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Relativistic Many-Body Bound Systems

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Relativistic Many-Body Bound Systems

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Foreword

When Yukawa introduced the mesons in order to explain the short range character of the nuclear forces, the field of nuclear physics split into two parts: nuclear structure and nuclear forces. Nuclear structure developed into non-relativistic nuclear physics and led to the creation of various models to describe the emerging wealth of nuclear data. The field of nuclear forces developed into high energy particle physics with its own immense body of phenomena and data.

For most nuclear phenomena the non-relativistic framework is fully adequate. However, this framework is too narrow in phenomena associated with exchange currents and with high momentum transfers. In such cases the presence of particles other than protrons and neutrons in the nuclei must be explicitly accounted for, and one must take recourse to the field of high energy physics. This monograph is devoted to the merger of nuclear and high energy physics, and to the formulation of the quantum field theory of nuclei. The main emphasis in this work is on providing the mathematical tools needed to obtain solutions to specific problems in a fully relativistic, consistent manner and up to a known, predetermined accuracy. I expect that this pioneering work should be of value to all who are involved in calculations of nuclear structure.

Richard W. Roberts
Director

ABSTRACT

The principles and the mathematical details of a fully relativistic nuclear theory are given. Since the concept of nuclear forces is a strictly non-relativistic construct, it must be abandoned and the forces must be replaced explicitly by their physical origin, i.e., by the interaction between nucleons and mesons. Thus, in this monograph the description of a nucleus has been formulated as a problem of relativistic quantum field theory which is solved by nuclear physics methods. To wit: The physics is described by specifying a Lagrangian which is a functional of the constituent fields (= of the parton fields). The solutions for the physical systems then are obtained in a time-independent treatment as expansions in the parton fields: both particles and nuclei are composite systems, made up of parton configurations, which define a representation of the Hamiltonian (associated with the specified Lagrangian). The Hamiltonian is truncated by omitting all configurations having a diagonal element exceeding that of the lowest configuration by a pre-determined value, E_{max}, and is diagonalized. All formulae needed to carry out this program are derived and given in full detail for spin 0, 1/2, and 1 parton fields for PS, PV, and φ^4 interactions. Particular attention is devoted to the center-of-mass position coordinate which in relativistic kinematics is a non-separable many-body operator. Finally, the configurations up to E_{max} = 1 GeV are listed for the nucleon, the deuteron, and the pion.

Key words: Composite particles; interacting quantum fields; nuclear structure; particle structure; relativistic bound systems; relativistic nuclear physics.

CONTENTS

CHAPTER I

OVERVIEW AND GENERAL FORMALISM

I.1 - INTRODUCTION

With the advent of accelerators for hadrons and electrons in the 0.2 - 2. GeV range, the development of intermediate energy nuclear physics opens a whole new field of studies. Namely the postulate of the microscopic non-relativistic theory of nuclei, that is nuclei are made up only of protons and neutrons interacting via two-body potentials, has to be replaced by an explicit description of the mesic degrees of freedom and of the admixture of baryonic resonances in the nuclear medium.

In the present work the mathematical tools and the equations needed to compute fully relativistically many-body bound hadronic systems will be written down. As is well known the relativistic description of nuclei entails inescapably the dynamical description of the low energy structure of the nucleons and the mesons themselves. Thus the hadrons with baryonic number 0 or 1 will have to be described in a consistent manner in terms of constituent particles the nature of which (quantum numbers, masses, coupling constants ...) constitutes the various phenomenological models one may want to develop.[1]

Since the nuclear physicist is interested in the characteristics (form factors, magnetic and electric multipole moments...) of bound systems, our basic tool will be the Lagrangian field theory[2-7] solved in a time independent formalism in order to obtain relativistic stationary wave functions. The Hamiltonian matrix will be constructed in a truncated Hilbert space on discretized many-body configurations of correct relativistic behaviour, proper statistics and given total angular and isospin quantum numbers. The truncation criterium will be provided by the energy: all those configurations are included which yield a diagonal element of the Hamiltonian less or equal to the truncation energy.

The usual divergences of strong interaction theories will show up in the present treatment via the dependence of the solutions upon the cut-off energy chosen to truncate the functional space. Hence the background assumptions of this work are the following. As is well known, owing to vacuum fluctuations, a physical particle has necessarily a composite structure. In usual field theory this structure cannot be easily described because the relevant expressions diverge. One has to resort to renormalization techniques which by definition preclude even the posing of the question of the structure of a free particle. In this point of view the only observables are the S-matrix elements and energies. On the other hand it could be that the non-relativistic concept of a wave function in a truncated space together with the use of effective masses and coupling constants may be generalizable also to a fully relativistic system. Thus our work rests on the following basic hypothesis: there is a dimension of the configurational space on the one hand large enough to render the main physical characteristics of the solutions little dependent upon the truncation energy (although the input masses and coupling constants will be of course a function of the cut-off energy) and on the other hand, small enough so that the problem is still of manageable magnitude. If this assumption should be proven out, then the discussion in the present framework of the properties of bound hadronic systems (mesons, nucleons, deuteron, nuclei,...) such as form factors, electromagnetic characteristics, coherent photoproduction, photodisintegration,etc.,... will be analogous (using the language of nuclear physicists) to the treatment of nuclei with dressed particle energies and effective forces within the non-relativistic microscopic theory of nuclear structure.

Thus in short the physical hadrons, i.e., the mesons, the nucleons and also the light nuclei will be described in terms of a set of Boson and Fermion fields, the parton fields, which interact via the usual interaction Lagrangians of field theory. In other words the physical particles will be composite systems, i.e., configuration mixtures. The configurations are made up of different combinations of partons coupled to have the strong quantum numbers of the system. In the present model the partons carry the quantum numbers of the physical hadrons. The high energy microscopic structure of the partons (e.g., whether or not they are made up of quarks) cannot be accounted for within the limitations of the truncated space and are contained in the model masses and coupling constants. For example the wave function of the physical proton contains the following parton configurations:

i) a single "proton,"

ii) a "proton" plus one or several "π-mesons," "ρ-mesons," "ω-mesons"...

iii) a "proton" plus one or several "nucleon"-"anti-nucleon" pairs etc., ... Here the quotation marks denote the parton which has the quantum numbers of the corresponding physical particle. From now on, for typographical convenience, we shall adopt the convention of dropping the quotation marks and, when necessary, of adding the word "physical" if the physical particle rather than the parton is meant.

The overall procedure which will be followed thus is: write down the Lagrangian in terms of the parton masses and coupling constants; expand the parton fields in sets of discrete orthonormal modes; construct the correctly symmetrized and anti-symmetrized many-body relativistic state vectors with the quantum numbers of the physical system under consideration; compute and diagonalize the secular Hamiltonian matrix in the time independent Schrödinger picture using the representation defined by this set of states. The partons of the theory are partially undressed. The higher the truncation energy, the more they are undressed. For a truncation energy of say 1 GeV above the nucleon ground state, it will be shown later that this choice leads already to configurations containing up to 6 pions in intermediate states. All irreducible diagrams which can be constructed with up to 6 pions are thus treated to all orders by the secular problem with only the 1 GeV restriction on virtual intermediate state energies. The dimensions of the discretized energy matrix here are of ~ 120 for reasonable basis size parameter. It will then be interesting to test how much of say the anomalous magnetic moment, the form factors etc... of the physical nucleon will be described by this non perturbative approach. The question then is, to which accuracy it is possible to describe in a consistent way the physical data in terms of parton dynamics on the basis of a field theoretic relativistic Lagrangian. Of course, the answer to this question can only be given by the amount of low energy data which will be reproduced with a given set of effective masses and coupling constants.

The present work intends to lay the mathematical groundwork for such a phenomenological program. We treat in detail the spin 0, 1/2 and 1 fields and their interactions. For low values of the truncation energies it is improbable that the Δ_{33} will emerge as a physical pion-nucleon scattering resonance or that the effect of that state in hadronic systems can be accounted for. In order to achieve this, one has very likely to use a (3/2,3/2) parton field in the Lagrangian. Such a field is not included here. With the present tools a large number of applications and calculations must be carried out before either dismissing the postulates of the present description or proving that it gives a reasonable line of attack on the formidable problem of intermediate energy nuclear dynamics.

I.2 - SURVEY OF THE WORK

In the next sections of this introductory chapter, we shall go into more détail concerning the present theoretical approach to hadron dynamics. First, in Section I.3, we review a time-independent formulation of Lagrangian field theory, which is, of course, non-covariant but which yields a secular problem particularly well suited to the description of stationary states. In this connection one has to use discretized fields, i.e., wave packets over the energy. This in turn brings in one of the fundamental problems of the treatment of composite systems: the center of mass motion. The difficulty is augmented in the case of systems made of relativistic particles, because, as shown in the Appendix, the C.M. coordinate becomes a many-body operator.[8] The removal of the C.M. spurious energy is discussed in Section I.4.

In order to allow the use of the well-known angular momentum techniques[9-12] for many-body systems, the free field expansion will have to be constructed in tensorial form both in angular and isospin space. This requires a number of definitions and basic formulae which are given in Chapter II. In particular, a consistent set of phases is defined for the creation and annihilation operators, invariant vectors and matrix elements. This way powerful graphical techniques can be used to construct invariant quantities. The angular momentum coupling diagrams, whose rules are briefly recalled, vastly simplify the calculations. Their use throughout this work shall help the reader to follow and check the results in a convenient and concise way.

Chapter III is devoted to the construction with wave packets over the energy variable of the discretized expansions of the free fields in tensorial forms. The invariant matrix elements of the free Hamiltonian in the new basis are also computed in this chapter. The obtained tensorial forms of the field expansions are checked to verify the proper commutation (anti-commutation) relations and equations of motion. In this work we treat the cases of spin 0, 1/2 and 1 fields, namely the partons with the quantum numbers of the nucleons (1/2,1/2), the π-mesons (0,1), the ρ-mesons (1,1) and the ω-meson (1,0). For the spin 1 field we adopt the framework of the 8-component theory of Hayward[13] in which the reduction of the relativistic spin 1 field into its tensorial components of spin 1 and spin 0 is simple.

In Chapter IV the treatment of the center of mass motion is carried through in detail. This entails the calculation of the invariant matrix elements of the C.M. momentum and position operators. The latter being a many-body operator a method is given to transform it into a sum of products of one-body operators.

In Chapter V we calculate on the orthonormal discrete basis sets of Chapter III the invariant matrix elements of the simplest Hamiltonians of the usual strong interaction Lagrangians, namely πNN (PS and PV couplings), ωNN, ρNN, ρππ and $\omega\pi^3$ (V coupling) and π^4.

As illustrations of the many applications of this general formalism, Chapter VI presents some possible parton models of the nucleon, the deuteron and the pion. The partons constituting these systems are assumed to have the quantum numbers of N,π,ρ and ω. The lists of all the configurations which can be built up to the truncation energy of ~1 GeV above the ground states of the physical systems are given. It turns out that the dimensions of the various secular problems are quite reasonable ($\sim 60\times 60$ for the deuteron, 120×120 for the nucleon) although they may involve as many as six pions. Furthermore in these spaces the symmetrization of the bosons presents no difficulty since the most complicated redundant configurations are of the type $(1p)^3$. All other configurations with a number of bosons larger than 3 involve at most the trivial $(0s)^n$ structure which is treated in detail in Chapter II. All the matrix elements needed for these models have been given in the preceding chapters.

I.3 - FIELD THEORY FOR STATIONARY SYSTEMS

I.3.1 - The Schrödinger picture

The general relations of field theory needed throughout this work can be found in any textbook on the subject, see for example refs. [2-7].

In field theory, as is well known, one works with field operators and state vectors. The wave functions of quantum mechanics are matrix elements of field operators between state vectors. The dynamics of the system is contained in the Lagrangian which is a functional of the field operators and their derivatives. Conserved quantities are constructed from the field operators.

One utilizes three pictures, viz. the Heisenberg, the Schrödinger and the interaction pictures. In the Schrödinger picture the dynamical behaviour of a system is described by a time dependent state vector $|\Psi_n^{(S)}(t)\rangle$ which is a solution of the Schrödinger equation:

$$\frac{\partial}{\partial t} |\Psi_n^{(S)}(t)\rangle = - i H^{(S)} |\Psi^{(S)}(t)\rangle \quad . \tag{1.1}$$

The Hamiltonian is defined as the component T_{44} of the energy momentum tensor, and is given for the cases of interest here in Chapter III. In the Schrödinger picture, it does not depend on time for closed systems. In the Heisenberg picture the state vectors $\Psi_n^{(H)}$ do not depend on time while the dynamical variables exhibit the time dependence. For the Hamiltonian

$$H^{(H)}(t) = e^{iH^{(S)}t} H^{(S)} e^{-iH^{(S)}t} \equiv H^{(S)} \quad . \tag{1.2}$$

From now on we set $H = H^{(S)} = H^{(H)}(t)$. Any solution of Eq.(1.1) is of the form

$$|\Psi_n^{(S)}(t)\rangle = e^{-iHt}|\Psi_n^{(H)}\rangle = e^{-iE_n t}|\Psi_n^{(H)}\rangle \quad . \tag{1.3}$$

In treating interacting fields we have to decompose the Hamiltonian into two parts

$$H = H_o + V \tag{1.4}$$

where H_o is the Hamiltonian of the free fields and V the interacting Hamiltonian In terms of Hamiltonian densities

$$H_o = - \int d^3x (T_o(x))_{44} = \int d^3x\, \mathscr{H}_o(x) \quad , \tag{1.5}$$

$$V = - \int d^3x (T_I(x))_{44} = \int d^3x\, \mathcal{V}(x) \quad . \tag{1.6}$$

In these expressions the energy densities $\mathcal{H}_o(x)$ and $\mathcal{V}(x)$ are combination of the basic field functions taken at a certain fixed instant of time, let us say $t = 0$.

In the Schrödinger picture the free field operators can be expanded as

$$\Phi^{(S)}(x) = \sum_i (\varphi_i(x) a_i + \varphi_i^*(x) a_i^+) \tag{1.7}$$

for Bosons and

$$\psi^{(S)}(x) = \sum_j (u_j(x) b_j + v_j(x) c_j^+) + \text{c.c.} \tag{1.8}$$

for Fermions. Here the sums are understood to be over the possible discrete quantum numbers of the fields (magnetic and isospin projection quantum numbers) and over the continuous field energy variable.

The Boson operators a_i^+, a_i create or annihilate a particle in state i respectively, while the Fermion operators b_j, c_j^+ annihilate a particle or create an anti-particle in state j respectively. They fulfill the relations

$$[a_i , a_{i'}^+]_- = \delta_{ii'} \quad , \tag{1.9}$$

$$[b_j , b_{j'}^+]_+ = \delta_{jj'} \quad . \tag{1.10}$$

In the Heisenberg picture the corresponding free field operator expansions are

$$\Phi^{(H)}(x,t) = \sum_i \left(e^{-i \varepsilon_i t} \varphi_i(x) a_i + e^{i \varepsilon_i t} \varphi_i^*(x) a_i^+ \right) \quad , \tag{1.11}$$

$$\psi^{(H)}(x,t) = \sum_j \left(e^{-i \varepsilon_j t} u_j(x) b_j + e^{i \varepsilon_j t} v_j(x) c_j^+ \right) + \text{c.c.} \tag{1.12}$$

These fields are solutions of the equations of motion derived from the free field Lagrangian and they will be constructed in tensorial form in Chapter III.

I.3.2 - State vectors

Because of the separation of the total Hamiltonian, Eq. (1.4), the free fields play a special role. They are used to construct a basis set of state vectors, i.e., to define a Fock space. We shall define now the basis state vectors $|\alpha(t)\rangle^{(S)}$ and the basis configurational state vectors $|\nu(t)\rangle^{(S)}$ in this Fock space, as well as the corresponding wave functions $r_\alpha^{(S)}(t)$ and $R_\nu^{(S)}(t)$ in the ordinary position-momentum space.

i) Basis state vectors

A particular state of a many-body system for $V = 0$ can be specified in the occupation number representation by the basis state vector

$$|\alpha\rangle^{(H)} = s_\alpha^{(H)} |0\rangle \quad , \tag{1.13}$$

$$|\alpha(t)\rangle^{(S)} = s_\alpha^{(S)}(t)\,|0\rangle = e^{-i H_o t}\, s_\alpha^{(H)} |0\rangle \quad , \tag{1.14}$$

where $s_\alpha^{(H)}$ is a product of Fermion and Boson creation operators. The index α stands for all the quantum numbers which are needed to specify the occupied single particle states. For Fermions, as usual, this product of creation operators is given in a standard or lexicographic order. The ket $|0\rangle$ denotes the vacuum and with the definition

$$\langle 0|0\rangle = 1 \quad , \tag{1.15}$$

the basis state vectors(1.13) and (1.14) form a complete orthonormal basis of the Fock space. Simple examples are for a single Fermion in state α

$$s_\alpha^{(H)} |0\rangle = b_\alpha^+ |0\rangle \quad , \tag{1.16}$$

$$s_\alpha^{(S)}(t)\,|0\rangle = e^{-i \varepsilon_\alpha t}\, b_\alpha^+ |0\rangle \quad , \tag{1.17}$$

and for a system made of a Boson in state i and a Fermion in state j

$$s_\alpha^{(H)} |0\rangle = a_i^+ b_j^+ |0\rangle \quad , \tag{1.18}$$

$$s_\alpha^{(S)}(t_1,t_2)\,|0\rangle = e^{-i \varepsilon_i t_1 - i \varepsilon_j t_2}\, a_i^+ b_j^+ |0\rangle \quad , \tag{1.19}$$

etc... . Later on we shall discretize the Hilbert space by introducing, as already mentionned, wave packets over the energies ε_i, ε_j ... . At this point however the indices i, j, as shown, still specify discrete and energy quantum numbers.

ii) Basis configurational state vectors

They are the proper linear combination of the above basis state vectors needed to construct eigenstates $|\nu\rangle^{(H)}$ of simultaneously H_o, $\vec{J}^{\,2}$ (the total angular momentum), $\vec{T}^2$ (the total isospin) and all other relevant constants of the motion

$$|\nu\rangle^{(H)} = \mathcal{S}_\nu^{(H)}|0\rangle = \sum_\alpha C_\nu^\alpha s_\alpha^{(H)}|0\rangle . \qquad (1.20)$$

The coefficients C_ν^α are the elements of a unitary transformation constructed to yield an orthonormal basis

$$^{(H)}\langle\mu|\nu\rangle^{(H)} = \delta_{\mu\nu} . \qquad (1.21)$$

These coefficients are related of course to angular momentum coupling algebra and to coefficients of fractional parentage (CFP) as shown in Section II. For example the basis configurational state vectors for a system of total angular momentum $I\,M_I$ and total isospin $T\,M_T$ made of a Boson with quantum numbers $j_1\tau_1$ and a Fermion $j_2\tau_2$ is

$$\mathcal{S}^{(H)}_{IM_I,\,TM_T}|0\rangle = \sum_{\substack{m_1m_2\\ \kappa_1\kappa_2}} (j_1j_2m_1m_2|IM_I)(\tau_1\tau_2\kappa_1\kappa_2|TM_T) \times a^+_{j_1m_1,\tau_1\kappa_1}\; b^+_{j_2m_2,\tau_2\kappa_2}|0\rangle . \qquad (1.22)$$

Since in the linear combinations entering $\mathcal{S}_\nu^{(H)}$ the sums are only over magnetic quantum numbers, when going to the Schrödinger picture the time dependence is simply

$$|\nu(t_1,t_2,\ldots t_N)\rangle^{(S)} = \mathcal{S}_\nu^{(S)}(t_1,t_2,\ldots t_N)|0\rangle = e^{-i\sum_j \varepsilon_j t_j}\,\mathcal{S}_\nu^{(H)}|0\rangle . \qquad (1.23)$$

I.3.3 - Wave functions

The Schrödinger basis wave function corresponding for example to a basis state vector with M Bosons and N - M Fermions is given by

$$\begin{aligned} r_\alpha^{(S)}&(x_1t_1,x_2t_2\ldots x_Mt_M\,;\,x_{M+1}t_{M+1}\ldots x_Nt_N)\\ &= \langle 0|\Phi^{(S)}(x_1)\ldots\Phi^{(S)}(x_M)\,\psi^{(S)}(x_{M+1})\ldots\psi^{(S)}(x_N)\,s_\alpha^{(S)}(t_1\ldots t_N)|0\rangle\\ &= \langle 0|\Phi^{(H)}(x_1t_1)\ldots\Phi^{(H)}(x_Mt_M)\,\psi^{(H)}(x_{M+1}t_{M+1})\ldots\psi^{(H)}(x_Nt_N)\,s_\alpha^{(H)}|0\rangle \qquad (1.24)\end{aligned}$$

and likewise for a basis configurational wave function

$$R_\nu^{(S)}(x_1 t_1 \ldots x_N t_N) = \langle 0 | \Phi^{(S)}(x_1) \ldots \psi^{(S)}(x_N) \mathcal{S}_\nu^{(S)}(t_1 \ldots t_N) | 0 \rangle \,. \qquad (1.25)$$

These wave functions are automatically symmetric (or antisymmetric) under the exchange of the Boson (or Fermion) coordinates.

The basis wave functions thus constructed are more general than we need, in that each particle has its own time coordinate t_i. For our purposes it is sufficient to have them defined at a single common time, i.e., on a single space-like hypersurface, see for example ref. [7], chapter II. In other words, we shall require the description of the system in a single frame of reference, viz. the laboratory system. At this point the treatment ceases to be fully covariant, but, of course, it remains fully relativistic. In principle one can compute quantities which are relativistically correct with the resulting non-covariant wave functions, provided of course that the time-independent formulation is used throughout. This is in contrast to the usual time-dependent formulation wh where one would require the full information contained in Eq. (1.25).

