four-momenta of the system.

A rotation in the $x_\mu x_\nu$ plane is generated by the operator

$$\Lambda_{\alpha\beta}{}^{(\mu\nu)} = \delta_{\alpha\beta} + \omega_{\alpha\beta}{}^{(\mu\nu)} \tag{2.8}$$

where the $\omega_{\alpha\beta}{}^{(\mu\nu)}$ is an infinitesimal Lorentz "six vector," an antisymmetric tensor where

$$\omega_{\alpha\beta} = -\omega_{\beta\alpha} \tag{2.9}$$

$$\omega_{\alpha\beta}{}^{(\mu\nu)} = \frac{i}{2}(\varepsilon_{\mu\nu}Z_{\mu\nu})_{\alpha\beta} \tag{2.10}$$

where the generator, $Z_{\mu\nu}$, is a hermitian antisymmetric tensor having the structure constants

$$(Z_{\mu\nu})_{\alpha\beta} = -i(\delta_{\alpha\mu}\delta_{\beta\nu} - \delta_{\alpha\nu}\delta_{\beta\mu}) \tag{2.11}$$

The constant, $\varepsilon_{\mu\nu}$, represents the magnitude of "rotation" in the $x_\mu x_\nu$ plane and is of the form

$$\varepsilon_{\mu\nu} = (\vartheta_{23},\vartheta_{31},\vartheta_{12},i\Omega_{14},i\Omega_{24},i\Omega_{34}) \tag{2.12}$$

i.e., the angles in the space-like planes are real, while the "angles" in the space-time planes are imaginary.

The operator in the Hilbert space of the physical states which represents an infinitesimal rotation is denoted as $L(0,\Lambda)$ and is defined by

$$L(0,\Lambda) = 1 + \frac{i}{2}\varepsilon_{\mu\nu}J_{\mu\nu} \tag{2.13}$$

with the nonhermitian operator, $J_{\mu\nu}$, obeying the relations

$$J_{ij}{}^{\dagger} = J_{ij} = -J_{ji} \tag{2.14}$$

$$\beta J_{k4}{}^{\dagger}\beta = -J_{k4} = J_{4k} \tag{2.15}$$

The J_{23}, J_{31}, and J_{12} are the three components of the total angular momentum **J**. The finite rotation operator $L(0,\Lambda)$ is also found by exponentiation

$$L(0,\Lambda) = e^{i/2(\varepsilon_{\mu\nu}J_{\mu\nu})} \tag{2.16}$$

The operator $L(0,\Lambda)$ is not unitary, however its inverse can be found from the relation (2.4)

$$L^{-1}(0,\Lambda) = \beta L^{\dagger}(0,\Lambda)\beta \tag{2.17}$$

The following commutation relations hold among $J_{\mu\nu}$ and P_λ

$$[J_{\mu\nu},J_{\lambda\rho}] = i(\delta_{\mu\lambda}J_{\nu\rho} + \delta_{\nu\rho}J_{\mu\lambda} - \delta_{\mu\rho}J_{\nu\lambda} - \delta_{\nu\lambda}J_{\mu\rho}) \tag{2.18}$$

$$[J_{\mu\nu},P_\lambda] = i(\delta_{\mu\lambda}P_\nu - \delta_{\nu\lambda}P_\mu) \tag{2.19}$$

$$[P_\mu,P_\nu] = 0 \tag{2.20}$$

The generators for translations form an Abelian group, while the generators for "rotations" form a non-Abelian group obeying a Lie algebra. The problem of finding this representation of the Lorentz group is equivalent to finding all of the representations of the commutation relations (2.18–2.20).

The operator $J_{\mu\nu}$ can be expressed as the sum of two parts; $L_{\mu\nu}$ which operates on the variables x_μ or p_μ and $S_{\mu\nu}$ which operates on the intrinsic spin variables.

$$J_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu} \tag{2.21}$$

where

$$L_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu \tag{2.22}$$

The three spatial operators L_{23}, L_{31}, and L_{12} act on the orbital variables alone; **L** is the orbital angular momentum operator. Likewise S_{23}, S_{31}, and S_{12} act on the internal spin variables alone; **S** is the spin vector of the particle. The total angular momentum is

$$\mathbf{J} = \mathbf{L} + \mathbf{S} \tag{2.23}$$

Once a representation is found, the task is to find all of the invariants of the group in order to test for irreducibility. Clearly, only scalar or pseudoscalar operators can be invariants, and one must construct these invariant quantities from the generators.

One invariant is the Lorentz scalar

$$P \cdot P \equiv P_\mu P_\mu \tag{2.24}$$

This invariant obviously commutes with each of the generators P_μ and $J_{\mu\nu}$.

Another invariant has as one of its constituents the Pauli-Lubanski spin operator, a pseudovector,

$$w_\lambda = -\frac{i}{2}\epsilon_{\lambda\mu\nu\rho}J_{\mu\nu}P_\rho \tag{2.25}$$

The orbital term, $L_{\mu\nu}$, in w_λ drops out identically so that

$$w_\lambda = -\frac{i}{2}\epsilon_{\lambda\mu\nu\rho}S_{\mu\nu}P_\rho \tag{2.26}$$

The commutation relations involving w_λ and the generators, $J_{\mu\nu}$ and P_μ, are

$$[J_{\mu\nu},w_\lambda] = i(\delta_{\mu\lambda}w_\nu - \delta_{\nu\lambda}w_\mu) \tag{2.27}$$

$$[P_\mu,w_\lambda] = 0 \tag{2.28}$$

$$[w_\mu,w_\nu] = \epsilon_{\mu\nu\lambda\rho}w_\lambda P_\rho \tag{2.29}$$

and also

$$w_\mu P_\mu = 0 \tag{2.30}$$

This second invariant is the quantity [4, 18]

$$w_\mu w_\mu = -\frac{1}{2}P_\lambda P_\lambda S_{\mu\nu}S_{\mu\nu} + P_\mu S_{\mu\lambda}P_\nu S_{\nu\lambda} \tag{2.31}$$

Although $w_\mu w_\mu$ commutes with the generators $J_{\mu\nu}$ and P_μ, a requirement that it be a Lorentz scalar puts rather strong requirements on the properties of $S_{\mu\nu}$. If we designate

$$\mathbf{M} = (S_{23},S_{31},S_{12}) \tag{2.32}$$

$$\mathbf{N} = (S_{14},S_{24},S_{34}) \tag{2.33}$$

then $w_\mu w_\mu$ will be a scalar provided that

$$\mathbf{M} = \Theta\mathbf{N}; \qquad \mathbf{N} = \Theta\mathbf{M}; \qquad \Theta^2 = 1 \tag{2.34}$$

where Θ is some unitary hermitian operatoı that commutes with **M** and **N**.

This may be readily seen if we express w_μ in vector form

$$\mathbf{w} = -i\mathbf{P} \times \mathbf{N} + P_0\mathbf{M} \tag{2.35}$$

$$w_4 = i\mathbf{P}\cdot\mathbf{M} \tag{2.36}$$

Then

$$w_\mu w_\mu = -(\mathbf{P}\cdot\mathbf{P})(\mathbf{N}\cdot\mathbf{N}) + (\mathbf{P}\cdot\mathbf{N})(\mathbf{P}\cdot\mathbf{N}) - (\mathbf{P}\cdot\mathbf{M})(\mathbf{P}\cdot\mathbf{M}) + P_0^2(\mathbf{M}\cdot\mathbf{M}) - iP_0\mathbf{P}\cdot(\mathbf{N}\times\mathbf{M} - \mathbf{M}\times\mathbf{N}) \tag{2.37}$$

This quantity is a Lorentz scalar only when conditions (2.34) are obeyed, then

$$w_\mu w_\mu = -P_\mu P_\mu \mathbf{M}^2 = -P_\mu P_\mu \mathbf{N}^2 = -\frac{1}{2}P_\mu P_\mu(\mathbf{M}^2 + \mathbf{N}^2) \tag{2.38}$$

We shall consider subsequently two distinct representations of the operators $S_{\mu\nu}$, one representation obeying the condition (2.34), and the other not. The basis states in a Hilbert space are, of course, defined as the eigenvectors of a complete set of commuting operators. In order to get unique eigenvalues in the case of a nonzero mass we evaluate the operators $P_\mu P_\mu$, $w_\mu w_\mu$, P_μ and w_3 in a rest frame $\hat{p}_\mu$ where $\hat{p}_\mu = 0, 0, 0, im)$. We have

$$w_\mu = m(M_1,M_2,M_3,0) \tag{2.39}$$

$$w_\mu w_\mu = m^2\mathbf{M}^2 \tag{2.40}$$

The eigenvalues of the operators in the rest system for an irreducible representation are given by

$$P_\mu P_\mu \,|\,\hat{p}s\lambda\rangle = -m^2\,|\,\hat{p}s\lambda\rangle \tag{2.41}$$

$$w_\mu w_\mu \,|\,\hat{p}s\lambda\rangle = m^2 s(s+1)\,|\,\hat{p}s\lambda\rangle \tag{2.42}$$

$$w_3\,|\,\hat{p}s\lambda\rangle = m\lambda\,|\,\hat{p}s\lambda\rangle \tag{2.43}$$

$$P_\mu\,|\,\hat{p}s\lambda\rangle = \pm im\,|\,\hat{p}s\lambda\rangle \tag{2.44}$$

This set of eigenvalues gives a physical characterization of the eigenstate by specification of the mass, the spin, the component of spin along the z axis and the sign of the energy. Unfortunately, this Abelian set of operators does not suffice to characterize the differences between distinct irreducible representations of $S_{\mu\nu}$ describing the same spin in a way that is independent of the Lorentz frame of reference.

There are no other invariant quantities that may be formed from functions of the ten generators of the inhomogeneous Lorentz group. It should be noted that none of the invariants $P_\mu P_\mu$, $w_\mu w_\mu$, P_μ, or w_3 contain the coordinates x_μ. Group theory is stating succinctly that under translations there is no invariant with a coordinate dependence.

We may describe a physical state in a particular way that does not depend on a specific relativistic equation of motion. Following Jacob and Wick [19] we may define the nonzero mass state $|\,ps\lambda\rangle$ in terms of the state $|\,\hat{p}s\lambda\rangle$ in the rest system by the sequence of operations

$$|\,ps\lambda\rangle = D^{(J)-1}(\phi,\theta,-\phi)e^{\Omega_{34}N_3}\,|\,\hat{p}s\lambda\rangle \tag{2.45}$$

The rest state $|\hat{p}s\lambda\rangle$ has its spin component λ along the z axis. The boost operation $e^{\Omega_{34}N_3}$, depending upon N_3 (since L_{12} is zero in the rest system), yields a state with momentum p along the z direction. The particular spatial rotation

$$D^{(J)-1}(\phi,\theta,-\phi) = e^{-i\phi J_3}e^{-i\theta J_2}e^{i\phi J_3} = e^{-i\mathbf{n}\cdot\mathbf{J}} \tag{2.46}$$

yields a state with momentum $\mathbf{p} = \mathrm{p}(\cos\phi\,\sin\theta,\ \sin\phi\,\sin\theta,\ \cos\theta)$ without changing the helicity component, λ. We denote this particular sequence of homogeneous Lorentz transformations and the corresponding operator as $h(p)$ and $L(h(p))$ respectively, i.e.,

$$L(h(p)) = D^{(J)-1}(\phi,\theta,-\phi)e^{\Omega_{34}N_3} \tag{2.47}$$

The general Lorentz transformation and the corresponding operator are still denoted $x' = \Lambda x + a$ and $L(a,\Lambda)$ respectively.

According to the definitions (2.45, 2.47) we also have

$$|p's\lambda\rangle = L(h(p'))\,|\hat{p}s\lambda\rangle \tag{2.48}$$

Furthermore, we have

$$p' = \Lambda p = \Lambda h(p)\hat{p} = h(p')\hat{p} \tag{2.49}$$

hence

$$h^{-1}(\Lambda p)\Lambda h(p)\hat{p} = \hat{p} \tag{2.50}$$

The sequence of transformations $h^{-1}(\Lambda p)\Lambda h(p)$ corresponds to an ordinary spatial rotation of $\hat{p}$. Thus we can write the corresponding operator relation

$$\begin{aligned} L(0,\Lambda)\,|ps\lambda\rangle &= L(0,\Lambda h(p))\,|\hat{p}s\lambda\rangle \\ &= L(0,h(\Lambda p))L(0,h^{-1}(\Lambda p)\Lambda h(p))\,|\hat{p}s\lambda\rangle \end{aligned} \tag{2.51}$$

where we have employed (2.2) for the composition of successive Lorentz transformations. The nature of the operator $L(0,h^{-1}(\Lambda p)\Lambda h(p))$ is that of a Wigner rotation operator which we denote as $D^{(J)}(h^{-1}(\Lambda p)\Lambda h(p))$. Combining (2.47 and 2.51) we obtain

$$L(0,\Lambda)\,|ps\lambda\rangle = |\,\Lambda p,s,\sigma\rangle D_{\sigma\lambda}{}^{(J)}(h^{-1}(\Lambda p)\Lambda h(p)) \tag{2.52}$$

where the momentum states are related by (2.49).

This well established, but still surprising to many, Lorentz transformation has nothing to do with the representations of the complete operator $J_{\mu\nu}$ but rather involves only the familiar representations of the spatial rotation group depending only on J_{ij}.

The transformation properties of the state vector under translations are more straightforward. We have

$$L(a,1)\,|ps\lambda\rangle = e^{-ia_\mu P_\mu}\,|ps\lambda\rangle \tag{2.53}$$

We may represent the combined proper inhomogeneous transformation $x' = \Lambda x + a$ on the physical states as

$$L(a,\Lambda)\,|ps\lambda\rangle = e^{-ia_\mu P_\mu}\,|\,\Lambda p,s,\sigma\rangle D_{\sigma\lambda}{}^{(J)}(h^{-1}(\Lambda p)\Lambda h(p)) \tag{2.54}$$

The adjoint of the above is

$$\langle ps\lambda \mid L^{\dagger}(a,\Lambda)\beta\eta = \langle ps\lambda \mid \beta\eta L^{-1}(a,\Lambda) \tag{2.55}$$

The norm is preserved with the rotation matrix obeying (2.4)

$$D^{(J)-1}(h^{-1}(\Lambda p)\Lambda h(p)) = \beta D^{(J)\dagger}(h^{-1}(\Lambda p)\Lambda h(p))\beta \tag{2.56}$$

We may factor out of $L(a,\Lambda)$ that part which operates on the intrinsic spin variables which we designate as $D^{(s)}(\Lambda)$ so that

$$L(a,\Lambda) = e^{i/2(\epsilon_{\mu\nu}L_{\mu\nu})-ia_{\mu}P_{\mu}}D^{(s)}(\Lambda) \tag{2.57}$$

where

$$D^{(s)}(\Lambda) = e^{i/2(\epsilon_{\mu\nu}S_{\mu\nu})} \tag{2.58}$$

The operator $D^{(s)}(\Lambda)$ is not unitary but obeys (2.4) also

$$D^{(s)-1}(\Lambda) = \beta D^{(s)\dagger}(\Lambda)\beta \tag{2.59}$$

The projection of a physical state vector onto a four dimensional coordinate space defines a wave-function

$$\varphi(x) = \langle x \mid ps\lambda\rangle \tag{2.60}$$

and

$$\begin{aligned} \varphi'(x) &= \langle x \mid L(a,\Lambda) \mid ps\lambda\rangle \\ &= \langle \Lambda^{-1}(x-a) \mid D^{(s)}(\Lambda) \mid ps\lambda\rangle \\ &= D^{(s)}(\Lambda)\varphi(\Lambda^{-1}(x-a)) \end{aligned} \tag{2.61}$$

or relabelling the coordinates $x \longrightarrow x'$

$$\varphi'(x') = D^{(s)}(\Lambda)\varphi(x) \tag{2.62}$$

The operator, $D^{(s)}(\Lambda)$, effects the Lorentz transformation from the coordinate system x to $x' = \Lambda x + a$. The group representation of $D^{(s)}(\Lambda)$ is completely determined in its infinitesimal form, since the hermitian operators, $S_{\mu\nu}$, in the generators obey a Lie algebra similar to that of (2.14).

Using the relations (2.32, 2.33), we can write the commutation relations involving the components of $S_{\mu\nu}$ as

$$[M_i,M_j] = i\epsilon_{ijk}M_k \tag{2.63}$$

$$[M_i,N_j] = i\epsilon_{ijk}N_k \tag{2.64}$$

$$[N_i,N_j] = i\epsilon_{ijk}M_k \tag{2.65}$$

When (2.58) is expressed in terms of **M**, and **N**, the operators **M** generate spatial rotations and the operators **N** generate boosts in the basis (2.62).

Two invariants that commute with each of the M_i and N_i may be specified and will serve to classify the representations of this algebra. These invariants are given by

$$F = \tfrac{1}{2}(\mathbf{M}^2 + \mathbf{N}^2) \tag{2.66}$$

$$G = \mathbf{M}\cdot\mathbf{N} \tag{2.67}$$

While the F and G are not among the set of operators $P_\mu P_\mu$, $w_\mu w_\mu$, P_μ, and w_3 that characterize representations of this inhomogeneous Lorentz group, they commute with each of the set. Furthermore the state vectors $|ps\lambda\rangle$ are eigenvectors of all of the foregoing operators. We may use F and G to supplement the set $w_\mu w_\mu$, $P_\mu P_\mu$, P_μ, and w_3 to obtain a maximal Abelian set of operators whose eigenvalues completely specify a particular representation of the inhomogeneous Lorentz group whether or not condition (2.34) is obeyed.

The commutation relations (2.63–2.65) obey the same algebra as that for the group O(4) which is used to describe the subgroup of homogeneous Lorentz transformations.

Although we intend in this paper to use the representation having the algebra of O(4) in a Minkowski space, it is helpful to use the classification scheme of the covering group, SL(2c), even if we do not employ the representations of the latter group.

The commutation relations (2.63–2.65) may be decoupled by the introduction of two new hermitian quantities

$$\mathbf{S}^{(+)} = \tfrac{1}{2}(\mathbf{M} + \mathbf{N}) \tag{2.68}$$

$$\mathbf{S}^{(-)} = \tfrac{1}{2}(\mathbf{M} - \mathbf{N}) \tag{2.69}$$

with the commutation relations

$$[S_i^{(+)},S_j^{(+)}] = i\epsilon_{ijk}S_k^{(+)} \tag{2.70}$$

$$[S_i^{(-)},S_j^{(-)}] = i\epsilon_{ijk}S_k^{(-)} \tag{2.71}$$

$$[S_i^{(+)},S_j^{(-)}] = 0 \tag{2.72}$$

The operator $D^{(s)}(\Lambda)$ defined in (2.58) becomes

$$D^{(s)}(\Lambda) = e^{i(\boldsymbol{\vartheta}+i\boldsymbol{\Omega})\cdot\mathbf{S}^{(+)}}e^{i(\boldsymbol{\vartheta}-i\boldsymbol{\Omega})\cdot\mathbf{S}^{(-)}} \tag{2.73}$$

The operator $D^{(s)}(\Lambda)$ looks formally like the product of two independent rotations through a complex angle.

The $\mathbf{S}^{(+)}$ and $\mathbf{S}^{(-)}$ are independently $2s^{(+)} + 1$ and $2s^{(-)} + 1$ dimensional representations of the algebra of the rotation group O(3), denoted by $\{\mathbf{S}^{(+)2},S_3^{(+)}\}$ and $\{\mathbf{S}^{(-)2},S_3^{(-)}\}$, respectively. Taken together they form a $(2s^{(+)} + 1)(2s^{(-)} + 1)$ dimensional irreducible representation, $(s^{(+)},s^{(-)})$, defined for any value of $s^{(+)}$ and $s^{(-)}$ integer or half integer. The representation $(s^{(+)},s^{(-)})$ exhaust all finite dimensional irreducible representation of the group $O(3) \otimes O(3)$ which is homomorphic to SL(2c). Only the (0,0) representation is unitary.

It will be shown that under the operation of space inversion the operator $\mathbf{N}$ changes sign while $\mathbf{M}$ does not. As a consequence, under space inversion $\mathbf{S}^{(+)}$ becomes $\mathbf{S}^{(-)}$ and vice versa. Any irreducible representation of well defined parity must be given by $(s^{(+)},s^{(-)}) \oplus (s^{(-)},s^{(+)})$ and will have a dimension $2(2s^{(+)} + 1)(2s^{(-)} + 1)$ whenever $s^{(+)}$ differs from $s^{(-)}$.

The invariants F and G expressed in terms of the new operators $\mathbf{S}^{(+)}$ and $\mathbf{S}^{(-)}$ are

$$F = \mathbf{S}^{(+)2} + \mathbf{S}^{(-)2} \tag{2.74}$$

$$G = \mathbf{S}^{(+)2} - \mathbf{S}^{(-)2} \tag{2.75}$$

In what follows we shall use the labels $(s^{(+)},s^{(-)}) \oplus (s^{(-)},s^{(+)})$ to characterize a particular representation but will use the basis vectors of the operators obeying the representation (2.63–2.67).

The eigenvalues of F and G for a specific irreducible representation $(s^{(+)},s^{(-)})$ are given by

$$F \,|\, ps^{(+)}\lambda^{(+)}s^{(-)}\lambda^{(-)}\rangle = [s^{(+)}(s^{(+)} + 1) + s^{(-)}(s^{(-)} + 1)] \,|\, ps^{(+)}\lambda^{(+)}s^{(-)}\lambda^{(-)}\rangle \tag{2.76}$$

$$G \,|\, ps^{(+)}\lambda^{(+)}s^{(-)}\lambda^{(-)}\rangle = [s^{(+)}(s^{(+)} + 1) - s^{(-)}(s^{(-)} + 1)] \,|\, ps^{(+)}\lambda^{(+)}s^{(-)}\lambda^{(-)}\rangle \tag{2.77}$$

The irreducible or completely reducible representations of the Lorentz group can be classified according to whether p_μ is time-like, light-like, space-like, or if all components of p_μ are equal to zero. The last case will not be considered further since the only physical system of interest corresponds to the vacuum state where the representation is the trivial identity that is one dimensional.

We have already treated in some detail the case in which p_μ is time-like corresponding to $-p_\mu p_\mu = m^2 \geq 0$. There are two representations for each value $P_\mu P_\mu$, $w_\mu w_\mu$, $\mathbf{M}^2 + \mathbf{N}^2$, and $\mathbf{M}\cdot\mathbf{N}$; one for each sign of the energy $p_0 = \pm\sqrt{m^2 + \mathbf{p}^2}$. The spectrum of eigenvalues of w_3 are most conveniently obtained in the rest system according to (2.37). This is true only for the case of nonzero mass in which we are able to transform to the rest frame. In other frames in which $p_\mu = (\mathbf{p},p_4)$ we can use the helicity operator $(\mathbf{J}\cdot\mathbf{P})/|\,\mathbf{p}\,|$ to characterize the basis state. This is compatible with the description of a frame with motion along the z axis since $(\mathbf{J}\cdot\mathbf{P})/|\,\mathbf{p}\,|$ commutes with $\mathbf{J}\cdot\mathbf{n}$ and hence the operator $D^{(J)-1}(\phi,\theta,-\phi)$. Here,

$$\frac{\mathbf{J}\cdot\mathbf{P}}{|\,\mathbf{p}\,|}\,|\,ps\lambda\rangle = \lambda\,|\,ps\lambda\rangle \tag{2.78}$$

The subgroup of homogeneous transformations which keep one of the four components of p_μ unchanged is called by Wigner a "little group." The little group that leaves p_4 invariant is the three dimensional rotation group. For any fixed momentum component the operators w_μ are generators of the little group. Since $w_\mu P_\mu = 0$, only three of the generators are linearly independent. The operators $L(\Lambda')$ (where $\Lambda' p_c = p_c$) form an irreducible representation of the little group and determine the irreducible representation $L(a,\Lambda)$ of the inhomogeneous Lorentz group.

There is a second class of representations which are of physical interest, namely those corresponding to the light-like or zero rest mass case in which $p_\mu p_\mu = 0$. Here the invariants $P_\mu P_\mu$, and $w_\mu w_\mu$ do not suffice to characterize the representation since they are equal to zero. The invariants $\mathbf{M}^2 + \mathbf{N}^2$ and $\mathbf{M}\cdot\mathbf{N}$ are independent of the mass.

For a massless particle, there does not exist any coordinate frame in which all but one component of p_μ vanish. There is a frame, however, in which $p_\mu = (0,0,p,ip)$. In this frame the components of w_μ are obtained using (2.35, 2.36)

$$w_1 = P(M_1 + iN_2) \tag{2.79}$$

$$w_2 = P(M_2 - iN_1) \tag{2.80}$$

$$w_3 = PM_3 \tag{2.81}$$

$$w_4 = iPM_3 \tag{2.82}$$

We can put these in the form of raising and lowering operators

$$w_+ = w_1 + iw_2 = P(M_+ + N_+) \tag{2.83}$$

$$w_- = w_1 - iw_2 = P(M_- - N_-) \tag{2.84}$$

where $M_\pm = M_1 \pm iM_2$, etc.

We also have $w_\mu = \lambda P_\mu$ from $w_\mu P_\mu = 0$, so that

$$\lambda = \frac{w_3}{P} = M_3 \tag{2.85}$$

We can write

$$w_\mu w_\mu = w_+ w_- \tag{2.86}$$

since in this particular Lorentz frame $w_3{}^2 = -w_4{}^2$, and $[w_1,w_2] = 0$.

The $w_\pm$ and λ obey the commutation relations

$$[\lambda,w_+] = w_+ \tag{2.87}$$

$$[\lambda,w_-] = -w_- \tag{2.88}$$

$$[w_+,w_-] = 0 \tag{2.89}$$

The basis states may be identified by finding a representation of this algebra. This algebra is not semi-simple because the elements w_+ and w_- form an Abelian sub algebra. In order that the basis states form a finite set it is necessary that the expectation values of w_+ and w_- be zero. This latter requirement establishes the expectation values of λ, the helicity operator, to be $\pm s$ for a particle of spin s, either integer or half integer. The matrix element of $w_+ w_-$ is also zero, and hence that of $w_\mu w_\mu$ are both zero for this zero mass case. We may denote such a basis state still as $|p,s,\lambda\rangle$ so that

$$w_\mu |ps\lambda\rangle = p_\mu \lambda |ps\lambda\rangle \tag{2.90}$$

Only one eigenvalue of λ, either $+s$ or $-s$, survives for each irreducible representation after taking the expectation value, and hence λ is an invariant. Particular illustrations of the basis states will be given after a specific representation of the operators is developed. Here the maximal Abelian set of commuting operators P_μ, w_μ, $\mathbf{M}^2 + \mathbf{N}^2$, and $\mathbf{M} \cdot \mathbf{N}$ serves to define the basis states.

The nonsingular case, in which the matrix elements of $w_+ w_-$ or $w_\mu w_\mu$ do not vanish, corresponds to the zero-mass, continuous-spin representation [17]. This representation does not seem to have any physical significance and is not considered further.

We are limited to only one physically permissible irreducible representation of the two dimensional little group, a rotation in the plane perpendicular to the direction of propagation. The vector $p_\mu = (0,0,p,ip)$ is unchanged by this rotation.

The basis states for a zero mass particle of any momentum p' and helicity λ, propagating in a specified direction, may be obtained by a Lorentz transformation $L(a,\Lambda)$ from the particular frame $p_\mu = (0,0,p,ip)$ that we have considered. Here

$$p_\mu{}' = \Lambda_{\mu\nu} p_\nu \tag{2.91}$$

We can conveniently choose Λ to be a "boost" of p along its initial direction of propagation to a value p' followed by a spatial rotation that takes the initial direction into the final direction of propagation along p' where $\mathbf{p}' = p'(\cos\phi \sin\theta, \sin\phi \sin \theta, \cos\theta)$. So that Λ is given by

$$\Lambda_{\mu\nu} = D_{\mu\rho}{}^{(J)-1}(\phi,\theta,-\phi)[e^{\Omega_{34}Z_{34}}]_{\rho\nu} \tag{2.92}$$

and where the boost operator $e^{\Omega_{34}Z_{34}}$ is given by

$$e^{\Omega_{34}Z_{34}} = [I - Z_{34}{}^2 + Z_{34}{}^2(\cosh\Omega_{34} + Z_{34} \sinh\Omega_{34})] \tag{2.93}$$

$$\Omega_{34} = \ln \frac{|\mathbf{p}'|}{|\mathbf{p}|} \tag{2.94}$$

The quantity Z_{34} is the hermitian antisymmetric tensor defined in (2.11).

The spatial rotation operator $D_{\mu\rho}{}^{(J)-1}$ is the usual rotation operator of the little group where p_4 is unchanged in the rotation.

The representation of a zero mass state in any Lorentz frame is characterized by the eigenvalue $p_\mu\lambda$ in (2.90). No proper Lorentz transformation will relate states having opposite sign eigenvalues of λ.

There is a third class of representations where p_μ is space-like, $-p_\mu p_\mu = m^2 \leq 0$. Here, the particles would have imaginary mass and would propagate outside the light cone. Although these tachyonical representations are admissable, there is no evidence that particles corresponding to these representations exist physically.

We shall defer until section 6 the discussion of the improper Lorentz transformations, space and time inversion, after specific representations are developed. These transformations, along with charge conjugation, are treated in a manner similar to those treated in textbooks on field theory. The group theoretical treatment of associated conceptual problems and phase questions is nontrivial [20].

Time inversion is actually the product of two distinct operations: Wigner time reversal or simply time reversal, an operation in which all velocities are reversed, and charge conjugation, an operation in which all particles are changed into their antiparticles. Charge conjugation is a relativistic quantum-mechanical concept only. Both time reversal and charge conjugation are discontinuous operations like time inversion.

3. Representation of Operators and Eigenvectors

Our task is to develop a covariant description of particles and fields of discrete spin and arbitrary mass. We would expect that there would exist $2s + 1$ independent states for a particle of spin s and non-zero mass in the rest system. On the other hand, if the mass of the particle is zero, there will exist only two independent states for a particle of spin s corresponding to the two states of maximum helicity.

We want to construct an explicit representation of these state vectors in a Hilbert space and a set of linear operators that will directly account for all observable properties of the particle in a straightforward way. Our ansatz is to postulate that for a particle of spin s the state vector has as elements, $2s + 1$ space-like components and $2s - 1$ time-like components, the latter being required to give the proper number of degrees of freedom for an observable in any Lorentz frame. Further requirements of relativistic covariance with components of unique parity may double the number of components so that the state vector in our Hilbert space in the general case will have $2(2s + 1 + 2s - 1) = 8s$ components. We can designate the subset of $2s + 1$ space-like components as $|s_>,\sigma\rangle$ and the subset of $2s - 1$ time-like components as $|s_<,\sigma\rangle$. The time-like components have some characteristics of a spin $s - 1$.

We may represent the total state vector as a column matrix of $8s$ components.

$$
|p,s,\sigma\rangle = \begin{vmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{vmatrix}
\begin{matrix} \left.\vphantom{\begin{matrix}\cdot\\\cdot\\\cdot\\\cdot\\\cdot\end{matrix}}\right\} 2s+1 \quad \text{space-like components} \\ \left.\vphantom{\begin{matrix}\cdot\\\cdot\\\cdot\end{matrix}}\right\} 2s-1 \quad \text{time-like and ``small'' components} \\ \left.\vphantom{\begin{matrix}\cdot\\\cdot\\\cdot\\\cdot\\\cdot\\\cdot\\\cdot\end{matrix}}\right\} 4s \quad \text{``small'' components} \end{matrix}
\tag{3.1}
$$

The $8s$ dimension of the state vector in the Hilbert space implies that the linear operator $D^{(s)}(\Lambda)$ introduced in (2.58) will be represented by an $8s$ by $8s$ dimensional matrix. This operator is also a "Lorentz six vector"; it is the generator for the three space-like rotations and the three space-time rotations or "boosts." This implies that $S_{\mu\nu}$ and any irreducible submatrix of $S_{\mu\nu}$ shall behave as a special tensor operator of rank one.

We introduce, as constituents of the spin operator, $S_{\mu\nu}$, and other operators to be defined subsequently, two operators of tensor rank one. These quantities we shall designate as $S_i^{(s)}$ and $S_i^{(t)}$. The superscript (s) designates an operator that permutes space-like components with one another and time-like components with one another. The superscript (t) designates an operator that permutes space-like components with time-like components.

The operators $S_i^{(s)}$ and $S_i^{(t)}$ may be defined in terms of Wigner operators [21,22] of rank one. A Wigner operator of SU(2) is designated by the notation

$$
\left\langle 2k \begin{matrix} k+\kappa \\ \\ k+q \end{matrix} 0 \right\rangle
$$

and is an irreducible tensor operator of type $T(k,q)$ [23], where $q = k, k-1, \cdots, -k$. It is an operator which effects the shift κ on the label s of a generic state vector $|s,\sigma\rangle$ as well as the shift q on the component σ. More precisely, it is defined by

$$
\left\langle 2k \begin{matrix} k+\kappa \\ \\ k+q \end{matrix} 0 \right\rangle |s,\sigma\rangle = |s+\kappa,\sigma+q\rangle\langle s+\kappa,\sigma+q\,|\,s\sigma kq\rangle \tag{3.2}
$$

In particular, κ may take any of the values $k, k-1, \cdots, -k$, so that there are $(2k+1)^2$ Wigner operators of rank k, one for each value of κ and q. Note that because the vector coupling coefficients are defined to be zero if the triangle conditions on $\mathbf{s}$, $\mathbf{k}$, and $\mathbf{s}+\mathbf{k}$ are not satisfied certain irreducible representation spaces of $|s,\sigma\rangle$ will be annihilated by a given Wigner operator, i.e., each Wigner operator has an associated null space.

We restrict our representation subspace of dimension $4s$ to those of s and $s + \kappa$, both of which can have values s and $s - 1$ which we may denote as

$$\left| \begin{matrix} s,\sigma \\ \\ s-1,\sigma \end{matrix} \right\rangle = | s_{\gtrless},\sigma \rangle \tag{3.3}$$

The operator $S_q^{(s)}$ that operates in this subspace of the eigenvectors is defined as

$$S_q^{(s)} = \begin{vmatrix} \left\langle \begin{matrix} & 1 & \\ 2 & & 0 \\ & 1+q & \end{matrix} \right\rangle_R & 0 \\ 0 & \left\langle \begin{matrix} & 1 & \\ 2 & & 0 \\ & 1+q & \end{matrix} \right\rangle_R \end{vmatrix} \tag{3.4}$$

The subscript R on the Wigner operator indicates a "renormalized" Wigner operator from that defined in (3.2) in such a way that the denominators of the vector coupling coefficients are eliminated and the sign of the $q = +1$ components are reversed. This establishes a generalization of the usual angular momentum operators in our representation subspace that includes time-like components and is in accordance with the established Condon and Shortley phase convention [23].

With these definitions we can reexpress (3.2) in this generalized and renormalized form.

$$S_+^{(s)} | s_{\gtrless},\sigma \rangle = [(s + \sigma + \tfrac{1}{2}(1 + b))(s - \sigma - \tfrac{1}{2}(1 - b))]^{1/2} | s_{\gtrless},\sigma + 1 \rangle \tag{3.5}$$

$$S_-^{(s)} | s_{\gtrless},\sigma \rangle = [(s - \sigma + \tfrac{1}{2}(1 + b))(s + \sigma - \tfrac{1}{2}(1 - b))]^{1/2} | s_{\gtrless},\sigma - 1 \rangle \tag{3.6}$$

$$S_0^{(s)} | s_{\gtrless},\sigma \rangle = \sigma | s_{\gtrless},\sigma \rangle \tag{3.7}$$

where the nonhermitian raising and lowering operators, $S_+^{(s)}$ and $S_-^{(s)}$, may be defined in terms of hermitian operators $S_1^{(s)}$ and $S_2^{(s)}$, by

$$S_\pm^{(s)} = S_1^{(s)} \pm iS_2^{(s)} \tag{3.8}$$

and where b is the eigenvalue of a diagonal, hermitian, and unitary operator B on the states $| s_>,\sigma \rangle$ or $| s_<,\sigma \rangle$ such that

$$B | s_>,\sigma \rangle = | s_>,\sigma \rangle \tag{3.9}$$

$$B | s_<,\sigma \rangle = - | s_<,\sigma \rangle \tag{3.10}$$

We will denote this operator B as the metric operator.

Successive operations on $| s_{\gtrless},\sigma \rangle$ with $S_q^{(s)}$ for various values of $q = +, 0, -$ establishes that

$$\mathrm{s}^{(s)2} | s_{\gtrless},\sigma \rangle = s(s + b) | s_{\gtrless},\sigma \rangle \tag{3.11}$$

and that

$$[S_i^{(s)},S_j^{(s)}] = i\epsilon_{ijk}S_k^{(s)} \tag{3.12}$$

Furthermore

$$[S_i^{(s)},B] = 0 \tag{3.13}$$

We will have occasion to use a shorthand notation

$$\hat{S}_i^{(s)} = BS_i^{(s)} \tag{3.14}$$

The operator $S_q^{(t)}$ that operates in the subspace of the eigenvectors $| s_{\gtrless},\sigma \rangle$ is defined as

$$S_q^{(t)} = B \begin{vmatrix} 0 & \left\langle 2 \begin{matrix} 2 \\ \\ 1+q \end{matrix} 0 \right\rangle_R \\ \begin{matrix} 0 \\ \left\langle 2 \begin{matrix} \\ \\ 1+q \end{matrix} 0 \right\rangle_R \end{matrix} & 0 \end{vmatrix} \tag{3.15}$$

The conditions of renormalization are the same as those for $S_q^{(s)}$, including the Condon and Shortley phase convention. The operator B is introduced in order to make $S_q^{(t)}$ have hermiticity and reality conditions similar to $S_q^{(s)}$.

Using this definition we have

$$S_+^{(t)} \,|\, s_{\gtrless},\sigma\rangle = -b[(s + b\sigma + \tfrac{1}{2}(1 + b))(s + b\sigma - \tfrac{1}{2}(1 - b))]^{1/2} \,|\, s_{\lessgtr},\sigma + 1\rangle \tag{3.16}$$

$$S_-^{(t)} \,|\, s_{\gtrless},\sigma\rangle = b[(s - b\sigma + \tfrac{1}{2}(1 + b))(s - b\sigma - \tfrac{1}{2}(1 - b))]^{1/2} \,|\, s_{\lessgtr},\sigma - 1\rangle \tag{3.17}$$

$$S_0^{(t)} \,|\, s_{\gtrless},\sigma\rangle = [(s + \sigma)(s - \sigma)]^{1/2} \,|\, s_{\lessgtr},\sigma\rangle \tag{3.18}$$

where, similar to (3.8),

$$S_\pm^{(t)} = S_1^{(t)} \pm iS_2^{(t)} \tag{3.19}$$

Note that the operation of $S_q^{(t)}$ on $| s_{\gtrless},\sigma\rangle$ always changes $s_>$ into $s_<$ and vice versa. Furthermore

$$\{S_i^{(t)},B\} = 0 \tag{3.20}$$

Again we can use the shorthand notation to indicate the product of B and $S_i^{(t)}$

$$\hat{S}_i^{(t)} = BS_i^{(t)} = -S_i^{(t)}B \tag{3.21}$$

By successive application of $S_q^{(t)}$ we establish that

$$\mathbf{S}^{(t)2} \,|\, s_{\gtrless},\sigma\rangle = s(2s - b) \,|\, s_{\gtrless},\sigma\rangle \tag{3.22}$$

We note that in this representation subspace only one of the six operators $S_q^{(s)}$ and $S_q^{(t)}$ is diagonal, namely $S_0^{(s)}$. Successive application of $S_q^{(t)}$ and $S_q^{(s)}$ establishes the additional commutation relations from straightforward algebraic manipulations

$$[S_i^{(s)},S_j^{(t)}] = [S_i^{(t)},S_j^{(s)}] = i\epsilon_{ijk}S_k^{(t)} \tag{3.23}$$

$$[S_i^{(t)},S_j^{(t)}] = i\epsilon_{ijk}(2sB - 1)S_k^{(s)} \tag{3.24}$$

The latter commutation relation is rather unexpected. The six operators $S_i^{(s)}$ and $S_i^{(t)}$ do *not* obey a Lie algebra, a fact that will have interesting consequences.

Certain anticommutation relations may also be established in the same way as the commutation relations were established

$$\{S_i^{(s)},S_j^{(s)}\} + \{S_i^{(t)},S_j^{(t)}\} = 2s^2\delta_{ij} \tag{3.25}$$

$$\{S_i^{(s)},S_j^{(t)}\} - \{S_i^{(t)},S_j^{(s)}\} = 2si\epsilon_{ijk}\hat{S}_k^{(t)} \tag{3.26}$$

We note that neither $\mathbf{S}^{(s)2}$ nor $\mathbf{S}^{(t)2}$ commute with all of the components $S_i^{(s)}$ and $S_i^{(t)}$, but that the sum $\mathbf{S}^{(s)2} + \mathbf{S}^{(t)2}$ does, however. Adding (3.11) and (3.22) we obtain

$$(\mathbf{S}^{(s)2} + \mathbf{S}^{(t)2}) \mid s_{\gtrless},\sigma\rangle = 3s^2 \mid s_{\gtrless},\sigma\rangle \tag{3.27}$$

and recalling

$$S_0^{(s)} \mid s_{\gtrless},\sigma\rangle = \sigma \mid s_{\gtrless},\sigma\rangle \tag{3.7}$$

We have here defined a basis for this representation: the simultaneous eigenvectors of $(\mathbf{S}^{(s)2} + \mathbf{S}^{(t)2})$ and $S_0^{(s)}$.

The values of s, or equivalently $s_>$, are subject to the restriction that $s_>$ is fixed for a given representation and may take on one of the values 0, 1/2, 1, 3/2, 2, $\cdots$ The value of $s_<$ equals $s_> - 1$ except in the case when $s = 0, 1/2$ where the components corresponding to $s_<$ are absent.

The $4s$-dimensional representation whose basis is given by the eigenvectors $\mid s_{\gtrless},\sigma\rangle$, where $\sigma = s_{\gtrless}, s_{\gtrless} - 1, s_{\gtrless} - 2, \cdots, -s_{\gtrless}$, is in a group-theoretical sense irreducible. It is clear that, by successive use of the operators $S_q^{(s)}$ and $S_q^{(t)}$, with $q = +, 0$, and $-$, we may transform any vector in this set into any other. We shall denote this representation by $\{s_> \oplus s_<\}$. This irreducible representation can be regarded, generally speaking, as a reducible representation of the spatial rotation group. Here $\{s_> \oplus s_<\}$ regarded as a representation of the spatial rotation group transforms as $D_{\text{rot}}^{(s)}(\vartheta)$, a relativistic operator, which decomposes into the representation, e.g.,

$$D_{\text{rot}}^{(s)}(\vartheta_{31}) \rightarrow D^{(s)}(0,\vartheta,0) \oplus D^{(s-1)}(0,\vartheta,0) \tag{3.28}$$

where the operators on the right hand side are the usual nonrelativistic D functions. In nonrelativistic quantum mechanics we are concerned only with the $D^{(s)}$ representation since all time-like states that transform under the $D^{(s-1)}$ representation are neglected.

For a given eigenvector $\mid s_{\gtrless},\sigma\rangle$, there is a limit to the number of successive raising or lowering operations $S_\pm^{(s)}$ and $S_\pm^{(t)}$. This limit follows from the associated null spaces of the Wigner operators. Inspection of (3.5–3.7) and (3.16–3.18) allows us to write

$$S_+^{(s)} \mid s_>,s\rangle = 0; \qquad S_-^{(s)} \mid s_>,-s\rangle = 0 \tag{3.29}$$

$$S_+^{(s)} \mid s_<,s-1\rangle = 0; \qquad S_-^{(s)} \mid s_<,-(s-1)\rangle = 0 \tag{3.30}$$

$$S_+^{(t)} \mid s_>,s\rangle = 0; \qquad S_-^{(t)} \mid s_>,-s\rangle = 0 \tag{3.31}$$

$$S_+^{(t)} \mid s_>,s-1\rangle = 0; \qquad S_-^{(t)} \mid s_>,-(s-1)\rangle = 0 \tag{3.32}$$

$$S_+^{(t)} \mid s_<,s-1\rangle = -[2s(2s-1)]^{1/2} \mid s_>,s\rangle \tag{3.33}$$

$$S_-^{(t)} \mid s_<, -(s-1)\rangle = [2s(2s-1)]^{1/2} \mid s_>, -s\rangle \tag{3.34}$$

$$S_0^{(t)} \mid s_>, \pm s\rangle = 0 \tag{3.35}$$

Thus, we are able to generate only the $4s$ eigenvectors in our representation basis by successive use of the operators $S_q^{(s)}$ and $S_q^{(t)}$.

It should be strongly reiterated that the $S_q^{(s)}$ and $S_q^{(t)}$ are not in themselves spin operators, but rather should be regarded as operators having certain group properties in a subspace of other higher-dimensional operators which well may have different group properties.

The invariants of the group involving $S_q^{(s)}$ and $S_q^{(t)}$ are given by

$$\mathbf{S}^{(s)}\cdot\mathbf{S}^{(s)} + \mathbf{S}^{(t)}\cdot\mathbf{S}^{(t)} = 3s^2 I \tag{3.36}$$

$$\mathbf{S}^{(s)}\cdot\mathbf{S}^{(t)} = 0 \tag{3.37}$$

where I is the unit matrix of dimension $4s$ by $4s$.

The matrix elements of the operators $S_i^{(s)}$, $S_i^{(t)}$, B, and I in the representation $\{s_> \oplus s_<\}$ define the matrix representation of these quantities which are listed in table I for values of $s_>$ corresponding to 1/2, 1, 3/2, and 2.

It is sometimes convenient to use the quantities

$$\boldsymbol{\sigma}^{(s)} = \frac{1}{s}\mathbf{S}^{(s)}; \qquad \boldsymbol{\sigma}^{(t)} = \frac{1}{s}\mathbf{S}^{(t)} \tag{3.38}$$

The commutation and anticommutation relations written in terms of these quantities are

$$[\sigma_i^{(s)},\sigma_j^{(s)}] = i\frac{1}{s}\epsilon_{ijk}\sigma_k^{(s)} \tag{3.39}$$

$$[\sigma_i^{(s)},\sigma_j^{(t)}] = i\frac{1}{s}\epsilon_{ijk}\sigma_k^{(t)} \tag{3.40}$$

$$[\sigma_i^{(t)},\sigma_j^{(t)}] = i\epsilon_{ijk}(2B - s^{-1})\sigma_k^{(s)} \tag{3.41}$$

$$\{\sigma_i^{(s)},\sigma_j^{(s)}\} + \{\sigma_i^{(t)},\sigma_j^{(t)}\} = 2\delta_{ij} \tag{3.42}$$

$$\{\sigma_i^{(s)},\sigma_j^{(t)}\} - \{\sigma_i^{(t)},\sigma_j^{(s)}\} = 2i\epsilon_{ijk}\hat{\sigma}_k^{(t)} \tag{3.43}$$

$$[\sigma_i^{(s)},\sigma_j^{(s)}] + [\sigma_i^{(t)},\sigma_j^{(t)}] = 2i\epsilon_{ijk}\hat{\sigma}_k^{(s)} \tag{3.44}$$

$$[\sigma_i^{(s)},\sigma_j^{(t)}] - [\sigma_i^{(t)},\sigma_j^{(s)}] = 0 \tag{3.45}$$

$$[\sigma_i^{(s)},B] = 0 \tag{3.46}$$

$$\{\sigma_i^{(t)},B\} = 0 \tag{3.47}$$

These relations are generalizations of the familiar Pauli algebra for the spin 1/2 σ matrices.[4]

[4] For spin 1/2 there are no time-like components, furthermore $S_i^{(t)}$ and $\sigma_i^{(t)}$ are null operators. Such results as (3.36) are not contradictory since $3s^2 = s(s+1)$ when $s = 0, 1/2$.

Combining some of these commutation relations allows us to find additional relations that are often useful

$$\sigma_i^{(s)}\sigma_j^{(s)} + \sigma_i^{(t)}\sigma_j^{(t)} = \delta_{ij} + i\epsilon_{ijk}\hat{\sigma}_k^{(s)} \tag{3.48}$$

$$\sigma_i^{(s)}\sigma_j^{(t)} - \sigma_i^{(t)}\sigma_j^{(s)} = i\epsilon_{ijk}\hat{\sigma}_k^{(t)} \tag{3.49}$$

If we introduce two arbitrary vectors **A** and **C** it is straightforward to show that

$$(\boldsymbol{\sigma}^{(s)}\cdot\mathbf{A})(\boldsymbol{\sigma}^{(s)}\cdot\mathbf{C}) + (\boldsymbol{\sigma}^{(t)}\cdot\mathbf{A})(\boldsymbol{\sigma}^{(t)}\cdot\mathbf{C}) = \mathbf{A}\cdot\mathbf{C} + i\hat{\boldsymbol{\sigma}}^{(s)}\cdot\mathbf{A}\times\mathbf{C} \tag{3.50}$$

$$(\boldsymbol{\sigma}^{(s)}\cdot\mathbf{A})(\boldsymbol{\sigma}^{(t)}\cdot\mathbf{C}) - (\boldsymbol{\sigma}^{(t)}\cdot\mathbf{A})(\boldsymbol{\sigma}^{(s)}\cdot\mathbf{C}) = i\hat{\boldsymbol{\sigma}}^{(t)}\cdot\mathbf{A}\times\mathbf{C} \tag{3.51}$$

Again, these are generalizations of some familiar relations of the spin 1/2 Pauli algebra.

We now form several anticommuting unitary operators of dimension $8s$ by $8s$ from the components $\sigma_i^{(s)}$, $\sigma_i^{(t)}$, and B. We define:

$$\alpha_i = \begin{vmatrix} \sigma_i^{(t)} & \hat{\sigma}_i^{(s)} \\ \hat{\sigma}_i^{(s)} & \sigma_i^{(t)} \end{vmatrix} \tag{3.52}$$

$$\beta = \begin{vmatrix} B & 0 \\ 0 & -B \end{vmatrix} \tag{3.53}$$

These operators obey the anticommutation relations

$$\{\alpha_i,\alpha_j\} = 2\delta_{ij} \tag{3.54}$$

$$\{\alpha_i,\beta\} = 0 \tag{3.55}$$

which may be proved straighforwardly using (3.42–3.47).

Indeed, there are a number of familiar relations that can be developed. If we define

$$\gamma_i = -i\beta\alpha_i = \begin{vmatrix} -i\hat{\sigma}_i^{(t)} & -i\sigma_i^{(s)} \\ i\sigma_i^{(s)} & i\hat{\sigma}_i^{(t)} \end{vmatrix} \tag{3.56}$$

$$\gamma_4 = \beta \tag{3.57}$$

We have the generalization to any spin of expressions for the Dirac γ matrices in the Dirac-Pauli representation. These generalized matrices obey the anticommutation and commutation relations characteristic of a Clifford algebra [24]:

$$\{\gamma_\mu,\gamma_\nu\} = 2\delta_{\mu\nu} \tag{3.58}$$

$$[\gamma_\mu,\gamma_\nu] = 2i\sigma_{\mu\nu} \tag{3.59}$$

We should emphasize strongly that this $\sigma_{\mu\nu}$ is *not*, in general, proportional to the spin operator of the particle. The spin operator is yet to be determined.

We may also define a symmetric operator:

$$\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 = \begin{vmatrix} 0 & -I \\ -I & 0 \end{vmatrix} \tag{3.60}$$

There are additional relations that follow immediately:

$$[\sigma_{\mu\nu},\gamma_\lambda] = 2i\delta_{\mu\lambda}\gamma_\nu - 2i\delta_{\nu\lambda}\gamma_\mu \tag{3.61}$$

$$\{\sigma_{\mu\nu},\gamma_\lambda\} = 2i\epsilon_{\mu\nu\lambda\rho}\gamma_\rho\gamma_5 \tag{3.62}$$

$$[\sigma_{\mu\nu},\gamma_5] = 0 \tag{3.63}$$

$$\{\sigma_{\mu\nu},\gamma_5\} = 2\epsilon_{\mu\nu\lambda\rho}\sigma_{\rho\lambda} \tag{3.64}$$

$$[\sigma_{\mu\nu},\gamma_\lambda\gamma_5] = 2i\delta_{\mu\lambda}\gamma_\nu\gamma_5 - 2i\delta_{\nu\lambda}\gamma_\mu\gamma_5 \tag{3.65}$$

$$\{\sigma_{\mu\nu},\gamma_\lambda\gamma_5\} = 2i\epsilon_{\mu\nu\lambda\rho}\gamma_\rho \tag{3.66}$$

$$\sigma_{ij} = -\epsilon_{ijk}\gamma_5\alpha_k \tag{3.67}$$

$$\sigma_{k4} = \alpha_k \tag{3.68}$$

$$\gamma_\mu A_\mu \gamma_\nu C_\nu = A_\mu C_\mu + i\sigma_{\mu\nu}A_\mu C_\nu \tag{3.69}$$

All of the γ_μ and γ_5 are unitary hermitian operators in this Pauli-Dirac representation; γ_1 and γ_3 are imaginary and antisymmetric while γ_2, γ_4, and γ_5 are real and symmetric. All have trace zero [25].

We now introduce an $8s$ component wave function $\varphi(x)$ which is the projection of the state vector (3.1) onto coordinate space. For the moment we consider the $8s$ component column matrix to describe a one-particle state at coordinate $x = (\mathbf{x},it)$.

If we operate on this wave function with the scalar product of the γ_μ operator with the four-gradient operator ∂_μ we obtain another function $\chi(x)$.

$$\chi(x) = \gamma_\mu\partial_\mu\varphi(x) \tag{3.70}$$

The function $\chi(x)$ will also be an $8s$ component column matrix but it does not necessarily have the same Lorentz transformation characteristics as $\varphi(x)$.

The adjoint wave function is defined as the projection of the ajoint state vector, $\langle ps\lambda \mid \beta\eta$, onto coordinate space. The definition is similar to usual Pauli-Dirac adjoint but modified by an operator η that reverses the sign of those states that are subject to gauge conditions or, equivalently, those states that have time like components present in the rest system for fields with nonzero mass. This operator η introduces an indefinite metric into the Hilbert space in such a way that the norm and the expectation value of the four-momenta for each orthogonal quantum state is positive definite. This adjoint wave function is given in terms of $\varphi^\dagger(x)$ by

$$\bar{\varphi}(x) = \varphi^\dagger(x)\gamma_4\eta \tag{3.71}$$

and the adjoint to $\chi(x)$ is

$$\bar{\chi}(x) = \chi^\dagger(x)\gamma_4\eta \tag{3.72}$$

The properties of the operator $\gamma_\mu \partial_\mu$ under hermitian conjugation together with the commutation relations yield

$$\bar{\chi}(x) = -\bar{\varphi}(x)\partial_\mu\gamma_\mu \tag{3.73}$$

Placing ∂_μ to the left of the γ_μ implies that it operates towards the left.

We now consider a proper imhomogeneous orthochronous infinitesimal Lorentz transformation as defined by (2.62) of the basis states $\varphi(x)$ and $\chi(x)$ so that

$$\varphi'(x') = D^{(s,\varphi)}(\Lambda)\varphi(x) = (1 + i/2\, \varepsilon_{\mu\nu}S_{\mu\nu}{}^{(\varphi)})\varphi(x) \tag{3.74}$$

and

$$\chi'(x') = D^{(s,\chi)}(\Lambda)\chi(x) = (1 + i/2\, \varepsilon_{\mu\nu}S_{\mu\nu}{}^{(\chi)})\chi(x) \tag{3.75}$$

where

$$x_\mu' = x_\mu + \varepsilon_{\mu\nu}x_\nu + a_\mu$$

The spin operators $S_{\mu\nu}{}^{(\varphi)}$ and $S_{\mu\nu}{}^{(\chi)}$ are as yet undefined other than that they are the appropriate spin operators for $\varphi(x)$ and $\chi(x)$, respectively.

The relation (3.70) under this Lorentz transformation becomes

$$\begin{aligned}\chi'(x') &= D^{(s,\chi)}(\Lambda)\gamma_\lambda\partial_\lambda D^{(s,\varphi)-1}(\Lambda)D^{(s,\varphi)}(\Lambda)\varphi(x)\\ &= \left[\gamma_\lambda\partial_\lambda + \frac{i}{2}\varepsilon_{\mu\nu}(S_{\mu\nu}{}^{(\chi)}\gamma_\lambda - \gamma_\lambda S_{\mu\nu}{}^{(\varphi)})\partial_\lambda\right]\varphi'(x')\\ &= \gamma_\mu\partial_\mu{}'\varphi'(x')\end{aligned} \tag{3.76}$$

where

$$\partial_\mu' = \partial_\mu + \varepsilon_{\mu\nu}\partial_\nu$$

The functional relationship between $\chi(x)$ and $\varphi(x)$ is the same in the primed and unprimed system provided that

$$S_{\mu\nu}{}^{(\chi)}\gamma_\lambda - \gamma_\lambda S_{\mu\nu}{}^{(\varphi)} = i\delta_{\mu\lambda}\gamma_\nu - i\delta_{\nu\lambda}\gamma_\mu \tag{3.77}$$

This requirement (3.77) is the condition for covariance in the Lagrangian formulation to be discussed shortly. Note that (3.77) is not a commutation relation unless there is an equality between $S_{\mu\nu}{}^{(\chi)}$ and $S_{\mu\nu}{}^{(\varphi)}$. The commutation relations of $S_{\mu\nu}{}^{(\chi)}$ or $S_{\mu\nu}{}^{(\varphi)}$ with γ_λ when $S_{\mu\nu}{}^{(\chi)} \neq S_{\mu\nu}{}^{(\varphi)}$ are rather complicated and do not usually occur in practice. The $S_{\mu\nu}{}^{(\chi)}$ and $S_{\mu\nu}{}^{(\varphi)}$ always satisfy, individually, the Lie algebra of relations (2.18)

$$[S_{\mu\nu}{}^{(\varphi)},S_{\lambda\rho}{}^{(\varphi)}] = i[\delta_{\mu\lambda}S_{\nu\rho}{}^{(\varphi)} + \delta_{\nu\rho}S_{\mu\lambda}{}^{(\varphi)} - \delta_{\mu\rho}S_{\nu\lambda}{}^{(\varphi)} - \delta_{\nu\lambda}S_{\mu\rho}{}^{(\varphi)}] \tag{3.78}$$

$$[S_{\mu\nu}{}^{(\chi)},S_{\lambda\rho}{}^{(\chi)}] = i[\delta_{\mu\lambda}S_{\nu\rho}{}^{(\chi)} + \delta_{\nu\rho}S_{\mu\lambda}{}^{(\chi)} - \delta_{\mu\rho}S_{\nu\lambda}{}^{(\chi)} - \delta_{\nu\lambda}S_{\mu\rho}{}^{(\chi)}] \tag{3.79}$$

Inspection of the commutation relations (3.78, 3.79) suggests that one of the possibilities for the representation of $S_{\mu\nu}$ is given by

$$S_{ij} = \epsilon_{ijk} \begin{vmatrix} S_k^{(s)} & 0 \\ 0 & S_k^{(s)} \end{vmatrix} \tag{3.80}$$

$$S_{k4} = \begin{vmatrix} 0 & S_k^{(s)} \\ S_k^{(s)} & 0 \end{vmatrix} \tag{3.81}$$

We may associate (3.80, 3.81) with the spin operator for the χ wave function, i.e., $S_{\mu\nu}^{(\chi)}$. We can also in certain instances make the opposite association, i.e., with the φ wave function, but we will not give the arguments here.

The condition (3.77) then determines uniquely $S_{\mu\nu}^{(\varphi)}$ since both $S_{\mu\nu}^{(\chi)}$ and γ_λ are already determined in terms of the operators $S_i^{(s)}$, $S_i^{(t)}$, and B. We find

$$S_{ij}^{(\varphi)} = S_{ij}^{(\chi)} = \epsilon_{ijk} \begin{vmatrix} S_k^{(s)} & 0 \\ 0 & S_k^{(s)} \end{vmatrix} \tag{3.82}$$

$$S_{k4}^{(\varphi)} = \sigma_{k4} - S_{k4}^{(\chi)} = \begin{vmatrix} \sigma_k^{(t)} & \hat{\sigma}_k^{(s)} - S_k^{(s)} \\ \hat{\sigma}_k^{(s)} - S_k^{(s)} & \sigma_k^{(t)} \end{vmatrix} \tag{3.83}$$

These expressions for $S_{\mu\nu}^{(\varphi)}$ satisfy the commutation relations (3.78), as they must. We see that the space components $S_{ij}^{(\chi)} = S_{ij}^{(\varphi)}$ so that χ and φ wave functions always have the same spatial rotation properties. Relation (3.77) is a commutation relation for these spatial components for any spin; it is also a commutation relation for all components in the case of spin 1/2. Here $\hat{\sigma}^{(s)} = 2S^{(s)}$ while $\sigma^{(t)}$ is a null matrix so that $S_{\mu\nu}^{(\varphi)} = S_{\mu\nu}^{(\chi)}$. For spin 1/2 alone do the χ and φ wave functions have the same Lorentz transformation properties.

Other relations important in establishing covariance conditions are given by the commutation relations

$$[S_{\mu\nu}^{(\varphi)}, \sigma_{\lambda\rho}] = i[\delta_{\mu\lambda}\sigma_{\nu\rho} + \delta_{\nu\rho}\sigma_{\mu\lambda} - \delta_{\mu\rho}\sigma_{\nu\lambda} - \delta_{\nu\lambda}\sigma_{\mu\rho}] \tag{3.84}$$

$$[S_{\mu\nu}^{(\chi)}, \sigma_{\lambda\rho}] = [S_{\mu\nu}^{(\varphi)}, \sigma_{\lambda\rho}] \tag{3.85}$$

and, by taking the hermitian conjugate of (3.77)

$$S_{\mu\nu}^{(\varphi)}\gamma_\lambda - \gamma_\lambda S_{\mu\nu}^{(\chi)} = i\delta_{\mu\lambda}\gamma_\nu - i\delta_{\nu\lambda}\gamma_\mu \tag{3.86}$$

We may apply the criteria of (2.66, 2.67) to test these representations for irreducibility. We can define

$$\mathbf{M}^{(\varphi)} = (S_{23}^{(\varphi)}, S_{31}^{(\varphi)}, S_{12}^{(\varphi)}) \tag{2.28'}$$

$$\mathbf{N}^{(\varphi)} = (S_{14}^{(\varphi)}, S_{24}^{(\varphi)}, S_{34}^{(\varphi)}) \tag{2.29'}$$

and

$$\mathbf{M}^{(\chi)} = (S_{23}{}^{(\chi)}, S_{31}{}^{(\chi)}, S_{12}{}^{(\chi)}) \tag{2.28''}$$

$$\mathbf{N}^{(\chi)} = (S_{14}{}^{(\chi)}, S_{24}{}^{(\chi)}, S_{34}{}^{(\chi)}) \tag{2.29''}$$

Considering, for the moment, particles of nonzero mass, we have for the φ wave function, from (2.66, 2.67),

$$F^{(\varphi)} = \tfrac{1}{2}(\mathbf{M}^{(\varphi)2} + \mathbf{N}^{(\varphi)2}) = (s^2 + \tfrac{1}{2}) \begin{vmatrix} I & 0 \\ 0 & I \end{vmatrix} \tag{3.87}$$

$$G^{(\varphi)} = \mathbf{M}^{(\varphi)} \cdot \mathbf{N}^{(\varphi)} = (s^2 - 1)\gamma_5 \tag{3.88}$$

Both $F^{(\varphi)}$ and $G^{(\varphi)}$ commute with all of the generators of the inhomogeneous Lorentz group $J_{\mu\nu}{}^{(\varphi)}$ and P_μ, and $F^{(\varphi)}$ is a multiple of the unit matrix.

The corresponding quantities $F^{(\chi)}$ and $G^{(\chi)}$ are

$$F^{(\chi)} = \tfrac{1}{2}(\mathbf{M}^{(\chi)2} + \mathbf{N}^{(\chi)2}) = \begin{vmatrix} s(s+B) & 0 \\ 0 & s(s+B) \end{vmatrix} \tag{3.89}$$

$$G^{(\chi)} = \mathbf{M}^{(\chi)} \cdot \mathbf{N}^{(\chi)} = -\gamma_5 F^{(\chi)} \tag{3.90}$$

The invariants $F^{(\chi)}$ and $G^{(\chi)}$ also commute with all of the generators of the inhomogeneous Lorentz group. While $F^{(\chi)}$ is not a multiple of the unit matrix it is completely decomposable into two independent multiples of the unit matrix.

We can introduce a change of representation to display the structure of the invariants $F^{(\varphi)}$, $G^{(\varphi)}$, $F^{(\chi)}$, and $G^{(\chi)}$ in a more familiar way by the introduction of the hermitian quantities $\mathbf{S}^{(+)}$ and $\mathbf{S}^{(-)}$ defined in (2.68, 2.69). The eigenvalues of the operators F and G in this new representation are given by (2.67, 2.77)

$$F^{(\varphi)} \rightarrow s^{(+)}(s^{(+)} + 1) + s^{(-)}(s^{(-)} + 1) \tag{3.91}$$

$$G^{(\varphi)} \rightarrow s^{(+)}(s^{(+)} + 1) - s^{(-)}(s^{(-)} + 1) \tag{3.92}$$

Equating $F^{(\varphi)}$ and $G^{(\varphi)}$ in this representation with the corresponding $F^{(\varphi)}$ and $G^{(\varphi)}$ in the $O(4)$ representation (3.87, 3.88) allows us to solve for $s^{(+)}$ and $s^{(-)}$ in terms of s. Retaining only the positive values, we obtain $s^{(+)} = s - 1/2$ and $s^{(-)} = 1/2$. The space of the φ wave functions involves a $(s - 1/2, 1/2) \oplus (1/2, s - 1/2)$ representation. The φ transforms according to a $D^{(s-1/2,1/2)}(\Lambda) \oplus D^{(1/2,s-1/2)}(\Lambda)$ representation and has $8s$ components except for the special case when $s = 1$ where the irreducible representation has four components. A polar vector or an axial vector transforms as $D^{(1/2,1/2)}(\Lambda)$.

In a similar way we can compare the invariants for the χ wave function

$$F^{(\chi)} \rightarrow s^{(+)}(s^{(+)} + 1) + s^{(-)}(s^{(-)} + 1) \tag{3.91'}$$

$$G^{(\chi)} \rightarrow s^{(+)}(s^{(+)} + 1) - s^{(-)}(s^{(-)} + 1) \tag{3.92'}$$

with the corresponding invariants in the O(4) representation. We obtain the values $s^{(+)} = s$ and $s^{(-)} = 0$ when $b = +1$ and $s^{(+)} = s - 1$ and $s^{(-)} = 0$ when $b = -1$. The space of the χ wave function or state vector involves a completely reducible $(s,0) \oplus (0,s) + (s - 1,0) \oplus (0,s - 1)$ representation. The χ state transforms according to a $D^{(s,0)}(\Lambda) \oplus D^{(0,s)}(\Lambda) + D^{(s-1,0)}(\Lambda) \oplus D^{(0,s-1)}(\Lambda)$ representation and has $8s$ components that can be divided into two irreducible groups of $4s + 2$ and $4s - 2$ components. The special case when $s = 1$ has three independent groups a Lorentz six vector of six components, a scalar of one component and a pseudoscalar of one component.

The operation of $M_{\pm}^{(\chi)}$, $M_0^{(\chi)}$, $N_{\pm}^{(\chi)}$, and $N_0^{(\chi)}$ transforms any vector in a particular irreducible subspace into another vector in the same subspace.

It is the linear independence of these two irreducible subspaces of the χ state vector that provides a gauge freedom in the description of fields with spin greater than one half. This gauge freedom will be further elaborated upon in section 4.