Hence, from now on in the Schrödinger picture all times are made equal

$$t_i = t \quad \text{for all } i \quad ,$$

so that with the definition of the configurational energy $\mathcal{E}_\nu$

$$\mathcal{E}_\nu = \sum_{i=1}^{M+N} \varepsilon_i \quad , \qquad (1.26)$$

the time dependence of the configurational state vectors and wave functions in the Schrödinger picture become respectively

$$|\nu(t)\rangle^{(S)} = \mathcal{S}_\nu^{(S)}(t) = e^{-i\mathcal{E}_\nu t} \mathcal{S}_\nu^{(H)} |0\rangle \quad , \qquad (1.27)$$

and

$$R_\nu^{(S)}(x_1 \ldots x_{M+N} ; t) = e^{-i\mathcal{E}_\nu t} \langle 0 | \Phi^{(S)}(x_1) \ldots \psi^{(S)}(x_{M+N}) \mathcal{S}_\nu^{(H)} | 0 \rangle \,. \qquad (1.28)$$

I.3.4 - The secular problem

An approximation scheme can now be developed, based on the Schrödinger representation, along similar lines as in ordinary non-relativistic quantum mechanics. A formal solution $\Psi_n^{(S)}(t)$ of the Schrödinger equation with the complete

Hamiltonian (1.4) is assumed to be obtainable from an expansion on the complete set of the configurational wave functions $R_\nu^{(S)}(t)$

$$\Psi_n^{(S)}(t) = \sum_\nu x_\nu^n(t)\, R_\nu^{(S)}(t) = \sum_\nu x_\nu^n(t)\, e^{-i\mathscr{E}_\nu t}\, R_\nu^{(H)} \quad , \tag{1.29}$$

or in state vector form

$$|\Psi_n^{(S)}(t)\rangle = \sum_\nu x_\nu^n(t)\, e^{-i\mathscr{E}_\nu t}\, \mathscr{B}_\nu^{(H)}|0\rangle \quad . \tag{1.30}$$

This expansion generally involves basis state vectors $\mathscr{B}_\nu^{(H)}|0\rangle$ with a given number of baryons but with different numbers M of mesons. The indices ν thus stand for N,M as well as for all the other quantum numbers defining the basis state. The strong quantum numbers in ν (total $\vec{J},\vec{T}$, baryonic number, parity, $\mathscr{G}$ parity, etc...) are of course those of the composite system under consideration.

The Ansatz (1.30) means that the solution of the problem of interacting fields is represented by an expansion in terms of free fields. As is well known this procedure, strictly speaking, may not be legitimate in relativistic field theory in contrast to non-relativistic quantum mechanics because of the problem of divergences. We shall return to this point below.

For a stationary solution the Schrödinger equation

$$-iH^{(S)}|\Psi_n^{(S)}(t)\rangle = -iE_n|\Psi_n^{(S)}(t)\rangle \tag{1.31}$$

entails that the amplitudes $x_\nu^n(t)$ must be of the form

$$x_\nu^n(t) = e^{-i(E_n - \mathscr{E}_\nu)t}\, x_\nu^n \quad . \tag{1.32}$$

Inserting (1.32) into Eq.(1.31) we obtain the secular equation

$$\langle 0|\mathscr{B}_\nu^{(H)} \sum_{\nu'} H\, \mathscr{B}_{\nu'}^{(H)}|0\rangle\, x_\nu^n = E_n\, x_{\nu'}^n \tag{1.33}$$

namely

$$\sum_\nu {}^{(H)}\langle \nu\,|H - E_n|\nu'\rangle^{(H)}\, x_\nu^n = 0 \quad . \tag{1.34}$$

This is a continuous matrix since the sum over ν' contains an integration over the free field energies.

Since we are aiming at a formulation of the secular problem which will lead to the diagonalization of finite matrices, we must now re-express Eq. (1.30) in terms of discretized basis states.

Before doing it let us note that discretization of relativistic fields is somewhat different from the non-relativistic case. Namely in non-relativistic theories one can use a fictitious potential to generate a complete set of eigenstates with discrete energy eigenvalues. The solution of the true Hamiltonian then can be expanded in terms of these discrete states. In a relativistic theory one cannot use the non-relativistic concept of potential. Consequently one has to start from the free field solutions themselves for basis states, as it has been done above. The way to achieve a discrete secular problem is then to construct wave packets over the energy. This will be done in detail in chapter III. by applying a unitary transformation on the field operators,

$$A^{+}_{\kappa} = \int d\varepsilon \; f_{\kappa}(\varepsilon) \; a^{+}_{\varepsilon} \quad ,$$
$$B^{+}_{\kappa} = \int d\varepsilon \; f_{\kappa}(\varepsilon) \; b^{+}_{\kappa} \quad , \quad \text{etc...} \; , \tag{1.35}$$

where the energy functions $f_{\kappa}(\varepsilon)$ with discrete index κ form an orthonormal set

$$\int d\varepsilon \; f_{\kappa}(\varepsilon) \; f_{\kappa'}(\varepsilon) = \delta_{\kappa\kappa'} \quad ,$$
$$\sum_{\kappa} f_{\kappa}(\varepsilon) \; f_{\kappa}(\varepsilon') = \delta(\varepsilon-\varepsilon') \quad . \tag{1.36}$$

A convenient choice is for example harmonic oscillator functions. Inserting the discretized fields (1.35) into the definitions of basis state vectors, section 1.3.2., by means of the inverse transformations

$$a^{+}_{\varepsilon} = \sum_{\kappa} f_{\kappa}(\varepsilon) \; A^{+}_{\kappa} \quad ,$$
$$b^{+}_{\varepsilon} = \sum_{\kappa} f_{\kappa}(\varepsilon) \; B^{+}_{\kappa} \quad , \quad \text{etc...,}$$

we shall obtain discretized basis configurational states $|r\rangle$. The corresponding amplitudes $X^{(r)}_{n}$ for the solution $\Psi^{(S)}_{n}(t)$ are now given by the completely discretized system

$$\sum_{r'} \left\{ \langle r|H_o - E_n|r'\rangle + \langle r|V|r'\rangle \right\} X^{(r')}_{n} = 0 \quad . \tag{1.37}$$

In words, the time independent description of quantum field theory leads to a secular equation which is formally indistinguishable from that of the classical theory.

I.3.5 - Nature of the solutions

The solution $|\Psi_n^{(S)}(t)\rangle$ contains an undefined number of hadrons, i.e. it is expanded on configurational states with different numbers of Bosons. Unless the employed truncation energy is above the baryon - antibaryon threshold, the number of baryons of course is identical in all components, but their intrinsic quantum numbers may be different. The meaning of a stationary solution is then one in which the relative admixtures of parton components do not change in time. A pictorial representation of the effect of the creation and annihilation operators in time is given in figure 1.1. There a system is described as a mixture of a single Fermion f with different numbers of Bosons b. Thus at time t the state vector $|\Psi_n^{(S)}(t)\rangle$ is a mixture of the components f, fb, fbb ... with amplitudes $X_n^{(r)}$ (r= 1,2,3 ... respectively). After the infinitesimal time Δt, several processes have taken place due to the time evolution Hamiltonian, where a single Boson has been created or absorbed. These processes are associated with the field interaction matrix elements $\langle r'|V|r\rangle$. At time $t+\Delta t$ the configurational mixture must, however, remain unchanged in order for the state vector to be stationary. We have

$$\Psi_n^{(S)}(t+\Delta t) = \Psi_n^{(S)}(t) - i\,\Delta t\, H\, \Psi_n^{(S)}(t) \quad . \tag{1.38}$$

Substituting the expansion (I.30) we get

$$X_n^{(r)} e^{-i E_n \Delta t} = X_n^{(r)}(1 - i E_n \Delta t) = X_n^{(r)} - i\,\Delta t \sum_{r'} \langle r|V|r'\rangle X_n^{(r')} \quad , \tag{1.39}$$

which is the relation between the amplitudes given by Eq.(I.37). Assuming for simplicity H_o to be diagonal on the representation r

$$E_n X_n^{(1)} = \mathscr{E}_1 X_n^{(1)} + \langle 1|V|2\rangle X_n^{(2)} \quad ,$$

$$E_n X_n^{(2)} = \langle 2|V|1\rangle X_n^{(1)} + \mathscr{E}_2 X_n^{(2)} + \langle 2|V|3\rangle X_n^{(3)} \quad \text{etc} \ldots \tag{1.40}$$

Graphically (see fig.1.1) when going from t to $t+\Delta t$ the component r = 1 may either go into the component r = 1 via the free field Hamiltonian H_o with the amplitude $\mathscr{E}_1 X_n^{(1)}$ or into r = 2 via an interaction V where a Boson is created, (amplitude $\langle 2|V|1\rangle X_n^{(1)}$). Likewise the component r = 2 may either go to r = 1 via the absorption of a meson (amplitude $\langle 1|V|2\rangle X_n^{(2)}$) into r = 2 via H_o or into r = 3 via the creation of a Boson (amplitude $\langle 3|V|2\rangle X_n^{(2)}$) etc. In order for the composite state to be stationary all the amplitude flows (indicated by the arrows) must be such that they leave all the ratios $X_n^{(r)}/X_n^{(r')}$ unchanged.

This is precisely the meaning of the linear relations (1.37) or (1.40).

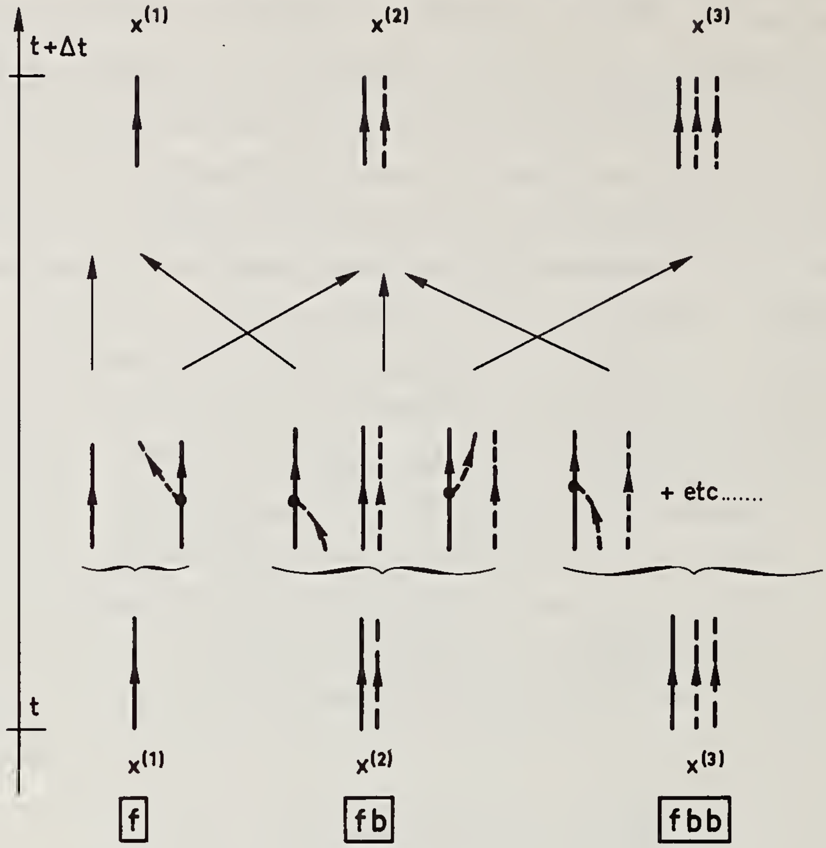


Figure 1.1

I.3.6 - Comparison with time dependent perturbation treatments

There are fundamental differences between the present approximation scheme, based on the hierarchy of the basis configurational states in the expansion of Eq.(1.29) with a cut-off energy or truncation of the Hilbert space, and the usual time-dependent perturbation treatment generally carried out in the interaction picture:

i) The energy matrix of the secular problem requires only the calculation of the interaction operator V at a single time point, between of course many-body basis state vectors. The proper statistics between the particles is readily achieved with well known techniques. This is in contrast to the ill-defined nature of the chronological products of interaction operators at contact points, see for example ref.[7] page 220, and to the rapidly increasing difficulty in calculating irreducible diagrams of increasing order.

ii) Upon diagonalization of the energy matrix all processes which can exist in the chosen configuration space are automatically generated and treated to all orders, with only the restriction of the cut-off energy. In a time-dependent treatment only selected irreducible graphs are calculated.

iii) The solutions of the Schrödinger equation are of course unitary in the truncated space.

iv) The divergences of the strong interaction theories which in the time-dependent treatments show up when integrating over the four-momenta of the intermediate virtual states, appear here as a divergent dependence of the solutions upon the cut-off energy of the truncated space.

I.4 - EXTRACTION OF THE CENTER OF MASS MOTION

As usual when working in a truncated Hilbert space, the solutions of the secular problem of Eq.(1.37) are each a mixture of states with different center of mass motion. In order to obtain solutions with a well-defined C.M. motion, viz. Os, a standard procedure of non-relativistic quantum mechanics consists in adding an artificial C.M. energy operator to the Hamiltonian

$$\mathcal{H}_{C.M.} = \frac{1}{2}\,\xi\,(\vec{P}^2 + \Omega^2\,\vec{R}^2) \quad , \tag{1.41}$$

where $\vec{P}$ and $\vec{R}$ are the center of mass momentum and position operators. This C.M. Hamiltonian is used without ascribing a physical meaning to it but as a device to split apart the solutions of $H+\mathcal{H}_{CM}$ into groups with each a given C.M. motion. This splitting increases with increasing values of the parameter ξ. Non-relativistically this procedure is exact besides the difficulties associated with truncation. Note that a large value of ξ does not imply a high velocity of the C.M. motion. The parameter ξ only changes the scale of the C.M. level spacing. It is the frequency or size parameter Ω which determines the velocity of the C.M. motion.

We shall use this method in the present work. Relativistically the C.M. momentum is simply

$$\vec{P} = \vec{p}_1 + \vec{p}_2 + \vec{p}_3 \ldots \tag{1.42}$$

and the evaluation of the matrix elements of $\vec{P}^2$ on the relativistic basis state vectors presents no particular difficulty. However in relativistic kinematics [8] the relation

$$\vec{R} = \Big(\sum_i m_i\,\vec{x}_i\Big)\Big/M \tag{1.43}$$

$$M = \sum_i m_i \tag{1.44}$$

is replaced as shown in the appendix by

$$\vec{R} = \left(\sum_i \varepsilon_i \vec{x}_i\right) \Big/ \left(\sum_k \varepsilon_k\right) \quad , \tag{1.45}$$

where the ε_i's are the energies of the partons making up the configuration

$$\varepsilon_k = \sqrt{(p_k^2 + m_k^2)} \quad . \tag{1.46}$$

One sees that the operator

$$\vec{R}^2 = \sum_i \frac{(\varepsilon_i \vec{x}_i)^2}{(\sum_k \varepsilon_k)^2} + \sum_{i \neq j} \frac{\varepsilon_i \varepsilon_j \vec{x}_i \cdot \vec{x}_j}{(\sum_k \varepsilon_k)^2} \tag{1.47}$$

in the C.M. Hamiltonian (1.41) is in fact not only a sum of many-body operators because of the denominators $(\sum_k \varepsilon_k)^2$ but is non-separable in the individual energy variables. This typically relativistic difficulty can be circumvented by the use of the transformation

$$\frac{1}{\sum_k \varepsilon_k} = \int_o^\infty dz \; e^{-z(\sum_k \varepsilon_k)} \quad ,$$

$$\frac{1}{(\sum_k \varepsilon_k)^2} = \int_o^\infty \int_o^\infty dz_1 \, dz_2 \; e^{-(z_1+z_2)(\sum_k \varepsilon_k)} \quad , \tag{1.48}$$

which factorizes, after substitution, each term of (1.47) into products of one- or two-body operators. Thus $(z = z_1 + z_2)$

$$\vec{R}^2(z) = \sum_i \left(\prod_{k \neq i} e^{-z\varepsilon_k}\right)\left(\varepsilon_i^2 \, e^{-z\varepsilon_i} \, \vec{x}_i^{\,2}\right)$$
$$+ \sum_{i \neq j} \left(\prod_{k \neq i,j} e^{-z\varepsilon_k}\right)\left(\varepsilon_i \, e^{-z\varepsilon_i} \, \vec{x}_i\right) \cdot \left(\varepsilon_j \, e^{-z\varepsilon_j} \, \vec{x}_j\right) \quad . \tag{1.49}$$

The invariant matrix elements of the pseudo-C.M. Hamiltonian (1.41) can thus be evaluated simply as shown in detail in Chapter IV.

There is, however, another difficulty which arises with relativistic kinematics. This is linked to retardation effects. To be specific, let us consider a problem in which we want to describe a composite particle (for example the physical nucleon) in its ground state. For high value of ξ the center of mass motion of the solutions of the pseudo Hamiltonian

$$\mathcal{H} = H + \mathcal{H}_{CM} \tag{1.50}$$

is well defined. However, this motion is not free but confined by the artificial potential (1.41). In such a potential the particle is accelerated and, because of retardation effects, the different parts of the particle experience different accelerations (analogous difficulties beset the classical model of an extended electron). In non-relativistic mechanics the potential can act simultaneously on all parts of the particle: the particle can be accelerated "as a whole." Because of the inevitable retardation effects, in relativistic mechanics the difference in the acceleration of the different parts of the particle will lead to deformation of the particle, i.e., to admixture of excited states (of the baryon resonances if the particle is a baryon) to the ground state of the particle. Thus relativistically, if no care is exercised, the purification of the C.M. motion by the adjunction of the potential (1.41) in turn entails the mixing in of intrinsic excited states. The magnitude of these admixtures will be small in the limit of a slow C.M. motion, i.e., for small values of the parameter Ω in (1.41).

Thus the solutions of the pseudo Hamiltonian (1.50)

$$\mathcal{H} \, \Psi_n(\xi,\Omega) = E_n \, \Psi_n(\xi,\Omega) \quad , \tag{1.53}$$

owing to completeness are in principle of the general form

$$\Psi_n(\xi,\Omega) = \sum_{a,\alpha} C^{(n)}_{a\,\alpha} \, \varphi_a \, \chi_\alpha \quad , \tag{1.54}$$

where φ_α are a set of wave functions of the C.M. motion while the χ_α are a set of wave functions for the internal motion. Now from the above discussion we make the hypothesis that for a sufficiently high value of ξ and low value of Ω, the lowest eigenstates are of the form

$$\Psi_n(\xi,\Omega) \sim \varphi_o \, \chi_n \quad , \tag{1.55}$$

where φ_o describes a Os C.M. motion. Of course, the errors introduced by the hypothesis (1.55) as well as the independency of the final results as a function of the parameters ξ and Ω must be discussed on the numerical solutions.

Finally we define (E_V is the energy of the physical vacuum;cf (D.11))

$$\langle\Psi_n|\sqrt{P^2+M_n^2}|\Psi_n\rangle = E_n - \langle\Psi_n|\mathscr{H}_{CM}|\Psi_n\rangle - E_V$$

$$= \tilde{E}_n \qquad (1.56)$$

In the limit of a slow C.M. motion $P^2 \ll M_n^2$, we obtain

$$\tilde{E}_n = M_n + \frac{\langle\Psi_n|P^2|\Psi_n\rangle}{2\,M_n} - \frac{5}{3}\,\frac{(\langle\Psi_n|P^2|\Psi_n\rangle)^2}{(2\,M_n)^3} \qquad (1.57)$$

i.e., upon inversion

$$M_n = \tilde{E}_n - \frac{\langle\Psi_n|P^2|\Psi_n\rangle}{2\,\tilde{E}_n} + \ldots \qquad (1.58)$$

which yields the mass spectrum M_n of the composite system, which is the physical quantity of interest.

CHAPTER II

DEFINITIONS AND FORMULAE

II.1 - PHASES

II.1.1 - Tensorial sets

The tensorial harmonics are defined as contrastandard tensors[9],

$$Y_m^{[\ell]}(\theta\varphi) \equiv \hat{r}_m^{[\ell]} = (-i)^{\ell}\, Y_{\ell m}(\theta\varphi) \tag{2.1}$$

with

$$Y_m^{[\ell]}(\theta\varphi) = \sum_{m'} Y_{m'}^{[\ell]}(\theta'\varphi')\, \mathscr{D}^{\ell}_{m'm}(\alpha\beta\gamma) \quad . \tag{2.2}$$

Here the rotation matrix is defined as in Fano-Racah[10] and Edmonds[11],

$$\mathscr{D}^{j}_{m'm}(\alpha\beta\gamma) = e^{im'\gamma}\, d^{j}_{m'm}(\beta)\, e^{im\alpha} \tag{2.3}$$

where the coordinate system Ox'y'z' is obtained from Oxyz by a rotation through the Euler angles α,β and γ in that order.

The time reversed tensors are the standard sets

$$Y_m^{(\ell)}(\theta\varphi) = \hat{r}_m^{(\ell)} = (-)^{\ell+m}\, Y_{-m}^{[\ell]} \tag{2.4}$$

with

$$Y_m^{(\ell)}(\theta\varphi) = \sum_{m'} Y_{m'}^{(\ell)}(\theta'\varphi')\, \mathscr{D}^{\ell *}_{m'm}(\alpha\beta\gamma) \quad . \tag{2.5}$$

Thus for the case of vector operators we have

$$J_0^{[1]} = -i\, J_z$$

$$J_1^{[1]} = \frac{1}{\sqrt{2}}\,(i\, J_x - J_y)$$

$$J_{-1}^{[1]} = \frac{1}{\sqrt{2}}\,(-i\, J_x - J_y) \quad . \tag{2.6}$$

These forms fulfill the relations

$$
\begin{aligned}
[J_x, J_y] &= i\, J_z \\
[J_z, J_x] &= i\, J_y \\
[J_y, J_z] &= i\, J_x \\
[J_1^{[1]}, J_{-1}^{[1]}] &= J_o^{[1]} \\
[J_1^{[1]}, J_o^{[1]}] &= i\, J_1^{[1]} \\
[J_{-1}^{[1]}, J_o^{[1]}] &= -i\, J_{-1}^{[1]} \quad .
\end{aligned}
\tag{2.7}
$$

$$\vec{J}^2 \equiv \sqrt{3}\,[J^{[1]}J^{[1]}]^{[0]} = J_1^{[1]}J_{-1}^{[1]} - J_o^{[1]}J_o^{[1]} + J_{-1}^{[1]}J_1^{[1]} \quad . \tag{2.8}$$

Next we then define the hermitian conjugate tensorial set $\tilde{\varphi}_m^{(j)}(x)$ of the normalized set $\varphi_m^{[j]}(x)$ as

$$\int dx\, \varphi_m^{[j]+}\, \varphi_m^{[j]} = \int dx\, \tilde{\varphi}_m^{(j)}\, \varphi_m^{[j]} = 1 \quad . \tag{2.9}$$

For example for the spin 1/2 functions, from their explicit form

$$\chi_{1/2}^{[1/2]} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad ; \qquad \chi_{-1/2}^{[1/2]} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.10}$$

and from the defining relation,

$$\sum_\alpha \tilde{\chi}_s^{(1/2)}(\alpha)\, \chi_{s'}^{[1/2]}(\alpha) = \delta_{ss'} \tag{2.11}$$

we get the hermitian conjugate spin functions $\tilde{\chi}_s^{(1/2)}$

$$\tilde{\chi}_{1/2}^{(1/2)} = (1\ 0) \quad ; \qquad \tilde{\chi}_{-1/2}^{(1/2)} = (0\ 1) \quad . \tag{2.12}$$

II.1.2 - Creation and annihilation operators

We now give a consistent set of definitions for creation and annihilation operators of the usual field formalism which incorporate these transformation properties of standard and contrastandard tensors as well as the hermitian conjugation. Therefore we demand

$$
\begin{aligned}
a_m^{(j)++} &= a_m^{(j)} \quad ; \qquad \tilde{a}_m^{(j)++} = \tilde{a}_m^{(j)} \\
a_m^{[j]++} &= a_m^{[j]} \quad ; \qquad \tilde{a}_m^{[j]++} = \tilde{a}_m^{[j]}
\end{aligned}
\tag{2.13}
$$

where on the one hand $a_m^{(j)}$ and $a_m^{[j]}$ are respectively the standard and contrastandard annihilation operators which transform according to

$$a_m^{[j]} = \sum_{m'} \mathscr{D}^j_{m'm}(\alpha\beta\gamma)\ a_{m'}^{[j]}$$

$$a_m^{(j)} = \sum_{m'} \mathscr{D}^{j*}_{m'm}(\alpha\beta\gamma)\ a_{m'}^{(j)} \qquad , \tag{2.14}$$

and where on the other hand the contrastandard creation operators $\tilde{a}_m^{[j]}$ are defined as the hermitian conjugates of the standard annihilation operators. This definition results from the fact that hermitian conjugation changes the rotational transformation properties as it includes a time reversal operation. Thus

$$\tilde{a}_m^{[j]} = (-)^{2j}\ a_m^{(j)+}$$

$$\tilde{a}_m^{(j)} = a_m^{[j]+} \tag{2.15}$$

and

$$a_m^{(j)} = (-)^{2j}\ \tilde{a}_m^{[j]+}$$

$$a_m^{[j]} = \tilde{a}_m^{(j)+} \qquad . \tag{2.16}$$

These definitions are chosen to be consistent with our phase convention given above, i.e.,

$$a_m^{(j)} = (-)^{j+m}\ a_{-m}^{[j]} \quad ; \quad a_m^{[j]} = (-)^{j-m}\ a_{-m}^{(j)}$$

$$\tilde{a}_m^{(j)} = (-)^{j+m}\ \tilde{a}_{-m}^{[j]} \quad ; \quad \tilde{a}_m^{[j]} = (-)^{j-m}\ \tilde{a}_{-m}^{(j)} \qquad . \tag{2.17}$$

These relations may also be written

$$a_m^{(j)+} = (-)^{j+m}\ \tilde{a}_{-m}^{(j)} \quad ; \quad a_m^{[j]+} = (-)^{j+m}\ \tilde{a}_{-m}^{[j]}$$

$$\tilde{a}_m^{(j)+} = (-)^{j-m}\ a_{-m}^{(j)} \quad ; \quad \tilde{a}_m^{[j]+} = (-)^{j-m}\ a_{-m}^{[j]} \tag{2.18}$$

which shows that with our choice of phases the creation and annihilation operators transform exactly like tensorial sets.

II.2 - INVARIANT STATES AND MATRICES

The most general invariant state vector with amplitudes $W_m^{(j)}$ and normalized components $\psi_m^{[j]}$ is, with the notation $\hat{j} = \sqrt{2j+1}$

$$\Psi = \sum_m W_m^{(j)} \psi_m^{[j]} = \hat{j} \, [W^{[j]} \psi^{[j]}]^{[0]} \quad . \tag{2.19}$$

The normalization of Ψ imposes

$$[W^{[j]} W^{[j]}]^{[0]} = \frac{1}{\hat{j}} \quad . \tag{2.20}$$

Likewise the most general invariant operator $\Omega(J)$ of multipolarity J is

$$\underline{\Omega}(J) = \sum_M \omega_M^{(J)} \Omega_M^{[J]} = \hat{J} \, [\omega^{[J]} \Omega^{[J]}]^{[0]} \tag{2.21}$$

with normalized amplitudes

$$[\omega^{[J]} \omega^{[J]}]^{[0]} = \frac{1}{\hat{J}} \quad . \tag{2.22}$$

Its matrix elements between the normalized states Ψ_i and Ψ_f are

$$\langle \Psi_f | \underline{\Omega}(J) | \Psi_i \rangle = [\psi_f^{[j]} | \Omega^{[J]} | \psi_i^{[k]}] \, [W_f^{[j]} \omega^{[J]} W_i^{[k]}]^{[0]} \quad , \tag{2.23}$$

which defines the invariant matrix element $[\psi_f^{[j]} | \Omega^{[J]} | \psi_i^{[k]}]$. Its connexion with ordinary matrix elements is given by [9]

$$\langle \psi_{f\,m}^{[j]} | \Omega_M^{[J]} | \psi_{i\,m'}^{[k]} \rangle = \int d^3\vec{r} \;\; \psi_{f\,m}^{[j]*} \, \Omega_M^{[J]} \, \psi_{i\,m'}^{[k]}$$

$$= (-)^{j+m} (-)^{j-J+k} \begin{pmatrix} j & J & k \\ -m & M & m' \end{pmatrix} [\psi_f^{[j]} | \Omega^{[J]} | \psi_i^{[k]}] \quad , \tag{2.24}$$

and with the reduced matrix elements of Edmonds, Fano-Racah and Messiah [10-12] by

$$\langle \psi_f^{[j]} \| \Omega^{[J]} \| \psi_i^{[k]} \rangle = (-)^{j+J-k} \, [\psi_f^{[j]} | \Omega^{[J]} | \psi_i^{[k]}] \quad . \tag{2.25}$$

We shall need the following list of useful invariant matrix elements

$$[\psi^{[J]} | 1^{[0]} | \psi^{[J]}] \equiv [\psi^{[J]} | \psi^{[J]}] = \hat{J} \quad , \tag{2.26}$$

$$[\chi^{[1/2]} | \sigma^{[1]} | \chi^{[1/2]}] = i\sqrt{6} \quad , \tag{2.27}$$

$$[\psi^{[J]}|J^{[1]}|\psi^{[J]}] = i\hat{J}(J(J+1))^{1/2} \tag{2.28}$$

$$[Y^{[\ell_1]}|Y^{[\ell_2]}|Y^{[\ell_3]}] = (-)^{1/2(\ell_1+\ell_2+\ell_3)} \frac{\hat{\ell}_1\hat{\ell}_2\hat{\ell}_3}{(4\pi)^{1/2}} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.29}$$

$$\equiv [\ell_1|\ell_2|\ell_3] \quad .$$

Likewise defining

$$\varphi^{[\ell]}_{\alpha m}(\vec{r}) = f_{\alpha\ell}(r)\, Y^{[\ell]}_m(\hat{r}) \tag{2.30}$$

together with the unit vectors in coordinate and momentum spaces

$$\vec{r}^{[1]}_m = (4\pi/3)^{1/2}\, r\, Y^{[1]}_m(\hat{r}) \tag{2.31}$$

$$\vec{p}^{[1]}_m = -\, i\, \vec{\nabla}^{[1]}_m \quad , \tag{2.32}$$

we get the invariants

$$[\varphi^{[\ell]}_\alpha|\vec{r}^{[1]}|\varphi^{[\ell-1]}_\beta] = \sqrt{\ell}\,\langle f_{\alpha\ell}|r|f_{\beta\ell-1}\rangle \quad ,$$
$$[\varphi^{[\ell]}_\alpha|\vec{r}^{[1]}|\varphi^{[\ell+1]}_\beta] = \sqrt{\ell+1}\,\langle f_{\alpha\ell}|r|f_{\beta\ell+1}\rangle \quad , \tag{2.33}$$

$$[\varphi^{[\ell]}_\alpha|\vec{p}^{[1]}|\varphi^{[\ell-1]}_\beta] = i\sqrt{\ell}\,\langle f_{\alpha\ell}|\frac{\ell-1}{r} - \frac{\partial}{\partial r}|f_{\beta\ell-1}\rangle \quad ,$$
$$[\varphi^{[\ell]}_\alpha|\vec{p}^{[1]}|\varphi^{[\ell+1]}_\beta] = -\,i\sqrt{\ell+1}\,\langle f_{\alpha\ell}|\frac{\ell+2}{r} + \frac{\partial}{\partial r}|f_{\beta\ell+1}\rangle \quad . \tag{2.34}$$

The radial functions $f_{\alpha\ell}$ will frequently be spherical Bessel functions $j_\ell(\alpha r)$ in which case using the relations

$$j_{\ell+1}(x) = \left(\frac{\ell}{x} - \frac{\partial}{\partial x}\right) j_\ell(x)$$
$$j_{\ell-1}(x) = \left(\frac{\ell+1}{x} + \frac{\partial}{\partial x}\right) j_\ell(x) \tag{2.35}$$

we get

$$[\varphi^{[\ell]}_p|\vec{\nabla}^{[1]}|\varphi^{[\lambda]}_q] = \alpha_{\ell\lambda}\left(\frac{\pi}{2}\right)\frac{\delta(p-q)}{p^2}\, p \tag{2.36}$$

with the factor $\alpha_{\ell\lambda}$ which will often be used thereafter

$$\alpha_{\ell\lambda} = \begin{cases} +\sqrt{\ell+1} & \text{if } \lambda = \ell+1 \\ 0 & \text{if } \lambda = \ell \\ -\sqrt{\ell} & \text{if } \lambda = \ell-1 \end{cases} \quad . \tag{2.37}$$

II.3 - A GRAPHICAL REPRESENTATION OF ANGULAR MOMENTUM COUPLING

Considering four tensors, the basic recoupling transformation is

$$\left[[\varphi^{[a]}\varphi^{[b]}]^{[c]}[\varphi^{[d]}\varphi^{[e]}]^{[f]}\right]^{[g]}$$

$$= \sum_{hi} \begin{bmatrix} a & b & c \\ d & e & f \\ h & i & g \end{bmatrix} \left[[\varphi^{[a]}\varphi^{[d]}]^{[h]}[\varphi^{[b]}\varphi^{[e]}]^{[i]}\right]^{[g]} \quad , \qquad (2.38)$$

where the square 9-j symbol is related to the ordinary 9-j coefficient by

$$\begin{bmatrix} a & b & c \\ d & e & f \\ h & i & g \end{bmatrix} = \hat{c}\,\hat{f}\,\hat{h}\,\hat{i} \begin{Bmatrix} a & b & c \\ d & e & f \\ h & i & g \end{Bmatrix} \qquad (2.39)$$

This transformation is represented by the basic diagram of figure 2.1[9], where summation over all new quantum number appearing in the diagram is implied (in the present case the summation must go over h and i) .

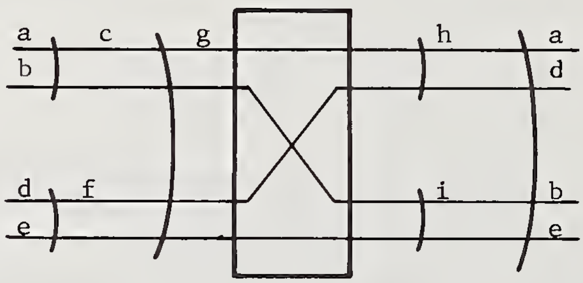
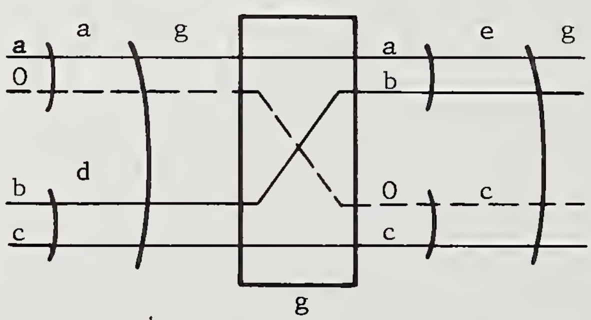


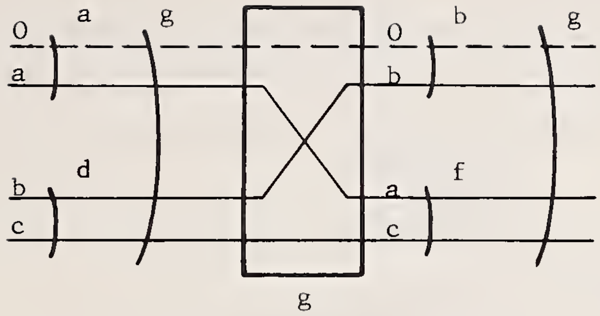
Fig. 2.1

Recoupling transformations for three tensors, viz. of the 6-j type, are represented by the same diagram with a dashed line representing a mock tensor of rank zero, as shown in figures 2.2a and 2.2b .



$$\Longrightarrow \begin{bmatrix} a & 0 & a \\ b & c & d \\ e & c & g \end{bmatrix}$$

Fig. 2.2a



$$\Longrightarrow \begin{bmatrix} 0 & a & a \\ b & c & d \\ b & f & g \end{bmatrix}$$

Fig. 2.2b

We frequently shall need the relations and special values of Table II.1

$$\begin{bmatrix} a & b & c \\ d & d & 0 \\ e & f & c \end{bmatrix} = \begin{bmatrix} a & d & e \\ b & d & f \\ c & 0 & c \end{bmatrix} = \frac{\hat{e}\,\hat{f}}{\hat{d}} (-)^{a+f+c+d} \begin{Bmatrix} a & b & c \\ f & e & d \end{Bmatrix}$$

$$\begin{bmatrix} a & a & 0 \\ b & c & d \\ e & f & d \end{bmatrix} = \begin{bmatrix} a & b & e \\ a & c & f \\ 0 & d & d \end{bmatrix} = \frac{\hat{e}\,\hat{f}}{\hat{a}} (-)^{e+c+a+d} \begin{Bmatrix} a & b & e \\ d & f & c \end{Bmatrix}$$

$$\begin{bmatrix} a & b & c \\ d & e & c \\ f & f & 0 \end{bmatrix} = \hat{c}\,\hat{f}\,(-)^{f+d+b+c} \begin{Bmatrix} a & b & c \\ e & d & f \end{Bmatrix}$$

$$\begin{bmatrix} a & b & c \\ d & 0 & d \\ e & b & f \end{bmatrix} = \begin{bmatrix} 0 & d & d \\ b & a & c \\ b & e & f \end{bmatrix} = \hat{c}\,\hat{e}\,(-)^{c+e+b+d} \begin{Bmatrix} a & b & c \\ f & d & e \end{Bmatrix}$$

$$\begin{bmatrix} a & a & 0 \\ 0 & b & b \\ a & c & b \end{bmatrix} = \begin{bmatrix} a & 0 & a \\ b & b & 0 \\ c & b & a \end{bmatrix} = \begin{bmatrix} a & b & c \\ 0 & b & b \\ a & 0 & a \end{bmatrix} = \frac{\hat{c}}{\hat{a}\,\hat{b}} (-)^{a+b-c}$$

$$\begin{bmatrix} a & a & 0 \\ b & a & c \\ c & 0 & c \end{bmatrix} = \frac{1}{\hat{a}^2} (-)^{a+c-b} \quad ; \quad \begin{bmatrix} 0 & a & a \\ b & b & 0 \\ b & c & a \end{bmatrix} = \begin{bmatrix} 0 & b & b \\ a & b & c \\ a & 0 & a \end{bmatrix} = \frac{\hat{c}}{\hat{a}\,\hat{b}}$$

$$\begin{bmatrix} a & a & 0 \\ a & b & c \\ 0 & c & c \end{bmatrix} = \frac{1}{\hat{a}^2} \quad ; \quad \begin{bmatrix} a & 0 & a \\ 0 & b & b \\ a & b & c \end{bmatrix} = 1$$

$$\begin{bmatrix} a & b & c \\ a & b & c \\ 0 & 0 & 0 \end{bmatrix} = \frac{\hat{c}}{\hat{a}\,\hat{b}} \qquad \begin{bmatrix} a & a & 0 \\ a & a & 0 \\ 0 & 0 & 0 \end{bmatrix} = \frac{1}{\hat{a}^2} \qquad \begin{bmatrix} a & a & 0 \\ 0 & a & a \\ a & 0 & a \end{bmatrix} = \frac{1}{\hat{a}^2} (-)^{2a}$$

Table II.1

The last line of Table II.1 corresponds to the common situation of final coupling into two invariant matrix elements. Finally for the interchange in the order of coupling two tensors we shall use the simplified diagram of figure 2.3.

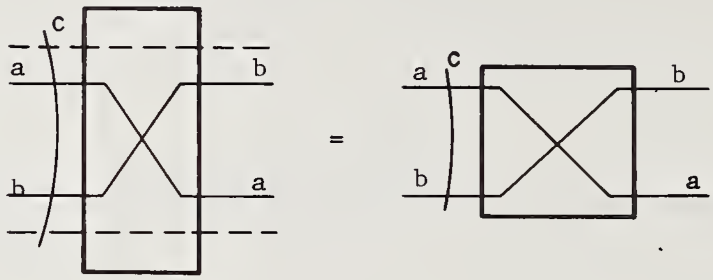


Fig. 2.3

which of course corresponds to the special square 9-j value

$$\begin{bmatrix} 0 & a & a \\ b & 0 & b \\ b & a & c \end{bmatrix} = (-1)^{a+b-c} \qquad (2.40)$$

When working both in angular momentum and isospin spaces we shall employ the double coupling and recoupling notation. Thus we shall write $\psi^{[JT]}_{m\ \tau}$ in that order and define the diagram of figure 2.4.

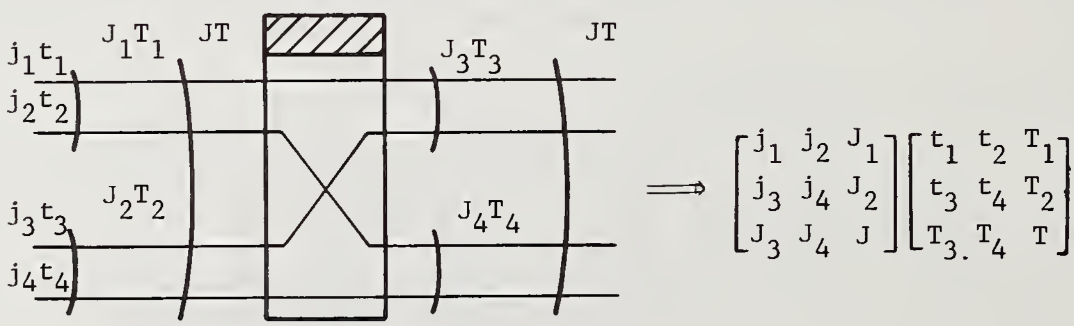


Fig. 2.4

We now turn to the diagrammatic representation of the invariant matrix elements which are given in figures 2.5a, or when working both in angular momentum and isospin spaces in figures 2.5b.

$$[\psi^{[j]}|\psi^{[j]}] \equiv \hat{j} =$$

$$[\psi_f^{[j]}|\Omega^{[J]}|\psi_i^{[k]}] =$$

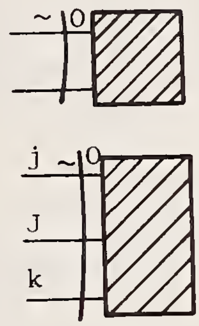


Fig. 2.5a

$$[\psi^{[jt]}|\psi^{[jt]}] \equiv \hat{j}\,\hat{t} =$$

$$[\psi_f^{[jt]}|\Omega^{[JT]}|\psi_i^{[kt']}] =$$

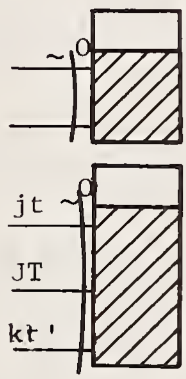


Fig. 2.5b

We have explicitly shown on these diagrams that the upper line corresponds to the bra function. The ~ sign will in general be omitted from the diagram with the understanding that the bra line must always enter a hatched box (invariant matrix element) at the top. An example of this rule, which in fact yields a phase only in the case of half-integer spins, is given below.

Finally we also need the insertion of a complete normalized set of states which is represented by the diagram of figure 2.6 which fulfills the

$$\delta^3(\vec{x}-\vec{y}) = \sum_{\ell\alpha} \hat{\ell}\, [\psi_\alpha^{[\ell]}(\vec{x})\,|\psi_\alpha^{[\ell]}(\vec{y})] =$$

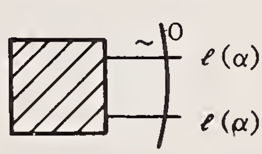


Fig. 2.6

completeness relation of figure 2.7. Here α denotes all other quantum numbers (besides ℓ) which define the basis set of states.