Equation (3.70) may be given a group theoretical classification. We recognize that a four-gradient operator ∂_μ is a four vector represented by the classification (1/2,1/2) and the structure of the γ matrices couples this four-gradient operator to the φ state vector in such a way that the resultant quantity transforms as the χ state vector. Then

$$\chi(x) = \gamma_\mu \partial_\mu \varphi(x) \tag{3.70}$$

is represented by the symbolic relation

$$(s,0) \oplus (0,s) + (s-1,0) \oplus (0,s-1) = (\tfrac{1}{2},\tfrac{1}{2}) \otimes [(s-\tfrac{1}{2},\tfrac{1}{2}) \oplus (\tfrac{1}{2},s-\tfrac{1}{2})] \tag{3.93}$$

There are two special cases of interest. Where $s = 1/2$ the representation $(s - 1,0 \oplus (0,s - 1)$ does not appear and (3.93) becomes

$$(\tfrac{1}{2},0) \oplus (0,\tfrac{1}{2}) = (\tfrac{1}{2},\tfrac{1}{2}) \otimes [(0,\tfrac{1}{2}) \oplus (\tfrac{1}{2},0)] \tag{3.94}$$

We see that the $\varphi(x)$ and $\chi(x)$ are basis states of representations having the same group structure and hence have the same transformation characteristics and that $F^{(\varphi)} = F^{(\chi)}$ and $G^{(\varphi)} = G^{(\chi)}$ whenever $s = 1/2$

When $s = 1$ (3.93) becomes

$$(1,0) \oplus (0,1) + (0,0) + (0,0) = (\tfrac{1}{2},\tfrac{1}{2}) \otimes [(\tfrac{1}{2},\tfrac{1}{2}) + (\tfrac{1}{2},\tfrac{1}{2})] \tag{3.95}$$

We have three independent irreducible quantities on the left hand side, a Lorentz six-vector, a scalar, and a pseudoscalar, while on the right hand side we have the independent quantities, a polar four-vector and an axial four-vector.

We are not limited to situations where the basic field is of the $(s - \tfrac{1}{2},\tfrac{1}{2}) \oplus (\tfrac{1}{2}.s - \tfrac{1}{2})$ representation. We also have the converse relation where the roles of the $\chi(x)$ and $\varphi(x)$ fields are interchanged. We may have, independently, the symbolic relationships.

$$(s-\tfrac{1}{2},\tfrac{1}{2}) \oplus (\tfrac{1}{2},s-\tfrac{1}{2}) = (\tfrac{1}{2},\tfrac{1}{2}) \otimes [(s,0) \oplus (0,s)] \tag{3.96}$$

and

$$(s-\tfrac{1}{2},\tfrac{1}{2}) \oplus (\tfrac{1}{2},s-\tfrac{1}{2}) = (\tfrac{1}{2},\tfrac{1}{2}) \otimes [(s-1,0) \oplus (0,s-1)] \tag{3.97}$$

Finite rotations and boosts of the $\varphi(x)$ and $\chi(x)$ functions representing a particular spin can be calculated from (2.12) using the expressions for $S_{\mu\nu}^{(\varphi)}$ and $S_{\mu\nu}^{(\chi)}$ and the explicit matrices in table I. Since $M^{(\varphi)} = M^{(\chi)}$, the rotation operator is the same for either wave function.

We label the argument of the operator $D^{(s)}(\Lambda)$ differently when making a specific rotation or boost. For a rotation ϑ_{ij} we write $D_{\text{rot}}^{(s)}(\vartheta_{ij})$ and for a boost Ω_{k4} we write $D_{\text{boost}}^{(s)}(\Omega_{k4})$. The rotation operator is

$$D_{\text{rot}}^{(s,\varphi)}(\vartheta_{ij}) = D_{\text{rot}}^{(s,\chi)}(\vartheta_{ij}) = \exp[i\vartheta_{ij} M_k^{(\varphi)}] \tag{3.98}$$

The operator for a boost along the k axis for the χ function is given by

$$D_{\text{boost}}{}^{(s,\chi)}(\Omega_{ij}) = \exp[-\Omega_{ij}N_k{}^{(\chi)}] \tag{3.99}$$

and a similar operator applies for the φ function. Each of these operators may be calculated by writing the exponential as an infinite power series and looking for algebraic combinations involving even powers of $S_{ij}{}^{(\chi)}$ or of $S_{k4}{}^{(\chi)}$ that become projection operators for states of a given spin component $|\lambda|$. This direct procedure becomes laborious as the value of s increases. An alternative method is to make use of homomorphism between the homogeneous Lorentz group and the group of unitary 2×2 matrices of SL(2c). The rotation and boost operator in the space of the χ function for any spin may be calculated from recursion formulae involving the matrices of spin 1/2. The reader is referred to works on group theory for details [26, 27].

The boost operator for the φ function may be conveniently calculated by noting that (3.83) allows us to write $D^{(s,\varphi)}(\Omega_{k4})$ as

$$D_{\text{boost}}{}^{(s,\varphi)}(\Omega_{k4}) = [\cosh\Omega_{k4} - \sigma_{k4}\sinh\Omega_{k4}][D_{\text{boost}}{}^{(s,\chi)-1}(\Omega_{k4})] \tag{3.100}$$

Specific rotation and boost operators for spins 1/2, 1, 3/2, and 2 calculated by the direct method are listed in table II.

That irreducible portion of the χ field with the classification $(s,0) \oplus (0,s)$ is related to fields discussed by Joos [28] and by Weinberg [29] who have developed extensively the transformation properties of fields with the SL(2c) group classification. While the properties of these fields are well understood within the framework of the inhomogeneous Lorentz group, no Lagrangian formulation can be achieved with this Joos-Weinberg representation alone, other than for the spin one-half case.

The rotation operators for a rotation about the y axis by the amount π will be of particular use. We denote this operator simply as D.

$$D \equiv D_{\text{rot}}{}^{(s)}(\vartheta_{31} = \pi) = \begin{vmatrix} d^{(s)}(\pi) & 0 \\ 0 & d^{(s)}(\pi) \end{vmatrix} \tag{3.101}$$

From table II we obtain

$$d^{(1/2)}(\pi) = \begin{vmatrix} \cdot & 1 \\ -1 & \cdot \end{vmatrix} \tag{3.102}$$

$$d^{(1)}(\pi) = \begin{vmatrix} \cdot & \cdot & 1 & \cdot \\ \cdot & -1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{vmatrix} \tag{3.103}$$

$$d^{(3/2)}(\pi) = \left|\begin{matrix} \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot \end{matrix}\right| \tag{3.104}$$

$$d^{(2)}(\pi) = \left|\begin{matrix} \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \end{matrix}\right| \tag{3.105}$$

The $d^{(s)}(\pi)$, and consequently D, are symmetric for integer spin and antisymmetric for half integer spin.

This operator D has the property[5]

$$D\tilde{\gamma}_\mu D^{-1} = \gamma_\mu \tag{3.106}$$

In order to obtain a physical picture of the operational meaning of the γ operators, it is instructive to look specifically at the spin one case. Here, we can also consider a cartesian as well as a spherical basis for the components.

Consider a four vector field $A(x)$. We can write the components in either basis:

$$\begin{aligned} A(x) &= A_\mu(x)\mathbf{e}_\mu \\ &= A_1(x)\mathbf{e}_1 + A_2(x)\mathbf{e}_2 + A_3(x)\mathbf{e}_3 + V(x)e_4 \end{aligned} \tag{3.107}$$

---

[5] This D is the operator that Pauli calls B; however, we have already usurped B for the designation of the metric operator. Arguments by Pauli require D to be antisymmetric for any representation of the γ matrices. This reasoning does not apply to matrices of dimension higher than the 4×4 Dirac matrices for spin 1/2 fields. For the spin 1/2 case in the Dirac-Pauli representation, $D = -\gamma_2\gamma_4\gamma_5 = i\sigma_{31}$.

or

$$A(x) = \sum_{\mu} (-1)^{\mu} A_{-\mu} \boldsymbol{\xi}_{\mu}$$

$$= -A_{-1}(x)\boldsymbol{\xi}_1 + A_0(x)\boldsymbol{\xi}_0 - A_1(x)\boldsymbol{\xi}_{-1} + V(x)\xi_4 \tag{3.108}$$

where the $\mathbf{e}_i$ are unit vectors along the cartesian axes and the $\boldsymbol{\xi}_m$ are unit vectors in a spherical representation; $e_4 = i$ and $\xi_4 = 1$.

$$\boldsymbol{\xi}_{+1} = -\frac{1}{\sqrt{2}}(\mathbf{e}_1 + i\mathbf{e}_2)$$

$$\boldsymbol{\xi}_0 = \mathbf{e}_3 \tag{3.109}$$

$$\boldsymbol{\xi}_{-1} = \frac{1}{\sqrt{2}}(\mathbf{e}_1 - i\mathbf{e}_2)$$

Consider a wave function $\varphi(x)$ in a cartesian and in a spherical basis where the components of $\varphi(x)$ are the components of a polar four vector, $A_{\mu} = (\mathbf{A}, iV)$ and an axial four-vector, $C_{\mu} = (\mathbf{C}, iW)$ in the cartesian basis.

We can write in a cartesian basis

$$\varphi_{\mathrm{cart}}(x) = \begin{vmatrix} A_1(x) \\ A_2(x) \\ A_3(x) \\ iV(x) \\ iC_1(x) \\ iC_2(x) \\ iC_3(x) \\ -W(x) \end{vmatrix} \equiv \vartheta(x) \tag{3.110}$$

The phase difference of A_{μ} and C_{μ} is chosen for convenience.

In a spherical basis[6]

$$\varphi(x) = \begin{vmatrix} -\frac{1}{\sqrt{2}}(A_1(x) - iA_2(x)) \\ A_3(x) \\ \frac{1}{\sqrt{2}}(A_1(x) + iA_2(x)) \\ V(x) \\ -\frac{i}{\sqrt{2}}(C_1(x) - iC_2(x)) \\ iC_3(x) \\ \frac{i}{\sqrt{2}}(C_1(x) + iC_2(x)) \\ iW(x) \end{vmatrix} \tag{3.111}$$

[6] There is some confusion in the literature as to the designation of spherical components. Those components directed along the unit vectors $\xi_{\pm 1}$ are

$$\mp \frac{1}{\sqrt{2}}(A, \mp iA_2),$$

respectively, according to the definition (3.108).

$$U = \begin{vmatrix} u & 0 \\ 0 & u \end{vmatrix} \tag{3.113}$$

The elements of u are defined by the matrix elements of the unit vectors in the two representations

$$u_{\mu\nu} = \mathbf{e}_\mu \cdot \boldsymbol{\xi}_\nu \tag{3.114}$$

giving

$$u = \begin{vmatrix} -\frac{1}{\sqrt{2}} & \cdot & \frac{1}{\sqrt{2}} & \cdot \\ -\frac{i}{\sqrt{2}} & \cdot & -\frac{i}{\sqrt{2}} & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & i \end{vmatrix} \tag{3.115}$$

The operators $S_{\mu\nu}{}^{(\varphi)}$, $S_{\mu\nu}{}^{(\chi)}$, γ_μ in the cartesian basis are all given by relations similar to

$$S_{\mu\nu}{}^{(\varphi)\prime} = US_{\mu\nu}{}^{(\varphi)}U^{-1} \tag{3.116}$$

The $S_{\mu\nu}{}^{(\varphi)\prime}$, $S_{\mu\nu}{}^{(\chi)\prime}$, and $\gamma_\mu{}'$ are defined in terms of the matrices $S_i{}^{(s)\prime}$, $S_i{}^{(t)\prime}$, and B' listed in table III using (3.56, 3.57, 3.80–3.83).

The matrices $S_i{}^{(s)\prime}$ and $S_i{}^{(t)\prime}$ are identical to the matrices $Z_{\mu\nu}$ which are the generators involved in the Lorentz transformation of coordinates and momenta, the components of a four vector. The correspondence is

$$S_i{}^{(s)\prime} = \tfrac{1}{2}\epsilon_{ijk}Z_{jk} \tag{3.117}$$

$$S_i{}^{(t)\prime} = Z_{i4} \tag{3.118}$$

They are the generators of the four-dimensional orthogonal group O(4) and obey the commutation relations (2.14).

The γ_μ in a cartesian representation are all real and symmetric[7]. It is instructive to write out the

[7] It is possible to interchange in each γ-matrix the 4th with the 8th row and the 4th with the 8th column hy a unitary transformation. The dimension of these transformed 8 hy 8 matrices may he reduced to 4 hy 4 matrices hy making use of the homomorphism of the 2 hy 2 suhmatrices:

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \rightarrow 1 \qquad \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} \rightarrow i$$

The resultant 4 hy 4 matrices correspond to the complex Dirac-Pauli γ-matrices for the spin one-half field. The real and symmetric 8 hy 8 γ-matrices are a real irreducible representation of the complex irreducible 4 hy 4 matrices.

components of (3.70) to indicate the operational meaning of $\gamma_\mu \partial_\mu$

$$\gamma_\mu' \partial_\mu = \begin{vmatrix} -i\hat{\sigma}^{(t)\prime} \cdot \nabla + B\partial_4 & -i\sigma^{(s)\prime} \cdot \nabla \\ i\sigma^{(s)\prime} \cdot \nabla & i\hat{\sigma}^{(t)\prime} \cdot \nabla - B\partial_4 \end{vmatrix} \tag{3.119}$$

In a cartesian representation

$$-i\sigma^{(s)\prime} \cdot \nabla \begin{vmatrix} A_1 \\ A_2 \\ A_3 \\ iV \end{vmatrix} = \begin{vmatrix} \partial_2 A_3 - \partial_3 A_2 \\ \partial_3 A_1 - \partial_1 A_3 \\ \partial_1 A_2 - \partial_2 A_1 \\ 0 \end{vmatrix} \tag{3.120}$$

$$-i\sigma^{(t)\prime} \cdot \nabla \begin{vmatrix} A_1 \\ A_2 \\ A_3 \\ iV \end{vmatrix} = \begin{vmatrix} -\partial_1 iV \\ -\partial_2 iV \\ -\partial_3 iV \\ \partial_1 A_1 + \partial_2 A_2 + \partial_3 A_3 \end{vmatrix} \tag{3.121}$$

The function $\chi_{\text{cart}}(x)$, related to $\varphi_{\text{cart}}(x)$ by (3.70), contains a Lorentz six-vector ($i\boldsymbol{\mathcal{E}},\boldsymbol{\mathcal{B}}$), a scalar $\mathcal{E}_0$, and a pseudoscalar $\mathcal{B}_0$, so that we can write (3.70) in terms of components, where again we have chosen phases for convenience

$$\chi_{\text{cart}}(x) = \begin{vmatrix} i\mathcal{E}_1 \\ i\mathcal{E}_2 \\ i\mathcal{E}_3 \\ -\mathcal{E}_0 \\ -\mathcal{B}_1 \\ -\mathcal{B}_2 \\ -\mathcal{B}_3 \\ -i\mathcal{B}_0 \end{vmatrix} = \gamma_\mu' \partial_\mu \begin{vmatrix} A_1 \\ A_2 \\ A_3 \\ iV \\ iC_1 \\ iC_2 \\ iC_3 \\ -W \end{vmatrix} \tag{3.122}$$

Carrying out the matrix multiplication in (3.122) we obtain the following relations:

$$\boldsymbol{\mathcal{E}} = -\nabla V - \partial_t \mathbf{A} + \nabla \times \mathbf{C} \tag{3.123}$$

$$\mathcal{E}_0 = \nabla \cdot \mathbf{A} + \partial_t V = \partial_\mu A_\mu \tag{3.124}$$

$$\boldsymbol{\mathcal{B}} = \nabla \times \mathbf{A} + \nabla W + \partial_t \mathbf{C} \tag{3.125}$$

$$\mathcal{B}_0 = -\nabla \cdot \mathbf{C} - \partial_t W = -\partial_\mu C_\mu \tag{3.126}$$

We will see in section 6 that, while these relations are invariant under proper Lorentz transformations, they change under the product of space inversion and charge conjugation. Since in this spin-one case, the four axial-vector components are independent of the polar-vector components, we may set $C_\mu = 0$ and obtain familiar relations that are invariant under all symmetry operations.

$$\boldsymbol{\mathcal{E}} = -\nabla V - \partial_t \mathbf{A} \tag{3.127}$$

$$\mathcal{E}_0 = \partial_\mu A_\mu \tag{3.128}$$

$$\mathcal{B} = \nabla \times \mathbf{A} \quad (3.129)$$

$$\mathcal{B}_0 = 0 \quad (3.130)$$

The four divergence, $\partial_\mu A_\mu$, is not required to be zero. Relations (3.127–3.130) are a specific example of the symbolic relation (3.95) for a spin-one polar vector field.

4. Lagrangian Formalism

A Lagrangian formalism provides a systematic way for obtaining the equations of motion of a field with an infinite number of degrees of freedom and for identifying and extracting the constants of the motion in a classical field theory. The equations of motion are obtained from Hamilton's action principle, which requires the action integral to be an extremum for arbitrary variations of the field function. There is also a conservation theorem and a constant of the motion corresponding to each continuous symmetry transformation that leaves the action integral unchanged and the equations of motion invariant in form. This conservation theorem permits observed selection rules in nature to be described directly in terms of the symmetry requirements of the Lagrangian.

These symmetries follow directly from the Lorentz transformations discussed in section 2. The proper transformations such as translations, rotations and boosts lead to familiar conservation laws. In addition, the improper transformations of space reflection, time reversal, and charge conjugation lead to important selection rules in interactions.

There are other invariance principles that arise from internal symmetries of the Lagrangian. These are based on an arbitrariness in the Lagrangian which manifests itself as the concept of gauge invariance. As was mentioned earlier in section 3 there are certain representations of the states in Hilbert space that are the direct sum of certain irreducible representations. The existence of these representations leads to a gauge invariance of the second kind. There is also an arbitrariness in the phase of the states in Hilbert space, where they occur in bilinear combinations. This phase arbitrariness leads to a gauge invariance of the first kind. The coexistence of these two types of gauge invariance further implies a particular form for interactions between fields.

The Lagrangian density describing a field of n components will depend, in general, on n field functions and their adjoints, if the latter are independent[8], and on the first order derivatives, with respect to the space-time coordinates, of the n field functions and their adjoints. This Lagrangian density is denoted as $\mathcal{L}(\varphi_\alpha, \bar{\varphi}_\alpha, \partial_\mu\varphi_\alpha, \partial_\mu\bar{\varphi}_\alpha)$ where $\alpha = 1 \cdots n$. The dependence of $\mathcal{L}$ on the space-time cordinates is only through the fields and their derivatives, and x_μ cannot appear explicitly in $\mathcal{L}$.

The action integral

$$I = \int L\,dt = \int \mathcal{L}\,d^4x \quad (4.1)$$

does not depend on the special choice of any reference frame, thus it must be a Lorentz scalar. Since the term $d^4x = dx_1\,dx_2\,dx_3\,dt$ is a Lorentz invariant, the Lagrangian density, $\mathcal{L}$, transforms as a scalar under the transformations of the proper Lorentz group. This condition makes it much easier to find the functional form of the Lagrangian density. In section 3 we demonstrated that the wave functions $\varphi(x)$ and $\chi(x)$ each spanned a particular representation space from which we can construct invariant bilinear combinations with the adjoints $\bar{\varphi}(x)$ and $\bar{\chi}(x)$, respectively. Furthermore, we want to have equations

[8] We shall always assume, unless we specifically indicate otherwise, that the adjoint field is a quantity that is linearly independent of the field, such that separate equations of motion can be obtained from the Lagrangian for a field and its adjoint. There are certain self adjoint fields, e.g., the photon, the neutral pion, or the neutral rho meson where $\bar{\varphi} = \bar{\varphi}$ or, in a quantized theory, where the wave functions are considered to be operators, $\bar{\varphi} = \varphi$. In this self adjoint case, we have only one set of equations of motion for the n fields. The Lagrangian will have an additional factor of 1/2 to avoid double counting for these self adjoint fields.

of motion linear in the field and exclude therefrom derivatives of higher than second order. These requirements shall be met if we demand that $\mathcal{L}$ shall contain bilinear quantities $\bar{\chi}(x)\chi(x)$ and $\bar{\varphi}(x)\varphi(x)$ and not powers of these bilinear quantities. Two other bilinear quantities occur in the case of spin 1/2 fields where $\chi(x)$ and $\varphi(x)$ span the same representation space, namely, $\bar{\chi}(x)\varphi(x)$ and $\bar{\varphi}(x)\chi(x)$.

Variation of the action integral with respect to the field functions, $\varphi(x)$ or $\bar{\varphi}(x)$, independently, while requiring the action integral to be stationary, yields the Euler-Lagrange equations of motion

$$\frac{\partial \mathcal{L}}{\partial \varphi(x)} - \partial_\mu \frac{\partial \mathcal{L}}{\partial\, \partial_\mu \varphi(x)} = 0 \tag{4.2}$$

$$\frac{\partial \mathcal{L}}{\partial \bar{\varphi}(x)} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\bar{\varphi}(x)\, \partial_\mu)} = 0 \tag{4.3}$$

One can construct a canonically conjugate momentum, $\pi(x)$, for each variable $\varphi(x)$

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}(x)} \tag{4.4}$$

and similarly

$$\bar{\pi}(x) = \frac{\partial \mathcal{L}}{\partial \dot{\bar{\varphi}}(x)} \tag{4.5}$$

From these we may calculate the Hamiltonian density

$$\mathcal{H} = \pi(x)\dot{\varphi}(x) + \dot{\bar{\varphi}}(x)\bar{\pi}(x) - \mathcal{L} \tag{4.6}$$

In (4.6) it is necessary to express $\dot{\varphi}(x)$ and $\dot{\bar{\varphi}}(x)$ as a function of $\pi(x)$ and $\bar{\pi}(x)$ by the use of (3.70, 3.73). Thus, the Hamiltonian density $(\pi_\alpha,\varphi_\alpha,\partial_k\varphi_\alpha,\bar{\pi}_\alpha,\bar{\varphi}_\alpha,\partial_k\bar{\varphi}_\alpha)$ is understood to be a function of the canonical momenta, the field variables, and their spatial derivatives. The time derivatives are eliminated. In a quantized theory we will require the field operators and their canonical momenta to obey the Heisenberg relations:

$$i[H,\varphi(x)] = \dot{\varphi}(x) \tag{4.7}$$

$$i[H,\pi(x)] = \dot{\pi}(x) \tag{4.8}$$

where $H = \int \mathcal{H}\, d^3x$.

It is natural to write a free Langrangian utilizing the lowest possible number of first order derivatives of the field. For spin 1/2, such a Lagrangian is

$$\mathcal{L} = -\tfrac{1}{2}[\bar{\varphi}(x)\chi(x) + \bar{\chi}(x)\varphi(x)] - m\bar{\varphi}(x)\varphi(x) \tag{4.9}$$

Writing our $\bar{\chi}$ and χ in terms of $\bar{\varphi}$ and φ, we have

$$\mathcal{L} = \tfrac{1}{2}[\bar{\varphi}(\partial_\mu\gamma_\mu - m)\varphi - \bar{\varphi}(\gamma_\mu\partial_\mu + m)\varphi] \tag{4.10}$$

The Euler-Lagrange equations yield the familiar Dirac equations of motion for the spin 1/2 field $\varphi(x)$ and its adjoint $\bar{\varphi}(x)$:

$$(\gamma_\mu \partial_\mu + m)\varphi(x) = 0 \tag{4.11}$$

$$\bar{\varphi}(x)(\partial_\mu \gamma_\mu - m) = 0 \tag{4.12}$$

The description of particles of fields with spins greater than one half requires a Lagrangian that is at least quadratic in the first order derivatives. We shall reserve the notation φ and $\bar{\varphi}$ for the wave functions of these fields with spins greater than 1/2 and denote the wave functions of the above first order equations (4.11, 4.12) by ψ and $\bar{\psi}$.

For any spin greater than 1/2 we can write the Lagrangian as

$$\mathcal{L} = \bar{\chi}(x)\chi(x) - m^2\bar{\varphi}(x)\varphi(x) \tag{4.13}$$

or when it is expressed only in terms of φ and $\bar{\varphi}$

$$\mathcal{L} = -\bar{\varphi}(x)\partial_\mu\gamma_\mu\gamma_\nu\partial_\nu\varphi(x) - m^2\bar{\varphi}(x)\varphi(x) \tag{4.14}$$

The Euler-Lagrange equations give

$$(\gamma_\mu\gamma_\nu\partial_\mu\partial_\nu - m^2)\varphi(x) = 0 \tag{4.15}$$

$$\bar{\varphi}(x)(\partial_\mu\partial_\nu\gamma_\mu\gamma_\nu - m^2) = 0 \tag{4.16}$$

We can use (3.58, 3.59) to write the relation

$$\gamma_\mu\gamma_\nu = \delta_{\mu\nu} + i\sigma_{\mu\nu} \tag{4.17}$$

Substituting this expression for $\gamma_\mu\gamma_\nu$ in (4.15) will simplify the expression since the term containing $\sigma_{\mu\nu}$ is zero, for we can interchange summation indices and commute the operators ∂_μ and ∂_ν and find that the term is equal to its negative, due to the antisymmetry $\sigma_{\mu\nu} = -\sigma_{\nu\mu}$.

This yields the Klein-Gordon equation for a free field in which there is no coupling between the various spin components

$$(\Box - m^2)\varphi(x) = 0 \tag{4.18}$$

$$\bar{\varphi}(x)(\Box - m^2) = 0 \tag{4.19}$$

It is more instructive to consider a charged particle with a spin greater than one half interacting with a massive polar vector field, $A_\mu(x)$. We use a gauge invariant type coupling with coupling constant g. This is the usual minimal coupling which will be discussed more fully later. Here

$$\partial_\mu \to \partial_\mu - igA_\mu(x) \tag{4.20}$$

The equation (3.70) becomes

$$\chi(x) = \gamma_\mu(\partial_\mu - igA_\mu(x))\varphi(x) \tag{4.21}$$

and equation (3.73) becomes

$$\bar{\chi}(x) = -\bar{\varphi}(x)(\partial_\mu + igA_\mu(x))\gamma_\mu \tag{4.22}$$

We can use this formalism to describe the electromagnetic field by considering the mass of the vector boson field to be zero.

In this case the neutral polar vector field denoted by $\vartheta(x)$ is best represented in a cartesian basis with components $A_\mu(x)$ since these are the components that occur in the interaction (4.21, 4.22) so that

$$\vartheta(x) = \begin{vmatrix} A_1(x) \\ A_2(x) \\ A_3(x) \\ A_4(x) \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix} \equiv \begin{vmatrix} \mathbf{A}(x) \\ A_4(x) \\ 0 \\ 0 \end{vmatrix} \tag{4.23}$$

The space-like components $A_k(x)$ are hermitian and the time-like component $A_4(x) = iV(x)$ is antihermitian, i.e., $V(x)$ is hermitian. We occasionally use a shorthand notation for vector fields in which we contract the space components 1,2,3 and 5,6,7 of $\vartheta(x)$ into a vector $\mathbf{A}(x)$ and a null pseudovector.

The Lagrangian can be written as

$$\begin{aligned} \mathfrak{L} = &-\bar{\varphi}(\partial_\mu + igA_\mu)\gamma_\mu\gamma_\nu(\partial_\nu - igA_\nu)\varphi - m_\varphi{}^2\bar{\varphi}\varphi \\ &- \tfrac{1}{2}\bar{\vartheta}\partial_\mu\gamma_\mu\gamma_\nu\partial_\nu\vartheta - \tfrac{1}{2}m_A{}^2\bar{\vartheta}\vartheta \end{aligned} \tag{4.24}$$

The field ϑ is a neutral field and is self adjoint, i.e., $\bar{\bar{\vartheta}} = \bar{\vartheta}$ so in essence we have three independent fields $\bar{\varphi}$, φ, and ϑ. Variation of the action integral with respect to $\bar{\varphi}$ gives, from the Euler-Lagrange equation,

$$-(\partial_\mu - igA_\mu)\gamma_\mu\gamma_\nu(\partial_\nu - igA_\nu)\varphi - m_\varphi{}^2\varphi = 0 \tag{4.25}$$

Again from the relation $\gamma_\mu\gamma_\nu = \delta_{\mu\nu} + i\sigma_{\mu\nu}$, the term containing the $\sigma_{\mu\nu}$ operator can be written in an asymmetrical form by changing summation labels

$$i\sigma_{\mu\nu}(\partial_\mu - igA_\mu)(\partial_\nu - igA_\nu) = \frac{g}{2}\sigma_{\mu\nu}F_{\mu\nu} \tag{4.26}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{4.27}$$

Thus we obtain

$$\left[-(\partial_\mu - igA_\mu)^2 - \frac{g}{2}\sigma_{\mu\nu}F_{\mu\nu} + m_\varphi{}^2\right]\varphi = 0 \tag{4.28}$$

and a similar equation for the adjoint

$$\bar{\varphi}\left[-(\partial_\mu + igA_\mu)^2 + \frac{g}{2}\sigma_{\mu\nu}F_{\mu\nu} + m_\varphi{}^2\right] = 0 \tag{4.29}$$

These are equations of motion for φ and $\bar{\varphi}$ in a massive vector boson field A_μ with coupling constant g. It was stated that these equations of motion applied to particles or fields having spin greater than 1/2. They also may apply to particles of spin 1/2 since the solution of the Dirac equation,

$$[\gamma_\mu(\partial_\mu - igA_\mu) + m]\psi = 0 \tag{4.30}$$

is also a solution of (4.28). Indeed, particles of spin 1/2 can be treated by a second order theory with alteration in the normalization conditions and with certain redundancies [30, 31].

The Lagrangian (4.24), when φ represents a charged vector boson, is equivalent to the formulation of Lee and Yang [32] with their parameter ξ equal to unity.

Some caution should be exercised in the application of the equations of motion (4.28, 4.29) since the equations contain only the components corresponding to those used in the variation of the action integral. This is of importance primarily in the case of the spin-one field where only the upper four components of $\varphi(x)$ are involved for a polar vector field and the lower four for an axial vector field. Here, only part of the matrix operator $\sigma_{\mu\nu}$ is effective and we can make the association $\sigma_{\mu\nu} = S_{\mu\nu}^{(\varphi)}$. In general, for fields having spins greater than unity, all of the operator $\sigma_{\mu\nu}$ contributes, and it contains terms not contained in the corresponding spin operator $S_{\mu\nu}^{(\varphi)}$. We may consider a field of spin s to have an anomalous magnetic moment of $1/s$ by consideration of the nonrelativistic limit of these equations of motion[9].

The equations of motion (4.28, 4.29) obey all of the Velo-Zwanziger criteria that the solutions are of a causal nature for any spin field. The equations remain hyperbolic and the characteristic surfaces which determine the maximum propagation velocity assure this propagation on or within the light cone.

When the variation of $\mathcal{L}$ is performed with respect to the field A_μ we obtain

$$[-\gamma_\lambda\gamma_\nu\partial_\lambda\partial_\nu\vartheta(x) + m_A{}^2\vartheta(x)]_\mu = -ig[\bar{\varphi}\gamma_\mu\gamma_\nu(\partial_\nu - igA_\nu)\varphi - \bar{\varphi}(\partial_\nu + igA_\nu)\gamma_\nu\gamma_\mu\varphi] \tag{4.31}$$

We must specify the μ component of the quantity on the left side since the variation was performed with respect to that component. The quantity on the right hand side is the four-vector current j_μ of the φ field and can be written as

$$j_\mu = -ig[\bar{\varphi}\gamma_\mu\chi + \bar{\chi}\gamma_\mu\varphi] \tag{4.32}$$

thus we have for the μ component of $\vartheta(x)$

$$[-\Box + m_A{}^2]A_\mu(x) = j_\mu(x) \tag{4.33}$$

This is the equation of motion for the massive vector field A_μ coupled to a vector current source, $j_\mu(x)$.

There is other information to be obtained from (4.31), however. Remembering the example (3.70) there is another quantity $\chi_{(\vartheta)}$ that is related to ϑ, a four vector field, by

$$\chi_{(\vartheta)}(x) = \gamma_\mu\partial_\mu\vartheta(x) \tag{4.34}$$

[9] The nonrelativistic limit may be achieved hy letting $E \to W + M$ and neglecting higher order terms where $gV \ll m$ and $W \ll m$. We then obtain the Schrödinger equation after dividing hy $2m$ and dropping the rest energy term as well as including only the space-like components in $\varphi(x)$ and neglecting the time-like and small components. We have effectively

$$\left[\frac{1}{2m}(\mathbf{p} - g\mathbf{A}(x))^2 - (W - gV(x)) - \frac{g}{2m}\mathbf{\sigma}^{(s)}\cdot\mathcal{B}\right]\varphi(x)^{NR} = 0$$

where $\mathbf{p} = -i\nabla$ and $W = -\partial_4$

The components of $\chi_{(\vartheta)}(x)$ are those of a Lorentz six-vector $(i\mathcal{E},\mathcal{B})$ and of a scalar $\mathcal{E}_0$

$$\chi_{(\vartheta)}(x) = \begin{vmatrix} i\mathcal{E}_1(x) \\ i\mathcal{E}_2(x) \\ i\mathcal{E}_3(x) \\ -\mathcal{E}_0(x) \\ -\mathcal{B}_1(x) \\ -\mathcal{B}_2(x) \\ -\mathcal{B}_3(x) \\ 0 \end{vmatrix} = \begin{vmatrix} i\mathcal{E}(x) \\ -\mathcal{E}_0(x) \\ -\mathcal{B}(x) \\ 0 \end{vmatrix} \tag{4.35}$$

Rewriting (4.31) as

$$[-\gamma_\lambda\partial_\lambda\chi_{(\vartheta)}(x) + m_A{}^2\vartheta(x)]_\mu = j_\mu(x) \tag{4.36}$$

we obtain the component equations

$$\nabla \times \mathcal{B} - \partial_t\mathcal{E} - \nabla\mathcal{E}_0 + m_A{}^2\mathbf{A} = \mathbf{j} \tag{4.37}$$

$$\nabla\cdot\mathcal{E} + \partial_t\mathcal{E}_0 + m_A{}^2V = \rho \tag{4.38}$$

where $i\rho = j_4$. Although these are the only equations that are obtained by a variational principle, it is of interest to calculate the lower four quantities on the left hand side of (4.36) where $\mu = 5,6,7,8$. The right hand side is identically zero. We get

$$\nabla \times \mathcal{E} + \partial_t\mathcal{B} = 0 \tag{4.39}$$

$$\nabla\cdot\mathcal{B} = 0 \tag{4.40}$$

These are not equations of motion but rather identities which can be shown to follow from relations (3.129, 3.130).

The four equations (4.37–4.40) are a generalization of Maxwell's equations for a massive vector field which is applicable to the description of an iosbaric singlet neutral vector meson. In a following paper we shall discuss a generalization to an octet of non-Abelian massive vector meson fields of the Yang-Mills type [33].

The equations (4.37–4.40) bear a resemblance to the component equations of the Proca equation in the limit $\mathcal{E}_0 = \partial_\mu A_\mu \rightarrow 0$. However we shall see in section 5 that $\mathcal{E}_0$ vanishes only in one particular gauge. Its presence is required in a gauge invariant formalism. The Proca equation is not gauge invariant. Moreover, if $m = 0$ these equations are not identical to the classical Maxwell equations. Much of the difficulty in quantizing the Maxwell field is avoided if it is recognized that the Maxwell equations are a classical limit of a more general quantum mechanical zero-mass equation that contains the term $\partial_\mu A_\mu$.

It should be made clear that the free Lagrangian for this neutral boson field,

$$\mathcal{L} = -\tfrac{1}{2}\tilde{\vartheta}\partial_\mu\gamma_\mu\gamma_\nu\partial_\nu\vartheta - \tfrac{1}{2}m_A{}^2\tilde{\vartheta}\vartheta \tag{4.41}$$

is exactly equivalent to the Lagrangian,

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F_{\mu\nu} - \tfrac{1}{2}(\partial_\mu A_\mu)(\partial_\nu A_\nu) - \tfrac{1}{2}m_A{}^2A_\mu A_\mu \tag{4.42}$$

which includes the term involving the four-divergence of the field, first introduced, ad hoc, by Fermi [34] in order to achieve a separation of the longitudinal and time-like components for the photon. The above Lagrangian (4.42) without the Fermi term, $-1/2(\partial_\mu A_\mu)(\partial_\nu A_\nu)$, is the Proca Lagrangian. It will be shown in a following paper that solutions of this Proca equation do not make physical sense in simple potential problems.

To show that the Lagrangian (4.24) is gauge invariant we must, for the moment, assume that the fields φ and ϑ are distinguishable and commute with one another, i.e., they are Abelian fields. We consider a gauge transformation of the first kind on the φ field. This transformation is a unitary transformation

$$\varphi'(x) = G\varphi(x) = e^{ig\lambda(x)}\varphi(x) \tag{4.43}$$

where $\lambda(x)$ is a hermitian scalar field with the same physical mass as the φ field. A change in gauge means a change in phase, a change that is, at first sight, devoid of any physical consequences for a bilinear quantity involving $\bar{\varphi}$ and φ. Also we have

$$\varphi(x) = G^{-1}\varphi'(x) = e^{-ig\lambda(x)}\varphi'(x) \tag{4.44}$$

In order that the action integral be invariant under this gauge change of $\varphi(x)$ we must simultaneously make a gauge change of the second kind on $\vartheta(x)$, such that

$$\vartheta'(x) = \vartheta(x) + \gamma_\mu\partial_\mu\Lambda(x) \tag{4.45}$$

$$\vartheta(x) = \vartheta'(x) - \gamma_\mu\partial_\mu\Lambda(x) \tag{4.46}$$

where $\Lambda(x)$ is a scalar field function of the form

$$\Lambda(x) = \begin{vmatrix} 0 \\ 0 \\ 0 \\ -\lambda(x) \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix} \equiv \begin{vmatrix} 0 \\ -\lambda(x) \\ 0 \\ 0 \end{vmatrix} \tag{4.47}$$

and where $\lambda(x)$ is the same scalar quantity as in (4.43). Then $\vartheta'(x)$ is of the form

$$\vartheta'(x) = \begin{vmatrix} \mathbf{A}(x) + \nabla\lambda(x) \\ A_4(x) + \partial_4\lambda(x) \\ 0 \\ 0 \end{vmatrix} \tag{4.48}$$

It is more convenient to use an equivalent form for the Lagrangian where the components of the ϑ field are explicitly labelled.

$$\mathfrak{L} = -\bar{\varphi}(\partial_\mu + igA_\mu)\gamma_\mu\gamma_\nu(\partial_\nu - igA_\nu)\varphi - m_\varphi{}^2\bar{\varphi}\varphi - \tfrac{1}{4}F_{\mu\nu}F_{\mu\nu} - \tfrac{1}{2}\mathcal{E}_0\mathcal{E}_0 - \tfrac{1}{2}m_A{}^2A_\mu A_\mu \tag{4.49}$$

The gauge change equivalent to (4.45) is

$$A_\mu(x) \rightarrow A_\mu'(x) = A_\mu(x) + \partial_\mu\lambda(x) \tag{4.50}$$

The simultaneous gauge transformation has the invariance properties

$$G^{-1}(\partial_\mu - igA_\mu')\varphi'(x) = (\partial_\mu - igA_\mu)\varphi(x) \tag{4.51}$$

The gauge transformed Lagrangian expressed in terms of the original fields φ, $\bar{\varphi}$, and A_μ is

$$\mathcal{L}' = -\bar{\varphi}(\partial_\mu + igA_\mu)\gamma_\mu\gamma_\nu(\partial_\nu - igA_\nu)\varphi - m_\varphi{}^2\bar{\varphi}\varphi$$

$$- \tfrac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \tfrac{1}{2}(\partial_\mu A_\mu)(\partial_\nu A_\nu) - \tfrac{1}{2}m_A{}^2 A_\mu A_\mu$$

$$- (\partial_\nu A_\nu)(\partial_\mu\partial_\mu\lambda(x)) - \tfrac{1}{2}(\partial_\mu\partial_\mu\lambda(x))^2 - m_A{}^2 A_\mu(\partial_\mu\lambda(x)) - \tfrac{1}{2}m_A{}^2(\partial_\mu\lambda(x))^2 \qquad (4.52)$$

If we impose the condition that the gauge field obey the equation of motion,

$$(-\Box + m_A{}^2)\lambda(x) = 0 \qquad (4.53)$$

then

$$\mathcal{L}' = \mathcal{L} - m_A{}^2\partial_\mu[A_\mu\lambda(x) + \tfrac{1}{2}\lambda(x)(\partial_\mu\lambda(x))] \qquad (4.54)$$

This latter term contributes nothing to the action integral since the term is a perfect differential. Thus

$$\int \mathcal{L}'\, d^4x = \int \mathcal{L}\, d^4x \qquad (4.55)$$

for any physically meaningful local field[10]. The equations of motion of $\varphi'(x)$ and $\varphi(x)$ have the same physical content. The A_μ fields are altered in their component structure but no physical characteristics of interaction terms are changed. This point will be elaborated upon in a following paper. We have here a *restricted* gauge transformation, since $\lambda(x)$ is not entirely arbitrary but subject to condition (4.53).

To show that the current (4.32) is a conserved quantity, we make use of the symmetry properties of the Lagrangian (4.49) under this gauge transformation, which is considered to be infinitesimal. This gauge group is a continuous group, and we can make use of variational methods which will yield a conservation law resulting from the internal symmetry properties.

From (4.48, 4.50, and 4.54) we have

$$\delta\varphi(x) = \varphi'(x) - \varphi(x) = ig\lambda(x)\varphi(x) \qquad (4.56)$$

$$\delta\bar{\varphi}(x) = \bar{\varphi}'(x) - \bar{\varphi}(x) = -ig\lambda(x)\bar{\varphi}(x) \qquad (4.57)$$

$$\delta A_\mu(x) = A_\mu'(x) - A_\mu(x) = \partial_\mu\lambda(x) \qquad (4.58)$$

$$\delta\mathcal{L} = \mathcal{L}' - \mathcal{L} = -\mathcal{E}_0(x)\partial_\mu\partial_\mu\lambda(x) - m_A{}^2 A_\mu(x)\partial_\mu\lambda(x) \qquad (4.59)$$

We can also express the variation of $\mathcal{L}$ as

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\varphi_\alpha}\delta\varphi_\alpha + \frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi_\alpha}\delta\partial_\mu\varphi_\alpha + \delta\bar{\varphi}_\alpha\frac{\partial\mathcal{L}}{\partial\bar{\varphi}_\alpha} + \delta\partial_\mu\bar{\varphi}_\alpha\frac{\partial\mathcal{L}}{\partial\partial_\mu\bar{\varphi}_\alpha} + \frac{\partial\mathcal{L}}{\partial A_\mu}\delta A_\mu + \frac{\partial\mathcal{L}}{\partial\partial_\nu A_\mu}\delta\partial_\nu A_\mu \qquad (4.60)$$

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[10] The requirement of constancy of the action integral is all that is required hy most symmetry arguments. This is a less restrictive condition than the often imposed requirement of form invariance of the Lagrangian density

$$\mathcal{L}(\varphi_\alpha', \bar{\varphi}_\alpha', A_\beta', \partial_\mu\varphi_\alpha', \partial_\mu\bar{\varphi}_\alpha', F_{\mu\nu}) = \mathcal{L}(\varphi_\alpha, \bar{\varphi}_\alpha, A_\beta, \partial_\mu\varphi_\alpha, \partial_\mu\bar{\varphi}_\alpha, F_{\mu\nu})$$

To satisfy this latter condition the masses of the A field and the gauge field $\lambda(x)$ must he zero. The equations of motion are form invariant in either case.

Arguments that are not hased on the spatial integration of $\mathcal{L}$, such as those made in the application of Noether's theorem, require the retention of the perfect differential term.

Equating (4.59) and (4.60) and making use of the Euler-Langrange equations we obtain

$$\partial_\mu\{ig\lambda(x)[-\bar{\varphi}(\partial_\nu + igA_\nu)\gamma_\nu\gamma_\mu\varphi + \bar{\varphi}\gamma_\mu\gamma_\nu(\partial_\nu - igA_\nu)\varphi]$$
$$-F_{\mu\nu}\partial_\nu\lambda(x) - \mathcal{E}_0\partial_\mu\lambda(x)\} + \mathcal{E}_0\partial_\mu\partial_\mu\lambda(x) + m_A{}^2A_\mu\partial_\mu\lambda(x) = 0 \qquad (4.61)$$

which reduces to

$$\{ig(\bar{\chi}\gamma_\mu\varphi + \bar{\varphi}\gamma_\mu\chi) + \partial_\nu F_{\mu\nu} - \partial_\mu\mathcal{E}_0 + m_A{}^2A_\mu\}\partial_\mu\lambda(x) + \{\partial_\mu ig(\bar{\chi}\gamma_\mu\varphi + \bar{\varphi}\gamma_\mu\chi)\}\lambda(x) = 0 \qquad (4.62)$$

The coefficient of $\partial_\mu\lambda(x)$ is zero since it is the μ component of the generalized "Maxwell" equations (4.38, 4.39). Since the function $\lambda(x)$ is nonzero we have

$$\partial_\mu ig(\bar{\chi}\gamma_\mu\varphi + \bar{\varphi}\gamma_\mu\chi) = 0 \qquad (4.63)$$

or

$$\partial_\mu j_\mu = 0 \qquad (4.64)$$

This conserved current can be separated into two parts. Writing out the current of (4.32) in terms of $\bar{\varphi}$ and φ

$$j_\mu = -ig[\bar{\varphi}\gamma_\mu\gamma_\nu(\partial_\nu - igA_\nu)\varphi - \bar{\varphi}(\partial_\nu + igA_\nu)\gamma_\nu\partial_\mu\varphi] \qquad (4.65)$$

We can write this as the sum of two terms, a "conduction current," $j_\mu{}^{(c)}$, which does not involve the spin components explicitly and a moment current, $j_\mu{}^{(m)}$, which does, by using the familiar relation $\gamma_\mu\gamma_\nu = \delta_{\mu\nu} + i\sigma_{\mu\nu}$

$$j_\mu = j_\mu{}^{(c)} + j_\mu{}^{(m)} \qquad (4.66)$$

where

$$j_\mu{}^{(c)} = -ig\bar{\varphi}[(\partial_\mu - igA_\mu)\varphi] + ig[\bar{\varphi}(\partial_\mu + igA_\mu)]\varphi \qquad (4.67)$$

$$j_\mu{}^{(m)} = g\partial_\nu(\bar{\varphi}\sigma_{\mu\nu}\varphi) \qquad (4.68)$$

The four-divergence of the moment current is zero because of the antisymmetry of $\sigma_{\mu\nu}$:

$$\partial_\mu j_\mu{}^{(m)} = g\partial_\mu\partial_\nu(\bar{\varphi}\sigma_{\mu\nu}\varphi) = 0 \qquad (4.69)$$

The moment current is conserved separately; therefore the conduction current must be conserved also.

The four "Maxwell" equations (4.38–4.41) are, of course, invariant under the gauge transformation

$$A_\mu(x) \rightarrow A_\mu{}'(x) = A_\mu(x) + \partial_\mu\lambda(x) \qquad (4.50)$$

From the definitions of $\boldsymbol{\mathcal{E}}$, $\boldsymbol{\mathcal{B}}$ and $\mathcal{E}_0$ we have $\boldsymbol{\mathcal{E}}' = \boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{B}}' = \boldsymbol{\mathcal{B}}$ while

$$\mathcal{E}_0{}' = \mathcal{E}_0 + \partial_\mu\partial_\mu\lambda(x) \qquad (4.70)$$

Involving the condition (4.53) we see that the Maxwell conditions can be written in the new gauge in an invariant form.

$$\nabla \times \mathcal{B}' - \partial_t \mathcal{E}' - \nabla \mathcal{E}_0' + m_A^2 \mathbf{A}' = \mathbf{j}' \tag{4.38'}$$

$$\nabla \cdot \mathcal{E}' + \partial_t \mathcal{E}_0' + m_A^2 V' = \rho' \tag{4.39'}$$

$$\nabla \times \mathcal{E}' + \partial_t \mathcal{B}' = 0 \tag{4.40'}$$

$$\nabla \cdot \mathcal{B}' = 0 \tag{4.41'}$$

In section 3 it was shown that we have form invariance of the quantities $\chi = \gamma_\mu \partial_\mu \varphi$ and $\bar{\chi} = -\bar{\varphi}\partial_\mu \gamma_\mu$ and thus we have form invariance of the Lagrangian under Lorentz transformations. The invariance of the action integral under any continuous symmetry group will induce a conservation law for certain physical quantities which can be derived for any physical system once the Lagrangian is known. This conservation law is known as Noether's theorem [35] and is written in terms of the variation of the coordinates and the fields:

$$\int d^3x\, \partial_\mu \left[\left(\mathcal{L}\delta_{\mu\nu} - \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} \partial_\nu \varphi - \bar{\varphi}\partial_\nu \frac{\partial \mathcal{L}}{\partial \bar{\varphi} \partial_\mu} \right) \delta x_\nu + \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} \delta \varphi + \delta \bar{\varphi} \frac{\partial \mathcal{L}}{\partial \bar{\varphi} \partial_\mu} + \delta \Omega_\mu \right] = 0 \tag{4.71}$$

The translation group is a continuous group that leads to the energy-momentum conservation law. The translation

$$x_\nu \rightarrow x_\nu' = x_\nu + a_\nu \tag{4.72}$$

corresponds to a variation of the coordinates

$$\delta x_\nu = a_\nu \tag{4.73}$$

Some care must be exercised in the expression of the variation of the functions $\varphi(x)$ and $\bar{\varphi}(x)$. Expression (4.71) is applicable for the definitions

$$\delta \varphi(x) = \varphi'(x') - \varphi(x) \tag{4.74}$$

$$\delta \bar{\varphi}(x) = \bar{\varphi}'(x') - \bar{\varphi}(x) \tag{4.75}$$

For pure translations, the relation (2.62) indicates that

$$\delta \varphi(x) = \delta \bar{\varphi}(x) = 0 \tag{4.76}$$

Other formulations of Noether's theorem use different definitions for the variations of $\varphi(x)$ and $\bar{\varphi}(x)$ [36].

The fact that two Lagrange density functions lead to the same set of equations of motion if they differ by a four divergence, $\mathcal{L}$ and $\mathcal{L} + \partial_\mu \Omega_\mu$ may be included in Noether's theorem with the condition that

$$\int d^3x\, \partial_\mu \delta \Omega_\mu = 0 \tag{4.77}$$

Inserting (4.74, 4.75) into (4.71) and making use of the arbitrariness of a_ν, we obtain

$$\partial_\mu \left[\delta_{\mu\nu}\mathfrak{L} - \frac{\partial \mathfrak{L}}{\partial \partial_\mu \varphi} \partial_\nu \varphi - \bar{\varphi} \partial_\nu \frac{\partial \mathfrak{L}}{\partial \bar{\varphi} \partial_\mu} \right] = 0 \tag{4.78}$$

The conserved quantity in the square brackets is called the canonical energy-momentum density tensor $\mathfrak{Z}_{\mu\nu}$, thus

$$\partial_\mu \mathfrak{Z}_{\mu\nu} = 0 \tag{4.79}$$

The free Lagrangian (4.14) yields the specific energy-momentum tensor

$$\mathfrak{Z}_{\mu\nu} = (\bar{\chi}\chi - m_\varphi{}^2 \bar{\varphi}\varphi)\delta_{\mu\nu} - \bar{\chi}\gamma_\mu \partial_\nu \varphi + \bar{\varphi}\partial_\nu \gamma_\mu \chi \tag{4.80}$$

We denote the quantity P_ν as the total four-momentum of the field where

$$P_\nu = \int \mathcal{P}_\nu \, d^3x = -i \int \mathfrak{Z}_{4\nu} \, d^3x \tag{4.81}$$

The quantities $\mathcal{P}_k$ and $\mathcal{P}_0 = -i\mathcal{P}_4$ are the momentum and energy densities of the fields and their space integrals are constants of the motion by virtue of (4.79, 4.81), i.e.,

$$-i \int \partial_\mu \mathfrak{Z}_{\mu\nu} \, d^3x = -i \int (\partial_4 \mathfrak{Z}_{4\nu} + \partial_k \mathfrak{Z}_{k\nu}) \, d^3x \tag{4.82}$$

Using the Gauss theorem, the three dimension volume integral involving $\partial_k \mathfrak{Z}_{k\nu}$ vanishes if the fields are well behaved at infinity, thus

$$\frac{\partial}{\partial t} P_\nu = 0 \tag{4.83}$$

The momentum and energy densities are

$$\mathcal{P}_k = i(\bar{\chi}\gamma_4 \partial_k \varphi - \bar{\varphi}\partial_k \gamma_4 \chi) \tag{4.84}$$

$$\mathcal{P}_0 = \bar{\chi}\gamma_4 \partial_4 \varphi - \bar{\varphi}\partial_4 \gamma_4 \chi - \bar{\chi}\chi + m_\varphi{}^2 \bar{\varphi}\varphi \tag{4.85}$$

The latter expression is of course equivalent to the Hamiltonian density of (4.6) provided that the time derivatives of φ are eliminated by the usual procedure.

We may denote an energy current or energy flux density by a generalized Poynting vector ζ, where

$$\zeta_k = -i\mathfrak{Z}_{k4} = i(\bar{\chi}\gamma_k \partial_4 \varphi - \bar{\varphi}\partial_4 \gamma_k \chi) \tag{4.86}$$

The remaining portion of the canonical energy momentum tensor is the momentum current density or stress density tensor, $\mathfrak{Z}_{ij}$. From (4.79) we may establish the continuity relations that relate the time rate of change of energy density to the divergence of the energy flux density and the time rate of change of the momentum density to the divergence of the stress density in the field, i.e.,

$$\nabla \cdot \zeta + \frac{\partial}{\partial t} \mathcal{P}_0 = 0 \tag{4.87}$$

$$\nabla \cdot \mathfrak{T}_j + \frac{\partial}{\partial t} \mathcal{P}_j = 0 \tag{4.88}$$

where the dyadic $\mathfrak{T}_j$ is defined by

$$\mathfrak{T}_j = \mathbf{e}_1 \mathfrak{I}_{1j} + \mathbf{e}_2 \mathfrak{I}_{2j} + \mathbf{e}_3 \mathfrak{I}_{3j} \tag{4.89}$$

These relations are the continuity equations for the energy and momentum densities of the field.

The six parameter group of Lorentz rotations and boosts is the continuous group that leads to the angular momentum conservation laws. Here the transformation

$$x_\mu \rightarrow x_\mu' = x_\mu + \varepsilon_{\mu\nu} x_\nu \tag{4.90}$$

corresponds to the variation of the coordinates

$$\delta x_\mu = \varepsilon_{\mu\nu} x_\nu \tag{4.91}$$

while the variation of $\varphi(x)$ and $\bar{\varphi}(x)$ follow from expressions (2.62, 2.17)

$$\delta\varphi(x) = \varphi'(x') - \varphi(x) = \frac{i}{2} \varepsilon_{\mu\nu} S_{\mu\nu}{}^{(\varphi)} \varphi(x) \tag{4.92}$$

$$\delta\bar{\varphi}(x) = \bar{\varphi}'(x') - \varphi(x) = -\frac{i}{2} \varepsilon_{\mu\nu} \bar{\varphi}(x) S_{\mu\nu}{}^{(\varphi)} \tag{4.93}$$

Inserting these in the expression for Noether's theorem (4.71), we get

$$\int d^3x\, \partial_\mu \left[\left(\mathcal{L} \delta_{\mu\nu} - \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} \partial_\nu \varphi - (\bar{\varphi} \partial_\nu) \frac{\partial \mathcal{L}}{\partial \bar{\varphi} \partial_\mu} \right) x_\lambda + \frac{i}{2} \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} S_{\nu\lambda} \varphi - \frac{i}{2} \bar{\varphi} S_{\nu\lambda} \frac{\partial \mathcal{L}}{\partial \bar{\varphi} \partial_\mu} \right] \varepsilon_{\nu\lambda} = 0 \tag{4.94}$$

which we can write in terms of the canonical energy-momentum tensor after making use of the arbitrariness of the antisymmetrical quantity $\varepsilon_{\nu\lambda}$

$$\partial_\mu [x_\lambda \mathfrak{I}_{\mu\nu} - x_\nu \mathfrak{I}_{\mu\lambda} + i\bar{\chi} \gamma_\mu S_{\nu\lambda} \varphi + i\bar{\varphi} S_{\nu\lambda} \gamma_\mu \chi] = 0 \tag{4.95}$$

We can call the term in the square brackets $\mathfrak{M}_{\mu\nu\lambda}$, the angular momentum density tensor

$$\partial_\mu \mathfrak{M}_{\mu\nu\lambda} = 0 \tag{4.96}$$

The conserved angular momentum is defined as

$$J_{\nu\lambda} = i \int \mathfrak{M}_{4\nu\lambda}\, d^3x \tag{4.97}$$

From (4.96) and the Gauss theorem we can write

$$\frac{\partial}{\partial t} J_{\nu\lambda} = 0 \tag{4.98}$$

Using the Lagrangian (4.14) and the definition (4.92, 4.93) we can write $J_{\nu\lambda}$ in an obvious form: The sum of the space integrals of an orbital angular momentum density and a spin angular momentum density

$$J_{\nu\lambda} = \int d^3x[x_\nu \mathcal{P}_\lambda - x_\lambda \mathcal{P}_\nu - \bar{\chi}\gamma_4 S_{\nu\lambda}\varphi - \bar{\varphi}S_{\nu\lambda}\gamma_4\chi] \tag{4.99}$$

Neither the spin angular momentum nor the orbital angular momentum is separately a constant of the motion, only the total angular momentum is. This is true of all relativistic fields having a spin other than zero.

The tensor character of $\mathfrak{I}_{\mu\nu}$ under proper Lorentz transformations is a consequence of the Lagrangian density $\mathfrak{L}$ transforming as a scalar. This canonical energy momentum density tensor is not in general symmetric. It may be symmetrized by proper choice of the $\delta\Omega$ in Noether's theorem (4.71) using a generalization of method due to Belinfante [37]. The symmetrical tensor is given by

$$\Theta_{\mu\nu} = \mathfrak{I}_{\mu\nu} - \partial_\lambda G_{\lambda\mu\nu} \tag{4.100}$$

where

$$G_{\lambda\mu\nu} = \frac{i}{2}[\bar{\chi}\gamma_\lambda S_{\mu\nu}\varphi + \bar{\chi}\gamma_\mu S_{\nu\lambda}\varphi + \bar{\chi}\gamma_\nu S_{\mu\lambda}\varphi + \bar{\varphi}S_{\mu\nu}\gamma_\lambda\chi + \bar{\varphi}S_{\nu\lambda}\gamma_\mu\chi + \bar{\varphi}S_{\mu\lambda}\gamma_\nu\chi] \tag{4.101}$$

We are assured

$$\partial_\mu \Theta_{\mu\nu} = 0 \tag{4.102}$$

since the tensor $G_{\lambda\mu\nu}$ is antisymmetric in the first two indices

$$\partial_\mu \partial_\lambda G_{\lambda\mu\nu} = 0 \tag{4.103}$$

Furthermore, the angular momentum tensor defined in terms of $\Theta_{\mu\nu}$ becomes

$$\mathfrak{M}_{\mu\nu\lambda}' = x_\lambda \Theta_{\mu\nu} - x_\nu \Theta_{\mu\lambda} \tag{4.104}$$

which has the property that

$$\partial_\mu \mathfrak{M}_{\mu\nu\lambda}' = \partial_\mu \mathfrak{M}_{\mu\nu\lambda} \tag{4.105}$$

The ten quantities

$$P_\nu = -i\int \Theta_{4\nu}\, d^3x = -i\int \mathfrak{I}_{4\nu}\, d^3x \tag{4.106}$$

$$J_{\mu\nu} = i\int \mathfrak{M}_{4\mu\nu}'\, d^3x = i\int \mathfrak{M}_{4\mu\nu}\, d^3x \tag{4.107}$$

are conserved as a consequence of the invariance of $\mathfrak{L}$ under inhomogeneous Lorentz transformations. The total four momentum P_ν and the operator $J_{\mu\nu}$ are not changed, however, the density distribution features of the energy and momentum given by $\mathfrak{I}_{\mu\nu}$ are lost.

There is another interesting feature that can be obtained from the canonical energy-momentum tensor for interacting fields. Consider the Lagrangian (4.24) involving a charged field of nonzero mass with spin greater than one half interacting with a neutral vector boson field.

This canonical energy-momentum tensor is

$$\mathfrak{I}_{\mu\nu} = \mathfrak{L}\delta_{\mu\nu} - \frac{\partial \mathfrak{L}}{\partial \partial_\mu \varphi}\partial_\nu \varphi - (\bar{\varphi}\partial_\nu)\frac{\partial \mathfrak{L}}{\partial \bar{\varphi}\partial_\mu} - \frac{\partial \mathfrak{L}}{\partial \partial_\mu \vartheta}\partial_\nu \vartheta \tag{4.108}$$

We make a separation of this tensor, in a particular way, into parts which we shall denote as $\mathfrak{I}_{\mu\nu}{}^{(\varphi)}$ and $\mathfrak{I}_{\mu\nu}{}^{(\vartheta)}$. The total $\mathfrak{I}_{\mu\nu}$ always obeys the conservation law $\partial_\mu \mathfrak{I}_{\mu\nu} = 0$. These parts can be written in a gauge invariant form by adding and subtracting a term $j_\mu A_\nu$ to the canonical energy momentum tensor $\mathfrak{I}_{\mu\nu}$ yielding the separation:

$$\mathfrak{I}_{\mu\nu}{}^{(\varphi)} = (\bar{\chi}\chi - m_\varphi{}^2\bar{\varphi}\varphi)\delta_{\mu\nu} - \bar{\chi}\gamma_\mu(\partial_\nu - igA_\nu)\varphi + \bar{\varphi}(\partial_\nu + igA_\nu)\gamma_\mu\chi \tag{4.109}$$

while

$$\mathfrak{I}_{\mu\nu}{}^{(\vartheta)} = (-\tfrac{1}{2}\tilde{\vartheta}\partial_\lambda\gamma_\lambda\gamma_\rho\partial_\rho\vartheta - \tfrac{1}{2}m_A\tilde{\vartheta}\vartheta)\delta_{\mu\nu} + \tfrac{1}{2}\tilde{\vartheta}\partial_\lambda\gamma_\lambda\gamma_\mu\partial_\nu\vartheta + \tfrac{1}{2}\tilde{\vartheta}\partial_\nu\gamma_\mu\gamma_\lambda\partial_\lambda\vartheta + j_\mu A_\nu \tag{4.110}$$

where

$$j_\mu = -ig(\bar{\chi}\gamma_\mu\varphi + \bar{\varphi}\gamma_\mu\chi)$$

and

$$\chi = \gamma_\lambda(\partial_\lambda - igA_\lambda)\varphi$$

Explicitly calculating $\partial_\mu \mathfrak{I}_{\mu\nu}{}^{(\varphi)}$ and $\partial_\mu \mathfrak{I}_{\mu\nu}{}^{(\vartheta)}$ and making use of the "Maxwell equations" (4.37–4.40) we obtain

$$\begin{aligned} \partial_\mu \mathfrak{I}_{\mu\nu}{}^{(\varphi)} &= -ig(\bar{\chi}\gamma_\mu\varphi + \bar{\varphi}\gamma_\mu\chi)(\partial_\nu A_\mu - \partial_\mu A_\nu) \\ &= j_\mu F_{\nu\mu} = f_\nu \end{aligned} \tag{4.111}$$

and

$$\begin{aligned} \partial_\mu \mathfrak{I}_{\mu\nu}{}^{(\vartheta)} &= -ig(\bar{\chi}\gamma_\mu\varphi + \bar{\varphi}\gamma_\mu\chi)(\partial_\mu A_\nu - \partial_\nu A_\mu) \\ &= j_\mu F_{\mu\nu} = -f_\nu \end{aligned} \tag{4.112}$$

The quantity $\mathbf{f}$ is the relativistic Lorentz force and $f_0 = -if_4$ is the rate of change of energy density or work done.

$$\mathbf{f} = j_0\mathcal{E} + \mathbf{j} \times \mathcal{B} \tag{4.113}$$

$$f_0 = \mathbf{j}\cdot\mathcal{E} \tag{4.114}$$

This result is quite well known in classical electromagnetism. The point made here is that the inclusion of the term $\mathcal{E}_0 = \partial_\mu A_\mu$ in the Lagrangian and in the "Maxwell equations" for the massive vector field plays no role in the forces or rate of change of energy. The dynamics do not depend on the gauge, only the kinematics do. Similar results are obtained for a charged Dirac field interacting with a neutral boson field.

5. Free Field Solutions and the Quantization of the Field

We consider first the solutions of the wave equation

$$(-\gamma_\mu\partial_\mu\gamma_\nu\partial_\nu + m_\varphi{}^2)\varphi(x) = 0 \tag{4.15}$$

or, with the substitution $\chi(x) = \gamma_\nu\partial_\nu\varphi(x)$, the equation

$$-\gamma_\mu\partial_\mu\chi(x) + m_\varphi{}^2\varphi(x) = 0 \tag{4.36'}$$

The solutions for a field of spin, s, and nonzero mass, m, are most easily obtained by choosing a Lorentz frame that corresponds to the rest system of the field. In this rest frame only the diagonal portion, $\gamma_4\partial_4$, of the operator $\gamma_\mu\partial_\mu$ will contribute, and the wave function is of a simple form. We denote the wave function in the rest system as $\varphi^{(r)\prime}(x')$ where

$$\varphi^{(r)\prime}(x') = N\begin{vmatrix}\eta^{(r)}\\ \\ 0\end{vmatrix}e^{-imt'} \tag{5.1}$$

The quantity $\eta^{(r)}$ is a column submatrix of $4s$ elements in which the only nonzero element is the r^{th}, which has a unit occupation number. The quantity 0 is a column submatrix with $4s$ null elements.

The normalization N depends upon whether the wave function $\varphi(x)$ satisfies a first order Lagrangian, of the form (4.11), which is unique to the spin one-half field and which we denote by $\psi(x)$, or of the second order from (4.18) applicable to spins greater than one-half. The requirement that the Hamiltonian for a free particle at rest in any intrinsic state should be equal to the rest energy of the particle allows us to write:

$$N = \frac{1}{\sqrt{V'}} \qquad \text{spin } \tfrac{1}{2} \tag{5.2}$$

$$N = \frac{1}{\sqrt{2mV'}} \qquad \text{spin} > \tfrac{1}{2} \tag{5.3}$$

We want to transform from the rest system with four-momentum $p_\mu' = (0,0,0,im)$ to a state of four-momentum $p_\mu = (0,0,p,iE)$. In the unprimed system we have the space-time dependence of the wave functions determined by the relations

$$t' = t\cosh\Omega - z\sinh\Omega = t\frac{E}{m} - z\frac{p}{m} \tag{5.4}$$

$$V' = V\cosh\Omega = \frac{E}{m}V \tag{5.5}$$

The latter follows from the space-time contraction of the normalization volume along the direction of motion.

The solutions corresponding to a field in motion with velocity $p/E = \tanh\Omega$ may be readily found by making a Lorentz boost so that

$$\varphi^{(r)}(x) = D_{\text{boost}}{}^{(s,\varphi)-1}(\Omega_{34})\varphi^{(r)\prime}(x') \tag{5.6}$$

The operator $D_{\text{boost}}{}^{(s,\varphi)-1}(\Omega_{34})$ for a boost along the z axis can be obtained from table II and (3.100). The solutions for a plane wave travelling in another direction may be obtained by subsequently applying the operator $D_{\text{rot}}{}^{(J)}(\phi,\theta,-\phi)$ defined in (2.46) and reference [19]. As examples, we shall list here those solutions propagating in the z direction[11] in order to exhibit the structure of different spin fields.

The special plane wave solutions can be written as

$$\psi^{(r)}(x) = \sqrt{\frac{m}{EV}}\, u^{(r)}(p)e^{i(pz-Et)} \tag{5.7}$$

$$\varphi^{(r)}(x) = \frac{1}{\sqrt{2EV}}\, u^{(r)}(p)e^{i(pz-Et)} \tag{5.8}$$

for the first order and second order wave equations, respectively. The (r) superscript refers to the $2s+1$ space-like states and the $2s-1$ time-like states in the rest system. The spinors $u^{(r)}(p)$ are listed in table IV for fields of spin $\frac{1}{2}$, 1, $\frac{3}{2}$, and 2.