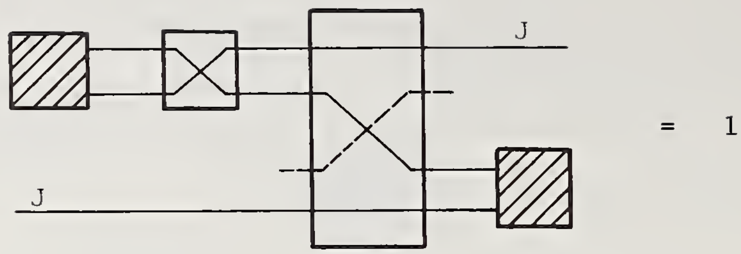


Fig. 2.7

The crossing sign box in Figure 2.7 is not needed for integer angular momentum.

We also will frequently need the reduction formula :

$$[Y^{[\ell_1]}(\hat{r})\ Y^{[\ell_2]}(\hat{r})]^{[\ell]}_m = Q^{\ell}_{\ell_1\ell_2}\ Y^{[\ell]}_m(\hat{r})$$

with

$$Q^{\ell}_{\ell_1\ell_2} = \frac{1}{\sqrt{4\pi}}\ \hat{\ell}_1\hat{\ell}_2 \begin{pmatrix} \ell_1 & \ell_2 & \ell \\ 0 & 0 & 0 \end{pmatrix} (-)^{1/2(\ell_1+\ell_2+\ell)} \qquad (2.41)$$

and which is represented by the diagram of figure 2.8 .

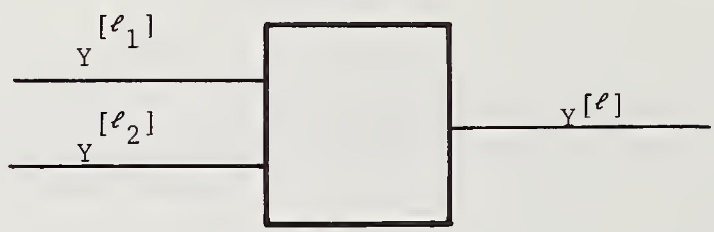


Fig. 2.8

II.4 - VECTOR ALGEBRA

Here we collect the invariant forms of vector algebra and vector analysis. They are based on the invariant representations

$$\vec{V} = V_x e_x + V_y e_y + V_z e_z = \sum_m V^{(1)}_m\ e^{[1]}_m$$

$$= \hat{1}\ [V^{[1]}\ e^{[1]}]^{[0]} \equiv \hat{1}\ [e^{[1]}\ V^{[1]}]^{[0]} \qquad (2.42)$$

$$\vec{\nabla} = \hat{1}\ [\nabla^{[1]}\ e^{[1]}]^{[0]} \qquad (2.43)$$

where $e^{[1]}$ is the unit vector related to the Cartesian unit vector by

$$e_1^{[1]} = (i\, e_x - e_y) / \sqrt{2}$$
$$e_0^{[1]} = -\, i\, e_z \qquad (2.44)$$
$$e_{-1}^{[1]} = (-i\, e_x - e_y) / \sqrt{2} \quad .$$

Thus we get for div, rot. and grad the following expressions :

i) $$\operatorname{div} \vec{V} = \vec{\nabla}.\vec{V} = \nabla_x V_x + \nabla_y V_y + \nabla_z V_z$$
$$= \hat{1}\, [\nabla^{[1]}\, V^{[1]}]^{[0]} \quad , \qquad (2.45)$$

i.e. the divergence of a vector is obtained by replacing in the vector expression (2.42) the components of the unit vector by the respective components of the gradient. Hence for

$$\vec{F} = [\ldots\ldots\, e^{[1]}\, \ldots\ldots] \qquad (2.46)$$

we get generally

$$\operatorname{div} \vec{F} = [\ldots\ldots\, \nabla^{[1]}\, \ldots\ldots] \quad . \qquad (2.47)$$

ii) $$\operatorname{rot} \vec{V} = \vec{\nabla}\times\vec{V} = (\vec{\nabla}\times\vec{V})_x e_x + (\vec{\nabla}\times\vec{V})_y e_y + (\vec{\nabla}\times\vec{V})_z e_z$$
$$= \hat{1}\sqrt{2}\, [[\nabla^{[1]}\, V^{[1]}]^{[1]}\, e^{[1]}]^{[0]}$$
$$\equiv \hat{1}\sqrt{2}\, [\nabla^{[1]}\, V^{[1]}\, e^{[1]}]^{[0]} \quad , \qquad (2.48)$$

where we have used the Fano-Racah[10] notation for the invariant triple product. The factor $\sqrt{2}$ arises because of the different normalization implied by the Cartesian cross product and the coupling of two vectors to $\ell = 1$. Generally for the above function $\vec{F}$ we get

$$\operatorname{rot} \vec{F} = \sqrt{2}\, [\ldots [e^{[1]}\, \nabla^{[1]}]^{[1]} \ldots] \quad . \qquad (2.49)$$

iii) $$\operatorname{grad} f = \vec{\nabla} f = \hat{1}\, [\nabla^{[1]}\, f\, e^{[1]}]^{[0]} \quad . \qquad (2.50)$$

iv) Special cases

We now give the formulae for a few special cases which will be relevant latter on. Let us define the function

$$\varphi_{pm}^{[\lambda]} = j_\lambda(pr)\; Y_m^{[\lambda]}(\hat{r}) \tag{2.51}$$

and the vector spherical multipole J with amplitudes $A_m^{[J]}$

$$\vec{\rho} = \hat{J}\,[A^{[J]}\; e^{[1]}\; \varphi_p^{[\lambda]}]^{[0]} \quad . \tag{2.52}$$

We get from Eq.(2.47)

$$\text{div}\,\vec{\rho} = \hat{J}\,[A^{[J]}\; \nabla^{[1]}\; \varphi_p^{[\lambda]}]^{[0]} \quad , \tag{2.53}$$

which diagrammatically is represented by the diagram of figure 2.9

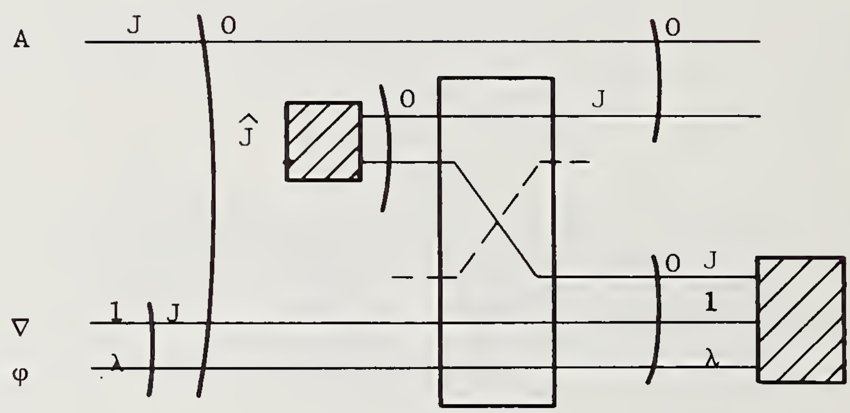


Fig. 2.9

which yields

$$\begin{aligned}\text{div}\,\vec{\rho} &= \sum_{JJ'} \hat{J}\,\hat{J}' \begin{bmatrix} J' & J' & 0 \\ 0 & J & J \\ J' & 0 & J \end{bmatrix} \frac{2}{\pi} \int q^2 dq\; [\varphi_q^{[J]} |\nabla^{[1]}| \varphi_p^{[\lambda]}]^{[0]} [A^{[J]}\; \varphi_p^{[J]}]^{[0]} \\ &= \delta_{JJ'}\, \alpha_{J\lambda}\, p\, [A^{[J]}\; \varphi_p^{[J]}]^{[0]} \quad . \end{aligned} \tag{2.54}$$

From now on we shall read off directly from the diagram the completeness summations and integrations, carry out directly the triangular condition $(J'J\,0) = \delta_{JJ'}$ and the integration over the continuous variable q which leads to $p = q$.

Likewise we get

$$\mathrm{rot}\,\vec{\rho} = \sqrt{2}\,\hat{J}\,[A^{[J]}[e^{[1]}\nabla^{[1]}]^{[1]}\varphi_p^{[\lambda]}]^{[0]} , \qquad (2.55)$$

which is evaluated with the diagram of figure 2.10 .

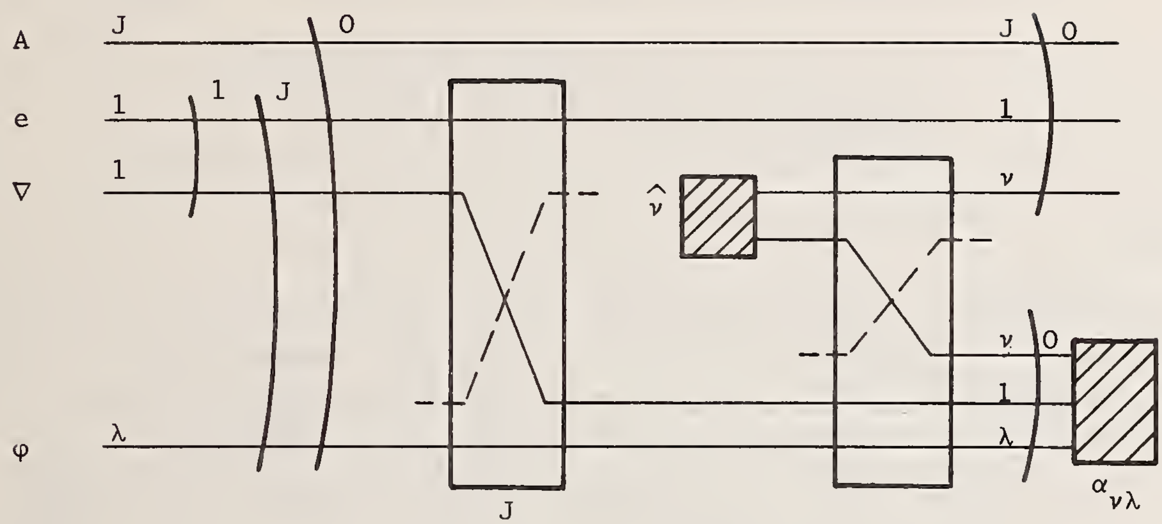


Fig. 2.10

Thus we get

$$\mathrm{rot}\,\vec{\rho} = \hat{J} \sum_{\nu} \begin{bmatrix} 1 & 1 & 1 \\ 0 & \lambda & \lambda \\ 1 & \nu & J \end{bmatrix} \hat{\nu} \begin{bmatrix} \nu & \nu & 0 \\ 0 & \nu & \nu \\ \nu & 0 & \nu \end{bmatrix} p\, \alpha_{\nu\lambda} [A^{[J]}\, e^{[1]} \varphi_p^{[\nu]}]^{[0]}$$

$$= -\,p \begin{cases} \sqrt{J}\,[A^{[J]} e^{[1]} \varphi^{[J]}]^{[0]} & \text{for } \lambda = J+1 , \\ \sqrt{J}\,[A^{[J]} e^{[1]} \varphi^{[J+1]}]^{[0]} + \sqrt{J+1}\,[A^{[J]} e^{[1]} \varphi^{[J-1]}]^{[0]} & \text{for } \lambda = J , \\ \sqrt{J+1}\,[A^{[J]} e^{[1]} \varphi^{[J]}]^{[0]} & \text{for } \lambda = J-1 . \end{cases} \qquad (2.56)$$

We have made use of the explicit values of the 6-j symbol entering the expression (2.56)

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & \lambda & \lambda \\ 1 & \nu & J \end{bmatrix} = (-)^{J+\lambda}\,\hat{1}\,\hat{\nu} \begin{Bmatrix} 1 & 1 & 1 \\ J & \nu & \lambda \end{Bmatrix} ; \qquad \nu = J,\ J\pm 1 . \qquad (2.57)$$

In a similar fashion

$$\mathrm{rot}\ \mathrm{rot}\ \vec{\rho} = 2\hat{J}\ [A^{[J]}[[[e^{[1]}\nabla^{[1]}]^{[1]}\nabla^{[1]}]^{[1]}\varphi^{[\lambda]}]^{[J]}]^{[0]} \quad , \qquad (2.58)$$

which from the diagram of figure 2.11 yields

$$\mathrm{rot}\ \mathrm{rot}\ \vec{\rho} = 2\hat{J}\sum_{\nu\mu}\begin{bmatrix}1 & 1 & 1\\ 0 & \lambda & \lambda\\ 1 & \nu & J\end{bmatrix}\hat{\nu}\begin{bmatrix}\nu & \nu & 0\\ 0 & \nu & \nu\\ \nu & 0 & \nu\end{bmatrix} p\ \alpha_{\nu\lambda}\begin{bmatrix}1 & 1 & 1\\ 0 & \nu & \nu\\ 1 & \mu & J\end{bmatrix}\hat{\mu}$$

$$\times\begin{bmatrix}\mu & \mu & 0\\ 0 & \mu & \mu\\ \mu & 0 & \mu\end{bmatrix} p\ \alpha_{\mu\nu}\ [A^{[J]}e^{[1]}\varphi^{[\mu]}]^{[0]} \qquad (2.59)$$

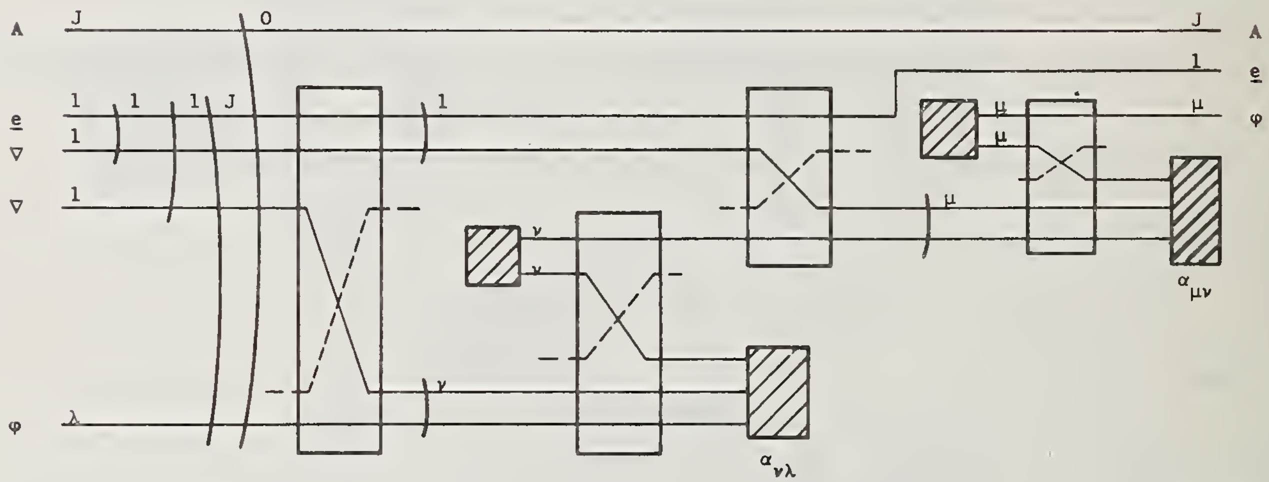


Fig. 2.11

After simplification this expression becomes

$$\mathrm{rot}\ \mathrm{rot}\ \vec{\rho} = \frac{p^2}{\hat{J}}\begin{cases}\sqrt{J}\left\{\sqrt{J}\ [A^{[J]}e^{[1]}\varphi^{[J+1]}]^{[0]} + \sqrt{J+1}\ \times [A^{[J]}e^{[1]}\varphi^{[J-1]}]^{[0]}\right\} & \text{if } \lambda = J+1\\ (2J+1)\ [A^{[J]}e^{[1]}\varphi^{[J]}]^{[0]} & \text{if } \lambda = J\\ \sqrt{J+1}\left\{\sqrt{J}\ [A^{[J]}e^{[1]}\varphi^{[J+1]}]^{[0]} + \sqrt{J+1}\ \times [A^{[J]}e^{[1]}\varphi^{[J-1]}]^{[0]}\right\} & \text{if } \lambda = J-1\end{cases} \quad . \qquad (2.60)$$

Finally the gradient is given by the formula

$$\text{grad}\left\{\hat{J}\,[A^{[J]}\varphi_p^{[J]}]^{[0]}\right\} = \sum_{\lambda = J \pm 1} p\,\alpha_{\lambda J}\,[A^{[J]}e^{[1]}\varphi_p^{[\lambda]}]^{[0]} \quad . \tag{2.61}$$

II.5 - STATES AND MATRIX ELEMENTS IN THE ANGULAR MOMENTUM COUPLED FOCK REPRESENTATION

II.5.1 - Commutators in a coupled scheme

Symmetrization or antisymmetrization of many-particle states of good angular momentum is carried out by fractional parentage coefficients. In field theory on the other hand the observable quantities are given in terms of field operators expressed with annihilation and creation operators well adapted to a single Slater determinant representation . Thus we must now give the tools to cast the fields into a form suitable to the angular momentum representation.

The creation and annihilation operators of section II.1.2 obey the commutation or anticommutation relations

$$\left\{a_m^{(j)},\, a_{m'}^{(j)\dagger}\right\}_{\pm} = \delta_{mm'} \quad , \tag{2.62}$$

or identically

$$(-)^{2j}\left\{a_m^{(j)},\, \tilde{a}_{m'}^{[j]}\right\}_{\pm} = \delta_{mm'} \quad . \tag{2.63}$$

In order to go to a coupled scheme we note that this expression may be written as

$$(-)^{2j}(-)^{j+m}\; a_{-m}^{[j]}\,\tilde{a}_{m'}^{[j]} = -\,\eta\,(-)^{2j}(-)^{j+m}\;\tilde{a}_{m'}^{[j]}\,a_{-m}^{[j]} + \delta_{mm'} \quad , \tag{2.64}$$

with $\eta = -1$ for Fermions and $+1$ for Bosons. After summation over the magnetic quantum numbers, with the restriction $m = m'$

$$[a^{[j]}\,\tilde{a}^{[j]}]^{[0]} = [\tilde{a}^{[j]}\,a^{[j]}]^{[0]} + (-)^{2j}\,\hat{j} \quad , \tag{2.65}$$

since $(-)^{2j} = \eta$. The commutator or anticommutator bracket notation $\{\ \}_{\pm}$ in eq.(2.63) is here of course to distinguish it from the coupling brackets $[\ \ ]$. Taking away the restriction $m = m'$ and coupling to an angular momentum $I \neq 0$ we get

$$[a^{[j]}\,\tilde{a}^{[j]}]_M^{[I]} = \eta\,(-)^{2j-I}\,[\tilde{a}^{[j]}\,a^{[j]}]_M^{[I]} + (-)^{2j}\,\hat{j}\,\delta_{I0} \tag{2.66}$$

with of course $\eta\,(-)^{2j} = 1$. This commutator or anticommutator relation in angular momentum representation shall be represented graphically by figure 2.12,

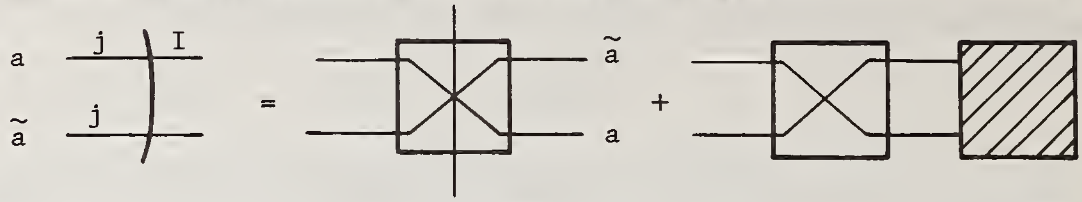


Fig. 2.12

where the square boxes are the usual crossing phases and the slash in the first one represents the factor η . (Recall that the overlap box, Fig.2.5a, implies a Kronecker factor δ_{I0}.)

II.5.2 - State vectors and wave functions

The basis configurational state vectors for n particles, defined in Eq. (1.20), in the occupation number representation, shall be constructed and used in the invariant form

$$|\Phi_n^{\alpha I}\rangle_{O.N.} = \hat{I}\,[W^{[I]}\,\tilde{S}^{[I]}_{n\,\alpha}]^{[0]}\,|0\rangle \quad , \tag{2.67}$$

where $|0\rangle$ is the reference vacuum state, $\tilde{S}^{[I]}_{n\alpha}$ a linear combination of products of creation operators $\tilde{a}^{[j]}$ chosen such as to yield a state of quantum number $I\alpha$. These state vectors are normalized according to

$$\langle\Phi_n^{\alpha I}\,|\Phi_n^{\beta I'}\rangle_{O.N.} = \delta_{\alpha\beta}\,\delta_{II'} \quad . \tag{2.68}$$

Hence

$$\langle 0|S^{[I]}_{n\alpha}\,\tilde{S}^{[I']}_{n\beta}\,|0\rangle = \delta_{\alpha\beta}\,\delta_{II'} \quad , \tag{2.69}$$

since

$$[W^{[I]}\,|W^{[I]}] = \frac{1}{\hat{I}} \quad \text{or} \quad \sum_M |W^{[I]}_M|^2 = 1 \quad . \tag{2.70}$$

In the R representation the corresponding invariant wave function is

$$\Phi_n^{\alpha I}(R) \equiv \mathcal{N}\,\langle 0|\{\Psi\}^n\,\hat{I}\,[W^{[I]}\,\tilde{S}^{[I]}]^{[0]}\,|0\rangle \quad , \tag{2.71}$$

where Ψ is the field operator and

$$\{\Psi\}^n = \Psi(x_n)\,\Psi(x_{n-1})\ldots\,\Psi(x_1) \quad . \tag{2.72}$$

We consider now a few special cases.

i) One particle state vector

In the occupation number representation it is

$$|\Phi_1^I\rangle_{O.N.} = \hat{I}\ [W^{[I]}\ \tilde{a}^{[I]}]^{[0]}|0\rangle \quad ,$$

$$_{O.N.}\langle\Phi_1^I| = \hat{I}\ \langle 0|[\tilde{W}^{[I]}\ a^{[I]}]^{[0]} \quad . \qquad (2.73)$$

The factor $\hat{I}$ ensures normalization as shown by the norm diagram of figure 2.13

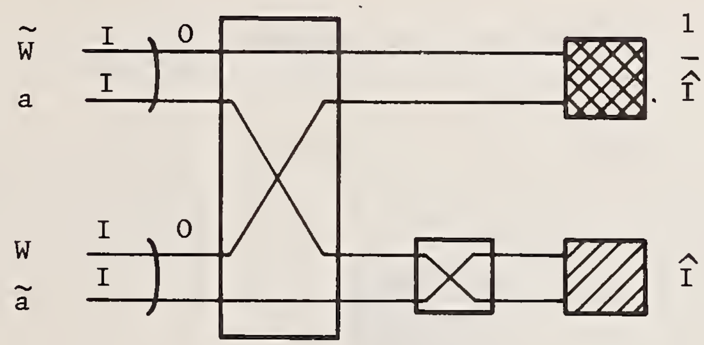


Fig. 2.13

In the R representation

$$\Phi_1^I(x) = \mathcal{N}\ \hat{I}\ \langle 0|\psi(x)\ [W^{[I]}\ \tilde{a}^{[I]}]^{[0]}|0\rangle$$

$$= \hat{I}\ [W^{[I]}\ \varphi^{[I]}(x)]^{[0]} \quad . \qquad (2.74)$$

The norm $\mathcal{N}$ is equal to unity if the single particle wave function $\varphi^{[I]}(x)$ entering in the field $\psi(x) \approx a\ \varphi + \ldots\ldots$ is assumed to be normalized. This expression can be read off the diagram of figure 2.14. We have introduced the cross-hatched box for the termination of the amplitudes having a value $1/\hat{I}$ according to 2.20.