In a similar manner, the negative energy solutions may be obtained in the rest system with four-momentum $p_\mu = (0,0,0,-im)$ and then Lorentz boosted to the system moving with velocity $-p/E$. There is an important difference however, the first order wave equation for spin $\frac{1}{2}$,

$$(\gamma_\mu\partial_\mu + m)\psi(x) = 0$$

requires that the solution in the rest system be of the form

$$\psi^{(r)\prime}(x') = \frac{1}{\sqrt{V'}}\begin{vmatrix} 0 \\ \eta^{(r)} \end{vmatrix} e^{imt'} \tag{5.9}$$

The $\eta^{(r)}$ occupies the lower two components rather than the upper two for the positive energy solutions. Higher spin fields obey a second order wave equation and there is an arbitrariness as to whether we have the $\eta^{(r)}$ in the upper $4s$ components or the lower $4s$ components of the wave functions. We know that certain neutral bosons will be their own antiparticles and furthermore that a boson antiparticle will have the same parity as the particle, hence we require the $\eta^{(r)}$ to be "upstairs" for bosons. On the other hand the parity of fermion antiparticles is opposite so, that according to arguments to be made subsequently, we require that $\eta^{(r)}$ be "downstairs" for fermions. We have solutions in the rest system of the form

$$\varphi^{(r)\prime}(x') = \frac{1}{\sqrt{2mV'}}\begin{vmatrix} \eta^{(r)} \\ 0 \end{vmatrix} e^{imt'} \qquad \text{bosons} \tag{5.10}$$

$$\varphi^{(r)\prime}(x') = \frac{1}{\sqrt{2mV'}}\begin{vmatrix} 0 \\ \eta^{(r)} \end{vmatrix} e^{imt'} \qquad \text{fermions} \tag{5.11}$$

[11] We could have been more general by boosting along an arbitrary direction followed by an arbitrary spatial rotation. This would greatly increase the complication without making the physical picture more cogent.

We shall not write the solutions in the system with four-momentum $p_\mu = (0,0,p,-iE)$, however, since these negative energy solutions will later be replaced by antiparticle solutions.

Rather, we make a Lorentz transformation to a system of opposite momentum so that the four-momentum is $p_\mu = (0,0,-p,-iE)$. This is the same transformation as in (5.6) since $v = p/E$. These negative energy and momentum solutions are of the form

$$\psi^{(r)}(x) = \sqrt{\frac{m}{EV}}\, u_-^{(r)}(-p)e^{-i(pz-Et)} \tag{5.12}$$

$$\varphi^{(r)}(x) = \frac{1}{\sqrt{2EV}}\, u_-^{(r)}(-p)e^{-i(pz-Et)} \tag{5.13}$$

and are related to the positive energy and momentum antiparticle solutions by a unitary transformation which, in turn, can be related to the positive energy particle solutions (5.7, 5.8) by a charge conjugation transformation to be discussed in the next section. These negative energy and momentum spinors $u_-^{(r)}(-p)$ are listed in table V expressed in terms of the true antiparticle spinors $v^{(r)}(p)$ for spins $\frac{1}{2}$, 1, $\frac{3}{2}$, and 2.

Free-field solutions in a spherical basis and solutions corresponding to fields in a central potential will be treated in a following paper.

The wave function $\chi^{(r)}(x)$ that is related to $\varphi^{(r)}(x)$ by the relation $\chi^{(r)}(x) = \gamma_\mu \partial_\mu \varphi^{(r)}(x)$ may be written as

$$\chi^{(r)}(x) = \frac{1}{\sqrt{2EV}}\, w^{(r)}(p)e^{i(pz-Et)} \tag{5.14}$$

for free positive energy fields. We have

$$w^{(r)}(p) = i\gamma_\mu p_\mu u^{(r)}(p) \tag{5.15}$$

or in the rest system

$$w^{(r)}(0) = -m\gamma_4 u^{(r)}(0) \tag{5.16}$$

The spinors $w^{(r)}(p)$ may be obtained in two equivalent ways; by calculating from (5.15) using the explicit representation of the γ matrices or by performing a Lorentz boost from the rest system to that with velocity p/E by use of the expressions in table II appropriate for the χ function. The spinors $w^{(r)}(p)$ obtained are listed in table VI for fields of spin $\frac{1}{2}$, 1, $\frac{3}{2}$, and 2. The spinors corresponding to the negative energy and momentum solutions for $\chi^{(r)}(x)$ may be obtained from

$$w_-^{(r)}(-p) = -i\gamma_\mu p_\mu u_-^{(r)}(-p) \tag{5.17}$$

These are also related to the positive energy and momentum antiparticle spinors $y^{(s)}(p)$ by a unitary transformation to be discussed in the next section. Here, we have a relation similar to (5.15) for anti-particles

$$y^{(r)}(p) = -i\gamma_\mu p_\mu v^{(r)}(p) \tag{5.18}$$

These are listed in table VII for fields of spin $\frac{1}{2}$, 1, $\frac{3}{2}$, and 2, together with the relation between $y^{(r)}(p)$ and $w_-^{(r)}(-p)$.

Our special plane wave solutions were obtained from a complete and orthogonal set of solutions in the rest frame by making a Lorentz boost to a momentum p along the z axis. It is easy to demonstrate by direct calculation that the solutions in the boosted Lorentz frame also constitute a complete and orthogonal set. We have listed these completeness and orthogonality relations for fields of spin $\frac{1}{2}$, 1, $\frac{3}{2}$, and 2 in tables VIII, IX, X, and XI, respectively. It is important to recall the rules for the idenfinite metric of the Hilbert space, Relations (3.71, 3.72) imply, for example, that

$$\bar{u}^{(r)}(p) = u^{(r)\dagger}(p)\gamma_4\eta$$

The first $2s + 1$ spinors obey the relation

$$\bar{u}^{(r)}(p) = u^{(r)\dagger}(p)\gamma_4 \qquad \text{for } r = 1, \cdots, (2s+1)$$

and the latter $2s - 1$ spinors obey the relation

$$\bar{u}^{(r)}(p) = -u^{(r)\dagger}(p)\gamma_4 \qquad \text{for } r = (2s+2), \cdots, 4s$$

Similar adjoints are obtained for $\bar{w}^{(r)}(p)$, $\bar{v}^{(r)}(p)$, and $\bar{y}^{(r)}(p)$ by use of the same rules.

All of the norms and completeness relations are positive definite with this indefinite metric of the Hilbert space. The completeness relations for fields of spin greater than 2 are tedious but straightforward to calculate. For example, we may use the relation (5.6) to obtain

$$\sum_{r=1}^{4s} u^{(r)}(p)\bar{u}^{(r)}(p) = D_{\text{boost}}{}^{(s,\varphi)-1}(\Omega) \begin{vmatrix} I & 0 \\ 0 & 0 \end{vmatrix} D_{\text{boost}}{}^{(s,\varphi)}(\Omega)$$

The free field wave function $\varphi^{(r)}(x)$ has $4s$ orthogonal solutions each having $8s$ components in the general case. Of these $4s$ orthogonal solutions only $2s + 1$ represent independent dynamical degrees of freedom since $2s - 1$ depend on gauge conditions and may be arbitrarily removed by a gauge transformation of the second kind.

$$\varphi'(x) = \varphi(x) + \gamma_\mu \partial_\mu \Lambda(x) \tag{4.45'}$$

The function $\Lambda(x)$ satisfies the wave equation

$$(-\Box + m_\varphi{}^2)\Lambda(x) = 0 \tag{4.53'}$$

and has the Lorentz transformation properties of a field of spin $s - 1$ if it is the gauge field of spin s. In a parity conserving theory $\Lambda(x)$ has the group representation $\{(s-1,0) \oplus (0,s-1)\}$. The Lorentz transformation operators will be given by those of χ vector since that irreducible portion that operates on $\Lambda(x)$ corresponds to the same representation. We may define a positive energy spinor $z^{(r)}(p)$ by the solution

$$\Lambda^{(r)}(x) = \frac{1}{\sqrt{2EV}} z^{(r)}(p) e^{i(pz-Et)} \tag{5.19}$$

The spinors may be determined in the same manner as $w^{(r)}(p)$ by making a Lorentz boost from the rest system. These spinors are tabulated in table XII. It will be observed that $w^{(r)}(p) = m^2 z^{(r)}(p)$ where r labels only those states that correspond to the time like states in the rest system. The negative energy and momentum gauge spinors may be obtained in a similar manner.

We shall denote the gauge where we have the $4s$ (canonical) degrees of freedom as the Feynman gauge and that gauge where $2s + 1$ degrees of freedom survive as the Lorentz gauge. These are generalizations of the nomenclature for the spin 1 electromagnetic field. The Lorentz gauge corresponds in that case to $\partial_\mu A_\mu(x) = 0$.

The plane wave solutions describing zero mass fields require special consideration. There is no Lorentz transformation corresponding to a boost to the limiting velocity c. This implies the zero mass solutions cannot be obtained from the solutions for a massive field by taking the limiting value as $m \to 0$. Inspection of tables IV and VI indicate that those spinor components having $|\lambda| < s$ become infinite, i.e., $u^{(r)}(p) \to \infty$ while at the same time $w^{(r)}(p) \to 0$ in such a way that certain bilinear combinations such as $\bar{u}^{(r)}(p)u^{(r)}(p)$ or $\bar{w}^{(r)}(p)\gamma_4 u^{(r)}(p)$ do not depend on the limiting value of the mass. This limiting procedure is clearly not acceptable because the expectation value of the helicity operator $w_3/|p|$ defined by (2.78) would be

$$-\chi^{(\lambda')}(x)\gamma_4 J_3 \varphi^{(\lambda)}(x) - \varphi^{(\lambda')}(x) J_3 \gamma_4 \chi^{(\lambda)}(x) = \lambda\delta_{\lambda\lambda'} \tag{5.20}$$

for all possible values of λ; $-s \leq \lambda \leq s$. The zero mass case is characterized by a λ having only the expectation values of $+s$ or $-s$.

The spinors for the zero mass case in the particular Lorentz frame $p_\mu = (0,0,p,ip)$ may be obtained from the finite mass spinors of table IV by "renormalizing" those components having $|\lambda| < s$ such that they have propagation characteristics similar to those components having $|\lambda| = s$. Several interesting features emerge. The $4s$ independent and orthogonal solutions in the massive field case are reduced to $2s + 1$ orthogonal solutions in the zero mass case. The $2s - 1$ solutions that corresponded to a time-like field in the rest system become proportional to the $2s - 1$ solutions that corresponded to "longitudinal" fields in the rest system, i.e., those solutions having $|\lambda| < s$, and hence these sets are no longer orthogonal to each other. Of course, the concept of a rest system is meaningless and actually the introduction of the renormalization artifice is, in essence, the abandonment of this concept. The normalization constant is completely arbitrary since it may be changed at will by a gauge transformation of the second kind.

Some zero mass plane wave solutions are given in tables XIII for fields of spin $\frac{1}{2}$, 1, $\frac{3}{2}$, and 2. These solutions will have the same form in another Lorentz frame $p_\mu' = (0,0,p',ip')$ which can be shown by using the Lorentz boosts of table II and then regauging the fields. In this zero mass case, the parameter Ω is given by

$$\Omega = \ln\frac{p'}{p} \tag{2.73}$$

For $s > \frac{1}{2}$ the spinors $w^{(r)}(p) = i\gamma_\mu p_\mu u^{(r)}(p)$ are nonzero only for those components corresponding to $|\lambda| = s$. This result is independent of the gauge. We should expect that the helicity should be $+s$ or $-s$ for a field of any discrete spin s, however we obtain that result in this formulation only for the spins $\frac{1}{2}$ and 1.

A most surprising feature is that for $s > 1$ massless fields have zero norm. It follows that the helicity and all of the elements of the energy-momentum tensor for the free fields are zero.

The well-established zero-mass particles occurring in nature are those of spin $\frac{1}{2}$ and 1. The neutrino has spin $\frac{1}{2}$ and zero mass but appears in a different guise than that formulated here. The spinors of table XIII are modified by a factor $(2)^{-1/2}(1 + \gamma_5)$ [38] so that the four possible solutions are reduced to two: One helicity state for the neutrino and one for the antineutrino. These are shown in table XIV.

The photon has spin 1 and zero mass and may be described with the spinors of table XIII in several gauges. When the longitudinal and time-like components are eliminated we have the transverse gauge; when present we have the Lorentz gauge. There is a "nonphysical" gauge in which the longitudinal and time-like components are treated independently. This gauge, shown in table XV, is called the Feynman or covariant gauge. In order that this not be in violation of certain physical facts, certain formal subsidiary conditions [39, 40] must be applied at the expense of a cogent physical interpretation. This gauge, in particular, violates the characterizing group properties of a zero-mass field. It is ironical that the name covariant gauge is applied to a gauge that is not among the set of basis states in the Hilbert space of the inhomogeneous Lorentz group.

The conjectured quantum of gravitation, the graviton, is a massless boson of spin 2. Its source is the sixteen component energy-momentum tensor. This divergenceless tensor encompasses all physical quantities except gravity that contribute to the energy content of space. The Einstein field equations may be linearized in the weak field limit with the result that a wave equation for a field of sixteen uncoupled components is obtained. A zero mass plane wave travelling in a z direction will have at most four nonzero components with only two independent. This is just the case for the spin 2 boson in table XIII when all components but those having $\lambda = \pm 2$ have been gauged away. Our sixteen component spinor can be recast in tensor form but it is not necessary to do so here. The interesting feature is that the zero norm of our spin 2 massless boson leads to zero energy content of the free boson field. In the weak field limit the contribution of the field energy to the gravitational Hamiltonian is zero. A further consequence is that the free zero-mass spin 2 boson carries no energy or momentum[12]. This statement does not imply that there should be no gravitational radiation, since we are considering here only a linear theory. The experimental detection of gravitational waves is, at this time, still a matter of controversy.

Having determined a complete orthonormal set of positive and negative energy classical fields of discrete spin, it is possible to develop a quantized operator field theory obeying, in a systematic way, most of the customary assumptions of Lagrangian field theory.

In an operator field theory that is quantized by the canonical formalism, it is customary to require that the canonical boson commutation relations, or fermion anticommutation relations, among the field operators and their associated canonical momenta are such that the quantum mechanical equations of motion in the Heisenberg picture agree formally with those obtained from Hamilton's canonical equations for continuous classical systems. These relations are

$$\dot{\varphi}_\alpha(x) = i[H,\varphi_\alpha(x)] = \frac{\delta\mathcal{H}(x)}{\delta\pi_\alpha(x)} \tag{5.21}$$

$$\dot{\pi}_\alpha(x) = i[H,\pi_\alpha(x)] = -\frac{\delta\mathcal{H}(x)}{\delta\varphi_\alpha(x)} \tag{5.22}$$

where

$$H = \int \mathcal{H}(x)\, d^3x \tag{5.23}$$

and the quantity $\delta\mathcal{H}(x)/\delta\pi(x)$ is the functional derivative defined by

$$\frac{\delta\mathcal{H}(x)}{\delta\pi_\alpha(x)} = \frac{\partial\mathcal{H}(x)}{\partial\pi_\alpha(x)} - \partial_k \frac{\partial\mathcal{H}(x)}{\partial\partial_k\pi_\alpha(x)} \tag{5.24}$$

There are similar relations involving the adjoint fields and their canonical momenta. The normalization is determined such that the Hamiltonian has the value

$$H = N\omega \tag{5.25}$$

where N is the total number of particles or quanta present.

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[12] This feature is also implicit in the linearized Einstein equation, following directly from the Bianchi identities.

It should be emphasized that this requirement does *not* always lead to the *postulated* [41] equal time commutation relations between canonically related fields and momenta in the general case. The relations usually invoked

$$[\varphi_\alpha(x),\pi_\beta(x')]_{\pm \atop x_0=x_0'} = i\delta_{\alpha\beta}\delta(\mathbf{x}-\mathbf{x}') \tag{5.26}$$

$$[\bar{\varphi}_\alpha(x),\bar{\pi}_\beta(x')]_{\pm \atop x_0=x_0'} = i\delta_{\alpha\beta}\delta(\mathbf{x}-\mathbf{x}') \tag{5.27}$$

$$[\bar{\varphi}_\alpha(x),\varphi_\beta(x')]_{\pm \atop x_0=x_0'} = 0 \tag{5.28}$$

$$[\bar{\pi}_\alpha(x),\pi_\beta(x')]_{\pm \atop x_0=x_0'} = 0 \tag{5.29}$$

are not always valid. Simultaneous satisfaction of relations (5.21, 5.22) and (5.26–5.29) cannot be obtained for field operators corresponding to particles having spin $s \geqq \frac{3}{2}$.

The failure of the canonical quantization method for $s \geqq \frac{3}{2}$ does not imply that these fields cannot be quantized, but rather that we must give a particle interpretation of the field $\varphi(x)$ as the basic postulate. The Hamiltonian (5.23) is a function of one or more fields which are linear combinations of the creation and annihilation operators. With this particle interpretation along with Lorentz invariance and locality the usual connections between spin and statistics [42, 43] follow with the conventional choice for commutation relations involving the creation and annihilation operators of the boson fields and anti-commutation relations involving the creation and annihilation operators of the fermion fields. Crossing symmetry follows from the same assumptions.

Care must be exercised in the construction of Hamiltonian density (4.6) to include only the canonical conjugate momenta corresponding to each component of the field operator $\varphi(x)$ and to maintain the ordering of the operators in the expressions. Consider the Lagrangian for a free field with spin greater than one half:

$$\mathfrak{L} = -\bar{\varphi}\partial_\mu\gamma_\mu\gamma_\nu\partial_\nu\varphi - m_\varphi{}^2\bar{\varphi}\varphi \tag{4.14}$$

We introduce the generalized coordinates

$$\varphi_\alpha(x) = \mathfrak{P}_\alpha\varphi(x) \tag{5.30}$$

$$\bar{\varphi}_\alpha(x) = \bar{\varphi}(x)\mathfrak{P}_\alpha \tag{5.31}$$

where $\mathfrak{P}_\alpha$ is an idempotent projection operator that projects the α component of $\varphi(x)$. The canonically conjugate momenta are given by

$$\pi_\alpha(x) = \frac{\partial\mathfrak{L}}{\partial\dot{\bar{\varphi}}_\alpha(x)} = i\bar{\varphi}(x)\partial_\mu\gamma_\mu\gamma_4\mathfrak{P}_\alpha = -i\bar{\chi}(x)\gamma_4\mathfrak{P}_\alpha \tag{5.32}$$

$$\bar{\pi}_\alpha(x) = \frac{\partial\mathfrak{L}}{\partial\dot{\bar{\varphi}}_\alpha(x)} = i\mathfrak{P}_\alpha\gamma_4\gamma_\mu\partial_\mu\varphi(x) = i\mathfrak{P}_\alpha\gamma_4\chi(x) \tag{5.33}$$

then

$$\mathfrak{H} = \pi_\alpha\dot{\varphi}_\alpha + \dot{\bar{\varphi}}_\alpha\bar{\pi}_\alpha - \mathfrak{L} \tag{5.34}$$

We must write $\dot{\varphi}_\alpha$, $\dot{\bar{\varphi}}_\alpha$ and $\mathfrak{L}$ in terms of the canonical coordinates so that the Hamiltonian does not explicitly depend on the time derivatives of the fields. The $\mathfrak{P}_\alpha$ is a diagonal operator that will commute with γ_4 Then

$$\dot{\varphi}_\alpha = \pi_\alpha + \mathfrak{P}_\alpha \sigma_{4k} \partial_k \varphi \tag{5.35}$$

$$\dot{\bar{\varphi}}_\alpha = \bar{\pi}_\alpha + \bar{\varphi} \partial_k \sigma_{k4} \mathfrak{P}_\alpha \tag{5.36}$$

$$\mathfrak{L} = \pi_\alpha \bar{\pi}_\alpha - m_\varphi{}^2 \bar{\varphi}\varphi - \bar{\varphi}\partial_k \gamma_k (1 - \mathfrak{P}_\alpha) \gamma_j \partial_j \varphi \tag{5.37}$$

where we have used the relation $\gamma_\mu \gamma_\nu = \delta_{\mu\nu} + i\sigma_{\mu\nu}$ so that the Hamiltonian density is

$$\mathfrak{H} = \pi_\alpha \bar{\pi}_\alpha + \pi_\alpha \mathfrak{P}_\alpha \sigma_{4k} \partial_k \varphi + \bar{\varphi} \partial_k \sigma_{k4} \mathfrak{P}_\alpha \bar{\pi}_\alpha + m_\varphi{}^2 \bar{\varphi}\varphi + \bar{\varphi}\partial_k \gamma_k (1 - \mathfrak{P}_\alpha) \gamma_j \partial_j \varphi \tag{5.38}$$

Inspection of tables III and V shows us that

$$\mathfrak{P}_\alpha = \begin{vmatrix} I & 0 \\ 0 & 0 \end{vmatrix}_{\alpha\alpha'} \qquad \text{spin 1, polar vector} \tag{5.39}$$

$$\mathfrak{P}_\alpha = \begin{vmatrix} 0 & 0 \\ 0 & I \end{vmatrix}_{\alpha\alpha'} \qquad \text{spin 1, axial vector} \tag{5.40}$$

$$\mathfrak{P}_\alpha = \begin{vmatrix} I & 0 \\ 0 & I \end{vmatrix}_{\alpha\alpha'} \qquad \text{spin} > 1 \tag{5.41}$$

The last term in (5.38) drops out for all but spin one fields.
It is straightforward to show that Hamilton's canonical equations

$$\dot{\varphi}_\alpha(x) = \frac{\partial \mathfrak{H}(x)}{\partial \pi_\alpha(x)} - \partial_k \frac{\partial \mathfrak{H}(x)}{\partial \partial_k \pi(x)} \tag{5.21}$$

$$\dot{\pi}_\alpha(x) = -\frac{\partial \mathfrak{H}(x)}{\partial \varphi_\alpha(x)} + \partial_k \frac{\partial \mathfrak{H}(x)}{\partial \partial_k \varphi_\alpha(x)} \tag{5.22}$$

are satisfied and are compatible with the equations of motion (4.15).

To establish contact with the particle interpretation we want to Fourier analyze the field operators $\varphi(x)$ and $\pi(x)$. We start from the classical fields and decompose them into expansion coefficients involving the plane wave solutions of momentum p times certain creation and annihilation operators $a^\dagger(p)$ and $a(p)$ for particles and $b^\dagger(p)$ and $b(p)$ for antiparticles, respectively.

We will consider, in the main, non-hermitian fields where a mathematical operation of conjugation will be associated with particles and antiparticles. Only one type of quantum will be associated with a real hermitian field, while the complex non-hermitian field has twice as many degrees of freedom and describes particles which carry opposite "charges." Whenever complex or hermitian conjugation is involved, it is convenient to start at once with non-hermitian fields so as not to be confused by irrelevant identies. On the other hand, zero-mass boson fields which carry no "charge" must be real hermitian fields. In the following we do not wish to join the cult of the arbitrary complex phase factor. We shall write all non-hermitian fields such that the hermitian field may be obtained by letting

$$-(-1)^{2s} b^{(r)\dagger}(\mathbf{p}) v^{(r)}(\mathbf{p}) \rightarrow a^{(r)\dagger}(\mathbf{p}) u^{(r)\dagger}(\mathbf{p})$$

and

$$-(-1)^{2s} b^{(r)}(\mathbf{p}) \bar{v}^{(r)}(\mathbf{p}) \rightarrow a^{(r)}(\mathbf{p}) \bar{u}^{(r)\dagger}(\mathbf{p})$$

This phase choice[13] shall also guarantee crossing symmetry where all particles are replaced by antiparticles.

The usual ascribed commutation relations for bosons and anticommutation relations for fermions occur among these creation and annihilation operators

$$[a^{(r)}(\mathbf{p}), a^{(s)\dagger}(\mathbf{p}')]_{\pm} = [b^{(r)}(\mathbf{p}), b^{(s)\dagger}(\mathbf{p}')]_{\pm} = \delta_{rs}\delta(\mathbf{p} - \mathbf{p}') \tag{5.42}$$

$$[a^{(r)}(\mathbf{p}), a^{(s)}(\mathbf{p}')]_{\pm} = [a^{(r)\dagger}(\mathbf{p}), a^{(s)\dagger}(\mathbf{p}')]_{\pm} = 0 \tag{5.43}$$

$$[b^{(r)}(\mathbf{p}), b^{(s)}(\mathbf{p}')]_{\pm} = [b^{(r)\dagger}(\mathbf{p}), b^{(s)\dagger}(\mathbf{p}')]_{\pm} = 0 \tag{5.44}$$

Rather than express these quantized fields as infinite series of plane wave solutions of momentum p, we make use of the relationship

$$\frac{1}{\sqrt{V}} \sum_{p=-\infty}^{\infty} \rightarrow \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} d^3p \tag{5.45}$$

to express the quantized fields in integral form.

These quantized fields are:

For spin $\frac{1}{2}$:

$$\psi(x) = \frac{m^{1/2}}{(2\pi)^{3/2}} \int \frac{d^3p}{E_p^{1/2}} \{a^{(r)}(\mathbf{p}) u^{(r)} e^{ip\cdot x} + b^{(r)\dagger}(\mathbf{p}) v^{(r)}(\mathbf{p}) e^{-ip\cdot x}\} \tag{5.46}$$

$$\bar{\psi}(x) = \frac{m^{1/2}}{(2\pi)^{3/2}} \int \frac{d^3p}{E_p^{1/2}} \{a^{(r)\dagger}(\mathbf{p}) \bar{u}^{(r)}(\mathbf{p}) e^{-ip\cdot x} + b^{(r)}(\mathbf{p}) \bar{v}^{(r)}(\mathbf{p}) e^{ip\cdot x}\} \tag{5.47}$$

For spin ≥ 1:

$$\varphi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{(2E_p)^{1/2}} \{a^{(r)}(\mathbf{p}) u^{(r)} e^{ip\cdot x} - (-1)^{2s} b^{(r)\dagger}(\mathbf{p}) v^{(r)}(\mathbf{p}) e^{-ip\cdot x}\} \tag{5.48}$$

$$\bar{\varphi}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{(2E_p)^{1/2}} \{a^{(r)\dagger}(\mathbf{p}) \bar{u}^{(r)}(\mathbf{p}) e^{-ip\cdot x} - (-1)^{2s} b^{(r)}(\mathbf{p}) \bar{v}^{(r)}(\mathbf{p}) e^{ip\cdot x}\} \tag{5.49}$$

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[13] For bosons, this phase choice follows from the definition $v^{(r)}(\mathbf{p}) = \eta D\tilde{\bar{u}}^{(r)}(\mathbf{p})$. The properties of these spinors in the indefinite metric give the equivalent description $v^{(r)}(\mathbf{p}) = (Bd(\pi))_{rr'} u^{(r')}(\mathbf{p})$. Likewise we have the relationship $b^{(r)\dagger}(\mathbf{p}) = -(Bd(\pi))_{rr''} a^{(r'')\dagger}(\mathbf{p})$. We have $(Bd(\pi))_{rr'}(Bd(\pi))_{rr''} = (-1)^{2s}\delta_{r'r''}$, hence the expression for the hermitian field follows directly.

$$\chi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{(2E_p)^{1/2}} \{a^{(r)}(\mathbf{p})w^{(r)}(\mathbf{p})e^{ip\cdot x} - (-1)^{2s}b^{(r)\dagger}(\mathbf{p})y^{(r)}(\mathbf{p})e^{-ip\cdot x}\} \tag{5.50}$$

$$\bar{\chi}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{(2E_p)^{1/2}} \{a^{(r)\dagger}(\mathbf{p})\bar{w}^{(r)}(\mathbf{p})e^{-ip\cdot x} - (-1)^{2s}b^{(r)}(\mathbf{p})\bar{y}^{(r)}(\mathbf{p})e^{ip\cdot x}\} \tag{5.51}$$

For a zero mass field, only the spin $\frac{1}{2}$ form differs by the factor $\sqrt{m/E_p}$

$$\psi(x) = \frac{1}{(2\pi)^{3/2}} \int d^3p \{a^{(r)}(\mathbf{p})u^{(r)}(\mathbf{p})e^{ip\cdot x} + b^{(r)\dagger}(\mathbf{p})v^{(r)}(\mathbf{p})e^{-ip\cdot x}\} \tag{5.52}$$

$$\bar{\psi}(x) = \frac{1}{(2\pi)^{3/2}} \int d^3p \{a^{(r)\dagger}(\mathbf{p})\bar{u}^{(r)}(\mathbf{p})e^{-ip\cdot x} + b^{(r)}(\mathbf{p})\bar{v}^{(r)}(\mathbf{p})e^{ip\cdot x}\} \tag{5.53}$$

In all of these solutions the sum over the index (r) extends over the $4s$ orthogonal spinors, some of which may be zero, depending on the gauge. The adjoint spinor is now defined as

$$\bar{u}^{(r)}(\mathbf{p}) = \eta^{-1}u^{(r)\dagger}(\mathbf{p})\gamma_4\eta \tag{5.54}$$

$$\bar{v}^{(r)}(\mathbf{p}) = \eta^{-1}v^{(r)\dagger}(\mathbf{p})\gamma_4\eta \tag{5.55}$$

where η is the operator that reverses the sign of the last $2s - 1$ solutions to achieve the desired idenfinite metric.

In a quantized field theory the field amplitudes are operators on the state vectors in a Hilbert space. The physical observables are the matrix elements of these operators, and the requirements of symmetry and covariance shall be imposed upon these matrix elements rather than on the state vectors of the unquantized theory.

In a Lorentz transformation $x \rightarrow x' = \Lambda x + a$ we may expect the matrix element of the field operator, which is a c-number, to transform in the same manner as the unquantized field according to (2.53). We require

$$(\Phi',\varphi(x')\Phi') = D(\Lambda)(\Phi,\varphi(x)\Phi) \tag{5.56}$$

There exists an operator $L(a,\Lambda)$ that accomplishes the desired transformation of the state vectors

$$\Phi'(x) = L(a,\Lambda)\Phi(x) \tag{5.57}$$

From this we get the transformation property for the operator

$$L(a,\Lambda)\varphi(x)L^{-1}(a,\Lambda) = D^{-1}(\Lambda)\varphi(\Lambda x + a) \tag{5.58}$$

We first consider an infinitesimal displacement by the amount a_μ

$$L(a,1) = 1 - ia_\mu P_\mu \tag{5.59}$$

where P_μ is the total four momentum defined in (4.81). We make a Taylor expansion of the right hand side of (5.58) and obtain after cancellation

$$[P_\mu,\varphi(x)] = i\partial_\mu\varphi(x) \tag{5.60}$$

In a similar way we consider an infinitisimal Lorentz rotation by the amount $\varepsilon_{\mu\nu}$ with the $J_{\mu\nu}$ defined by (4.99)

$$L(0,\Lambda) = 1 + \frac{i}{2}\varepsilon_{\mu\nu}J_{\mu\nu} \tag{5.61}$$

$$D^{-1}(\Lambda) = 1 - \frac{i}{2}\varepsilon_{\mu\nu}S_{\mu\nu}{}^{(\varphi)} \tag{5.62}$$

Expanding the right hand side, as before, and keeping the lowest order terms we obtain

$$[J_{\mu\nu},\varphi(x)] = [i(x_\mu\partial_\nu - x_\nu\partial_\mu) - S_{\mu\nu}{}^{(\varphi)}]\varphi(x) \tag{5.63}$$

The matrix elements of certain observables, formed from products of the field operators such as the four-momentum densities or the charge density, may include undesirable singularities resulting from certain inherent properties of the vacuum. To remove these features of the vacuum we must either symmetrize or antisymmetrize the factors in the Hamiltonian and Lagrangian or alternatively postulate a normal ordering. For the free field theory the two methods are completely equivalent. For the case of interacting fields in the Heisenberg picture the normal product cannot be as straightforwardly defined, thus we invoke the procedure of symmetrizing the order of boson operators and antisymmetrizing the order of fermion operators [44].

The notation : : stands for a symmetrized or antisymmetrized product, e.g.,

$$:\bar{\chi}\gamma_\mu\partial_\nu\varphi: \equiv \tfrac{1}{2}[\bar{\chi}\gamma_\mu\partial_\nu\varphi \pm \tilde{\varphi}\partial_\nu\tilde{\gamma}_\mu\tilde{\bar{\chi}}] \tag{5.64}$$

The difference between $\bar{\chi}\gamma_\mu\partial_\nu\varphi$ and $:\bar{\chi}\gamma_\mu\partial_\nu\varphi:$ is always a c-number, and this is sufficient also to remove undesired zero-point singularities that appear when the product is not properly symmetrized.

The operator for the four-momentum of a field of any particular spin may be calculated directly from (4.81) using the solutions (5.48–5.51) and the spinors taken from tables IV to VII. We obtain

$$\mathsf{P}_\mu = \int d^3p\; p_\mu(a^{(r)\dagger}(\mathbf{p})a^{(r)}(\mathbf{p}) + b^{(r)\dagger}(\mathbf{p})b^{(r)}(\mathbf{p})) \tag{5.65}$$

$$\mathsf{P}_\mu = \int d^3p\; p_\mu(N_+{}^{(r)}(p) + N_-{}^{(r)}(p)) \tag{5.66}$$

The sum over (r) extends over the $2s+1$ space-like states and also the $2s-1$ time-like states, if they are present in the gauge considered. The positive definite property of this four-momentum operator P_μ follows from the adopted prescription of the indefinite metric of the Hilbert space of these time-like states.

The charge operator can be similarly expressed in this representation by direct calculation from (4.32)

$$\mathsf{Q} = \int d^3p : -ij_4(x):$$

$$= \int d^3p\; g(a^{(r)\dagger}(\mathbf{p})a^{(r)}(\mathbf{p}) - b^{(r)\dagger}(\mathbf{p})b^{(r)}(\mathbf{p})) \tag{5.67}$$

or

$$\mathbf{Q} = \int d^3p\, g(N_+^{(r)}(\mathbf{p}) - N_-^{(r)}(\mathbf{p})) \tag{5.68}$$

The effect of the symmetrization or antisymmetrization is to dispose of an infinite constant, the total charge of the negative energy states of the vacuum. The equations (5.66, 5.68) show the symmetric appearance of particles and antiparticles of equal masses and opposite *charges* in the quantized theory.

The angular momentum operators $\mathbf{J}_{\mu\nu}$ for fields of discrete spin are not as easily calculated as the operators for the four-momentum and charge, due to the complexities of the non-diagonal nature of the spin density terms in the angular momentum tensor. The task is not made simpler by using the angular momentum tensor that employs the symmetrized energy-momentum tensor. Furthermore, the particular Lorentz frame that we have chosen with the S_{12} component of the spin along the direction of motion for illustrative purposes (giving the spinors of tables III to VI) is not suitable for calculation of $\mathbf{J}_{\mu\nu}$ in the general case.