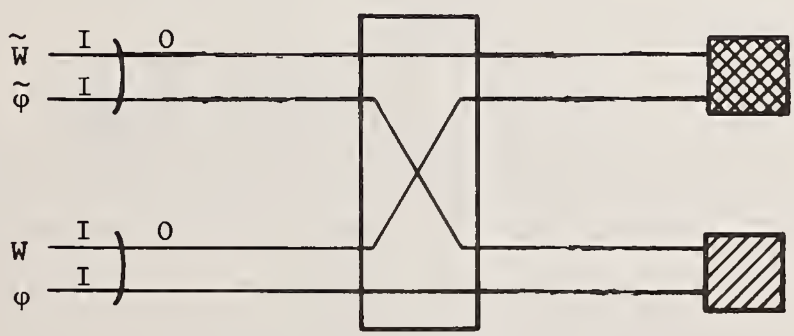


Fig. 2.14

ii) Two particle state vector

The state vector in the O.N. representation for two particles in shells j and k respectively is

$$|\Phi_2^I\rangle_{O.N.} = \mathcal{N}\,\hat{I}\,[W^{[I]}[\tilde{a}^{[j]}\,\tilde{a}^{[k]}]^{[I]}]^{[0]}|0\rangle \quad ,$$

$$_{O.N.}\langle\Phi_2^I| = (-)^{j+k-I}\,\mathcal{N}\,\hat{I}\,\langle 0|[\tilde{W}^{[I]}[a^{[k]}\,a^{[j]}]^{[I]}]^{[0]} \quad . \qquad (2.75)$$

The norm is given by the norm diagram (2.15).

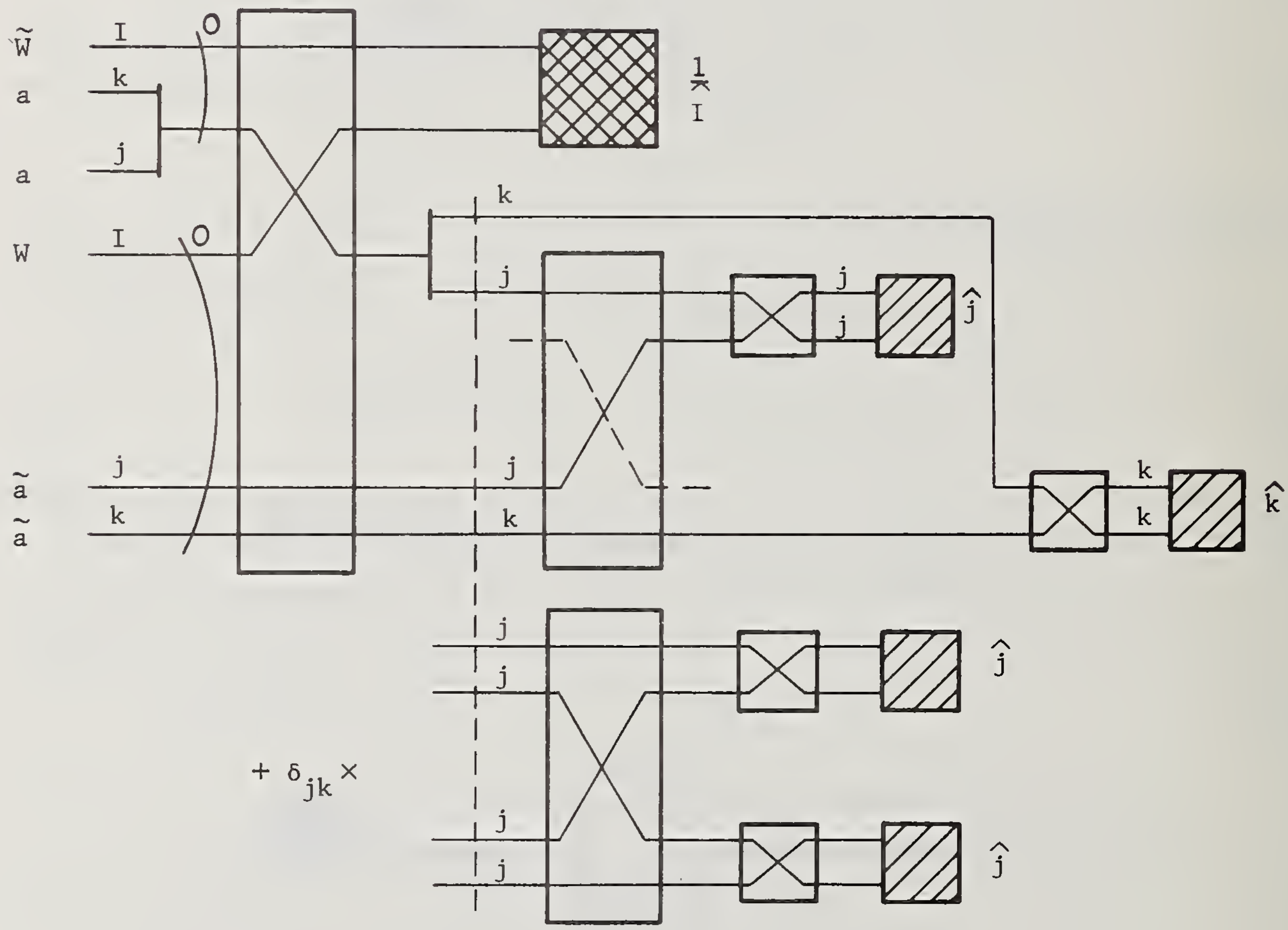


Fig. 2.15

Hence

$$|\Phi_2^I\rangle_{O.N.} = \frac{1}{\sqrt{1+\delta_{jk}}} \hat{I} \, [W^{[I]} [\tilde{a}^{[j]} \tilde{a}^{[k]}]^{[I]}]^{[0]} |0\rangle \tag{2.76}$$

or in our notation

$$\tilde{S}_2^{[I]} = \frac{1}{\sqrt{1+\delta_{jk}}} [\tilde{a}^{[j]} a^{[k]}]^{[I]} \quad . \tag{2.77}$$

iii) The s^n Boson state vector

The case of n identical bosons in a single s shell is very simple and will play later on an important role. Here only the isospin part needs symmetrization, and for isospin 1 particles all possible states are uniquely specified by the particle number and the total isospin quantum number T. This happens because each state (n,T) is generated at most by two states, namely the states (n-1, T-1) and (n-1, T+1) which yield , together with the normalization condition imposed on the two coefficients, a single possible state.

Thus in order to generate the successive (n,T) states let us consider the recursion in isospin quantum numbers

$$(A^n)^{[T]} |0\rangle = \frac{1}{\sqrt{n!}} \left\{ \alpha_n(T) \, [(A^{n-1})^{[T-1]} A^{[1]}]^{[T]} + \beta_n(T) \, [(A^{n-1})^{[T+1]} A^{[1]}]^{[T]} \right\} |0\rangle$$

$$\begin{aligned} (A^{n+1})^{[T]} |0\rangle = \frac{1}{\sqrt{n+1!}} \Big\{ \Big[\alpha_{n+1}(T) \Big\{ & \alpha_n(T-1) \, [(A^{n-1})^{[T-2]} A^{[1]}]^{[T-1]} A^{[1]}]^{[T]} \\ & + \beta_n(T-1) \, [(A^{n-1})^{[T]} A^{[1]}]^{[T-1]} A^{[1]}]^{[T]} \Big\} \\ + \beta_{n+1}(T) \Big\{ & \alpha_n(T+1) \, [(A^{n-1})^{[T]} A^{[1]}]^{[T+1]} A^{[1]}]^{[T]} \\ & + \beta_n(T+1) \, [(A^{n-1})^{[T+2]} A^{[1]}]^{[T+1]} A^{[1]}]^{[T]} \Big] \Big\} |0\rangle \quad . \end{aligned} \tag{2.78}$$

These four terms can be recoupled so as to bring out the symmetry character of the two last particles. It must be symmetric, namely the recoupled terms where the two last particles have $t = 1$ must be zero which yields the recursion relation

$$\alpha_{n+1}(T) \, \beta_n(T-1) \begin{bmatrix} 1 & 1 & T-1 \\ 0 & 1 & 1 \\ T & t & T \end{bmatrix} + \beta_{n+1}(T) \, \alpha_n(T+1) \begin{bmatrix} T & 1 & T+1 \\ 0 & 1 & 1 \\ T & t & T \end{bmatrix} = 0 \quad . \tag{2.79}$$

We now get the coefficients of the state together with the normalization condition

$$\alpha_{n+1}^{2}(T) + \beta_{n+1}^{2}(T) = 1 \quad .$$

For example for $n = 3$

$$\alpha_3(T=1) = \frac{\sqrt{5}}{3} \;, \qquad \beta_3(T=1) = \frac{2}{3} \;,$$

$$\alpha_3(T=3) = 1 \;, \qquad \beta_3(T=3) = 0 \quad . \tag{2.80}$$

iv) The general n particle state vector

Let us first consider the case of Fermions and denote (Φ_{n-1}) a single normalized Slater determinant for $n-1$ particles. The n particle antisymmetrized state in the R representation is

$$(\Phi_n) = \sum_i P_i \, \varphi_m^{[j]}(x_i)\,(\Phi_{n-1}) \tag{2.81}$$

where P_i is the appropriate permutation operator and phase. The normalization is

$$\langle (\Phi_n) | (\Phi_n) \rangle = n \quad . \tag{2.82}$$

A correctly antisymmetrized normalized coupled state is a linear combination of the form

$$\Phi_{n\,\alpha}^{[I]}(R) = \sum_{\beta L} (L\, j\, \beta |\, \} I\, \alpha)\, [\varphi^{[j]}(x)\; (\Phi_{n-1}^{\beta})^{[L]}]^{[I]} \;, \tag{2.83}$$

where the single particle CFP $(|\,\})$ has been introduced. In the occupation number representation the equivalent expression of Eq.(2.81) is

$$|\,(\Phi_n)\rangle_{O.N.} = \tilde{a}_m^{[j]}\, \tilde{S}_{n-1} |0\rangle \;, \tag{2.84}$$

with $\tilde{S}_{n-1}$ a normalized product of $n-1$ creation operators. Since all individual state indices $(j'm')$ in $\tilde{S}_{n-1}$ are such that $(j'm') \neq (jm)$ the norm is

$$\langle 0 | S_{n-1}\, a_m^{[j]}\, \tilde{a}_m^{[j]}\, \tilde{S}_{n-1} |0\rangle = 1 \quad . \tag{2.85}$$

Comparing with the expression (2.82) in R space, the coupled state vector in the occupation number representation is thus seen to be

$$\tilde{S}^{[I]}_{n\,\alpha}|0\rangle = \sqrt{n}\sum_{\beta L}(L\,j\,\beta|\}\,I\,\alpha)\,[\tilde{a}^{[j]}\,\tilde{S}^{[L]}_{n-1\,\beta}]^{[I]}|0\rangle \tag{2.86}$$

where the $\sqrt{n}$ factor corrects for the normalization of the usual CFP coefficients calculated in tables with the definitions of Eq.(2.82) and Eq.(2.83).

For Bosons the same result applies. This can be most easily seen from the fact that the norm is determined by the number of ways a particular coupling scheme appears. In that respect the only difference between Bosons and Fermions is the appearance of a minus sign in the contraction, i.e., the η factor in Eq.(2.66). The number of permutations (and the corresponding coupling schemes) are the same for both cases. Hence the relations (2.81) and (2.85) hold also for Boson and consequently the result (2.86) remains true. Of course the CFP's are different for Bosons and Fermions. In particular an arbitrary number of Bosons can occupy a given $N\ell j$ shell in contrast to Fermions for which the CFP's vanish for $n > 2j+1$.

II.5.3 - Matrix elements of Fermion-Boson systems

The field operators which will be constructed in the following chapters are of the general invariant form (see Eq.(2.21)).

$$\mathcal{O}^{\mathcal{L}\mathcal{T}} = \sum F(x,y\,;\lambda\mu\lambda'\mu'\mathcal{L}\mathcal{T})\,\hat{\mathcal{L}}\,\hat{\mathcal{T}}\,[\omega^{[\mathcal{L}\mathcal{T}]}\,\mathbf{B}_x^{[\lambda\mu]}\,\mathbf{A}_y^{[\lambda'\mu']}]^{[00]} \tag{2.87}$$

where $F(x,y\,;\lambda\mu\lambda'\mu'\mathcal{L}\mathcal{T})$ is the invariant matrix element for a given point process of multipolarities $\mathcal{L}$ and $\mathcal{T}$ in angular momentum and isospin spaces. $\mathbf{B}_x^{[\lambda\mu]}$ and $\mathbf{A}_y^{[\lambda'\mu']}$ are respectively linear combination of products of Boson and Fermion operators of multipolarities λ, λ' and isospin μ, μ'. For simplicity we shall limit here our expression to a scalar operator, for example the energy, for which $\mathcal{L}=0$, $\mathcal{T}=0$ and $\lambda=\lambda'$, $\mu=\mu'$.

$$\mathcal{O}^{00} = \sum F(x,y\,;\lambda\mu)\,[\mathbf{B}_x^{[\lambda\mu]}\,\mathbf{A}_y^{[\lambda\mu]}]^{[00]} \quad . \tag{2.88}$$

$F(x,y\,;\lambda\mu)$ is an invariant matrix element whose calculation is given in Chapter III (free field energy), Chapter IV (center of mass energy), Chapter VI (field interactions).

We consider now an initial state $|i,IT\rangle$ made of many Fermions coupled to J_iT_i and many Bosons coupled to $J_i'T_i'$:

$$|i\rangle = \hat{I}\,\hat{T}\Big[[W^{[IT]}[\tilde{\mathbf{Y}}_i^{[J_iT_i]}\,\tilde{\mathbf{X}}_i^{[J_i'T_i']}]^{[IT]}\Big]^{[00]}|0\rangle \quad , \tag{2.89}$$

where $\mathbf{Y}$ and $\mathbf{X}$ represent linear combinations of Fermion and Boson operators which are properly orthonormalized as discussed above, defining completely a configuration i in a given coupling scheme. We have a similar expression for the final state $|f\rangle$ and the matrix element is here,

$$\langle f|\mathcal{O}^{00}|i\rangle = \sum F(x,y\,;\lambda\mu)\;\hat{I}^2\,\hat{T}^2$$
$$\langle 0|\,[\tilde{W}_f^{[IT]}\,[\mathbf{Y}_f^{[J_fT_f]}\,\mathbf{X}_f^{[J_f'T_f']}]^{[IT]}]^{[00]}\,|\,[\mathbf{B}_x^{[\lambda\mu]}\,\mathbf{A}_y^{[\lambda\mu]}]^{[00]}\,|$$
$$[W_i^{[IT]}\,[\tilde{\mathbf{Y}}_i^{[J_iT_i]}\,\tilde{\mathbf{X}}_i^{[J_i'T_i']}]^{[IT]}]^{[00]}\,|0\rangle \qquad . \qquad (2.90)$$

The recoupling diagram is given in figure 2.16.

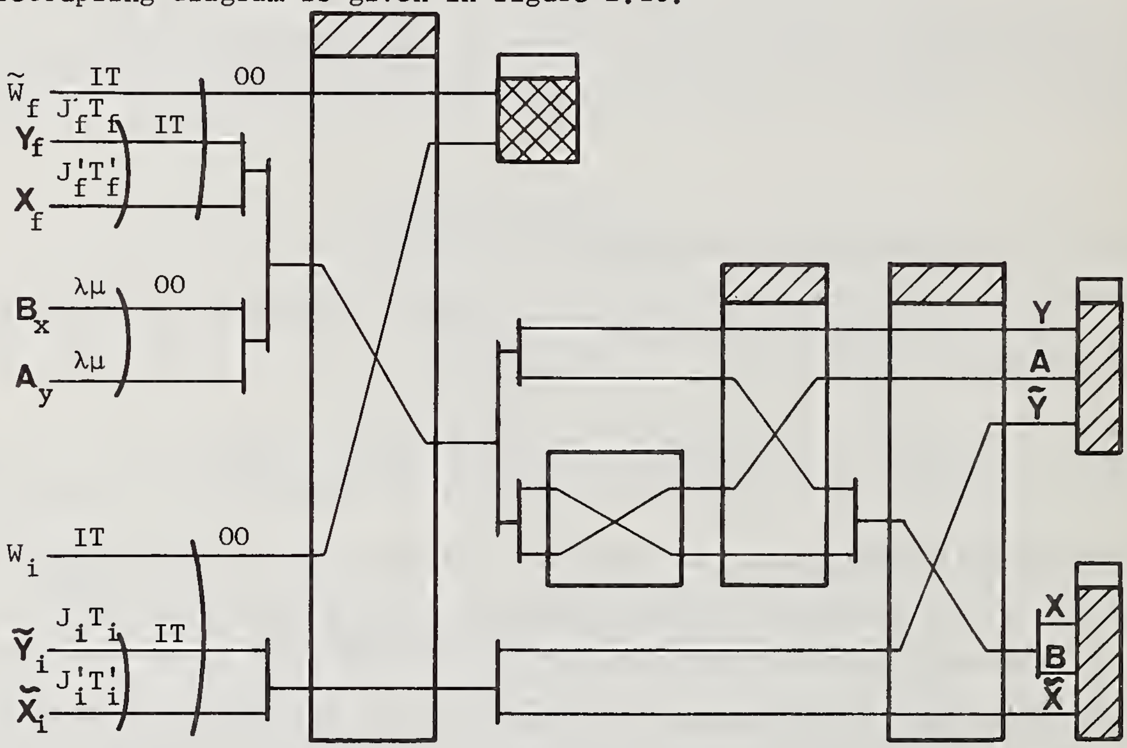


Fig. 2.16

This diagram yields,

$$\langle f|\mathcal{O}^{00}|i\rangle = \sum_{xy\lambda\mu} F(x,y\,;\lambda\mu)\begin{bmatrix} J_f & J_f' & I \\ \lambda & \lambda & 0 \\ J_i & J_i' & I \end{bmatrix}\begin{bmatrix} T_f & T_f' & T \\ \mu & \mu & 0 \\ T_i & T_i' & T \end{bmatrix}\frac{1}{\hat{J}_i\hat{J}_i'\,\hat{T}_i\hat{T}_i'}$$
$$\langle 0|\,[\,\mathbf{X}^{[J_f'T_f']}\,|\,\mathbf{B}^{[\lambda\mu]}\,|\,\mathbf{X}^{[J_i'T_i']}\,]^{[00]}\,|0\rangle$$
$$\langle 0|\,[\,\mathbf{Y}^{[J_fT_f]}\,|\,\mathbf{A}^{[\lambda\mu]}\,|\,\mathbf{Y}^{[J_iT_i]}\,]^{[00]}\,|0\rangle \qquad . \qquad (2.91)$$

Thus after evaluation of the basic invariant matrix elements F, the many body problem appears as the usual evaluation of mean values of complicated products of creation and annihilation operators according to well known technique. A few simple examples will be worked out in the following chapters.

CHAPTER III

FREE FIELD DISCRETIZED EXPANSIONS AND ENERGIES

III.1 - SPIN 0 FIELD

III.1.1 - Field equations

We consider a complex field $\Phi'_\kappa(r,t)$ of spin 0, isospin 1 and charge κ. The Lagrangian is

$$\mathscr{L} = - \sum_\kappa \partial_\lambda \Phi'^+_\kappa \partial_\lambda \Phi'_\kappa + m^2 \Phi'^+_\kappa \Phi'_\kappa \quad . \tag{3.1}$$

For $\kappa = 0$, $\Phi'^+_o = \Phi_o$. The conjugate field is

$$\pi'_\kappa(\vec{r},t) = \frac{\partial \mathscr{L}}{\partial \dot{\Phi}'_\kappa} = \dot{\Phi}'^+_\kappa \tag{3.2a}$$

$$\pi'^+_\kappa(\vec{r},t) = \frac{\partial \mathscr{L}}{\partial \dot{\Phi}'^+_\kappa} = \dot{\Phi}'_\kappa \quad . \tag{3.2b}$$

We shall work with real fields $\Phi(\vec{r},t)$ defined as

$$\Phi = \Phi' + \Phi'^+ \quad , \tag{3.3}$$

which yields, deleting the κ index

$$\mathscr{L} = - \frac{1}{2} \left\{ \partial_\lambda \Phi \partial_\lambda \Phi + m^2 \Phi^2 \right\} \tag{3.4}$$

and

$$\pi \equiv \pi^+ = i \partial_4 \Phi \quad . \tag{3.5}$$

The equations of motion (Klein-Gordon) are

$$- \partial^2_\lambda \Phi + m^2 \Phi = 0 \tag{3.6}$$

and the Hamiltonian

$$\begin{aligned} H &= : \pi \dot{\Phi} - \mathscr{L} : = : - \partial_4 \Phi \partial_4 \Phi - \mathscr{L} : \\ &= \frac{1}{2} : \left\{ - \partial_4 \Phi \partial_4 \Phi + \partial_x \Phi \partial_x \Phi + m^2 \Phi^2 \right\} : \quad , \end{aligned} \tag{3.7}$$

where $x = 1, 2, 3$.

III.1.2 - Multipole expansion of the field

The plane wave solutions of the free field equations are

$$\Phi(\vec{r},t) = \left(\frac{1}{2\pi}\right)^{3/2} \sum_{\kappa} \int d^3p \, \frac{1}{\sqrt{2E}} \Big\{ e^{i(\vec{p}\,\vec{r} - E\,t)} a^{(1)}_{\kappa \vec{p}} \, \eta^{[1]}_{\kappa} + e^{-i(\vec{p}\,\vec{r} - E\,t)} a^{(1)+}_{\kappa \vec{p}} \, \eta^{[1]+}_{\kappa} \Big\} \quad . \tag{3.8}$$

In this expression $\eta^{[1]}_{\kappa}$ is the isospin wave function normalized according to

$$\left(\tilde{\eta}^{(1)}_{\kappa} \; \eta^{[1]}_{\kappa'}\right) = \delta_{\kappa\kappa'} \quad . \tag{3.9}$$

The creation (annihilation) operators $a^{(1)}_{\kappa\vec{p}}$ $(a^{(1)+}_{\kappa\vec{p}})$ transform as scalars in orbital space and as vectors in isospin space. They verify the commutation relations,

$$\left\{ a^{(1)}_{\kappa\vec{p}} \, , \; a^{(1)+}_{\kappa'\vec{p}\,'} \right\}_{-} = \delta_{\kappa\kappa'} \, \delta^3(\vec{p} - \vec{p}\,') \quad . \tag{3.10}$$

The normalization in (3.8) is chosen to achieve the equal time commutator,

$$\left\{ \Phi(\vec{r},t), \, \pi(\vec{r}\,',t) \right\}_{-} = i \, 3 \, \delta^3(\vec{r} - \vec{r}\,') \quad . \tag{3.11}$$

The factor 3 corresponds to the summation over the isospin projection. This way each particle of a given charge κ is normalized to unity.

We introduce the multipole expansion of the field at $t = 0$,

$$\Phi(\vec{r}) = \sqrt{\frac{2}{\pi}} \sum_{\kappa} \int d^3p \, \frac{1}{\sqrt{2E}} \sum_{\ell,m} i^{\ell} j_{\ell}(pr) \Big\{ \hat{p}^{(\ell)}_{m} \, \hat{r}^{[\ell]}_{m} \, a^{(1)}_{\kappa,\vec{p}} \, \eta^{[1]}_{\kappa} + (-)^{\ell} \, \hat{p}^{[\ell]}_{m} \, \hat{r}^{(\ell)}_{m} \, \tilde{a}^{[1]}_{\kappa\vec{p}} \, \tilde{\eta}^{(1)}_{\kappa} \Big\} \quad , \tag{3.12}$$

where we use the notation,

$$\hat{p}^{[\ell]}_{m} = Y^{[\ell]}_{m}(\hat{p}) = (-i)^{\ell} \, Y_{\ell m}(\hat{p}) \quad . \tag{3.13}$$

We define multipole creation and annihilation operators by

$$\underline{A}^{(\ell 1)}_{m\,\kappa}(p) = \int d^2\hat{p} \; a^{(1)}_{\kappa\vec{p}} \, \hat{p}^{(\ell)}_{m} \tag{3.14}$$

$$\tilde{\underline{A}}^{[\ell 1]}_{m\kappa}(p) = \underline{A}^{(\ell 1)+}_{m\kappa}(p) \quad , \tag{3.15}$$

which obey the commutation relations

$$\left\{\underline{A}^{(\ell 1)}_{m\kappa}(p),\ \underline{A}^{(\ell' 1)^+}_{m'\kappa'}(p')\right\}_- = \delta_{\kappa\kappa'}\ \delta_{\ell\ell'}\ \delta_{mm'}\ \frac{\delta(p-p')}{p^2} \quad . \tag{3.16}$$

Herewith the field becomes

$$\Phi(\vec{r}) = \sqrt{\frac{2}{\pi}} \int p^2 dp\ \frac{1}{\sqrt{2E}} \sum_{\ell m \kappa} (i)^\ell\ j_\ell(pr)$$
$$\times \left\{\underline{A}^{(\ell 1)}_{m\kappa}(p)\ \hat{r}^{[\ell]}_m\ \eta^{[1]}_\kappa + (-)^\ell\ \underline{\tilde{A}}^{[\ell 1]}_{m\kappa}(p)\ \hat{r}^{(\ell)}_m\ \tilde{\eta}^{(1)}_\kappa\right\} , \tag{3.17}$$

or in a coupled form both in orbital and isospin space

$$\Phi(\vec{r}) = \sqrt{\frac{2}{\pi}} \int p^2 dp\ \frac{1}{\sqrt{2E}} \sum_{\ell} (i)^\ell\ j_\ell(pr)\ \hat{\ell}\ \hat{1}$$
$$\times \left\{[\underline{A}^{[\ell 1]}(p)\ \hat{r}^{[\ell]}\ \eta^{[1]}]^{[oo]} + (-)^\ell\ [\underline{\tilde{A}}^{[\ell 1]}(p)\ \hat{r}^{[\ell]}\ \tilde{\eta}^{[1]}]^{[oo]}\right\} . \tag{3.18}$$

III.1.3 - Discretization of the field

We now want to go from the plane wave basis of continuous index p to a discrete basis with a discrete index ν. To that aim we introduce a set of orthogonal functions $f_{\nu\ell}(p)$ which obey the relations :

$$\text{orthogonality} \qquad \int p^2 dp\ f_{\nu\ell}(p)\ f_{\mu\ell}(p) = \delta_{\mu\nu} \quad , \tag{3.19}$$

$$\text{completeness} \qquad \sum_\nu f_{\nu\ell}(p)\ f_{\nu\ell}(p') = \frac{\delta(p-p')}{p^2} \quad . \tag{3.20}$$

We thus can introduce discretized creation and annihilation operators $A^{[\ell 1]}_\nu$ defined as

$$\underline{A}^{[\ell 1]}(p) = \sum_\nu A^{[\ell 1]}_\nu\ f_{\nu\ell}(p) \quad , \tag{3.21}$$

or conversely

$$A^{[\ell 1]}_\nu = \int p^2 dp\ \underline{A}^{[\ell 1]}(p)\ f_{\nu\ell}(p) \quad . \tag{3.22}$$

They fulfill the commutation relations

$$\left\{A^{(\ell 1)}_{\nu m\kappa},\ A^{(\ell' 1)^+}_{\nu' m'\kappa'}\right\}_- = \delta_{\kappa\kappa'}\ \delta_{mm'}\ \delta_{\ell\ell'}\ \delta_{\nu\nu'} \quad . \tag{3.23}$$

Substituting the definition (3.21) into the multipole expansion (3.18) we get for the field,

$$\Phi(\vec{r}) = \frac{1}{\sqrt{2}} \sum_{\nu} \sum_{\ell} g_{\nu\ell}(r) \, (i)^{\ell} \, \hat{\ell} \, \hat{1} \Big\{ [A_{\nu}^{[\ell 1]} \, \hat{r}^{[\ell]} \, \eta^{[1]}]^{[oo]} + (-)^{\ell} \, [\tilde{A}_{\nu}^{[\ell 1]} \, \hat{r}^{[\ell]} \, \tilde{\eta}^{[1]}]^{[oo]} \Big\} \, , \tag{3.24}$$

$$\pi(\vec{r}) = \dot{\Phi}(\vec{r}) = \frac{i}{\sqrt{2}} \sum_{\nu} \sum_{\ell} h_{\nu\ell}(r) \, (-i)^{\ell} \, \hat{\ell} \, \hat{1} \Big\{ [\tilde{A}_{\nu}^{[\ell 1]} \, \hat{r}^{[\ell]} \, \tilde{\eta}^{[1]}]^{[oo]} - (-)^{\ell} \, [A_{\nu}^{[\ell 1]} \, \hat{r}^{[\ell]} \, \eta^{[1]}]^{[oo]} \Big\} \, , \tag{3.25}$$

where we have introduced the functions,

$$g_{\nu\ell}(r) = \sqrt{\frac{2}{\pi}} \int p^2 dp \, \frac{1}{\sqrt{E}} \, j_{\ell}(pr) \, f_{\nu\ell}(p) \, , \tag{3.26}$$

$$h_{\nu\ell}(r) = \sqrt{\frac{2}{\pi}} \int p^2 dp \, \sqrt{E} \, j_{\ell}(pr) \, f_{\nu\ell}(p) \quad . \tag{3.27}$$

By a straigthforward calculation one may demonstrate on these discretized forms of the fields that the field commutation relations (3.11) are indeed fulfilled, using the result

$$\sum_{\nu} g_{\nu\ell}(r) \, h_{\nu\ell}(r') = \frac{2}{\pi} \iint p^2 dp \, p'^2 dp' \, j_{\ell}(pr) \, j_{\ell}(p'r') \, f_{\nu\ell}(p) \, f_{\nu\ell}(p') = \frac{\delta(r-r')}{r^2} \quad . \tag{3.28}$$

The wave function for the spin 0 particle with quantum numbers ν, ℓ, m, κ is obtained from the field expansion,

$$\varphi_{\nu m \kappa}^{[\ell 1]}(\vec{r}) = \langle 0 | \Phi \, \tilde{A}_{\nu m \kappa}^{[\ell 1]} | 0 \rangle = \frac{1}{\sqrt{2}} \, (i)^{\ell} \, g_{\nu\ell}(r) \, \hat{r}_{m}^{[\ell]} \, \eta_{\kappa}^{[1]} \, , \tag{3.29}$$

which is normalized according to

$$i \int d^3 r \, \varphi_{\nu m \kappa}^{[\ell 1]*}(\vec{r}) \, \frac{\overleftrightarrow{\partial}}{\partial t} \, \varphi_{\nu' m' \kappa'}^{[\ell' 1]}(\vec{r}) = \delta_{\nu\nu'} \, \delta_{\ell\ell'} \, \delta_{mm'} \, \delta_{\kappa\kappa'} \quad . \tag{3.30}$$

Here we have the usual notation $f \frac{\overleftrightarrow{\partial}}{\partial t} g = f \dot{g} - \dot{f} g$.

III.1.4 - The discretizing unitary transformation

We choose for the discrete basis $f_{\nu\ell}(p)$ harmonic oscillator functions,

$$f_{\nu\ell}(p) = \alpha^{3/2} \, C_{\nu\ell} \, \exp\Big(-\tfrac{1}{2} \alpha^2 p^2\Big) (\alpha p)^{\ell} \, L_{\nu}^{(\ell+1/2)}(\alpha^2 p^2) \quad , \tag{3.31}$$

where α specifies the scale and where ν = 0, 1, 2, 3 ... etc, with

$$C_{\nu\ell} = \frac{1}{\pi^{1/4}} \left[\frac{2^{\nu+\ell+2}\ \nu!}{(2\nu+2\ell+1)!!} \right]^{1/2} . \tag{3.32}$$

The harmonic oscillator principal quantum number is $N = 2\nu + \ell$.

III.2 - THE FREE FIELD ENERGY FOR SPIN 0 PARTICLES

The Boson Hamiltonian is of the form, eq.(3.7)

$$\begin{aligned}
\int dr^3 \mathscr{H}_B &= \frac{1}{2} : \int d^3r \left\{ - \partial_4 \Phi \partial_4 \Phi + \partial_x \Phi \partial_x \Phi + m^2 \Phi^2 \right\} : \\
&= \frac{1}{2} \int d^3r\ \Phi (- \overleftarrow{\partial}_4 \overrightarrow{\partial}_4 - \nabla^2 + m^2) \Phi \\
&= \frac{1}{2} \int d^3r \sum_{\substack{\ell_1 \ell_2 \\ \nu_1 \nu_2}} \frac{i^{\ell_1+\ell_2}}{2} \hat{\ell}_1 \hat{\ell}_2 \hat{1}^2 \frac{2}{\pi} \int p^2 dp \int p'^2 dp' \\
&\times j_{\ell_1}(pr) f_{\ell_1 \nu_1}(p)\ 2(-)^{\ell_1} [\tilde{A}^{[\ell_1 1]}_{\nu_1}\ \hat{r}^{[\ell_1]}\ \tilde{\eta}^{[1]}]^{[oo]} (- \overleftarrow{\partial}_4 \overrightarrow{\partial}_4 - \nabla^2 + m^2) \\
&\times [A^{[\ell_2 1]}_{\nu_2}\ \hat{r}^{[\ell_2]}\ \eta^{[1]}]^{[oo]}\ j_{\ell_2}(p'r) f_{\ell_2 \nu_2}(p') \quad ,
\end{aligned} \tag{3.33}$$

where a factor 2 comes from the two identical contributions $:\tilde{A}A:$ and $:A\tilde{A}:$.
Using

$$(- \nabla^2 + m^2) j_\ell(pr)\ Y^{[\ell]}(\hat{r}) = (p^2 + m^2) j_\ell(pr)\ Y^{[\ell]}(\hat{r}) \quad , \tag{3.34}$$

we get,

$$\int d^3r\ \mathscr{H}_B = \sum_{\substack{\ell \\ \nu_1 \nu_2}} \hat{\ell}\ \hat{1} \int p^2 dp\ E\ f_{\ell \nu_1}(p)\ f_{\ell \nu_2}(p) \left[\tilde{A}^{[\ell 1]}_{\nu_1}\ A^{[\ell 1]}_{\nu_2} \right]^{[oo]}$$

and for the matrix element, defined between single particle states as in eq.(2.23)

$$\begin{aligned}
&\hat{\ell}_2\ \hat{\ell}_1\ \hat{1}^2 \langle 0 | [\tilde{W}^{[\ell_1 1]}_{\nu_1}\ A^{[\ell_1 1]}_{\nu_1}]^{[oo]} \int d^3r\ \mathscr{H}_B [W^{[\ell_2 1]}_{\nu_2}\ \tilde{A}^{[\ell_2 1]}_{\nu_2}] | 0 \rangle \\
&= \hat{\ell}\ \hat{1}\ [\tilde{W}^{[\ell 1]}_{\nu_1} | W^{[\ell 1]}_{\nu_2}] \int p^2 dp\ E\ f_{\nu_1 \ell}(p)\ f_{\nu_2 \ell}(p)\ \delta_{\ell_1 \ell_2 \ell} .
\end{aligned} \tag{3.35}$$

Thus, the free field energy invariant matrix element for spin 0 particle is

$$[\nu_1 \ell_1 | \int d^3r\ \mathscr{H}_B | \nu_2 \ell_2] = \delta_{\ell_1 \ell_2}\ \hat{\ell}_1 \hat{1} \int p^2 dp\ E\ f_{\nu_1 \ell_1}(p)\ f_{\nu_2 \ell_1}(p) \quad . \tag{3.36}$$

III.3 - SPIN 1/2 FIELD

III.3.1 - Field equations

We consider a field ψ for spin 1/2 and isospin 1/2 particles. Its components are denoted $\psi_{\alpha\kappa}(\vec{r},t)$ where α stands for the spinor index and κ for the isospin projection.

The Lagrangian is

$$\mathscr{L} = - \sum_{\kappa} (\bar{\psi}_\kappa \gamma_\mu \partial_\mu \psi_\kappa) + m(\bar{\psi}_\kappa \psi) \quad , \tag{3.37}$$

with

$$\bar{\psi}_\kappa = \psi^+ \gamma_4 \quad . \tag{3.38}$$

The conjugate field is

$$\pi_{\kappa,\alpha} = i\ (\bar{\psi}_\kappa \gamma_4)_\alpha = i\ \psi^+_{\kappa,\alpha} \quad . \tag{3.39}$$

The equation of motion (Dirac equation) is,

$$\frac{\partial\mathscr{L}}{\partial\bar{\psi}} = 0 = \gamma_\mu \partial_\mu \psi + m\,\psi \quad , \tag{3.40}$$

and the Hamiltonian

$$H = :\pi\dot{\psi} - \mathscr{L}: = \ :\sum_{i=1,2,3} \bar{\psi}\gamma_i \partial_i \psi + m\bar{\psi}\psi : \quad . \tag{3.41}$$

We choose the following representation for the 4×4 representation of the γ matrices,

$$\gamma_{i=x,y,z} = \begin{pmatrix} 0 & -i\sigma_i \\ i\sigma_i & 0 \end{pmatrix} \qquad \gamma_4 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{3.42}$$

with the usual definition ,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad . \tag{3.43}$$

Thus we have the anticommutation relations,

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\,\delta_{\mu\nu} \quad ; \qquad \gamma_\mu^2 = +1 \quad . \tag{3.44}$$

We work with the metric :

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad . \tag{3.45}$$

Hence the four vector scalar product $\gamma.x$ shall be denoted by

$$\gamma.x = \sum_{i=1,2,3} \gamma_i x_i + \gamma_4 x_4 \tag{3.46}$$

or

$$\gamma x = \sum_{i=1,2,3} \gamma_i x_i - \gamma_o x_o \quad , \tag{3.47}$$

with $x_4 = it$, $x_o = t$ and $\gamma_o = -i\,\gamma_4$. Likewise the time derivative is

$$\dot{\psi} \equiv \partial_o \psi = i\,\partial_4 \psi \quad . \tag{3.48}$$

Let us recall here a few useful relations for the 4×4 matrices $\underset{\sim}{\sigma}_i$ ($i,j,k = x,y,z$, cyclically) :

$$\underset{\sim}{\sigma}_i \underset{\sim}{\sigma}_j - \underset{\sim}{\sigma}_j \underset{\sim}{\sigma}_i = 2\,i\,\underset{\sim}{\sigma}_k (\varepsilon_{ijk}) \ , \tag{3.49}$$

$$\underset{\sim}{\sigma}_i \underset{\sim}{\sigma}_j + \underset{\sim}{\sigma}_j \underset{\sim}{\sigma}_i = 2\,\delta_{ij} \ , \tag{3.50}$$

$$\underset{\sim}{\sigma}_i = -\,i\,\gamma_j \gamma_k \quad . \tag{3.51}$$

The matrices

$$S_i = \frac{1}{4} (\gamma_j \gamma_k - \gamma_k \gamma_j) = \frac{i}{2} \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} \tag{3.52}$$

verify

$$\left\{ S_i , S_j \right\}_- = i\,S_k\,\varepsilon_{ijk} \quad . \tag{3.53}$$

Hence our representation of spin is

$$\vec{S} = \frac{i}{2} \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \quad . \tag{3.54}$$

We shall also need the matrix

$$\gamma_5 = \gamma_x \gamma_y \gamma_z \gamma_4 = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix} \tag{3.55}$$

which verifies the anticommutators

$$\gamma_5 \gamma_\nu + \gamma_\nu \gamma_5 = 0 \quad . \tag{3.56}$$

We define now the spherical tensor representation of these various matrices, which will be needed to carry out the calculations in angular momentum Hilbert space,

$$\underset{\sim}{\sigma}_o^{[1]} = -i\underset{\sim}{\sigma}_z = -\gamma_x\gamma_y \quad , \tag{3.57}$$

$$\underset{\sim}{\sigma}_1^{[1]} = \frac{1}{\sqrt{2}}(i\underset{\sim}{\sigma}_x - \underset{\sim}{\sigma}_y) = \frac{1}{\sqrt{2}}(\gamma_y\gamma_z + i\gamma_z\gamma_x) \quad , \tag{3.58}$$

$$\underset{\sim}{\sigma}_{-1}^{[1]} = \frac{-1}{\sqrt{2}}(i\underset{\sim}{\sigma}_x + \underset{\sim}{\sigma}_y) = \frac{1}{\sqrt{2}}(-\gamma_y\gamma_z + i\gamma_z\gamma_x) \quad . \tag{3.59}$$

We can check from the definitions,

$$\underset{\sim}{\sigma}_1^{[1]} = (-)^{1-m}\,\underset{\sim}{\sigma}_{-1}^{[1]} \quad , \tag{3.60}$$

$$\left\{\underset{\sim}{\sigma}_m^{[1]}, \underset{\sim}{\sigma}_n^{[1]}\right\}_+ = 0 \quad \text{for all m,n except} \quad , \tag{3.61}$$

$$\left\{\underset{\sim}{\sigma}_o^{[1]}, \underset{\sim}{\sigma}_o^{[1]}\right\}_+ = -2 \quad . \tag{3.62}$$

Likewise the spherical $\gamma_m^{[1]}$ are given by

$$\gamma_m^{[1]} = \begin{pmatrix} 0 & -i\,\sigma_m^{[1]} \\ i\,\sigma_m^{[1]} & 0 \end{pmatrix} \quad , \tag{3.63}$$

and

$$\left\{\gamma_m^{[1]}, \gamma_n^{[1]}\right\}_+ = 0 \quad \text{for all m,n except} \tag{3.64}$$

$$\left\{\gamma_o^{[1]}, \gamma_o^{[1]}\right\}_+ = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \quad . \tag{3.65}$$

III.3.2 - The plane wave solutions

The free field plane wave expansion of the solutions of the Dirac equation (3.40) is,

$$\psi_\alpha(\vec{r},t) = \left(\frac{1}{2\pi}\right)^{3/2} \int d^3p \sqrt{\frac{m}{E}} \sum_{\kappa,s} \Big\{ a_{s\,\kappa\,\vec{p}}^{(1/2\ 1/2)}\, \mathcal{U}_{s\,\alpha}^{[1/2]}(p)\ \eta_\kappa^{[1/2]} \exp(i(\vec{p}\vec{r} - Et))$$

$$+ \tilde{b}_{s\,\kappa\,\vec{p}}^{[1/2\ 1/2]}\, \tilde{\mathcal{V}}_{s\,\alpha}^{(1/2)}(p)\ \tilde{\eta}_\kappa^{(1/2)} \exp(-i(\vec{p}\vec{r} - Et)) \Big\} \quad , \tag{3.66}$$

where $a_{s\,\kappa\,\vec{p}}^{[1/2\ 1/2]}$ denotes the annihilation operator for a particle of spin 1/2, spin projection s, isospin 1/2, isospin projection κ, described by a plane wave of linear momentum $\vec{p}$ corresponding to the energy

$$E^2 = p^2 + m^2 \quad . \tag{3.67}$$

We have also introduced for the same energy the creation operator for an anti-particle of same quantum numbers s, κ, $\vec{p}$,

$$\tilde{b}^{[1/2\ 1/2]}_{s\,\kappa\vec{p}} = b^{(1/2\ 1/2)+}_{s\,\kappa\vec{p}} \quad . \tag{3.68}$$

Note that the phase $(-)^{2S+2T}$ from Eq.(2.15) is here equal to +1 . The isospin 1/2 functions are normalized according to

$$(\tilde{\eta}^{(1/2)}_{\kappa}\ \eta^{[1/2]}_{\kappa'}) = \delta_{\kappa\kappa'} \quad . \tag{3.69}$$

The spinors are solutions of the equations of motion (3.40),

$$(i\,\gamma_\lambda p_\lambda + m)\ \mathcal{U}^{[1/2]}_{s}(p) = 0 \quad , \tag{3.70}$$

$$(-i\,\gamma_\lambda p_\lambda + m)\ \mathcal{V}^{[1/2]}_{s}(p) = 0 \quad . \tag{3.71}$$

In a spherical representation they are of the form,

$$\mathcal{U}^{[1/2]}_{s}(p) = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} \chi^{[1/2]}_{s} \\ \dfrac{\vec{\sigma}\,.\,\vec{p}}{E+m} \times \chi^{[1/2]}_{s} \end{pmatrix} \quad , \tag{3.72}$$

$$\mathcal{V}^{[1/2]}_{s}(p) = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} \dfrac{\vec{\sigma}\,.\,\vec{p}}{E+m} \times \chi^{[1/2]}_{s} \\ \chi^{[1/2]}_{s} \end{pmatrix} \quad . \tag{3.73}$$

The explicit form of the spin 1/2 vectors $\chi^{[1/2]}_{s}$ in our representation are given in equation (2.10). The conjugate functions $\tilde{\chi}^{(1/2)}_{s}$ defined by the normalization (2.11) are given in equation (2.12).