If one starts from spinors formulated in an arbitrary Lorentz frame, the calculation of the operator $\mathbf{J}_{\mu\nu}$ can be done in a straightforward way, but it is very tedious for fields of spin greater than one and will not be done here. It is simpler to verify that relation (5.63) is indeed satisfied for all components of the operator $\mathbf{J}_{\mu\nu}$ with $\varphi(x)$ in the rest frame or for the components $\mathbf{J}_{12}$ and $\mathbf{J}_{34}$ when $\varphi(x)$ is boosted from the rest frame along the z axis corresponding to the spinors in table IV or by calculation of $\mathbf{J}_{\mu\nu}$ in the rest system.

Using the identity

$$[AB,C] = A[B,C] + [A,C]B = A\{B,C\} - \{A,C\}B$$

we are able to establish the relations

$$[a^{(r)\dagger}(\mathbf{p})a^{(r)}(\mathbf{p}),a^{(s)}(\mathbf{p}')] = -\delta_{rs}\delta^{(3)}(\mathbf{p} - \mathbf{p}')a^{(r)}(\mathbf{p}) \tag{5.69}$$

$$[a^{(r)\dagger}(\mathbf{p})a^{(r)}(\mathbf{p}),a^{(s)\dagger}(\mathbf{p}')] = \delta_{rs}\delta^{(3)}(\mathbf{p} - \mathbf{p}')a^{(r)\dagger}(\mathbf{p}) \tag{5.70}$$

These relations hold whether the $a^{(r)\dagger}(\mathbf{p})$ and $a^{(r)}(\mathbf{p})$ obey the commutation relations of bosons or the anticommutation relations of fermions. Thus one can establish by direct computation the commutation relations of $\mathbf{P}_\mu$, $\mathbf{J}_{\mu\nu}$ and $\mathbf{Q}$ for any field $\varphi(x)$ and $\bar{\varphi}(x)$

$$[\mathbf{P}_\mu,\varphi(x)] = i\partial_\mu\varphi(x) \tag{5.71}$$

$$[\mathbf{P}_\mu,\bar{\varphi}(x)] = i\partial_\mu\bar{\varphi}(x) \tag{5.72}$$

$$[\mathbf{J}_{\mu\nu},\varphi(x)] = [i(x_\mu\partial_\nu - x_\nu\partial_\mu) - S_{\mu\nu}^{(\varphi)}]\varphi(x) \tag{5.73}$$

$$[\mathbf{J}_{\mu\nu},\bar{\varphi}(x)] = [i(x_\mu\partial_\nu - x_\nu\partial_\mu) - S_{\mu\nu}^{(\varphi)}]\bar{\varphi}(x) \tag{5.74}$$

$$[\mathbf{Q},\varphi(x)] = -g\varphi(x) \tag{5.75}$$

$$[\mathbf{Q},\bar{\varphi}(x)] = g\bar{\varphi}(x) \tag{5.76}$$

The relations (5.71–5.74) can be considered to express the invariance of the quantized theory under infinitesimal translations, rotations, and boosts and the operators $\mathbf{P}_\mu$ and $\mathbf{J}_{\mu\nu}$ may be considered to be displacement and Lorentz rotation operators of the quantized theory.

All of the commutation relations of the inhomogeneous Lorentz group are reproduced in quantized operator form

$$[\mathbf{P}_\mu,\mathbf{P}_\nu] = 0 \tag{5.77}$$

$$[\mathbf{J}_{\mu\nu},\mathbf{P}_\lambda] = i\delta_{\mu\lambda}\mathbf{P}_\nu - i\delta_{\nu\lambda}\mathbf{P}_\mu \tag{5.78}$$

$$[\mathbf{J}_{\mu\nu},\mathbf{J}_{\lambda\rho}] = i(\delta_{\mu\lambda}\mathbf{J}_{\nu\rho} + \delta_{\nu\rho}\mathbf{J}_{\mu\lambda} - \delta_{\mu\rho}\mathbf{J}_{\nu\lambda} - \delta_{\nu\lambda}\mathbf{J}_{\mu\rho}) \tag{5.79}$$

The latter two relations may be readily established by showing that $\mathbf{P}_\lambda$ and $\mathbf{J}_{\lambda\rho}$ transform as a four vector and a Lorentz six vector, respectively, under the infinitesimal homogeneous Lorentz transformation $\{0,\Lambda\}$.

$$\mathbf{P}_\lambda' = L^{-1}(0,\Lambda)\mathbf{P}_\lambda L(0,\Lambda) = \mathbf{P}_\lambda - \frac{i}{2}\varepsilon_{\mu\nu}[\mathbf{J}_{\mu\nu},\mathbf{P}_\lambda]$$

$$\mathbf{J}_{\lambda\rho}' = L^{-1}(0,\Lambda)\mathbf{J}_{\lambda\rho}L(0,\Lambda) = \mathbf{J}_{\lambda\rho} - \frac{i}{2}\varepsilon_{\mu\nu}[\mathbf{J}_{\mu\nu},\mathbf{J}_{\lambda\rho}]$$

Furthermore, one can establish that

$$[\mathbf{Q},\mathbf{P}_\mu] = 0 \tag{5.80}$$

$$[\mathbf{Q},\mathbf{J}_{\mu\nu}] = 0 \tag{5.81}$$

Relations (5.77, 5.80) establish $\mathbf{P}_\mu$ and $\mathbf{Q}$ as constants of the motion. $\mathbf{J}_{\mu\nu}$ contains the coordinates explicitly so that we must calculate the time derivative by the usual Heisenberg relation.

$$\dot{\mathbf{J}}_{\mu\nu} = i[H,\mathbf{J}_{\mu\nu}] + \frac{\partial}{\partial t}\mathbf{J}_{\mu\nu} \tag{5.82}$$

Using an explicit representation for $\mathbf{J}_{\mu\nu}$ we see that the last term on the right cancels the nonzero value of the commutator so that

$$\dot{\mathbf{J}}_{\mu\nu} = 0 \tag{5.83}$$

and $\mathbf{J}_{\mu\nu}$ is also a constant of the motion.

We conclude that the postulated canonical commutation relations (5.26–5.29) are a special case holding only for spin zero and spin one fields[14] and do not necessarily provide the basis for a quantized theory, but rather, the particle interpretation of the field is the more fundamental postulate for fields of arbitrary spin. Schwinger [45] using a different line of reasoning, also concluded that the conventional canonical quantization approach fails for fields having a spin $s \geq \frac{3}{2}$.

We are now in a position to consider the vacuum expectation values of the field operators; of particular interest is the Feynman propagator which is readily evaluated. We are interested in a transition from vacuum to vacuum in which a particle is created and later absorbed and only certain

[14] The canonical commutation rules do not hold if the Lagrangian density for a spin one half field is of the form (4.10). Only if one considers a particular Lagrangian density of the form $\mathcal{L} = -\bar{\psi}(x)(\gamma_\mu\partial_\mu + m)\psi(x)$ in which the canonical momentum corresponding to $\bar{\psi}(x)$ is undefined, do we have the canonical commntation relations oheyed. Of course all physical phenomena are described hy either Lagrangian.

combinations of creation and annihilation operators give nonvanishing contributions. The Feynman propagator involves the vacuum expectation value of the time ordered product of the field functions $\varphi(x)$ or $\chi(x)$. For example, the propagator for the φ field is

$$D_F^{(\varphi)}(x,x') = \tfrac{1}{2}\langle T[\varphi(x)\bar{\varphi}(x')]\rangle_{\text{vac}} \tag{5.84}$$

where

$$\begin{aligned} T[\varphi(x)\bar{\varphi}(x')] &= \varphi(x)\bar{\varphi}(x') && \text{if } x_0 \geq x_0' \\ &= (-1)^{2s}\bar{\varphi}(x')\varphi(x) && \text{if } x_0 < x_0' \end{aligned}$$

The expressions (5.46–5.53) may be used to calculate this propagator by standard methods of field theory (e.g., see reference [41]) to yield the result

$$D_F^{(\varphi)}(x,x') = \frac{-i}{(2\pi)^4}\int d^4p \, \frac{u^{(r)}(\mathbf{p})\bar{u}^{(r)}(\mathbf{p})}{p^2 + m^2 - i\epsilon} e^{ip\cdot(x-x')} \tag{5.85}$$

where $d^4p = d^3p \, dp_0$ and ϵ is a small positive number which tends to zero after the integration, such that the contour of integration of the four variables $p_0, \cdots, p_3$ is along the real axis. The values for the quantities $u^{(r)}(\mathbf{p})\bar{u}^{(r)}(\mathbf{p})$, which give the matrix operators involving the spin sums, may be taken from tables VIII through XI for fields of spins $\frac{1}{2}$, 1, $\frac{3}{2}$ and 2 in some particular gauges. Unfortunately, the Lorentz gauges for spins $\frac{3}{2}$ and 2 are particularly complicated and are not listed. Similar arguments provide the propagator for the χ field

$$D_F^{(\chi)}(x,x') = \frac{-i}{(2\pi)^4}\int d^4p \, \frac{w^{(r)}(\mathbf{p})\bar{w}^{(r)}(\mathbf{p})}{p^2 + m^2 - i\epsilon} e^{ip\cdot(x-x')} \tag{5.86}$$

The values for the quantities $w^{(r)}(\mathbf{p})\bar{w}^{(r)}(\mathbf{p})$ are also taken from tables VIII through XI. If the χ field is considered the basic and not a derivative field we may set $-p_\mu p_\mu = m^2$ in the expressions for $w^{(r)}(\mathbf{p})\bar{w}^{(r)}(\mathbf{p})$. Here the results agree with the expressions given by Weinberg [29] for the propagators corresponding to that portion of the χ field in the $(s,0) \oplus (0,s)$ representation. Other invariant Δ functions for fields of any spin may be calculated with the procedures developed in this paper, however these quantities lie outside the scope of the considerations of this present work.

6. Discrete Symmetries

Thus far we have considered only those symmetries and invariance properties that are obtained from transformations that differ infinitesimally from the identity transformation. Here, we investigate those improper Lorentz transformations that do not contain the identity, namely space reflection and time reversal, and an additional transformation, charge conjugation, under which each state is mapped into one where all particles are replaced by their antiparticles, the other properties of the state being essentially unchanged. A satisfactory formulation of charge conjugation cannot be made without quantization of the field such that required commutation relations are obeyed. It is important in this symmetry transformation that all physical observables involving bilinear products of the field operators be symmetrized or antisymmetrized according to (5.64). This symmetrization or antisymmetrization is implicitly understood and is written explicitly only when required.

The Lagrangian (4.24) for a charged non-hermitian field of any discrete spin interacting with a neutral hermitian vector boson field provides us with a convenient model to study these discrete transformations. The Lagrangian, the equations of motion, and the equal time Heisenberg commutation relations shall remain invariant under these transformations. The form invariance of the relations (4.21, 4.22) establishes the properties of the operations for space inversion, time reversal, and charge conjugation.

The first of these discrete symmetries to be considered is that of space inversion, or the parity transformation in which $\mathbf{x} \to -\mathbf{x}$, while $x_4 \to x_4$. This transformation is the subset of Lorentz transformations $x' = \Lambda x$ with $\det \Lambda = -1$ and $\Lambda_{44} = 1$. We may characterize this transformation by defining a unitary operator $\mathcal{P}$ that has the property

$$\mathcal{P}\mathcal{L}(\mathbf{x},t)\mathcal{P}^{-1} = \mathcal{L}^{(P)}(\mathbf{x},t) = \mathcal{L}(-\mathbf{x},t) \tag{6.1}$$

The action of this unitary operator $\mathcal{P}$ on the field operators is defined as

$$\mathcal{P}\varphi(\mathbf{x},t)\mathcal{P}^{-1} = \varphi^{(P)}(\mathbf{x},t) = P\varphi(-\mathbf{x},t) \tag{6.2}$$

$$\mathcal{P}\chi(\mathbf{x},t)\mathcal{P}^{-1} = \chi^{(P)}(\mathbf{x},t) = P\chi(-\mathbf{x},t) \tag{6.3}$$

where P is a unitary matrix of dimension $8s$ by $8s$. Applying this transformation to (4.21) we obtain

$$P\chi(-\mathbf{x},t) = P[-\gamma_k(\partial_k + igA_k(-\mathbf{x},t)) + \gamma_4(\partial_4 - igA_4(-\mathbf{x},t))]P^{-1}P\varphi(-\mathbf{x},t) \tag{6.4}$$

We can write this in the space-inverted system as

$$\chi^{(P)}(\mathbf{x},t) = \gamma_\mu(\partial_\mu - igA_\mu{}^{(P)}(\mathbf{x},t))\varphi^{(P)}(\mathbf{x},t) \tag{6.5}$$

provided that

$$P\gamma_k P^{-1} = -\gamma_k \tag{6.6}$$

$$P\gamma_4 P^{-1} = \gamma_4 \tag{6.7}$$

and that

$$\mathbf{A}^{(P)}(\mathbf{x},t) = \mathcal{P}\mathbf{A}(\mathbf{x},t)\mathcal{P}^{-1} = -\mathbf{A}(-\mathbf{x},t) \tag{6.8}$$

$$A_4{}^{(P)}(\mathbf{x},t) = \mathcal{P}A_4(\mathbf{x},t)\mathcal{P}^{-1} = A_4(-\mathbf{x},t) \tag{6.9}$$

The matrix P that satisfies (6.6, 6.7) is

$$P = \eta_P\gamma_4 \tag{6.10}$$

where the constant η_P has modulus one but must be limited to the values

$$\eta_P = \pm 1 \tag{6.11}$$

since two successive space inversions returns the coordinate system to the original such that $P^2 = 1$. The choice of the $+$ or $-$ sign in (6.11) defines what is called the intrinsic parity of the particle described by this field. It is a specific rule of transformation of the field which creates the particle by operation on the vacuum state. This sign may be determined only when interactions are present between different particles. As we shall see the requirement (6.8, 6.9) determines $\eta_P = -1$ for a polar four-vector field. In general η_P may be determined only for boson field operators. Fermion field operators always occur in bilinear combinations such that only $|\eta_P|^2$ occurs.

The invariance of (4.21) under the parity transformation, and similarly also for the adjoint relation (4.22), assures that a scalar Lagrangian and the equations of motion will be invariant.

The invariance of the commutation or anticommutation relations may be verified by consideration of the transformation properties of the expansion coefficients for the quantized field

$$\mathcal{P}\varphi(\mathbf{x},t)\mathcal{P}^{-1} = \int \frac{d^3p}{(2\pi)^{3/2}} \frac{1}{(2E)^{1/2}} \{\mathcal{P}a^{(r)}(\mathbf{p})\mathcal{P}^{-1}u^{(r)}(\mathbf{p})e^{i(\mathbf{p}\cdot\mathbf{x}-Et)}$$

$$- (-1)^{2s}\mathcal{P}b^{(r)\dagger}(\mathbf{p})\mathcal{P}^{-1}v^{(r)}(\mathbf{p})e^{-i(\mathbf{p}\cdot\mathbf{x}-Et)}\}$$

$$= \int \frac{d^3p}{(2\pi)^{3/2}} \frac{1}{(2E)^{1/2}} \{a^{(r)}(\mathbf{p})Pu^{(r)}(\mathbf{p})e^{-i(\mathbf{p}\cdot\mathbf{x}+Et)}$$

$$- (-1)^{2s}b^{(r)\dagger}(\mathbf{p})Pv^{(r)}(\mathbf{p})e^{i(\mathbf{p}\cdot\mathbf{x}+Et)}\} \tag{6.12}$$

Changing $\mathbf{p}$ to $-\mathbf{p}$ in the right hand side of (6.12) and making use of the spinor properties

$$Pu^{(r)}(-\mathbf{p}) = \eta_P B_{rr'} u^{(r')}(\mathbf{p}) \tag{6.13}$$

$$Pv^{(r)}(-\mathbf{p}) = (-1)^{2s}\eta_P B_{rr'} v^{(r')}(\mathbf{p}) \tag{6.14}$$

we obtain

$$\mathcal{P}a^{(r)}(\mathbf{p})\mathcal{P}^{-1} = \eta_P a^{(r')}(-\mathbf{p})B_{r'r} \tag{6.15}$$

$$\mathcal{P}b^{(r)\dagger}(\mathbf{p})\mathcal{P}^{-1} = \eta_P(-1)^{2s}b^{(r')\dagger}(-\mathbf{p})B_{r'r} \tag{6.16}$$

The factor $(-)^{2s}$ is positive or negative whether bosons or fermions, respectively, are described by the field, and $B_{rr'}$ is the rr' component of the diagonal matrix B, the metric operator.

It is possible to get explicit representations of the operator $\mathcal{P}$ making use of the commutation relations of $a^{(r)}(\mathbf{p})$, $b^{(r)\dagger}(\mathbf{p})$ and their hermitian conjugates, however it is not necessary to pursue this here.

Similarly one may show for the adjoint field

$$\mathcal{P}\bar{\varphi}(\mathbf{x},t)\mathcal{P}^{-1} = \bar{\varphi}^{(P)}(\mathbf{x},t) = \bar{\varphi}^{(P)}(-\mathbf{x},t)P^{-1} \tag{6.17}$$

and

$$\mathcal{P}a^{(r)\dagger}(\mathbf{p})\mathcal{P}^{-1} = \eta_P{}^* a^{(r')\dagger}(-\mathbf{p})B_{r'r} \tag{6.18}$$

$$\mathcal{P}b^{(r)}(\mathbf{p})\mathcal{P}^{-1} = \eta_P{}^*(-1)^{2s}b^{(r')}(-\mathbf{p})B_{r'r} \tag{6.19}$$

The properties of the single particle states under space inversion follow immediately from the definitions (6.13, 6.14).

The requirement (6.8, 6.9) for the transformation properties of the components of the field $A_\mu(\mathbf{x},t)$ under space inversion anticipate the transformation properties of the neutral vector boson field described by $\vartheta(\mathbf{x},t)$. The remaining portion of the Lagrangian is also invariant provided that

$$\mathcal{P}\vartheta(\mathbf{x},t)\mathcal{P}^{-1} = \vartheta^{(P)}(\mathbf{x},t) = \eta_P{}^{(\vartheta)}\gamma_4\vartheta(-\mathbf{x},t) \tag{6.20}$$

Here $\eta_p^{(\vartheta)}$ is not arbitrary but has already been determined by the requirements (6.8, 6.9) so that $\eta_p^{(\vartheta)} = -1$. All transformation arguments for the $\vartheta(\mathbf{x},t)$ field parallel those for the $\varphi(\mathbf{x},t)$, if we recognize the fact that here the hermitian field is described by letting

$$-(-1)^{2s}b^{(r)\dagger}(\mathbf{p})v^{(r)}(\mathbf{p}) \longrightarrow a^{(r)\dagger}(\mathbf{p})u^{(r)\dagger}(\mathbf{p}) \quad \text{and} \quad -(-1)^{2s}b^{(r)}(\mathbf{p})\bar{v}^{(r)}(\mathbf{p}) \longrightarrow a^{(r)}(\mathbf{p})\bar{u}^{(r)\dagger}(\mathbf{p})$$

Next we consider the operation of time reversal in which $x_4 \longrightarrow -x_4$ while $\mathbf{x} \longrightarrow \mathbf{x}$. This transformation is the subset of Lorentz transformations $x' = \Lambda x$ with $\det \Lambda = -1$ and $\Lambda_{44} = -1$. We may characterize this transformation by defining an antiunitary operator $\mathfrak{T}$ that has the property that

$$\mathfrak{T}\mathfrak{L}(x,t)\mathfrak{T}^{-1} = \mathfrak{L}^{(T)}(\mathbf{x},t) = \mathfrak{L}(\mathbf{x},-t) \tag{6.21}$$

The action of this antiunitary operator $\mathfrak{T}$, so chosen that the commutation or anticommutation relations of the field operators remain invariant, is defined as

$$\mathfrak{T}\varphi(\mathbf{x},t)\mathfrak{T}^{-1} = \varphi^{(T)}(\mathbf{x},t) = T\tilde{\varphi}(\mathbf{x},-t) \tag{6.22}$$

$$\mathfrak{T}\chi(\mathbf{x},t)\mathfrak{T}^{-1} = \chi^{(T)}(\mathbf{x},t) = T\tilde{\chi}(\mathbf{x},-t) \tag{6.23}$$

where T is an $8s$ by $8s$ unitary matrix. Applying this transformation to the transpose of (4.22) we obtain

$$T\tilde{\chi}(\mathbf{x},-t) = T[-\tilde{\gamma}_k(\partial_k + igA_k(\mathbf{x},-t)) + \tilde{\gamma}_4(\partial_4 - igA_4(\mathbf{x},-t))]T^{-1}T\tilde{\varphi}(\mathbf{x},-t) \tag{6.24}$$

we can write this in the time reversed system as

$$\chi^{(T)}(\mathbf{x},t) = \gamma_\mu(\partial_\mu - igA_\mu^{(T)}(\mathbf{x},t))\varphi^{(T)}(\mathbf{x},t) \tag{6.25}$$

provided that

$$T\tilde{\gamma}_k T^{-1} = -\gamma_k \tag{6.26}$$

$$T\tilde{\gamma}_4 T^{-1} = \gamma_4 \tag{6.27}$$

and that the field $A_\mu^{(T)}(\mathbf{x},t)$ obeys the relations

$$\mathbf{A}^{(T)}(\mathbf{x},t) = \mathfrak{T}\mathbf{A}(\mathbf{x},t)\mathfrak{T}^{-1} = -\mathbf{A}(\mathbf{x},-t) \tag{6.28}$$

$$A_4^{(T)}(\mathbf{x},t) = \mathfrak{T}A_4(\mathbf{x},t)\mathfrak{T}^{-1} = A_4(\mathbf{x},-t) \tag{6.29}$$

A matrix T that fulfills the relations (6.26, 6.27) in our "Dirac-Pauli" representation is given by

$$T = \eta_T\eta D\gamma_4 \tag{6.30}$$

where D is the operator, corresponding to a rotation by the amount π about the y axis, with the properties (3.106). The operator T is symmetric or antisymmetric depending upon whether it corresponds to bosons of fermions respectively. The restriction on the value η_T is that $|\eta_T|^2 = 1$. Interaction terms determine the value of η_T for bosons but not fermions. The operator η is uniquely defined and determines the metric of the time reversed states.

The invariance of the commutation or anticommutation relations may be verified by considering the transformation properties of the expansion coefficients of the quantized fields

$$\mathfrak{I}\varphi(\mathbf{x},t)\mathfrak{I}^{-1} = \int \frac{d^3p}{(2\pi)^{3/2}} \frac{1}{(2E)^{1/2}} \{\mathfrak{I}a^{(r)}(\mathbf{p})\mathfrak{I}^{-1}u^{(r)}(\mathbf{p})e^{i(\mathbf{p}\cdot\mathbf{x}-Et)} - (-1)^{2s}\mathfrak{I}b^{(r)\dagger}(\mathbf{p})\mathfrak{I}^{-1}v^{(r)}(\mathbf{p})e^{-i(\mathbf{p}\cdot\mathbf{x}-Et)}\}$$

$$= \int \frac{d^3p}{(2\pi)^{3/2}} \frac{1}{(2E)^{1/2}} \{a^{(r)\dagger}(\mathbf{p})T\tilde{\bar{u}}^{(r)}(\mathbf{p})e^{-i(\mathbf{p}\cdot\mathbf{x}+Et)} - (-1)^{2s}b^{(r)}(\mathbf{p})T\tilde{\bar{v}}^{(r)}(\mathbf{p})e^{i(\mathbf{p}\cdot\mathbf{x}+Et)}\} \quad (6.31)$$

changing $\mathbf{p}$ to $-\mathbf{p}$ on the right hand side of (6.31) and making use of the spinor properties in table XVI to give

$$T\tilde{\bar{u}}^{(r)}(-\mathbf{p}) = \eta_T(-1)^{2s}(d^{(s)}(\pi))_{rr'}u^{(r')}(\mathbf{p}) \quad (6.32)$$

$$T\tilde{\bar{v}}^{(r)}(-\mathbf{p}) = \eta_T(-1)^{2s}(d^{(s)}(\pi))_{rr'}v^{(r')}(\mathbf{p}) \quad (6.33)$$

we obtain

$$\mathfrak{I}a^{(r)}(\mathbf{p})\mathfrak{I}^{-1} = \eta_T(-1)^{2s}a^{(r')\dagger}(-\mathbf{p})(d^{(s)}(\pi))_{r'r} \quad (6.34)$$

$$\mathfrak{I}b^{(r)\dagger}(\mathbf{p})\mathfrak{I}^{-1} = \eta_T(-1)^{2s}b^{(r')}(-\mathbf{p})(d^{(s)}(\pi))_{r'r} \quad (6.35)$$

The factor $(d^{(s)}(\pi))_{r'r}$ is the $r'r$ component of the rotation matrix $d^{(s)}(\pi)$ defined in (3.102–3.105).

As for the case of the parity operator it is possible to obtain a representation for the time-reversal operator, $\mathfrak{I}$, however it is not necessary to do so since its action is defined by (6.34, 6.35).

Similar considerations yield the time reversal analog of (4.21)

$$\chi_{(\vartheta)}{}^{(T)}(\mathbf{x},t) = \gamma_\mu\partial_\mu\vartheta^{(T)}(\mathbf{x},t) \quad (6.36)$$

where

$$\vartheta^{(T)}(\mathbf{x},t) = \eta_T{}^{(\vartheta)}\eta D\gamma_4\tilde{\bar{\vartheta}}(\mathbf{x},-t) \quad (6.37)$$

The phase $\eta_T{}^{(\vartheta)}$ is determined uniquely by the interaction term as in (6.25) to be $\eta_T{}^{(\vartheta)} = 1$. This calls for a closer examination. We have specified a cartesian basis for the $\vartheta(\mathbf{x},t)$ field. If it is described in a spherical basis, D is given by

$$D = \begin{vmatrix} d^{(s)}(\pi) & \cdot \\ \cdot & d^{(s)}(\pi) \end{vmatrix} \quad (3.101)$$

where

$$d^{(1)}(\pi) = \begin{vmatrix} \cdot & \cdot & 1 & \cdot \\ \cdot & -1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{vmatrix} \quad (3.103)$$

The relation (6.37) can be rewritten as

$$\vartheta^{(T)}(\mathbf{x},t) = DK\vartheta(\mathbf{x},-t) \tag{6.38}$$

where K is the complex conjugation operator. We make a unitary transformation to get to a cartesian representation using (3.112).

$$U\vartheta^{(T)}(\mathbf{x},t) = UDKU^{\dagger}U\vartheta(\mathbf{x},-t) \tag{6.39}$$

$$\vartheta_{\text{cart}}{}^{(T)}(\mathbf{x},t) = -K\vartheta_{\text{cart}}(\mathbf{x},-t) \tag{6.40}$$

This transformation results from the action of the complex conjugation operator on $U^{\dagger}$ so that $KU^{\dagger} = \tilde{U}K$, with the result

$$UD\tilde{U} = D_{\text{cart}} = -\begin{vmatrix} I & 0 \\ 0 & I \end{vmatrix} \tag{6.41}$$

In a cartesian representation all of the γ_{μ} are real and symmetric so that (3.106) is satisfied even though D_{cart} is the negative of the unit matrix.

Arguments similar to those made for the $\varphi(\mathbf{x},t)$ field hold for the time reversed creation and destruction operators of the quantized $\vartheta(\mathbf{x},t)$ field.

The commutation relations involving the quantized fields will be maintained provided the anti-unitary property of the time-reversal operator is observed. The order of all operators is reversed.

Charge conjugation, while not a Lorentz transformation, is a discrete symmetry similar to the improper Lorentz transformations, space inversion and time reversal. We may characterize this transformation by defining a linear unitary operator $\mathcal{C}$ with the property that

$$\mathcal{C}\mathcal{L}(\mathbf{x},t)\mathcal{C}^{-1} = \mathcal{L}^{(C)}(\mathbf{x},t) \tag{6.42}$$

where in the transformed Lagrangian the same dynamical state occurs but with all particles transformed into antiparticles and vice versa.

The action of this unitary operator, so chosen that the commutation or anticommutation relations of the field operators remain invariant, is defined as

$$\mathcal{C}\varphi(\mathbf{x},t)\mathcal{C}^{-1} = \varphi^{(C)}(\mathbf{x},t) = C\tilde{\tilde{\varphi}}(\mathbf{x},t) \tag{6.43}$$

$$\mathcal{C}\chi(\mathbf{x},t)\mathcal{C}^{-1} = \chi^{(C)}(\mathbf{x},t) = \pm C\tilde{\tilde{\chi}}(\mathbf{x},t) \tag{6.44}$$

where C is an $8s$ by $8s$ unitary matrix. The plus or minus sign represents an arbitrariness to be resolved depending upon our choice for the matrix C.

Applying this transformation to the transpose of (4.22) we obtain

$$C\tilde{\tilde{\chi}}(\mathbf{x},t) = C[-\tilde{\gamma}_{\mu}(\partial_{\mu} + igA_{\mu}(\mathbf{x},t))]C^{-1}C\tilde{\tilde{\varphi}}(\mathbf{x},t) \tag{6.45}$$

The invariance of the Lagrangian is explicit for it is quadratic in $\varphi(x)$ and its adjoint and in $\chi(x)$ and its adjoint. We can have two operators with the properties

$$C_F\tilde{\gamma}_{\mu}C_F = -\gamma_{\mu} \tag{6.46}$$

and

$$C_B \tilde{\gamma}_\mu C_B = \gamma_\mu \tag{6.47}$$

that leave the Lagrangian invariant. We know, however, that certain neutral boson fields are self conjugate with the same parity so that the matrix associated with boson fields is such that

$$\varphi^{(C)}(\mathbf{x},t) = C_B \tilde{\tilde{\varphi}}(\mathbf{x},t) \tag{6.48}$$

$$\chi^{(C)}(\mathbf{x},t) = -C_B \tilde{\tilde{\chi}}(\mathbf{x},t) \tag{6.49}$$

where

$$C_B = \eta_C{}^{(B)} \eta D \tag{6.50}$$

On the other hand the invariance of the first order Lagrangian and the assurance of the proper anticommutation relations require for fermions that

$$\varphi^{(C)}(\mathbf{x},t) = C_F \tilde{\tilde{\varphi}}(\mathbf{x},t) \tag{6.51}$$

$$\chi^{(C)}(\mathbf{x},t) = C_F \tilde{\tilde{\chi}}(\mathbf{x},t) \tag{6.52}$$

where

$$C_F = -\eta_C{}^{(F)} \eta D \gamma_5 \tag{6.53}$$

As in the case of space inversion and of time reversal, the phase factor $\eta_C{}^{(F)}$ cannot be determined for fermions but is limited to $| \eta_C{}^{(F)} |^2 = 1$. The interaction term involving the electromagnetic field, a vector boson, determines $\eta_C{}^{(B)} = 1$. The metric operator η is always defined[15].

If we consider $A_\mu(x)$ to be an external field not subject to the conjugation operation then our transformed relation is

$$\chi^{(C)}(x) = \gamma_\mu(\partial_\mu + igA_\mu(x))\varphi^{(C)}(x) \tag{6.54}$$

This is similar to (4.21) except for the reversed sign of the coupling coefficient g. On the other hand, if $A_\mu(x)$ is subjected to the conjugation operation then

$$A_\mu{}^{(C)}(x) = \mathcal{C} A_\mu(x) \mathcal{C}^{-1} = -A_\mu(x) \tag{6.55}$$

so that the transformed relation is

$$\chi^{(C)}(x) = \gamma_\mu(\partial_\mu - igA_\mu{}^{(C)}(x))\varphi^{(C)}(x) \tag{6.56}$$

The charge conjugation of $A_\mu(x)$, the component of $\vartheta(x)$, implies that

$$\vartheta^{(C)}(x) = \eta D \tilde{\tilde{\vartheta}}(x) \tag{6.57}$$

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[15] The relative forms and phases of relations (6.48–6.53) are identical to those obtained by the so-called operation "strong reflection" combined with the CPT theorem. Strong reflection implies a finite proper Lorentz rotation through an angle π in the x_1x_2 plane followed by a finite complex proper Lorentz rotation through an angle π in the x_3x_4 plane and a subsequent transposition of the order of all operators. The expressions for the complex rotation in the x_3x_4 plane may be obtained from the formulae in table II with the substitution $\Omega_{34} = -i\pi$.

In a cartesian system D is the negative unit matrix and the $\vartheta(x)$ field is self adjoint, $\vartheta_{\text{cart}}(x) = \eta\tilde{\tilde{\vartheta}}_{\text{cart}}(x)$, so that

$$\vartheta_{\text{cart}}^{(C)}(x) = -\vartheta_{\text{cart}}(x) \tag{6.58}$$

for a neutral vector boson field.

The transformation properties of the expansion coefficients of the quantized fields are determined by the relation

$$\mathcal{C}\varphi(x)\mathcal{C}^{-1} = \int \frac{d^3p}{(2\pi)^{3/2}} \frac{1}{(2E)^{1/2}} \{\mathcal{C}a^{(r)}(\mathbf{p})\mathcal{C}^{-1}u^{(r)}(\mathbf{p})e^{ip\cdot x} - (-1)^{2s}\mathcal{C}b^{(r)\dagger}(\mathbf{p})\mathcal{C}^{-1}v^{(r)}(\mathbf{p})e^{-ip\cdot x}\}$$

$$= \int \frac{d^3p}{(2\pi)^{3/2}} \frac{1}{(2E)^{1/2}} \{a^{(r)\dagger}(\mathbf{p})C\tilde{\bar{u}}^{(r)}(\mathbf{p})e^{-ip\cdot x} - (-1)^{2s}b^{(r)}(\mathbf{p})C\tilde{\bar{v}}^{(r)}(\mathbf{p})e^{ip\cdot x}\} \tag{6.59}$$

We can make use of the spinor properties of bosons in table XVI to give

$$C_B\tilde{\bar{u}}^{(r)}(\mathbf{p}) = \eta_C{}^{(B)}v^{(r)}(\mathbf{p}) \tag{6.60}$$

$$C_B\tilde{\bar{v}}^{(r)}(\mathbf{p}) = \eta_C{}^{(B)}u^{(r)}(\mathbf{p}) \tag{6.61}$$

or similarly, the spinor properties of fermions in table XVI to give

$$C_F\tilde{\bar{u}}(\mathbf{p}) = \eta_C{}^{(F)}v^{(r)}(\mathbf{p}) \tag{6.62}$$

$$C_F\tilde{\bar{v}}^{(r)}(\mathbf{p}) = \eta_C{}^{(F)}u^{(r)}(\mathbf{p}) \tag{6.63}$$

so that by comparison of the two schemes

$$\mathcal{C}a^{(r)}(\mathbf{p})\mathcal{C}^{-1} = -(-1)^{2s}\eta_C b^{(r)}(\mathbf{p}) \tag{6.64}$$

$$\mathcal{C}b^{(r)\dagger}(\mathbf{p})\mathcal{C}^{-1} = -(-1)^{2s}\eta_C a^{(r)\dagger}(\mathbf{p}) \tag{6.65}$$

Lagrangians of the type (4.24) are not the only ones that it is possible to write down. For any process we wish to describe involving several fields, we must consider that the Lagrangian consists of the individual Lagrangians of the free fields and interaction terms involving two or more fields. These interaction terms must be invariant under proper Lorentz transformations and at least invariant under the product of the improper transformations $\mathcal{P}\mathcal{C}\mathcal{T}$.