With the chosen normalization the spinors obey the following relations (the α indices here denote the spinor indices).

i) Orthogonality

$$\sum_{\alpha} \mathcal{U}^{[1/2]+}_{s\,\alpha}(p)\ \mathcal{U}^{[1/2]}_{s'\alpha}(p) = \frac{E}{m}\,\delta_{ss'} \quad . \tag{3.74}$$

In order to demonstrate this result we note that

$$\vec{\sigma}.\vec{p}\ \chi_s^{[1/2]} = \hat{1}[\sigma^{[1]}\vec{p}^{[1]}]^{[o]}\ \chi_s^{[1/2]} = -\hat{1}\begin{bmatrix}1 & 1 & 0\\ 0 & 1/2 & 1/2\\ 1 & 1/2 & 1/2\end{bmatrix}\frac{\hat{1}}{\hat{2}}\begin{bmatrix}1/2 & 1/2 & 0\\ 0 & 1/2 & 1/2\\ 1/2 & 0 & 1/2\end{bmatrix}$$

$$\times\ [1/2|\sigma|1/2][\vec{p}^{[1]}\ \chi^{[1/2]}]_s^{[1/2]} \qquad (3.75)$$

$$= i\sqrt{3}\ [\vec{p}^{[1]}\ \chi^{[1/2]}]_s^{[1/2]} \quad ,$$

from the coupling diagram of figure 3.1, which makes use of the graphical representation of completeness in spin space according to figure 2.7.

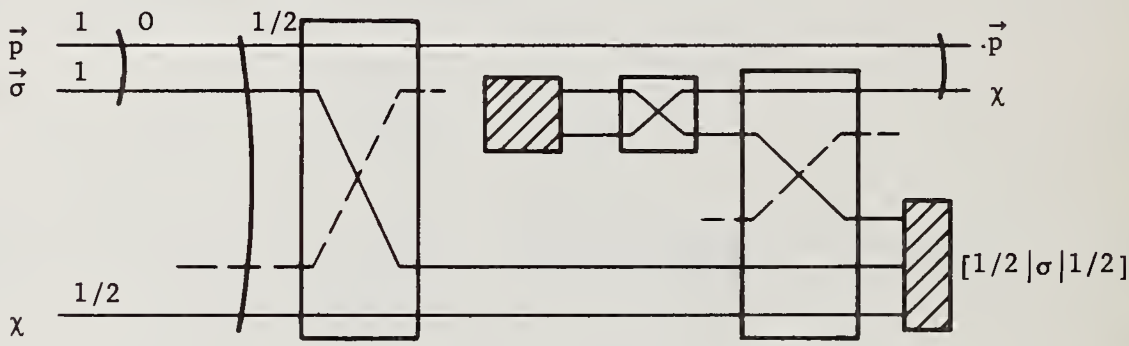


Figure 3.1

Furthermore

$$(\vec{\sigma}.\vec{p}\ \chi_{s'}^{[1/2]})^{+}(\vec{\sigma}.\vec{p}\ \chi_s^{[1/2]}) = \delta_{ss'}\ \frac{\hat{1}^2}{\hat{\frac{1}{2}}}\ [[\vec{p}\ \tilde{\chi}]^{[1/2]}\ [\vec{p}\ \chi]^{[1/2]}]^{[o]}$$

$$= p^2\ \delta_{ss'} \quad , \qquad (3.76)$$

hence

$$\sum_\alpha \mathcal{U}_{s\,\alpha}^{[1/2]+}(p)\ \mathcal{U}_{s'\,\alpha}^{[1/2]}(p) = \frac{E+m}{2m}\left[\chi_{s'}^{[1/2]+}\ \chi_s^{[1/2]} + \left(\frac{\vec{\sigma}.\vec{p}}{E+m}\ \chi_{s'}^{[1/2]}\right)^{+}\left(\frac{\vec{\sigma}.\vec{p}}{E+m}\ \chi_s^{[1/2]}\right)\right]$$

$$= \frac{E+m}{2m}\left(1 + \frac{p^2}{(E+m)^2}\right)\delta_{ss'} = \frac{E}{m}\ \delta_{ss'} \quad . \qquad (3.77)$$

Likewise :

$$\sum_{\alpha} \mathcal{V}^{[1/2]+}_{s\,\alpha}(p)\ \mathcal{V}^{[1/2]}_{s'\,\alpha}(p) = \frac{E}{m}\,\delta_{ss'} \quad , \tag{3.78}$$

$$\Big(\sum_{\alpha=1,2} - \sum_{\alpha=3,4}\Big)\ \tilde{\mathcal{U}}^{(1/2)}_{s\,\alpha}(p)\ \mathcal{U}^{[1/2]}_{s'\,\alpha}(p) = \delta_{ss'} \quad , \tag{3.79}$$

$$\Big(-\sum_{\alpha=1,2} + \sum_{\alpha=3,4}\Big)\ \tilde{\mathcal{V}}^{(1/2)}_{s\,\alpha}(p)\ \mathcal{V}^{[1/2]}_{s'\,\alpha}(p) = \delta_{ss'} \quad , \tag{3.80}$$

$$\Big(\sum_{\alpha=1,2} - \sum_{\alpha=3,4}\Big)\ \tilde{\mathcal{V}}^{(1/2)}_{s\,\alpha}(p)\ \mathcal{U}^{[1/2]}_{s'\,\alpha}(p) = 0 \tag{3.81}$$

$$\sum_{\alpha} \tilde{\mathcal{V}}^{(1/2)}_{s\,\alpha}(p)\ \mathcal{U}^{[1/2]}_{s'\,\alpha}(-p) = 0 \quad . \tag{3.82}$$

ii) Completeness

$$\sum_{s} \Big\{\mathcal{U}^{[1/2]}_{s\,\alpha}(p)\ \tilde{\mathcal{U}}^{(1/2)}_{s\,\beta}(p) - \mathcal{V}^{[1/2]}_{s\,\alpha}(p)\ \tilde{\mathcal{V}}^{(1/2)}_{s\,\beta}(p)\Big\} = (\gamma_4)_{\alpha\beta}$$

$$= \delta_{\alpha\beta} \times \begin{cases} 1 & \text{for } \alpha = 1,\,2 \\ -1 & \text{for } \alpha = 3,\,4 \end{cases} \tag{3.83}$$

$$\sum_{s} \mathcal{U}^{[1/2]}_{s\,\alpha}(p)\ \tilde{\mathcal{U}}^{(1/2)}_{s\,\beta}(p) = \left(\frac{-i\,\gamma p + m}{2m}\right)_{\alpha\beta} (\gamma_4)_{\beta\beta}$$

$$= \left(\frac{-i\,\gamma p + m}{2m}\right)_{\alpha\beta} \times \begin{cases} 1 & \text{for } \beta = 1,\,2 \\ -1 & \text{for } \beta = 3,\,4 \end{cases} \tag{3.84}$$

$$-\sum_{s} \mathcal{V}^{[1/2]}_{s\,\alpha}(p)\ \tilde{\mathcal{V}}^{(1/2)}_{s\,\beta}(p) = \left(\frac{i\,\gamma p + m}{2m}\right)_{\alpha\beta} (\gamma_4)_{\beta\beta} \quad . \tag{3.85}$$

These two relations (3.84) and (3.85) make use of,

$$-i\,\gamma p = \begin{pmatrix} E & -\vec{\sigma}.\vec{p} \\ \vec{\sigma}.\vec{p} & E \end{pmatrix} \quad . \tag{3.86}$$

It follows that the normalized field ψ Eq.(3.66) and its conjugate (3.39) fulfill the equal time anticommutation relations

$$\big\{\psi_\alpha(\vec{r}),\ \pi_\beta(\vec{r}\,')\big\}_+ = 2i\ \delta_{\alpha\beta}\ \delta^3(\vec{r}-\vec{r}\,') \quad , \tag{3.87}$$

where the factor 2 comes from the summation over the isospin components.

III.3.3 - Multipole expansion and discretization

The multipole expansion of the field at time $t=0$ is,

$$\psi_\alpha(\vec{r}) = \sqrt{\frac{2}{\pi}} \int d^3p \sqrt{\frac{m}{E}} \sum_{\kappa s} \sum_{\ell m} (i)^\ell \Big\{ a^{(1/2\ 1/2)}_{s\,\kappa\,\vec{p}}\ \mathcal{U}^{[1/2]}_{s\ \alpha}(p)\ \hat{p}^{(\ell)}_m \hat{r}^{[\ell]}_m j_\ell(pr)\ \eta^{[1/2]}_\kappa$$

$$+ (-)^\ell\ \tilde{b}^{[1/2\ 1/2]}_{s\,\kappa\,\vec{p}}\ \tilde{\mathcal{V}}^{(1/2)}_{s\ \alpha}(p)\ \hat{p}^{[\ell]}_m\ \hat{r}^{(\ell)}_m\ j_\ell(pr)\ \tilde{\eta}^{(1/2)}_\kappa \Big\} \qquad , \qquad (3.88)$$

or in a coupled form, separating large and small components,

$$\psi(\vec{r}) = \sqrt{\frac{1}{\pi}} \int \frac{d^3p}{\sqrt{E}} \sum_\ell \sum_{j=\ell\pm 1/2} (i)^\ell\ \frac{\hat{1}}{2}\ \hat{j}$$

$$\Bigg\{ \left[\left[a^{[1/2\ 1/2]}_{\vec{p}}\ \hat{p}^{[\ell]}\ \eta^{[1/2]} \right]^{[j0]} \begin{pmatrix} \sqrt{E+m}\ [\chi^{[1/2]}\ \hat{r}^{[\ell]}]^{[j]}\ j_\ell(pr) \\ -\sqrt{E-m}\ [\chi^{[1/2]}\ \hat{r}^{[\lambda]}]^{[j]}\ j_\lambda(pr) \end{pmatrix} \right]^{[0]}$$

$$+ (-)^\ell \left[\left[\tilde{b}^{[1/2\ 1/2]}_{\vec{p}}\ \hat{p}^{[\ell]}\ \tilde{\eta}^{[1/2]} \right]^{[j0]} \begin{pmatrix} -\sqrt{E-m}\ [\tilde{\chi}^{[1/2]}\ \hat{r}^{[\lambda]}]^{[j]}\ j_\lambda(pr) \\ \sqrt{E+m}\ [\tilde{\chi}^{[1/2]}\ \hat{r}^{[\ell]}]^{[j]}\ j_\ell(pr) \end{pmatrix} \right]^{[0]} \Bigg\} \qquad (3.89)$$

where $\lambda = \ell+1$ for $j = \ell+1/2$ and $\lambda = \ell-1$ for $j = \ell-1/2$.

The angular algebra which is used to obtain the small components is given in figure 3.2 neglecting isospin.

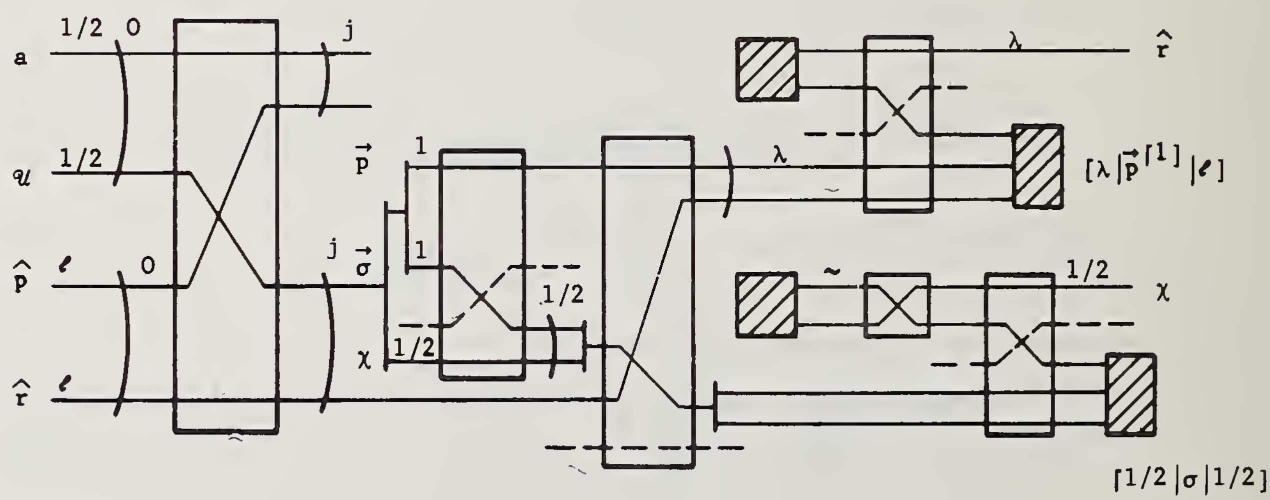


Figure 3.2

We define again multipole creation and annihilation operators as in eq. (3.14)

$$\underline{B}^{[j\ 1/2]}_{m\kappa\ell}(p) = \int d^2\hat{p}\ [a^{[1/2\ 1/2]}_{\kappa\ \vec{p}}\ \hat{p}^{[\ell]}]^{[j\ 1/2]}_{m} \quad , \tag{3.90}$$

which obey the anticommutation relations

$$\left\{\underline{B}^{[j\ 1/2]}_{m\kappa\ell}(p),\ \underline{B}^{[j'\ 1/2]+}_{m'\ \kappa'\ \ell'}(p')\right\}_+ = \delta_{jj'}\ \delta_{mm'}\ \delta_{\ell\ell'}\ \delta_{\kappa\kappa'}\ \frac{\delta(p-p')}{p^2} \quad , \tag{3.91}$$

and which upon discretization are substituted with (cf. eq. (3.21))

$$\underline{B}^{[j\ 1/2]}_{\ell}(p) = \sum_{\nu} B^{[j\ 1/2]}_{\nu\ell}\ f_{\nu\ell}(p) \quad , \tag{3.92}$$

omitting the m and κ indices. We have similar expressions for the antiparticle operators.

Furthermore we define the tensorial multipoles

$$Y^{[j]}_{1/2\ \ell\ m}(\hat{r}) = [\chi^{[1/2]}\ \hat{r}^{[\ell]}]^{[j]}_{m} \quad , \tag{3.93}$$

$$\tilde{Y}^{[j]}_{1/2\ \ell\ m}(\hat{r}) = [\breve{\chi}^{[1/2]}\ \hat{r}^{[\ell]}]^{[j]}_{m} \quad , \tag{3.94}$$

and the functions,

$$u_{\nu\ell}(r) = \sqrt{\frac{2}{\pi}} \int p^2 dp \sqrt{\frac{E+m}{E}}\ f_{\nu\ell}(p)\ j_{\ell}(pr) \quad , \tag{3.95}$$

$$v_{\nu\ell\lambda}(r) = \sqrt{\frac{2}{\pi}} \int p^2 dp \sqrt{\frac{E-m}{E}}\ f_{\nu\ell}(p)\ j_{\lambda}(pr) \quad , \tag{3.96}$$

which are normalized such that,

$$\sum_{\nu,\ell,\lambda} \left(u_{\nu\ell}(r)\ u_{\nu\ell}(r') + v_{\nu\ell\lambda}(r)\ v_{\nu\ell\lambda}(r')\right) = 2\ \frac{\delta(r-r')}{r^2} \quad . \tag{3.97}$$

Substituting these definitions in Eq. (3.89), we finally obtain the discretized field expansion,

$$\psi(\vec{r}) = \frac{1}{\sqrt{2}} \sum_{\nu\ell j} (i)^{\ell}\ \hat{\tfrac{1}{2}}\ \hat{j} \left\{ \left[\left[B^{[j\ 1/2]}_{\nu\ell}\ \eta^{[1/2]} \right]^{[0]} \begin{pmatrix} Y^{[j]}_{1/2\ \ell}(\hat{r})\ u_{\nu\ell}(r) \\ -Y^{[j]}_{1/2\ \lambda}(\hat{r})\ v_{\nu\ell\lambda}(r) \end{pmatrix} \right]^{[0]} + (-)^{\ell} \left[\left[\tilde{C}^{[j\ 1/2]}_{\nu\ \ell}\ \tilde{\eta}^{[1/2]} \right]^{[0]} \begin{pmatrix} -\tilde{Y}^{[j]}_{1/2\ \lambda}(\hat{r})\ v_{\nu\ell\lambda}(r) \\ \tilde{Y}^{[j]}_{1/2\ \ell}(\hat{r})\ u_{\nu\ell}(r) \end{pmatrix} \right]^{[0]} \right\} \tag{3.98}$$

where the $\tilde{C}$'s denote the antiparticle field. In each term the first coupling to [0] corresponds to total isospin and the second one to total angular momentum. It may be verified on this discretized form, together with the similar expression for the conjugate field, and using the relation (3.97) that the anticommutation relations are fulfilled.