It is worthwhile to study the transformation properties of covariant quantities of several fields.

Five bilinear covariants can be formed from the quantities $\varphi(x)$ and $\chi(x)$ and their adjoints. They transform as a scalar S, a polar vector V, an antisymmetric second rank tensor T, an axial vector A, and a pseudoscalar P. They are

$$f^{(S)}(x) = c_1{:}\bar{\varphi}(x)\varphi(x){:} + c_2{:}\bar{\chi}(x)\chi(x){:} \tag{6.66}$$

$$f_\mu{}^{(V)}(x) = {:}\bar{\chi}(x)\gamma_\mu\varphi(x){:} + {:}\bar{\varphi}(x)\gamma_\mu\chi(x){:} \tag{6.67}$$

$$f_{\mu\nu}{}^{(T)}(x) = c_1{:}\bar{\varphi}(x)\sigma_{\mu\nu}\varphi(x){:} + c_2{:}\bar{\chi}(x)\sigma_{\mu\nu}\chi(x){:} \tag{6.68}$$

$$f_\mu{}^{(A)}(x) = {:}\bar{\chi}(x)\gamma_\mu\gamma_5\varphi(x){:} + {:}\bar{\varphi}(x)\gamma_5\gamma_\mu\chi(x){:} \tag{6.69}$$

$$f^{(P)}(x) = c_1{:}\bar{\varphi}(x)\gamma_5\varphi(x){:} + c_2{:}\bar{\chi}(x)\gamma_5\chi(x){:} \tag{6.70}$$

where c_1 and c_2 are arbitrary constants. We have put these in symmetrized or antisymmetrized form depending on whether they are covariants of boson or fermion fields, respectively.

It is straighforward to calculate each of these covariants in another Lorentz frame x' by considering infinitesimal Lorentz transformations such as

$$\varphi'(x') = \left(1 + \frac{i}{2}\, \varepsilon_{\mu\nu} S_{\mu\nu}{}^{(\varphi)}\right) \varphi(x) \tag{3.74}$$

$$\chi'(x') = \left(1 + \frac{i}{2}\, \varepsilon_{\mu\nu} S_{\mu\nu}{}^{(x)}\right) \chi(x) \tag{3.75}$$

Relations (3.76, 3.77, 3.84, 3.85) are used to get

$$f^{(S)}(x) = f^{(S)\prime}(x') \tag{6.71}$$

$$f_\mu{}^{(V)}(x) = \Lambda_{\mu\nu}{}^{-1} f_\nu{}^{(V)\prime}(x') \tag{6.72}$$

$$f_{\mu\nu}{}^{(T)}(x) = \Lambda_{\mu\lambda}{}^{-1}\Lambda_{\nu\rho}{}^{-1} f_{\lambda\rho}{}^{(T)\prime}(x') \tag{6.73}$$

$$f_\mu{}^{(A)}(x) = \Lambda_{\mu\nu}{}^{-1} f_\nu{}^{(A)\prime}(x') \tag{6.74}$$

$$f^{(P)}(x) = f^{(P)\prime}(x') \tag{6.75}$$

in which

$$\Lambda_{\mu\nu}{}^{-1} = \delta_{\mu\nu} - \varepsilon_{\mu\nu} \tag{6.76}$$

We see that the proper Lorentz transformation properties of these bilinear covariants do not depend on the value of the spin of the field, whether integer or half integer. Each of these bilinear covariants has characteristic transformation properties, also independent of the spin value of the field, under the discrete operations $\mathcal{P}$, $\mathcal{C}$, and $\mathfrak{I}$. The properties follow in a straightforward way from the definitions (6.2, 6.22, 6.43). The transformation properties for bilinear covariants and for components of the energy-momentum tensor $\mathfrak{I}_{\mu\nu}$ are listed in table XVII for the operations $\mathcal{P}$, $\mathcal{C}$, $\mathfrak{I}$, and $\mathcal{P}\mathcal{C}\mathfrak{I}$. We note that they are identical to the transformation properties of bilinear covariants and the energy-momentum tensor involving Dirac spinors [46]. It is these properties that determine the intrinsic phase factors η_P, η_C and η_T for boson fields in interaction terms where a scalar product of the field and a bilinear covariant occurs.

Finally, we are in a position to extend the proper inhomogeneous Lorentz group, abstractly defined by the commutation relations (2.18–2.20), by adjoining the generating elements of the improper operations, space inversion and time reversal and also charge conjugation. We can make use of the specific commutation relations of the operator D and the matrices γ_4 and γ_5 with the spin matrices $S_{\mu\nu}{}^{(\varphi)}$ and $S_{\mu\nu}{}^{(x)}$ to obtain

$$PS_{ij}{}^{(\varphi)}P^{-1} = S_{ij}{}^{(\varphi)} \tag{6.77}$$

$$PS_{k4}{}^{(\varphi)}P^{-1} = -S_{k4}{}^{(\varphi)} \tag{6.78}$$

$$T\tilde{S}_{ij}{}^{(\varphi)}T^{-1} = -S_{ij}{}^{(\varphi)} \tag{6.79}$$

$$T\tilde{S}_{k4}^{(\varphi)}T^{-1} = S_{k4}^{(\varphi)} \tag{6.80}$$

$$C\tilde{S}_{\mu\nu}^{(\varphi)}C^{-1} = -S_{\mu\nu}^{(\varphi)} \tag{6.81}$$

where we have used the definitions (6.10, 6.30, 6.50, 6.53). There is a second set of similar relations where the corresponding components of $S_{\mu\nu}^{(\chi)}$ are substituted for those of $S_{\mu\nu}^{(\varphi)}$. The transformation properties of coordinates and momenta coupled with the relations (6.77–6.81) then determines the transformation properties of the generators of the proper inhomogeneous group to be

$$\mathcal{P}(1 - ia_kP_k)\mathcal{P}^{-1} = 1 + ia_kP_k \tag{6.82}$$

$$\mathcal{P}(1 - ia_4P_4)\mathcal{P}^{-1} = 1 - ia_4P_4 \tag{6.83}$$

$$\mathcal{P}\left(1 + \frac{i}{2}\vartheta_{ij}J_{ij}\right)\mathcal{P}^{-1} = 1 + \frac{i}{2}\vartheta_{ij}J_{ij} \tag{6.84}$$

$$\mathcal{P}(1 - \Omega_{k4}J_{k4})\mathcal{P}^{-1} = 1 + \Omega_{k4}J_{k4} \tag{6.85}$$

$$\mathfrak{I}(1 - ia_kP_k)\mathfrak{I}^{-1} = 1 - ia_kP_k \tag{6.86}$$

$$\mathfrak{I}(1 - ia_4P_4)\mathfrak{I}^{-1} = 1 + ia_4P_4 \tag{6.87}$$

$$\mathfrak{I}\left(1 + \frac{i}{2}\vartheta_{ij}J_{ij}\right)\mathfrak{I}^{-1} = 1 + \frac{i}{2}\vartheta_{ij}J_{ij} \tag{6.88}$$

$$\mathfrak{I}(1 - \Omega_{k4}J_{k4})\mathfrak{I}^{-1} = 1 + \Omega_{k4}J_{k4} \tag{6.89}$$

$$\mathcal{C}(1 - ia_\mu P_\mu)\mathcal{C}^{-1} = 1 - ia_\mu P_\mu \tag{6.90}$$

$$\mathcal{C}\left(1 + \frac{i}{2}\varepsilon_{\mu\nu}J_{\mu\nu}\right)\mathcal{C}^{-1} = 1 + \frac{i}{2}\varepsilon_{\mu\nu}J_{\mu\nu} \tag{6.91}$$

These relations hold whether the generators are in the Hilbert space of the φ field or of the χ field. Thus we may effectively write the commutators involving the space inversion operator $\mathcal{P}$ as

$$\{\mathcal{P},P_i\} = 0 \tag{6.92}$$

$$[\mathcal{P},P_4] = 0 \tag{6.93}$$

$$[\mathcal{P},J_{ij}] = 0 \tag{6.94}$$

$$\{\mathcal{P},J_{k4}\} = 0 \tag{6.95}$$

The commutators involving $\mathfrak{I}$, an antiunitary operator, will appear to be different than those for $\mathcal{P}$ because of the transposition involved. They are:

$$\{\mathfrak{I},P_i\} = 0 \tag{6.96}$$

$$[\mathfrak{I},P_4] = 0 \tag{6.97}$$

$$\{\mathfrak{I},J_{ij}\} = 0 \tag{6.98}$$

$$[\mathfrak{I},J_{k4}] = 0 \tag{6.99}$$

The commutators involving $\mathcal{C}$ are:

$$[\mathcal{C},P_\mu] = 0 \tag{6.100}$$

$$[\mathcal{C},J_{\mu\nu}] = 0 \tag{6.101}$$

The remaining commutators among $\mathcal{P}$, $\mathcal{C}$, and $\mathcal{T}$ will not be considered here, since phase relations are rather delicate and are treated in detail elsewhere for the spin one-half case [47] which corresponds, when expressed in the Dirac-Pauli representation, to our representation.

7. References

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Table I *The matrix operators* $S_i^{(s)}$, $S_i^{(t)}$, B, *and* I *corresponding to fields of spin* $\frac{1}{2}$, *1*, $\frac{3}{2}$, *and 2*

Spin 1/2:

$$S_1^{(s)} = \frac{1}{2}\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \quad S_2^{(s)} = \frac{1}{2}\begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix} \quad S_3^{(s)} = \frac{1}{2}\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} \quad S_i^{(t)} = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} \quad B = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \quad I = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

Spin 1:

$$S_1^{(s)} = \frac{1}{\sqrt{2}}\begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} \quad S_2^{(s)} = \frac{1}{\sqrt{2}}\begin{vmatrix} 0 & -i & 0 & 0 \\ i & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} \quad S_3^{(s)} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} \quad S_1^{(t)} = \frac{1}{\sqrt{2}}\begin{vmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{vmatrix}$$

$$S_2^{(t)} = \frac{1}{\sqrt{2}}\begin{vmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & -i & 0 \end{vmatrix} \quad S_3^{(t)} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix} \quad B = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} \quad I = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

Spin 3/2:

$$S_1^{(s)} = \frac{1}{2}\begin{vmatrix} 0 & \sqrt{3} & 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{vmatrix} \quad S_2^{(s)} = \frac{i}{2}\begin{vmatrix} 0 & -\sqrt{3} & 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 & 0 & 0 \\ 0 & 2 & 0 & -\sqrt{3} & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{vmatrix} \quad S_3^{(s)} = \frac{1}{2}\begin{vmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{vmatrix}$$

$$S_1^{(t)} = \frac{1}{\sqrt{2}}\begin{vmatrix} 0 & 0 & 0 & 0 & -\sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{3} \\ -\sqrt{3} & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & \sqrt{3} & 0 & 0 \end{vmatrix} \quad S_2^{(t)} = \frac{i}{\sqrt{2}}\begin{vmatrix} 0 & 0 & 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{3} \\ -\sqrt{3} & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -\sqrt{3} & 0 & 0 \end{vmatrix} \quad S_3^{(t)} = \sqrt{2}\begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{vmatrix}$$

$$B = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{vmatrix} \quad I = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

TABLE I *The matrix operators* $S_i^{(s)}$, $S_i^{(t)}$, B, *and* I *corresponding to fields of spin* $\frac{1}{2}$, *1*, $\frac{3}{2}$, *and 2*—Continued

$$S_1^{(s)} = \frac{1}{\sqrt{2}} \begin{vmatrix} 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & \sqrt{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{3} & 0 & \sqrt{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{vmatrix}$$

$$S_2^{(s)} = \frac{i}{\sqrt{2}} \begin{vmatrix} 0 & -\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & -\sqrt{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{3} & 0 & -\sqrt{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & -\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{vmatrix}$$

$$S_3^{(s)} = \begin{vmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{vmatrix}$$

$$S_1^{(t)} = \frac{1}{\sqrt{2}} \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & -\sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{6} \\ -\sqrt{6} & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\sqrt{3} & 0 & \sqrt{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & \sqrt{6} & 0 & 0 & 0 \end{vmatrix}$$

$$S_2^{(t)} = \frac{i}{\sqrt{2}} \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{6} \\ -\sqrt{6} & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\sqrt{3} & 0 & -\sqrt{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -\sqrt{6} & 0 & 0 & 0 \end{vmatrix}$$

$$S_3^{(t)} = \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} & 0 & 0 & 0 & 0 \end{vmatrix}$$

$$B = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{vmatrix}$$

$$I = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

TABLE II *Rotation and boost operators for fields of spin* ½, *1*, $\frac{3}{2}$, *and 2. Included are the hyperbolic functions involving the parameter* Ω *in terms of of* p, E, *and* m. *Included also are the projection operators for the polarization component* $|\sigma|$, *denoted by* $\mathfrak{P}_\sigma$

$$D^{(\frac{1}{2})}_{rot}(\vartheta_{ij}) = \cos\tfrac{1}{2}\vartheta_{ij} + i\,2\,S^{(\varphi)}_{ij}\sin\tfrac{1}{2}\vartheta_{ij} \tag{II.1}$$

$$D^{(1)}_{rot}(\vartheta_{ij}) = \left[1 - \left(S^{(\varphi)}_{ij}\right)^2\right] + \left(S^{(\varphi)}_{ij}\right)^2\left[\cos\vartheta_{ij} + i\,S^{(\varphi)}_{ij}\sin\vartheta_{ij}\right] \tag{II.2}$$

$$\begin{aligned} D^{(\frac{3}{2})}_{rot}(\vartheta_{ij}) &= \tfrac{1}{2}\left[\tfrac{9}{4} - \left(S^{(\varphi)}_{ij}\right)^2\right]\left[\cos\tfrac{1}{2}\vartheta_{ij} + i\,2\,S^{(\varphi)}_{ij}\sin\tfrac{1}{2}\vartheta_{ij}\right] \\ &+ \tfrac{1}{2}\left[\left(S^{(\varphi)}_{ij}\right)^2 - \tfrac{1}{4}\right]\left[\cos\tfrac{3}{2}\vartheta_{ij} + i\tfrac{2}{3}S^{(\varphi)}_{ij}\sin\tfrac{3}{2}\vartheta_{ij}\right] \end{aligned} \tag{II.3}$$

$$\begin{aligned} D^{(2)}_{rot}(\vartheta_{ij}) &= 1 - \tfrac{5}{4}\left(S^{(\varphi)}_{ij}\right)^2 + \tfrac{1}{4}\left(S^{(\varphi)}_{ij}\right)^4 \\ &+ \tfrac{1}{3}\left[4\left(S^{(\varphi)}_{ij}\right)^2 - \left(S^{(\varphi)}_{ij}\right)^4\right]\left[\cos\vartheta_{ij} + i\,S^{(\varphi)}_{ij}\sin\vartheta_{ij}\right] \\ &- \tfrac{1}{12}\left[\left(S^{(\varphi)}_{ij}\right)^2 - \left(S^{(\varphi)}_{ij}\right)^4\right]\left[\cos 2\vartheta_{ij} + \tfrac{i}{2}S^{(\varphi)}_{ij}\sin 2\vartheta_{ij}\right] \end{aligned} \tag{II.4}$$

$$D^{(\frac{1}{2})}_{boost}(\Omega_{k4}) = \cosh\tfrac{1}{2}\Omega_{k4} - 2S_{k4}\sinh\tfrac{1}{2}\Omega_{k4} \tag{II.5}$$

$$D^{(1,\chi)}_{boost}(\Omega_{k4}) = \left[1 - \left(S^{(\chi)}_{k4}\right)^2\right] + \left(S^{(\chi)}_{k4}\right)^2\left[\cosh\Omega_{k4} - S^{(\chi)}_{k4}\sinh\Omega_{k4}\right] \tag{II.6}$$

$$D^{(1,\varphi)}_{boost}(\Omega_{k4}) = \left[1 - \left(S^{(\varphi)}_{k4}\right)^2\right] + \left(S^{(\varphi)}_{k4}\right)^2\left[\cosh\Omega_{k4} - S^{(\varphi)}_{k4}\sinh\Omega_{k4}\right] \tag{II.7}$$

$$\begin{aligned} D^{(\frac{3}{2},\chi)}_{boost}(\Omega_{k4}) &= \tfrac{1}{2}\left[\tfrac{9}{4} - \left(S^{(\chi)}_{k4}\right)^2\right]\left[\cosh\tfrac{1}{2}\Omega_{k4} - 2\,S^{(\chi)}_{k4}\sinh\tfrac{1}{2}\Omega_{k4}\right] \\ &+ \tfrac{1}{2}\left[\left(S^{(\chi)}_{k4}\right)^2 - \tfrac{1}{4}\right]\left[\cosh\tfrac{3}{2}\Omega_{k4} - \tfrac{2}{3}S^{(\chi)}_{k4}\sinh\tfrac{3}{2}\Omega_{k4}\right] \end{aligned} \tag{II.8}$$

$$\begin{aligned} D^{(\frac{3}{2},\varphi)}_{boost}(\Omega_{k4}) &= \left[\cosh\Omega_{k4} - \sigma_{k4}\sinh\Omega_{k4}\right] \\ &\left\{\tfrac{1}{2}\left[\tfrac{9}{4} - \left(S^{(\chi)}_{k4}\right)^2\right]\left[\cosh\tfrac{1}{2}\Omega_{k4} + 2\,S^{(\chi)}_{k4}\sinh\tfrac{1}{2}\Omega_{k4}\right]\right. \\ &\left. + \tfrac{1}{2}\left[\left(S^{(\chi)}_{k4}\right)^2 - \tfrac{1}{4}\right]\left[\cosh\tfrac{3}{2}\Omega_{k4} + \tfrac{2}{3}S^{(\chi)}_{k4}\sinh\tfrac{3}{2}\Omega_{k4}\right]\right\} \end{aligned} \tag{II.9}$$

TABLE II *Rotation and boost operators for fields of spin* ½, *1,* $\frac{3}{2}$, *and 2. Included are the hyperbolic functions involving the parameter* Ω *in terms of* p, E, *and* m. *Included also are the projection operators for the polarization component* | σ |, *denoted by* $\mathfrak{P}_\sigma$—Continued

$$D^{(2,\chi)}_{boost}(\Omega_{k4}) = \left[1 - \frac{5}{4}\left(S^{(\chi)}_{k4}\right)^2 + \frac{1}{4}\left(S^{(\chi)}_{k4}\right)^4\right]$$

$$+ \frac{1}{3}\left[4\left(S^{(\chi)}_{k4}\right)^2 - \left(S^{(\chi)}_{k4}\right)^4\right]\left[\cosh\Omega_{k4} - S^{(\chi)}\sinh\Omega_{k4}\right]$$

$$- \frac{1}{12}\left[\left(S^{(\chi)}_{k4}\right)^2 - \left(S^{(\chi)}_{k4}\right)^4\right]\left[\cosh 2\Omega_{k4} - \frac{1}{2}S^{(\chi)}_{k4}\, 2\,\Omega_{k4}\right] \qquad \text{(II.10)}$$

$$D^{(2,\varphi)}_{boost}(\Omega_{k4}) = \left[\cosh\Omega_{k4} - \sigma_{k4}\sinh\Omega_{k4}\right]\cdot\left\{\left[1 - \frac{5}{4}\left(S^{(\chi)}_{k4}\right)^2 + \frac{1}{4}\left(S^{(2)}_{k4}\right)^4\right]\right.$$

$$+ \frac{1}{3}\left[4\left(S^{(\chi)}_{k4}\right)^2 - \left(S^{(\chi)}_{k}\right)^4\right]\left[\cosh\Omega_{k4} + S^{(\chi)}_{k4}\sinh\Omega_{k4}\right]$$

$$\left. - \frac{1}{12}\left[\left(S^{(\chi)}_{k4}\right)^2 - \left(S^{(\chi)}_{k4}\right)^4\right]\left[\cosh 2\Omega_{k4} + \frac{1}{2}S^{(\chi)}_{k4}\sinh 2\,\Omega_{k4}\right]\right\} \qquad \text{(II.11)}$$

with

$$\Omega_{k4} = \tanh^{-1}\frac{p_k}{E}$$

$$\cosh\frac{\Omega}{2} = \sqrt{\frac{E+m}{2m}} \qquad \sinh\frac{\Omega}{2} = \sqrt{\frac{E+m}{2m}}\,\frac{p}{E+m}$$

$$\cosh\Omega = \frac{E}{m} \qquad \sinh\Omega = \frac{p}{m}$$

$$\cosh\frac{3}{2}\Omega = \sqrt{\frac{E+m}{2m}}\,\frac{2E-m}{m} \qquad \sinh\frac{3}{2}\Omega = \sqrt{\frac{E+m}{2m}}\,\frac{2E+m}{m}\,\frac{p}{E+m}$$

$$\cosh 2\,\Omega = \frac{2E^2 - m^2}{m^2} \qquad \sinh 2\,\Omega = \frac{2Ep}{m^2}$$

Spin 1/2:

$$\mathfrak{P}_{\frac{1}{2}} = 1$$

Spin 1:

$$\mathfrak{P}_0 = 1 - \left(S^{(\varphi)}_{12}\right)^2$$

$$\mathfrak{P}_1 = \left(S^{(\varphi)}_{12}\right)^2$$

Spin 3/2:

$$\mathfrak{P}_{\frac{1}{2}} = \frac{1}{2}\left[\frac{9}{4} - \left(S^{(\varphi)}_{12}\right)^2\right]$$

$$\mathfrak{P}_{\frac{3}{2}} = -\frac{1}{2}\left[\frac{1}{4} - \left(S^{(\varphi)}_{12}\right)^2\right]$$

Spin 2:

$$\mathfrak{P}_0 = 1 + \frac{1}{4}\left(S^{(\varphi)}_{12}\right)^4 - \frac{5}{4}\left(S^{(\varphi)}_{12}\right)^2$$

$$\mathfrak{P}_1 = -\frac{1}{3}\left[\left(S^{(\varphi)}_{12}\right)^4 - 4\left(S^{(\varphi)}_{12}\right)^2\right]$$

$$\mathfrak{P}_2 = \frac{1}{12}\left[\left(S^{(\varphi)}_{12}\right)^4 - \left(S^{(\varphi)}_{12}\right)^2\right]$$

TABLE III *The matrix operators $S_i^{(s)\prime}$, $S_i^{(t)\prime}$, B', and I' corresponding to a field of spin 1 in a cartesian representation*

$$S_1^{(s)\prime} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} \qquad S_2^{(s)\prime} = \begin{vmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} \qquad S_3^{(s)\prime} = \begin{vmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

$$S_1^{(t)\prime} = \begin{vmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{vmatrix} \qquad S_2^{(t)\prime} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{vmatrix} \qquad S_3^{(t)\prime} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{vmatrix}$$

$B' = B$ $\qquad$ $I' = I$

Table IV *Plane wave, positive energy spinors* $u^{(r)}(\boldsymbol{p})$ *for fields of spin* $\frac{1}{2}$, *1*, $\frac{3}{2}$, *and 2 with a nonzero mass*

Spin 1/2:

$$u^{(1)}(p) = \sqrt{\frac{E+m}{2m}} \begin{vmatrix} 1 \\ 0 \\ \frac{p}{E+m} \\ 0 \end{vmatrix} \qquad u^{(2)}(p) = \sqrt{\frac{E+m}{2m}} \begin{vmatrix} 0 \\ 1 \\ 0 \\ \frac{-p}{E+m} \end{vmatrix}$$

Spin 1:

$$u^{(1)}(p) = \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix} \qquad u^{(2)}(p) = \frac{E}{m} \begin{vmatrix} 0 \\ 1 \\ 0 \\ \frac{p}{E} \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix} \qquad u^{(3)}(p) = \begin{vmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix} \qquad u^{(4)}(p) = \frac{E}{m} \begin{vmatrix} 0 \\ \frac{p}{E} \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix}$$

Spin 3/2:

$$u^{(1)}(p) = \sqrt{\frac{E+m}{2m}} \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{-p}{E+m} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix} \qquad u^{(2)}(p) = \sqrt{\frac{E+m}{2m}} \frac{1}{3m} \begin{vmatrix} 0 \\ 2E+m \\ 0 \\ 0 \\ 2\sqrt{2}p \\ 0 \\ 0 \\ \frac{-(2E-m)p}{E+m} \\ 0 \\ 0 \\ -2\sqrt{2}(E-m) \\ 0 \end{vmatrix} \qquad u^{(3)}(p) = \sqrt{\frac{E+m}{2m}} \frac{1}{3m} \begin{vmatrix} 0 \\ 0 \\ 2E+m \\ 0 \\ 0 \\ 2\sqrt{2}p \\ 0 \\ 0 \\ \frac{(2E-m)p}{E+m} \\ 0 \\ 0 \\ 2\sqrt{2}(E-m) \end{vmatrix}$$

$$u^{(4)}(p) = \sqrt{\frac{E+m}{2m}} \begin{vmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{p}{E+m} \\ 0 \\ 0 \end{vmatrix} \qquad u^{(5)}(p) = \sqrt{\frac{E+m}{2m}} \frac{1}{3m} \begin{vmatrix} 0 \\ 2\sqrt{2}p \\ 0 \\ 0 \\ (4E-m) \\ 0 \\ 0 \\ -2\sqrt{2}(E-m) \\ 0 \\ 0 \\ \frac{-(4E+m)p}{E+m} \\ 0 \end{vmatrix} \qquad u^{(6)}(p) = \sqrt{\frac{E+m}{2m}} \frac{1}{3m} \begin{vmatrix} 0 \\ 0 \\ 2\sqrt{2}p \\ 0 \\ 0 \\ (4E-m) \\ 0 \\ 0 \\ 2\sqrt{2}(E-m) \\ 0 \\ 0 \\ \frac{(4E+m)p}{E+m} \end{vmatrix}$$

Spin 2:

$$u^{(1)}(p) = \frac{E}{m}\begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{p}{E} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix} \qquad u^{(2)}(p) = \frac{1}{2m^2}\begin{vmatrix} 0 \\ E^2+m^2 \\ 0 \\ 0 \\ 0 \\ \sqrt{3}pE \\ 0 \\ 0 \\ 0 \\ -pE \\ 0 \\ 0 \\ 0 \\ -\sqrt{3}p^2 \\ 0 \\ 0 \end{vmatrix} \qquad u^{(3)}(p) = \frac{E}{m}\begin{vmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ \frac{p}{E} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix} \qquad u^{(4)}(p) = \frac{1}{2m^2}\begin{vmatrix} 0 \\ 0 \\ 0 \\ E^2+m^2 \\ 0 \\ 0 \\ 0 \\ \sqrt{3}pE \\ 0 \\ 0 \\ 0 \\ pE \\ 0 \\ 0 \\ 0 \\ \sqrt{3}p^2 \end{vmatrix}$$

$$u^{(5)}(p) = \frac{E}{m}\begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{p}{E} \\ 0 \\ 0 \\ 0 \end{vmatrix} \qquad u^{(6)}(p) = \frac{1}{2m^2}\begin{vmatrix} 0 \\ \sqrt{3}pE \\ 0 \\ 0 \\ 0 \\ 3E^2-m^2 \\ 0 \\ 0 \\ 0 \\ -\sqrt{3}p^2 \\ 0 \\ 0 \\ 0 \\ -3pE \\ 0 \\ 0 \end{vmatrix} \qquad u^{(7)}(p) = \frac{E}{m}\begin{vmatrix} 0 \\ 0 \\ \frac{p}{E} \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix} \qquad u^{(8)}(p) = \frac{1}{2m^2}\begin{vmatrix} 0 \\ 0 \\ 0 \\ \sqrt{3}pE \\ 0 \\ 0 \\ 0 \\ 3E^2-m^2 \\ 0 \\ 0 \\ 0 \\ \sqrt{3}p^2 \\ 0 \\ 0 \\ 0 \\ 3pE \end{vmatrix}$$

Table v *Plane wave, positive energy, charge conjugate spinors* $v^{(r)}(\mathbf{p})$ *for fields of spin* $\frac{1}{2}$, *1*, $\frac{3}{2}$, *and 2 and their relation to the negative energy and momentum spinors* $u_{-}^{(s)}(-\mathbf{p})$ *with a nonzero mass*

Spin 1/2:

$$v^{(1)}(p) = \sqrt{\frac{E+m}{2m}} \left|\begin{matrix} 0 \\ \frac{p}{E+m} \\ 0 \\ -1 \end{matrix}\right| \qquad v^{(2)}(p) = \sqrt{\frac{E+m}{2m}} \left|\begin{matrix} \frac{p}{E+m} \\ 0 \\ 1 \\ 0 \end{matrix}\right|$$

Spin 1:

$$v^{(1)}(p) = \left|\begin{matrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix}\right| \qquad v^{(2)}(p) = \frac{E}{m} \left|\begin{matrix} 0 \\ -1 \\ 0 \\ -\frac{p}{E} \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix}\right| \qquad v^{(3)}(p) = \left|\begin{matrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix}\right| \qquad v^{(4)}(p) = \frac{E}{m} \left|\begin{matrix} 0 \\ -\frac{p}{E} \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix}\right|$$

Spin 3/2:

$$v^{(1)}(p) = \sqrt{\frac{E+m}{2m}} \left|\begin{matrix} 0 \\ 0 \\ 0 \\ \frac{-p}{E+m} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{matrix}\right| \qquad v^{(2)}(p) = \sqrt{\frac{E+m}{2m}}\,\frac{1}{3m} \left|\begin{matrix} 0 \\ 0 \\ \frac{(2E-m)p}{E+m} \\ 0 \\ 0 \\ 2\sqrt{2}(E-m) \\ 0 \\ 0 \\ (2E+m) \\ 0 \\ 0 \\ 2\sqrt{2}p \end{matrix}\right| \qquad v^{(3)}(p) = \sqrt{\frac{E+m}{2m}}\,\frac{1}{3m} \left|\begin{matrix} 0 \\ \frac{(2E-m)p}{E+m} \\ 0 \\ 0 \\ 2\sqrt{2}(E-m) \\ 0 \\ 0 \\ -(2E+m) \\ 0 \\ 0 \\ -2\sqrt{2}p \\ 0 \end{matrix}\right|$$

$$v^{(4)}(p) = \sqrt{\frac{E+m}{2m}} \left|\begin{matrix} \frac{-p}{E+m} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix}\right| \qquad v^{(5)}(p) = \sqrt{\frac{E+m}{2m}}\,\frac{1}{3m} \left|\begin{matrix} 0 \\ 0 \\ 2\sqrt{2}(E-m) \\ 0 \\ 0 \\ \frac{(4E+m)p}{E+m} \\ 0 \\ 0 \\ 2\sqrt{2}p \\ 0 \\ 0 \\ (4E-m) \end{matrix}\right| \qquad v^{(6)}(p) = \sqrt{\frac{E+m}{2m}}\,\frac{1}{3m} \left|\begin{matrix} 0 \\ 2\sqrt{2}(E-m) \\ 0 \\ 0 \\ \frac{(4E+m)p}{E+m} \\ 0 \\ 0 \\ -2\sqrt{2}p \\ 0 \\ 0 \\ -(4E-m) \\ 0 \end{matrix}\right|$$

TABLE V *Plane wave, positive energy, charge conjugate spinors* $v^{(r)}(\boldsymbol{p})$ *for fields of spin* $\frac{1}{2}$, *1*, $\frac{3}{2}$, *and 2 and their relation to the negative energy and momentum spinors* $u_-^{(s)}(-\boldsymbol{p})$ *with a nonzero mass*—Continued

Spin 2:

$$v^{(1)}(p) = \frac{E}{m}\begin{pmatrix} 0\\0\\0\\0\\1\\0\\0\\0\\0\\0\\0\\0\\ \frac{p}{E}\\0\\0\\0 \end{pmatrix} \qquad v^{(2)}(p) = -\frac{1}{2m^2}\begin{pmatrix} 0\\0\\0\\E^2+m^2\\0\\0\\0\\\sqrt{3}pE\\0\\0\\0\\pE\\0\\0\\0\\\sqrt{3}p^2 \end{pmatrix} \qquad v^{(3)}(p) = \frac{E}{m}\begin{pmatrix} 0\\0\\1\\0\\0\\0\\ \frac{p}{E}\\0\\0\\0\\0\\0\\0\\0\\0\\0 \end{pmatrix} \qquad v^{(4)}(p) = -\frac{1}{2m^2}\begin{pmatrix} 0\\E^2+m^2\\0\\0\\0\\\sqrt{3}pE\\0\\0\\0\\-pE\\0\\0\\0\\-\sqrt{3}p^2\\0\\0 \end{pmatrix}$$

$$v^{(5)}(p) = \frac{E}{m}\begin{pmatrix} 1\\0\\0\\0\\0\\0\\0\\0\\-\frac{p}{E}\\0\\0\\0\\0\\0\\0\\0 \end{pmatrix} \qquad v^{(6)}(p) = -\frac{1}{2m^2}\begin{pmatrix} 0\\0\\0\\\sqrt{3}pE\\0\\0\\0\\3E^2-m^2\\0\\0\\0\\\sqrt{3}p^2\\0\\0\\0\\3pE \end{pmatrix} \qquad v^{(7)}(p) = \frac{E}{m}\begin{pmatrix} 0\\0\\ \frac{p}{E}\\0\\0\\0\\1\\0\\0\\0\\0\\0\\0\\0\\0\\0 \end{pmatrix} \qquad v^{(8)}(p) = -\frac{1}{2m^2}\begin{pmatrix} 0\\\sqrt{3}pE\\0\\0\\0\\3E^2-m^2\\0\\0\\0\\-\sqrt{3}p^2\\0\\0\\0\\-3pE\\0\\0 \end{pmatrix}$$

$$v^{(r)}(p) = (-1)^{2s}\left(Bd(\pi)\right)_{rr'}\, u_-^{(r')}(-p)$$

TABLE VI *Plane wave, positive energy spinors $w^{(r)}(\boldsymbol{p})$ for fields of spin $\frac{1}{2}$, 1, $\frac{3}{2}$, and 2 with a nonzero mass*

Spin 1/2:

$$w^{(1)}(p) = -m\,u^{(1)}(p) \qquad w^{(2)}(p) = -m\,u^{(2)}(p)$$

Spin 1:

$$w^{(1)}(p) = -E\begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \frac{p}{E} \\ 0 \\ 0 \\ 0 \end{vmatrix} \qquad w^{(2)}(p) = -m\begin{vmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix} \qquad w^{(3)}(p) = -E\begin{vmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ -\frac{p}{E} \\ 0 \end{vmatrix} \qquad w^{(4)}(p) = m\begin{vmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix}$$

Spin 3/2:

$$w^{(1)}(p) = -\sqrt{\frac{E+m}{2m}}\begin{vmatrix} 2E-m \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{(2E+m)p}{E+m} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix} \qquad w^{(2)}(p) = -\sqrt{\frac{E+m}{2m}}\,m\begin{vmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{p}{E+m} \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix} \qquad w^{(3)}(p) = -\sqrt{\frac{E+m}{2m}}\,m\begin{vmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{p}{E+m} \\ 0 \\ 0 \\ 0 \end{vmatrix}$$

$$w^{(4)}(p) = -\sqrt{\frac{E+m}{2m}}\begin{vmatrix} 0 \\ 0 \\ 0 \\ 2E-m \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{-(2E+m)p}{E+m} \\ 0 \\ 0 \end{vmatrix} \qquad w^{(5)}(p) = \sqrt{\frac{E+m}{2m}}\,m\begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{p}{E+m} \\ 0 \end{vmatrix} \qquad w^{(6)}(p) = \sqrt{\frac{E+m}{2m}}\,m\begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{p}{E+m} \end{vmatrix}$$

TABLE VI *Plane wave, positive energy spinors* $w^{(r)}(\mathbf{p})$ *for fields of spin* $\frac{1}{2}$, *1*, $\frac{3}{2}$, *and 2 with a nonzero mass*—Continued

Spin 2:

$$w^{(1)}(p) = -\frac{1}{m}\begin{vmatrix} 2E^2-m^2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 2Ep \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix} \qquad w^{(2)}(p) = -\begin{vmatrix} 0 \\ E \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ p \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix} \qquad w^{(3)}(p) = -\begin{vmatrix} 0 \\ 0 \\ m \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix} \qquad w^{(4)}(p) = -\begin{vmatrix} 0 \\ 0 \\ 0 \\ E \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -p \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix}$$

$$w^{(5)}(p) = -\frac{1}{m}\begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2E^2-m^2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -2Ep \\ 0 \\ 0 \\ 0 \end{vmatrix} \qquad w^{(6)}(p) = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ E \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ p \\ 0 \\ 0 \end{vmatrix} \qquad w^{(7)}(p) = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ m \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix} \qquad w^{(8)}(p) = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ E \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -p \end{vmatrix}$$

TABLE VII *Plane wave, positive energy, charge conjugate spinors* $y^{(r)}(\boldsymbol{p})$ *for fields of spin* $\frac{1}{2}$, *1*, $\frac{3}{2}$, *and 2 and their relation to the negative energy and momentum spinors* $w_{-}^{(s)}(-\boldsymbol{p})$ *with a nonzero mass*

Spin 1/2:

$$y^{(1)}(p) = -m\, v^{(1)}(p) \qquad y^{(2)}(p) = -m\, v^{(2)}(p)$$

Spin 1:

$$y^{(1)}(p) = E \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ -\frac{p}{E} \\ 0 \end{pmatrix} \qquad y^{(2)}(p) = -m \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad y^{(3)}(p) = E \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \frac{p}{E} \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad y^{(4)}(p) = m \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Spin 3/2:

$$y^{(1)}(p) = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{-(2E+m)p}{E+m} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ (2E-m) \\ 0 \\ 0 \end{pmatrix} \qquad y^{(2)}(p) = -\sqrt{\frac{E+m}{2m}}\, m \begin{pmatrix} 0 \\ 0 \\ \frac{-p}{E+m} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad y^{(3)}(p) = \sqrt{\frac{E+m}{2m}}\, m \begin{pmatrix} 0 \\ \frac{p}{E+m} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$y^{(4)}(p) = -\sqrt{\frac{E+m}{2m}} \begin{pmatrix} \frac{(2\ +m)p}{E+m} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ (2E-m) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad y^{(5)}(p) = \sqrt{\frac{E+m}{2m}}\, m \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{p}{E+m} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad y^{(6)}(p) = -\sqrt{\frac{E+m}{2m}}\, m \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{p}{E+m} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

TABLE VII *Plane wave, positive energy, charge conjugate spinors* $y^{(r)}(\boldsymbol{p})$ *for fields of spin* $\frac{1}{2}$, *1*, $\frac{3}{2}$, *and 2 and their relation to the negative energy and momentum spinors* $w_-^{(s)}(-\boldsymbol{p})$ *with a nonzero mass*—Continued

Spin 2:

$$y^{(1)}(p) = \frac{1}{m}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2E^2-m^2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -2Ep \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad y^{(2)}(p) = -\begin{pmatrix} 0 \\ 0 \\ 0 \\ E \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -p \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad y^{(3)}(p) = \begin{pmatrix} 0 \\ 0 \\ m \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad y^{(4)}(p) = -\begin{pmatrix} 0 \\ E \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ p \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$y^{(5)}(p) = \frac{1}{m}\begin{pmatrix} 2E^2-m^2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 2Ep \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad y^{(6)}(p) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ E \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -p \end{pmatrix} \qquad y^{(7)}(p) = -\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ m \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad y^{(8)}(p) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ E \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ p \\ 0 \\ 0 \end{pmatrix}$$

$$y^{(r)}(p) = (-1)^{2s}\left(Bd(\pi)\right)_{rr'}\, w_-^{(r')}(-p)$$

Table VIII *Completeness and orthogonality relations involving spinors of the spin ½ field with a nonzero mass*

$$\bar{u}^{(r)}(\underline{p})u^{(s)}(\underline{p}) = -\bar{v}^{(r)}(\underline{p})v^{(s)}(\underline{p}) = \delta_{rs} \quad \text{(VIII.1)}$$

$$\bar{u}^{(r)}(\underline{p})\gamma_4 u^{(s)}(\underline{p}) = \bar{v}^{(r)}(\underline{p})\gamma_4 v^{(s)}(\underline{p}) = \frac{E}{m}\delta_{rs} \quad \text{(VIII.2)}$$

$$\bar{u}^{(r)}(\underline{p})v^{(s)}(\underline{p}) = \bar{v}^{(r)}(\underline{p})u^{(s)}(\underline{p}) = 0 \quad \text{(VIII.3)}$$

$$\sum_{r=1}^{2} u^{(r)}(\underline{p})\bar{u}^{(r)}(\underline{p}) = \frac{1}{2}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \gamma_4\left[\frac{E + \gamma_5(\underline{M}^{(\chi)}, \underline{p})}{2m}\right] = \frac{-i\gamma_\mu p_\mu + m}{2m} \quad \text{(VIII.4)}$$

$$\sum_{r=1}^{2} v^{(r)}(\underline{p})\bar{v}^{(r)}(\underline{p}) = -\frac{1}{2}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \gamma_4\left[\frac{E + \gamma_5(\underline{M}^{(\chi)}, \underline{p})}{2m}\right] = \frac{-i\gamma_\mu p_\mu - m}{2m} \quad \text{(VIII.5)}$$

$$\sum_{r=1}^{2}\left(u^{(r)}(\underline{p})\bar{u}^{(r)}(\underline{p}) - v^{(r)}(\underline{p})\bar{v}^{(r)}(\underline{p})\right) = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \quad \text{(VIII.6)}$$

TABLE IX *Completeness and orthogonality relations involving spinors of the spin 1 field with a nonzero mass*

$$\bar{u}^{(r)}(\underline{p})u^{(s)}(\underline{p}) = \bar{v}^{(r)}(\underline{p})v^{(s)}(\underline{p}) = \delta_{rs} \qquad \text{(IX.1)}$$

$$\bar{w}^{(r)}(\underline{p})\gamma_4\, u^{(s)}(\underline{p}) = -\bar{y}^{(r)}(\underline{p})\gamma_4\, v^{(s)}(\underline{p}) = -E\,\delta_{rs} \qquad \text{(IX.2)}$$

$$\bar{w}^{(r)}(\underline{p})w^{(s)}(\underline{p}) = \bar{y}^{(r)}(\underline{p})y^{(s)}(\underline{p}) = -p_\mu p_\mu\,\delta_{rs} \qquad \text{(IX.3)}$$

$$\bar{u}^{(r)}(\underline{p})v^{(s)}(\underline{p}) = \bar{v}^{(r)}(\underline{p})u^{(s)}(\underline{p}) = \big(Bd(\pi)\big)_{rs} \qquad \text{(IX.4)}$$

$$\sum_{r=1}^{4} u^{(r)}(\underline{p})\bar{u}^{(r)}(\underline{p}) = \sum v^{(r)}(\underline{p})\bar{v}^{(s)}(\underline{p}) = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{(IX.5)}$$

$$\sum_{r=1}^{4} w^{(r)}(\underline{p})\bar{w}^{(r)}(\underline{p}) = -p_\mu p_\mu\left\{\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} + \gamma_4\left[\frac{\big(\underline{N}^{(\chi)}\cdot\underline{p}\big)^2 - E\big(\underline{N}^{(\chi)}\cdot\underline{p}\big)}{m^2}\right]\right\} \qquad \text{(IX.6)}$$

$$\sum_{r=1}^{3} u^{(r)}(\underline{p})\bar{u}^{(r)}(\underline{p}) = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\left\{\frac{1+\gamma_4}{2} + \gamma_4\left[\frac{\big(\underline{N}^{(\varphi)}\cdot\underline{p}\big)^2 - E\big(\underline{N}^{(\varphi)}\cdot\underline{p}\big)}{m^2}\right]\right\} \qquad \text{(IX.7)}$$

$$\sum_{r=1}^{3} w^{(r)}(\underline{p})\bar{w}^{(r)}(\underline{p}) = \frac{1}{2}\begin{pmatrix} I+B & 0 \\ 0 & I+B \end{pmatrix}\sum_{r=1}^{4} w^{(r)}(\underline{p})\,\bar{w}^{(r)}(\underline{p}) \qquad \text{(IX.8)}$$

In Cartesian Coordinates

$$\sum_{r=1}^{4} u_\alpha^{(r)}(\underline{p})\bar{u}_\beta^{(r)}(\underline{p}) = \delta_{\alpha\beta} \qquad \text{(IX.9)}$$

$$\sum_{r=1}^{3} u_\alpha^{(r)}(\underline{p})\bar{u}_\beta^{(r)}(\underline{p}) = \delta_{\alpha\beta} + \frac{p_\alpha p_\beta}{m^2} \qquad \text{(IX.10)}$$

TABLE X *Completeness and orthogonality relations involving spinors of the spin $\frac{3}{2}$ field with a nonzero mass*

$$\bar{u}^{(r)}(\underline{p})u^{(s)}(\underline{p}) = -\bar{v}^{(r)}(\underline{p})\,v^{(s)}(\underline{p}) = \delta_{rs} \tag{X.1}$$

$$\bar{w}^{(r)}(\underline{p})\,\gamma_4\,u^{(s)}(\underline{p}) = \bar{y}^{(r)}(\underline{p})\,\gamma_4\,v^{(s)}(\underline{p}) = -E\,\delta_{rs} \tag{X.2}$$

$$\bar{w}^{(r)}(\underline{p})w^{(s)}(\underline{p}) = -\bar{y}^{(r)}(\underline{p})\,y^{(s)}(\underline{p}) = -p_\mu p_\mu\,\delta_{rs} \tag{X.3}$$

$$\bar{u}^{(r)}(\underline{p})\,v^{(s)}(\underline{p}) = \bar{v}^{(r)}(\underline{p})\,u^{(s)}(\underline{p}) = 0 \tag{X.4}$$

$$\sum_{r=1}^{6} u^{(r)}(\underline{p})\bar{u}^{(r)}(\underline{p}) = \frac{1}{2}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\left(\frac{E + 2\underline{N}^{(\chi)}\cdot\underline{p}}{2m}\right) - \gamma_4\left(\frac{2\underline{N}^{(\chi)}\cdot\underline{p}}{3m}\right) + \frac{1}{6}\left(\frac{(2\underline{N}^{(\varphi)}\cdot\underline{p})^2 - \underline{p}^2}{m^2}\right)\left(\frac{i\gamma_\mu p_\mu}{2m}\right) \tag{X.5}$$

$$\sum_{r=1}^{6} v^{(r)}(\underline{p})\bar{v}^{(r)}(\underline{p}) = -\frac{1}{2}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\left(\frac{E + 2\underline{N}^{(\chi)}\cdot\underline{p}}{2m}\right) - \gamma_4\left(\frac{2\underline{N}^{(\chi)}\cdot\underline{p}}{3m}\right) + \frac{1}{6}\left(\frac{(2\underline{N}^{(\varphi)}\cdot\underline{p})^2 - \underline{p}^2}{m^2}\right)\left(\frac{i\gamma_\mu p_\mu}{2m}\right) \tag{X.6}$$

$$\sum_{r=1}^{6}\left(u^{(r)}(\underline{p})\bar{u}^{(r)}(\underline{p}) - v^{(r)}(\underline{p})\bar{v}^{(r)}(\underline{p})\right) = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \tag{X.7}$$

$$\sum_{r=1}^{6} w^{(r)}(\underline{p})\bar{w}^{(r)}(\underline{p}) = -p_\mu p_\mu\left\{\frac{1}{2}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\left[\frac{E - 2\underline{N}^{(\chi)}\cdot\underline{p}}{2m} + \frac{1}{6}\left(\frac{(2\underline{N}^{(\chi)}\cdot\underline{p})^2 - \underline{p}^2}{m^2}\right)\left(\frac{3E - 2\underline{N}^{(\chi)}\cdot\underline{p}}{2m}\right)\right]\right\} \tag{X.8}$$

$$\sum_{r=1}^{6} y^{(r)}(\underline{p})\bar{y}^{(r)}(\underline{p}) = -p_\mu p_\mu\left\{-\frac{1}{2}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\left[\frac{E - 2\underline{N}^{(\chi)}\cdot\underline{p}}{2m} + \frac{1}{6}\left(\frac{(2\underline{N}^{(\chi)}\cdot\underline{p})^2 - \underline{p}^2}{m^2}\right)\left(\frac{3E - 2\underline{N}^{(\chi)}\cdot\underline{p}}{2m}\right)\right]\right\} \tag{X.9}$$

$$\sum_{r=1}^{4} w^{(r)}(\underline{p})\bar{w}^{(r)}(\underline{p}) = \frac{1}{2}\begin{pmatrix} I+B & 0 \\ 0 & I+B \end{pmatrix}\sum_{r=1}^{6} w^{(r)}(\underline{p})\bar{w}^{(r)}(\underline{p}) \tag{X.10}$$

$$\sum_{r=1}^{4} y^{(r)}(\underline{p})\bar{y}^{(r)}(\underline{p}) = \frac{1}{2}\begin{pmatrix} I+B & 0 \\ 0 & I+B \end{pmatrix}\sum_{r=1}^{6} y^{(r)}(\underline{p})\bar{y}^{(r)}(\underline{p}) \tag{X.11}$$

TABLE XI *Completeness and orthogonality relations involving spinors of the spin 2 field with a nonzero mass*

$$\bar{u}^{(r)}(\underline{p})\, u^{(s)}(\underline{p}) = \bar{v}^{(r)}(\underline{p})\, v^{(s)}(\underline{p}) = \delta_{rs} \tag{XI.1}$$

$$\bar{w}^{(r)}(\underline{p})\, \gamma_4\, u^{(s)}(\underline{p}) = -\bar{y}^{(r)}(\underline{p})\, \gamma_4\, v^{(s)}(\underline{p}) = -E\, \delta_{rs} \tag{XI.2}$$

$$\bar{w}^{(r)}(\underline{p})\, w^{(s)}(\underline{p}) = \bar{y}^{(r)}(\underline{p})\, y^{(s)}(\underline{p}) = -p_\mu p_\mu\, \delta_{rs} \tag{XI.3}$$

$$\bar{u}^{(r)}(\underline{p})\, v^{(s)}(\underline{p}) = \bar{v}^{(r)}(\underline{p})\, u^{(s)}(\underline{p}) = \left(Bd(\pi)\right)_{rs} \tag{XI.4}$$

$$\sum_{r=1}^{8} u^{(r)}(\underline{p})\,\bar{u}^{(r)}(\underline{p}) = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\left(\frac{\left(\underline{N}^{(\chi)}\cdot\underline{p}\right)^2 + \left(\underline{N}^{(\chi)}\cdot\underline{p}\right)E}{m^2}\right) - \gamma_4\left(\frac{\left(\underline{N}^{(\varphi)}\cdot\underline{p}\right)^2 - \underline{p}^2 + 2\left(\underline{N}^{(\chi)}\cdot\underline{p}\right)E}{m^2}\right)$$
$$+ \frac{3}{2}\,\frac{\left(\underline{N}^{(\chi)}\cdot\underline{p}\right) i\gamma_\mu p_\mu}{m^2} + \left(\frac{4\left(\underline{N}^{(\chi)}\cdot\underline{p}\right)^2 - \left(\underline{N}^{(\chi)}\cdot\underline{p}\right)^4\left(\underline{p}^{-2}\right)}{3m^2}\right)\left(\frac{\left(\underline{N}^{(\varphi)}\cdot\underline{p}\right) i\gamma_\mu p_\mu}{m^2}\right) \tag{XI.5}$$

$$\sum_{r=1}^{8} v^{(r)}(\underline{p})\,\bar{v}^{(r)}(\underline{p}) = \sum_{r=1}^{8} u^{(r)}(\underline{p})\,\bar{u}^{(r)}(\underline{p}) \tag{XI.6}$$

$$\sum_{r=1}^{8} w^{(r)}(\underline{p})\,\bar{w}^{(r)}(\underline{p}) = -p_\mu p_\mu\left\{\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\left[\frac{\left(\underline{N}^{(\chi)}\cdot\underline{p}\right)^2 - \left(\underline{N}^{(\chi)}\cdot\underline{p}\right)E}{m^2}\right.\right.$$
$$\left.\left. + \frac{1}{3}\left(\frac{\left(\underline{N}^{(\chi)}\cdot\underline{p}\right)^2 - \underline{p}^2}{m^2}\right)\left(\frac{\left(\underline{N}^{(\chi)}\cdot\underline{p}\right)^2 - 2\left(\underline{N}^{(\chi)}\cdot\underline{p}\right)E}{m^2}\right)\right]\right\} \tag{XI.7}$$

$$\sum_{r=1}^{8} y^{(r)}(\underline{p})\,\bar{y}^{(r)}(\underline{p}) = \sum_{r=1}^{8} w^{(r)}(\underline{p})\,\bar{w}^{(r)}(\underline{p}) \tag{XI.8}$$

$$\sum_{r=1}^{5} w^{(r)}(\underline{p})\,\bar{w}^{(r)}(\underline{p}) = \frac{1}{2}\begin{pmatrix} (1+B) & 0 \\ 0 & (1+B) \end{pmatrix} \sum_{r=1}^{8} w^{(r)}(\underline{p})\,\bar{w}^{(r)}(\underline{p}) \tag{XI.9}$$

TABLE XII *Plane wave positive energy spinors* $z^{(r)}(\mathbf{p})$ *for the gauge fields corresponding to fields of spin 1,* $\frac{3}{2}$*, and 2 with a nonzero mass*

Spin 1:

$$z^{(4)}(p) = \frac{1}{m}\begin{vmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix}$$

Spin 3/2:

$$z^{(5)}(p) = \sqrt{\frac{E+m}{2m^3}}\begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{p}{E+m} \\ 0 \end{vmatrix} \qquad z^{(6)}(p) = \sqrt{\frac{E+m}{2m^3}}\begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{p}{E+m} \end{vmatrix}$$

Spin 2:

$$z^{(6)}(p) = \frac{E}{m^2}\begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{p}{E} \\ 0 \\ 0 \end{vmatrix} \qquad z^{(7)}(p) = \frac{1}{m}\begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix} \qquad z^{(8)}(p) = \frac{E}{m^2}\begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{p}{E} \end{vmatrix}$$

TABLE XIII *Plane wave positive energy spinors* $u^{(r)}(\mathbf{p})$ *and* $w^{(r)}(\mathbf{p})$ *for zero-mass fields of spin* $\frac{1}{2}$, *1*, $\frac{3}{2}$, *and 2 in the "physical gauges"*

Spin 1/2:

$$u^{(1)}(p) = \frac{1}{\sqrt{2}} \begin{vmatrix} 1 \\ 0 \\ +1 \\ 0 \end{vmatrix} \qquad u^{(2)}(p) = \frac{1}{\sqrt{2}} \begin{vmatrix} 0 \\ 1 \\ 0 \\ -1 \end{vmatrix} \qquad v^{(2)}(p) = \frac{1}{\sqrt{2}} \begin{vmatrix} +1 \\ 0 \\ 1 \\ 0 \end{vmatrix} \qquad v^{(1)}(p) = \frac{-1}{\sqrt{2}} \begin{vmatrix} 0 \\ -1 \\ 0 \\ 1 \end{vmatrix}$$

Spin 1:

$$u^{(1)}(p) = \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix} \qquad u^{(2)}(p) = \begin{vmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix} \qquad u^{(3)}(p) = \begin{vmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix} \qquad u^{(4)}(p) = \begin{vmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix}$$

$u^{(2)}(p) = u^{(4)}(p)$ as above, Lorentz gauge

$u^{(2)}(p) = u^{(4)}(p) = 0$ Transverse gauge

$$w^{(1)}(p) = -E \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{vmatrix} \qquad w^{(3)}(p) = -E \begin{vmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{vmatrix}$$

$w^{(2)}(p) = w^{(4)}(p) = 0$ in either gauge

TABLE XIII *Plane wave positive energy spinors* $u^{(r)}(\boldsymbol{p})$ *and* $w^{(r)}(\boldsymbol{p})$ *for zero-mass fields of spin* $\frac{1}{2}$, *1*, $\frac{3}{2}$, *and 2 in the "physical gauges"* — Continued

Spin 3/2:

$$u^{(1)}(p) = \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix} \qquad u^{(2)}(p) = \begin{vmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \sqrt{2} \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ -\sqrt{2} \\ 0 \end{vmatrix} \qquad u^{(3)}(p) = \begin{vmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ \sqrt{2} \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ \sqrt{2} \end{vmatrix} \qquad u^{(4)}(p) = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{vmatrix}$$

$$w^{(1)}(p) = -2E \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix} \qquad w^{(4)}(p) = -2E \begin{vmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{vmatrix}$$

$$\left.\begin{aligned} u^{(2)}(p) &= u^{(5)}(p) \\ u^{(3)}(p) &= u^{(6)}(p) \end{aligned}\right\} \quad \text{may be gauged away}$$

$$w^{(2)}(p) = w^{(5)}(p) = 0$$

$$w^{(3)}(p) = w^{(6)}(p) = 0$$

TABLE XIII *Plane wave positive energy spinors* $u^{(r)}(p)$ *and* $w^{(r)}(p)$ *for zero-mass fields of spin* $\frac{1}{2}$, *1*, $\frac{3}{2}$, *and 2 in the "physical gauges"*— Continued

Spin 2:

$$u^{(1)}(p) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad u^{(2)}(p) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ \sqrt{3} \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ -\sqrt{3} \\ 0 \\ 0 \end{pmatrix} \quad u^{(3)}(p) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad u^{(4)}(p) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ \sqrt{3} \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ \sqrt{3} \end{pmatrix} \quad u^{(5)}(p) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$w^{(1)}(p) = -2E \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad w^{(5)}(p) = -2E \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} u^{(2)}(p) &= u^{(6)}(p) \\ u^{(3)}(p) &= u^{(7)}(p) \\ u^{(4)}(p) &= u^{(8)}(p) \end{aligned}\right\} \quad \text{may be gauged away}$$

$$w^{(2)}(p) = w^{(6)}(p) = 0$$

$$w^{(3)}(p) = w^{(7)}(p) = 0$$

$$w^{(4)}(p) = w^{(8)}(p) = 0$$

TABLE XIV *Plane wave positive energy spinors for the zero-mass spin ½ parity nonconserving field*

$$\frac{1}{\sqrt{2}}(1+\gamma_5)u^{(1)}(p) = 0 \qquad \frac{1}{\sqrt{2}}(1+\gamma_5)u^{(2)}(p) = \begin{vmatrix} 0 \\ 1 \\ 0 \\ -1 \end{vmatrix}$$

$$\frac{1}{\sqrt{2}}(1+\gamma_5)v^{(1)}(p) = -\begin{vmatrix} 0 \\ -1 \\ 0 \\ 1 \end{vmatrix} \qquad \frac{1}{\sqrt{2}}(1+\gamma_5)v^{(2)}(p) = 0$$

$$\frac{1}{\sqrt{2}}(1+\gamma_5)\tilde{\bar{u}}^{(1)}(p) = \begin{vmatrix} 1 \\ 0 \\ -1 \\ 0 \end{vmatrix} \qquad \frac{1}{\sqrt{2}}(1+\gamma_5)\tilde{\bar{u}}^{(2)}(p) = 0$$

$$\frac{1}{\sqrt{2}}(1+\gamma_5)\tilde{\bar{v}}^{(1)}(p) = 0 \qquad \frac{1}{\sqrt{2}}(1+\gamma_5)\tilde{\bar{v}}^{(2)}(p) = -\begin{vmatrix} -1 \\ 0 \\ 1 \\ 0 \end{vmatrix}$$

Table XV *Plane wave positive energy spinors* $u^{(r)}(\boldsymbol{p})$ *for the zero-mass spin 1 field in the "nonphysical" covariant gauge*

$$u^{(1)}(p) = \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix} \qquad u^{(2)}(p) = \begin{vmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix} \qquad u^{(3)}(p) = \begin{vmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix} \qquad u^{(4)}(p) = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix}$$

$$w^{(1)}(p) = -E \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{vmatrix} \qquad w^{(2)}(p) = -E \begin{vmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix} \qquad w^{(3)}(p) = -E \begin{vmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{vmatrix} \qquad w^{(4)}(p) = E \begin{vmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix}$$

TABLE XVI *Some relationships between spinors in the indefinite metric*

$$u^{(r)}(-\underline{p}) = B_{rr'}\gamma_4 u^{(r')}(\underline{p}) \qquad \text{(XVI.1)}$$

$$v^{(r)}(-\underline{p}) = (-1)^{2s}B_{rr'}\gamma_4 v^{(r')}(\underline{p}) \qquad \text{(XVI.2)}$$

$$\eta\tilde{\bar{u}}^{(r)}(-\underline{p}) = \gamma_4 u^{(r)*}(-\underline{p}) = B_{rr'}u^{(r')}(\underline{p}) \qquad \text{(XVI.3)}$$

$$\eta\tilde{\bar{v}}^{(r)}(-\underline{p}) = \gamma_4 v^{(r)*}(-\underline{p}) = (-1)^{2s}B_{rr'}v^{(r')}(\underline{p}) \qquad \text{(XVI.4)}$$

$$D\eta\tilde{\bar{u}}^{(r)}(\underline{p}) = D\gamma_4 u^{(r)*}(\underline{p}) = v^{(r)}(\underline{p}) \qquad \text{(XVI.5)}$$

$$D\eta\tilde{\bar{v}}^{(r)}(\underline{p}) = D\gamma_4 v^{(r)*}(\underline{p}) = u^{(r)}(\underline{p}) \qquad \text{(XVI.6)}$$

for bosons only

$$-\gamma_5\,\eta\, D\tilde{\bar{u}}^{(r)}(\underline{p}) = -\gamma_5 D\gamma_4 u^{(r)*}(\underline{p}) = v^{(r)}(\underline{p}) \qquad \text{(XVI.7)}$$

$$-\gamma_5\,\eta\, D\tilde{\bar{v}}^{(r)}(\underline{p}) = -\gamma_5 D\gamma_4 v^{(r)*}(\underline{p}) = u^{(r)}(\underline{p}) \qquad \text{(XVI.8)}$$

for fermions only

where, e.g.

$$\bar{u}^{(r)}(\underline{p}) = u^{(r)\dagger}(\underline{p})\gamma_4\eta \equiv u^{(r')\dagger}(\underline{p})\gamma_4 B_{r'r}$$

$$\eta^2 = 1$$

$$B_{r'r}B_{rr''} = \delta_{r'r''}$$

TABLE XVII *Transformation properties of bilinear covariants and the energy-momentum tensor under space inversion, time reversal, and charge conjugation and their product*

Covariant	$\mathcal{P}$	$\mathcal{C}$	$\mathcal{T}$	$\mathcal{PCT}$
$f^{(S)}(\underline{x},t)$	$f^{(S)}(-\underline{x},t)$	$f^{(S)}(\underline{x},t)$	$f^{(S)}(\underline{x},-t)$	$f^{(S)}(-\underline{x},-t)$
$f_k^{(V)}(\underline{x},t)$	$-f_k^{(V)}(-\underline{x},t)$	$-f_k^{(V)}(\underline{x},t)$	$-f_k^{(V)}(\underline{x},-t)$	$-f_k^{(V)}(-\underline{x},-t)$
$f_4^{(V)}(\underline{x},t)$	$f_4^{(V)}(-\underline{x},t)$	$-f_4^{(V)}(\underline{x},t)$	$f_4^{(V)}(\underline{x},-t)$	$-f_4^{(V)}(-\underline{x},-t)$
$f_{ij}^{(T)}(\underline{x},t)$	$f_{ij}^{(T)}(-\underline{x},t)$	$-f_{ij}^{(T)}(\underline{x},t)$	$-f_{ij}^{(T)}(\underline{x},-t)$	$f_{ij}^{(T)}(-\underline{x},-t)$
$f_{4k}^{(T)}(\underline{x},t)$	$-f_{4k}^{(T)}(-\underline{x},t)$	$-f_{4k}^{(T)}(\underline{x},t)$	$f_{4k}^{(T)}(\underline{x},-t)$	$f_{4k}^{(T)}(-\underline{x},-t)$
$f_k^{(A)}(\underline{x},t)$	$f_k^{(A)}(-\underline{x},t)$	$f_k^{(A)}(\underline{x},t)$	$-f_k^{(A)}(\underline{x},-t)$	$-f_k^{(A)}(-\underline{x},-t)$
$f_4^{(A)}(\underline{x},t)$	$-f_4^{(A)}(-\underline{x},t)$	$f_4^{(A)}(\underline{x},t)$	$f_4^{(A)}(\underline{x},-t)$	$-f_4^{(A)}(-\underline{x},-t)$
$f^{(P)}(\underline{x},t)$	$-f^{(P)}(-\underline{x},t)$	$f^{(P)}(\underline{x},t)$	$-f^{(P)}(\underline{x},-t)$	$f^{(P)}(-\underline{x},-t)$
$\mathcal{T}_{ij}(\underline{x},t)$	$\mathcal{T}_{ij}(-\underline{x},t)$	$\mathcal{T}_{ij}(\underline{x},t)$	$\mathcal{T}_{ij}(\underline{x},-t)$	$\mathcal{T}_{ij}(-\underline{x},-t)$
$\mathcal{T}_{i4}(\underline{x},t)$	$-\mathcal{T}_{i4}(-\underline{x},t)$	$\mathcal{T}_{i4}(\underline{x},t)$	$-\mathcal{T}_{i4}(\underline{x},-t)$	$\mathcal{T}_{i4}(-\underline{x},-t)$
$\mathcal{T}_{44}(\underline{x},t)$	$\mathcal{T}_{44}(-\underline{x},t)$	$\mathcal{T}_{44}(\underline{x},t)$	$\mathcal{T}_{44}(\underline{x},-t)$	$\mathcal{T}_{44}(-\underline{x},-t)$

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16. ABSTRACT *(A 200-word or less factual summary of most significant information. If document includes a significant bibliography or literature survey, mention it here.)*

There are several difficulties that plague all existing relativistic equations of motion describing elementary fields having an intrinsic spin greater than one. While the free field equations can be shown to be explicitly covariant, the introduction of interactions gives rise to a phenomenon of noncausality. In the presence of interactions, the retarded solutions spread beyond the light cone and the influence travels faster than light. Furthermore, the solutions in certain simple potentials do not have a finite norm, violating the probabilistic requirements of quantum mechanics.

This paper develops a relativistic theory that is free of the aforementioned difficulties. This Lagrangian theory describes fields and particles with arbitrary mass and charge and having any discrete spin, integer or half integer. Apart from gauge conditions there are no subsidiary conditions.

A matrix formulation is used. The generators of the inhomogeneous Lorentz group for a field of any intrinsic spin and mass are defined in terms of Wigner operators of the group SU(2) and a metric operator. A maximal Abelian set of invariants is formed which defines two completely reducible representation bases of the inhomogeneous Lorentz group having distinct structures. A set of γ matrices, obeying a Clifford algebra, is also defined in terms of the Wigner operators and the metric operator. State vectors having different structures and Lorentz transformation properties can be related to one another by operators involving the γ matrices.

The equations of motion can be obtained from the Lagrangian by variational methods, and certain aspects of the canonical formalism can be used to quantize the fields. Invariance of the Lagrangian under infinitesimal displacements and rotations yield conservation laws and constants of the motion for pertinent physical observables. The metric of the Hilbert space of the states is uniquely defined for any spin field, assuring positive definite four momenta and charge.

The Dirac formulation for the spin one-half field and the Maxwell-Lorentz formulation for the electromagnetic field are special cases of this theory.

17. KEY WORDS *(six to twelve entries; alphabetical order; capitalize only the first letter of the first key word unless a proper name; separated by semicolons)*

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