III.4 - THE FREE FIELD ENERGY FOR SPIN 1/2 PARTICLES

The Fermion Hamiltonian is of the form (omitting the antiparticle part)

$$\int d^3r\, \mathcal{H}_N = \int d^3r\, \bar{\psi}\, \gamma.\nabla\psi + m\bar{\psi}\psi$$

$$= \frac{1}{2} \sum_{\substack{\nu_1\ell_1\lambda_1 j\\ \nu_2\ell_2\lambda_2}} i^{-\ell_1+\ell_2}\, \hat{j}^2\, \hat{\tfrac{1}{2}}^2 \left[\tilde{B}^{[j\ 1/2]}_{\nu_1\ell_1} \begin{pmatrix} \tilde{Y}^{[j]}_{1/2\ \ell_1}(\hat{r})\ u_{\nu_1\ell_1}(r) \\ -\tilde{Y}^{[j]}_{1/2\ \lambda_1}(\hat{r})\ v_{\nu_1\ell_1\lambda_1}(r) \end{pmatrix} \tilde{\eta}^{[1/2]}\right]^{[00]}$$

$$\times \begin{pmatrix} m & \vec{\sigma}.\vec{p} \\ \vec{\sigma}.\vec{p} & -m \end{pmatrix} \left[B^{[j\ 1/2]}_{\nu_2\ell_2} \begin{pmatrix} Y^{[j]}_{1/2\ \ell_2}(\hat{r})\ u_{\nu_2\ell_2}(r) \\ -Y^{[j]}_{1/2\ \lambda_2}(\hat{r})\ v_{\nu_2\ell_2\lambda_2}(r) \end{pmatrix} \eta^{[1/2]}\right]^{[00]}$$

$$= \frac{1}{2} \sum_{\substack{\nu_1\ell_1\lambda_1 j\\ \nu_2\ell_2\lambda_2}} \hat{\tfrac{1}{2}} \Bigg\{ \hat{j} \int r^2 dr\, m(u_{\nu_1\ell_1}(r)u_{\nu_2\ell_1}(r) - v_{\nu_1\ell_1\lambda_1}(r)v_{\nu_2\ell_1\lambda_1}(r))\,\delta_{\ell_1\ell_2}\delta_{\lambda_1\lambda_2}$$

$$- (i)^{-\ell_1+\ell_2} \frac{[1/2\,|\,\sigma\,|\,1/2]}{\hat{1}} \left(\begin{bmatrix} 1/2 & \ell_1 & j \\ 1/2 & \lambda_2 & j \\ 1 & 1 & 0 \end{bmatrix} \left[u_{\nu_1\ell_1} Y^{[\ell_1]} |\vec{p}^{[1]}| v_{\nu_2\ell_2\lambda_2} Y^{[\lambda_2]}\right] \right.$$

$$\left. - \begin{bmatrix} 1/2 & \ell_2 & j \\ 1/2 & \lambda_1 & j \\ 1 & 1 & 0 \end{bmatrix} \left[v_{\nu_1\ell_1\lambda_1} Y^{[\lambda_1]} |\vec{p}^{[1]}| u_{\nu_2\ell_2} Y^{[\ell_2]}\right] \right) \Bigg\} \left[\tilde{B}^{[j\ 1/2]}_{\nu_1\ell_1} B^{[j\ 1/2]}_{\nu_2\ell_2}\right]^{[00]} . \tag{3.99}$$

In order to get the $\delta_{\ell_1\ell_2}$ factor in the first contribution we have used the fact that $\delta_{\lambda_1\lambda_2}$ for a given $j = j_1 = j_2$ determines uniquely the ℓ value. The matrix elements of the first term are given by,

$$\int r^2 dr\, m(u_{\nu_1\ell_1}(r)u_{\nu_2\ell_1}(r) - v_{\nu_1\ell_1\lambda_1}(r)v_{\nu_2\ell_1\lambda_1}(r)) = \int p^2 dp\, \frac{2m^2}{E}\, f_{\nu_1\ell_1}(p) f_{\nu_2\ell_1}(p) , \tag{3.100}$$

while in the second term,

$$\left[u_{\nu_1\ell_1}\, Y^{[\ell_1]}\,|\vec{p}^{[1]}|\,v_{\nu_2\ell_2\lambda_2}\, Y^{[\lambda_2]}\right] =$$

$$= - i\sqrt{\lambda_2+1}\left(u_{\nu_1\ell_1}\left|\frac{\partial}{\partial r} - \frac{\lambda_2}{r}\right|v_{\nu_2\ell_2\lambda_2}\right)\delta_{\ell_1\ell_2}\,\delta_{\lambda_1\lambda_2} \quad , \qquad (3.101)$$

when $j = \lambda_2 + 1/2$, $\ell_1 = \ell_2 = \lambda_2 + 1$ and

$$= - i\sqrt{\lambda_2}\left(u_{\nu_1\ell_1}\left|\frac{\partial}{\partial r} + \frac{\lambda_2+1}{r}\right|v_{\nu_2\ell_2\lambda_2}\right)\delta_{\ell_1\ell_2}\,\delta_{\lambda_1\lambda_2} \quad , \qquad (3.102)$$

when $j = \lambda_2 - 1/2$, $\ell_1 = \ell_2 = \lambda_2 - 1$. Likewise

$$\left[v_{\nu_1\ell_1\lambda_1}\, Y^{[\lambda_1]}\,|\vec{p}^{[1]}|\,u_{\nu_2\ell_2}\, Y^{[\ell_2]}\right] =$$

$$= - i\sqrt{\lambda_2}\left(v_{\nu_1\ell_1\lambda_1}\left|\frac{\partial}{\partial r} + \frac{\ell_2+1}{r}\right|u_{\nu_2\ell_2}\right)\delta_{\ell_1\ell_2}\,\delta_{\lambda_1\lambda_2} \quad , \qquad (3.103)$$

when $j = \lambda_1 + 1/2$, $\ell_1 = \ell_2 = \lambda_2 + 1$ and

$$= - i\sqrt{\ell_2+1}\left(v_{\nu_1\ell_1\lambda_1}\left|\frac{\partial}{\partial r} - \frac{\ell_2}{r}\right|u_{\nu_2\ell_2}\right)\delta_{\ell_1\ell_2}\,\delta_{\lambda_1\lambda_2} \quad , \qquad (3.104)$$

when $j = \lambda_1 - 1/2$, $\ell_1 = \ell_2 = \lambda_2 - 1$.

Utilizing the relations (2.35) we get

$$\left(u_{\nu_1\ell_1}\left|\frac{\partial}{\partial r} - \frac{\lambda_2}{r}\right|v_{\nu_2\ell_2\lambda_2}\right) = - \int dp\,\frac{p^4}{E}\, f_{\nu_1\ell_1}(p)\, f_{\nu_2\ell_2}(p)\,\delta_{\ell_1\ell_2} \quad , \qquad (3.105)$$

$$\left(u_{\nu_1\ell_1}\left|\frac{\partial}{\partial r} + \frac{\lambda_2+1}{r}\right|v_{\nu_2\ell_2\lambda_2}\right) = \int dp\,\frac{p^4}{E}\, f_{\nu_1\ell_1}(p)\, f_{\nu_2\ell_2}(p)\,\delta_{\ell_1\ell_2} \quad , \qquad (3.106)$$

$$\left(v_{\nu_1\ell_1\lambda_1}\left|\frac{\partial}{\partial r} + \frac{\ell_2+1}{r}\right|u_{\nu_2\ell_2}\right) = \int dp\,\frac{p^4}{E}\, f_{\nu_1\ell_1}(p)\, f_{\nu_2\ell_2}(p)\,\delta_{\ell_1\ell_2} \quad , \qquad (3.107)$$

$$\left(v_{\nu_1\ell_1\lambda_1}\left|\frac{\partial}{\partial r} - \frac{\ell_2}{r}\right|u_{\nu_2\ell_2}\right) = - \int dp\,\frac{p^4}{E}\, f_{\nu_1\ell_1}(p)\, f_{\nu_2\ell_2}(p)\,\delta_{\ell_1\ell_2} \quad . \qquad (3.108)$$

Furthermore since

$$\begin{bmatrix} 1/2 & \ell & j \\ 1/2 & \lambda & j \\ 1 & 1 & 0 \end{bmatrix} = \begin{cases} -\dfrac{1}{\sqrt{6\ell}}\,\hat{j}\,\hat{1} & \text{if } \ell = \lambda + 1 \quad , \\[2ex] \dfrac{1}{\sqrt{6(\ell+1)}}\,\hat{j}\,\hat{1} & \text{if } \ell = \lambda - 1 \quad , \end{cases} \qquad (3.109)$$

the second term of eq. (3.99) has the same form for both cases, i.e. $\ell = \lambda \pm 1$, and we finally get,

$$\int d^3r\, \mathcal{H}_N = \sum_{\substack{\nu_1\nu_2\\ \ell j}} \left\{ \frac{\hat{1}}{2}\, \hat{j} \int p^2 dp\, \frac{m^2}{E}\, f_{\nu_1\ell}(p)\, f_{\nu_2\ell}(p) + \frac{\hat{1}}{2}\, \hat{j} \int dp\, \frac{p^4}{E}\, f_{\nu_1\ell}(p)\, f_{\nu_2\ell}(p) \right\} \left[\tilde{B}^{[j\ 1/2]}_{\nu_1\ell}\ B^{[j\ 1/2]}_{\nu_2\ell} \right]^{[00]}$$

$$= \sum_{\substack{\nu_1\nu_2\\ \ell j}} \frac{\hat{1}}{2}\, \hat{j} \int p^2 dp\, E\, f_{\nu_1\ell}(p)\, f_{\nu_2\ell}(p) \left[\tilde{B}^{[j\ 1/2]}_{\nu_1\ell}\ B^{[j\ 1/2]}_{\nu_2\ell} \right]^{[00]} . \quad (3.110)$$

Its matrix element between single particle states of the form (2.81) is

$$\hat{j}^2\, \frac{\hat{1}^2}{2} \langle 0 | \left[\tilde{W}^{[j\ 1/2]}_{\nu_1\ell_1} B^{[j\ 1/2]}_{\nu_1\ell_1} \right]^{[00]} \int d^3r\, \mathcal{H}_N \left[W^{[j\ 1/2]}_{\nu_2\ell_2} \tilde{B}^{[j\ 1/2]}_{\nu_2\ell_2} \right]^{[00]} | 0 \rangle$$

$$= \hat{j}\, \frac{\hat{1}}{2} \left[\tilde{W}^{[j\ 1/2]}_{\nu_1\ell}\ W^{[j\ 1/2]}_{\nu_2\ell} \right] \int p^2 dp\, E\, f_{\nu_1\ell}(p)\, f_{\nu_2\ell}(p)\, \delta_{\ell_1\ell_2}\, \delta_{\ell_1\ell} , \quad (3.111)$$

and its invariant matrix element (cf. the defining equation (2.23))

$$[\nu_1\ell_1 j_1 | \int d^3r\, \mathcal{H}_N | \nu_2\ell_2 j_2] = \delta_{j_1 j_2}\, \delta_{\ell_1\ell_2}\, \hat{j}\, \frac{\hat{1}}{2} \int p^2 dp\, E\, f_{\nu_1\ell_1}(p)\, f_{\nu_2\ell_1}(p) . \quad (3.112)$$

III.5 - SPIN 1 FIELD

III.5.1 - Field equations

We consider now the ρ and ω meson fields, the quantum numbers of which are

$I^\pi = 1^-$; $T = 1$; even $\mathcal{G}$ parity for the ρ meson ,

$I^\pi = 1^-$; $T = 0$; odd $\mathcal{G}$ parity for the ω meson .

In order for the equations of motion to yield back the Maxwell equations at zero mass, we shall work within the 8 component theory framework of Hayward[13]. In this framework the ω and ρ free field solutions are 4 component vectors, 3 components transforming like those of a spin 1 and the fourth like a spin 0,

$$\omega = \begin{pmatrix} \vec{\omega} \\ \omega_4 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \\ \vdots \end{pmatrix} \tag{3.113}$$

$$\rho = \begin{pmatrix} \vec{\rho} \\ \rho_4 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \\ \rho_4 \\ \vdots \end{pmatrix} \tag{3.114}$$

III.5.1a - The ω field Hamiltonian

For the neutral field we have*

$$\vec{\omega}^{+} = \vec{\omega} \quad ; \quad \omega_4^{+} = -\omega_4 \quad . \tag{3.115}$$

---

* In general

$$\omega = (\vec{\omega}, \omega_4, 0, 0) \quad , \quad \omega^{+} = \eta^{-1}(\vec{\omega}, -\omega_4, 0, 0)\eta \quad ,$$

$$\bar{\omega} = \omega^{+}\gamma_4 = \eta^{-1}(\vec{\omega}, \omega_4, 0, 0)\eta \quad .$$

In the Lorentz gauge $\eta = I$. However in the Feynman gauge defined as

$$\psi \rightarrow \psi + \partial\Lambda \quad ,$$

where $(\Box + m^2)\Lambda = 0$, $(\Box + m^2)\partial\Lambda = 0$,

$$\bar{\psi} = \eta^{-1}\psi^{+}\gamma_4\eta \quad \text{such that} \quad \bar{\psi}_{\mathscr{L}} = \psi^{+}_{\mathscr{L}}\gamma_4 \quad , \quad \bar{\psi}_{\mathscr{T}} = \psi^{+}_{\mathscr{T}}\gamma_4 \quad ,$$

$$\bar{\psi}_{Time} = -\psi^{+}_{T}\gamma_4 \quad \text{necessary for } \mathscr{H} \text{ invariant in the transformation .}$$

Here the indices $\mathscr{L}$, $\mathscr{T}$, Time denote the longitudinal, transversal and time-like solutions respectively.

The free Lagrangian for the ω field is

$$\mathscr{L}_\omega = -\frac{1}{2}\left\{\omega \overleftarrow{\gamma_\mu \partial_\mu} \; \overrightarrow{\gamma_\nu \partial_\nu}\, \omega - \frac{1}{2} m^2 \omega^2\right\} \tag{3.116}$$

The γ_μ are 8×8 matrices defined as

$$\gamma_\mu = -\, i\, \beta\, \alpha_\mu \tag{3.117}$$

$$\gamma_1 = \left(\begin{array}{cccc|cccc} \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot \end{array}\right) \quad \gamma_2 = \left(\begin{array}{cccc|cccc} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot \end{array}\right) \quad \gamma_3 = \left(\begin{array}{cccc|cccc} \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot \end{array}\right)$$

$$\gamma_4 = \left(\begin{array}{cccc|cccc} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{array}\right) \quad \gamma_5 = \left(\begin{array}{cccc|cccc} \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \hline -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \end{array}\right) \tag{3.118}$$

These definitions yield,

$$\gamma_\mu \partial_\mu = \left(\begin{array}{ccc|c|ccc|c} \partial_4 & \cdot & \cdot & -\partial_1 & \cdot & -\partial_3 & +\partial_2 & \cdot \\ \cdot & \partial_4 & \cdot & -\partial_2 & \partial_3 & \cdot & -\partial_1 & \cdot \\ \cdot & \cdot & \partial_4 & -\partial_3 & -\partial_2 & +\partial_1 & \cdot & \cdot \\ \hline -\partial_1 & -\partial_2 & -\partial_3 & -\partial_4 & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \partial_3 & -\partial_2 & \cdot & -\partial_4 & \cdot & \cdot & \partial_1 \\ -\partial_3 & \cdot & \partial_1 & \cdot & \cdot & -\partial_4 & \cdot & \partial_2 \\ \partial_2 & -\partial_1 & \cdot & \cdot & \cdot & \cdot & -\partial_4 & \partial_3 \\ \hline \cdot & \cdot & \cdot & \cdot & \partial_1 & \partial_2 & \partial_3 & \partial_4 \end{array}\right), \tag{3.119}$$

and ($V = \vec{V}, V_4$; $W = \vec{W}, W_4$) ($V_4 = i V_o$; $W_4 = i W_o$)

$$\gamma_\mu \partial_\mu \begin{pmatrix} V \\ W \end{pmatrix} = \begin{pmatrix} \partial_4 \vec{V} - \operatorname{grad} V_4 + \operatorname{rot} \vec{W} \\ -\operatorname{div} \vec{V} - \partial_4 V_4 \\ -\operatorname{rot} \vec{V} + \operatorname{grad} W_4 - \partial_4 \vec{W} \\ \operatorname{div} \vec{W} + \partial_4 W_4 \end{pmatrix} . \tag{3.120}$$

The resulting canonical momentum is then

$$\pi = \frac{\partial\mathscr{L}}{\partial\dot{\omega}} = \frac{i}{2}\,(\gamma_4\gamma_\mu\overrightarrow{\partial_\mu}\omega + \omega\gamma_\mu\overleftarrow{\partial_\mu}\gamma_4) = \begin{pmatrix} i\;\partial_4\vec{\omega} - i\;\vec{\nabla}.\omega_4 \\ i\;\vec{\nabla}.\vec{\omega} + i\;\partial_4\omega_4 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \vec{\pi} \\ \pi_4 \\ 0 \\ 0 \end{pmatrix} \quad . \tag{3.121}$$

We choose now the Lorentz gauge which eliminates the π_4 component,

$$\pi_4 = 0 \quad \text{Lorentz gauge} \quad , \tag{3.122}$$

i.e., the fourth component of the field is given by

$$\partial_4\omega_4 = \partial_t\omega_o = -\operatorname{div}\vec{\omega} \quad , \tag{3.123}$$

and $\vec{\pi}^+ = \vec{\pi}$, $\bar{\pi} = \pi$.

Let us define the "electric" and "magnetic" fields,

$$\vec{E} = -\,i(\partial_4\vec{\omega} - \operatorname{grad}\omega_4) = -\,\vec{\pi} \quad , \tag{3.124}$$

$$\vec{B} = \operatorname{rot}\vec{\omega} \quad , \tag{3.125}$$

which yields for the Lagrangian the well-known form (taking into account the Lorentz condition),

$$\begin{aligned} \mathscr{L} &= -\frac{1}{2}\left\{(\partial_4\vec{\omega})^2 - 2\partial_4\vec{\omega}\operatorname{grad}\omega_4 + (\operatorname{grad}\omega_4)^2 + (\operatorname{rot}\vec{\omega})^2 + m^2(\vec{\omega}^2 - \omega_4^2)\right\} \\ &= -\frac{1}{2}\left\{-E^2 + B^2 + m^2(\vec{\omega}^2 - \omega_4^2)\right\} \quad . \end{aligned} \tag{3.126}$$

Likewise the Hamiltonian is

$$\mathscr{H} = :\pi\dot{\omega} - \mathscr{L}: = \frac{1}{2}:\left\{-(\partial_4\vec{\omega})^2 + (\operatorname{grad}\omega_4)^2 + (\operatorname{rot}\vec{\omega})^2 + m^2(\vec{\omega}^2 - \omega_4^2)\right\}: \quad . \tag{3.127}$$

Hence for transverse solutions for which

$$(\omega_{\mathscr{T}})_4 = 0 \quad , \tag{3.128}$$

we get the usual form

$$\mathscr{H}_{\mathscr{T}} = \frac{1}{2}\,(E^2 + B^2 + m^2\,\vec{\omega}^2) \quad , \tag{3.129}$$

while for the longitudinal part, for which $B_{\mathscr{L}} = \operatorname{rot}\vec{\omega}_{\mathscr{L}} = 0$, there is no such simple counterpart.

III.5.1b - The ρ field Hamiltonian

The charged field is complex and we denote it with a prime index ρ' to make the distinction with the real field ρ that we shall construct later on.

The free Lagrangian is

$$\mathscr{L} = -\sum_{\kappa} \bar{\rho}'_{\kappa} \gamma_\mu \overleftarrow{\partial}_\mu \gamma_\nu \overrightarrow{\partial}_\nu \rho'_\kappa - m^2 \bar{\rho}'_\kappa \rho'_\kappa \quad , \tag{3.130}$$

where the summation is carried out over the three charge states with isospin projection κ . This summation and the index κ are deleted hereafter. We have

$$\rho' = \begin{pmatrix} \vec{\rho}' \\ \rho'_4 \\ 0 \\ 0 \end{pmatrix} \quad , \qquad \bar{\rho}' = \begin{pmatrix} \vec{\rho}'^{+} \\ -\rho_4'^{+} \\ 0 \\ 0 \end{pmatrix} \quad , \tag{3.131}$$

$$\pi' = \frac{\partial \mathscr{L}}{\partial \dot{\rho}'} = i \begin{pmatrix} \partial_4 \vec{\rho}'^{+} + \text{grad } \rho_4'^{+} \\ \text{div } \vec{\rho}'^{+} - \partial_4 \rho_4'^{+} \\ 0 \\ 0 \end{pmatrix} \quad , \tag{3.132}$$

$$\bar{\pi}' = \frac{\partial \mathscr{L}}{\partial \dot{\bar{\rho}}'} = i \begin{pmatrix} \partial_4 \vec{\rho}' - \text{grad } \rho'_4 \\ \text{div } \vec{\rho}' + \partial_4 \rho'_4 \\ 0 \\ 0 \end{pmatrix} \quad . \tag{3.133}$$

We again make use of the Lorentz gauge $\pi'_4 = 0$,

$$\text{div } \vec{\rho}'^{+} - \partial_4 \rho_4'^{+} = \text{div } \vec{\rho}' + \partial_4 \rho'_4 = 0 \quad , \tag{3.134}$$

and we introduce the fields

$$\vec{E}' = -i(\partial_4 \vec{\rho}' - \text{grad } \rho'_4) \quad , \qquad \vec{E}'^{+} = -i(\partial_4 \vec{\rho}'^{+} + \text{grad } \rho_4'^{+}) \ , \tag{3.135}$$

$$\vec{B}' = \text{rot } \vec{\rho}' \quad , \qquad \vec{B}'^{+} = \text{rot } \vec{\rho}'^{+} \quad , \tag{3.136}$$

which finally yields for the Lagrangian

$$\mathscr{L} = -\Big\{\partial_4 \vec{\rho}'^{+} \partial_4 \vec{\rho}' - \partial_4 \vec{\rho}'^{+} \text{grad } \rho'_4 + \text{grad } \rho_4'^{+} \partial_4 \vec{\rho}' - \text{grad } \rho_4'^{+} \text{grad } \rho'_4$$
$$+ \text{rot } \vec{\rho}'^{+} \text{rot } \vec{\rho}' + m^2 (\vec{\rho}'^{+} \vec{\rho}' - \rho_4'^{+} \rho'_4)\Big\}$$
$$= -\Big\{-\vec{E}'^{+} \vec{E}' + \vec{B}'^{+} \vec{B}' + m^2 (\vec{\rho}'^{+} \vec{\rho}' - \rho_4'^{+} \rho'_4)\Big\} \tag{3.137}$$

The Hamiltonian is thus,

$$\mathscr{H} = :\pi' \dot{\rho}' + \bar{\pi}' \dot{\bar{\rho}}' - \mathscr{L}: \quad . \tag{3.138}$$

Note that for the complex boson field $:\rho^+\rho: = \rho^+\rho$; $:\rho\rho^+: = \rho^+\rho$. We have the results,

$$\pi'\dot{\rho}' = -\partial_4\vec{\rho}'^+ \partial_4\vec{\rho}' - \text{grad}\ \rho_4'^+ \partial_4\vec{\rho}' \quad , \tag{3.139}$$

$$\bar{\pi}'\dot{\bar{\rho}}' = -\partial_4\vec{\rho}' \partial_4\vec{\rho}'^+ + \text{grad}\ \rho_4' \partial_4\vec{\rho}'^+ \quad ; \tag{3.140}$$

$$\mathscr{H} = :\Big\{-\partial_4\vec{\rho}'^+ \partial_4\rho' - \text{grad}\ \rho_4'^+ \text{grad}\ \rho_4' + \text{rot}\ \vec{\rho}'^+ \text{rot}\ \vec{\rho}' + m^2(\vec{\rho}'^+ \vec{\rho}' - \rho_4'^+ \rho_4')\Big\}: \quad . \tag{3.141}$$

We get again for the transverse solutions ($\rho_4' = 0$)

$$\mathscr{H} = \vec{E}'^+\vec{E}' + \vec{B}'^+\vec{B}' + m^2\ \vec{\rho}'^+\vec{\rho}' \quad . \tag{3.142}$$

For the longitudinal ones we must use the expression (3.141) with $\text{rot}\ \vec{\rho}' = 0$.

For convenience we shall however work now with real fields,

$$\rho = \rho'^+ + \rho' = (\vec{\rho}'^+ + \vec{\rho}'\ ;\ \rho_4'^+ + \rho_4'\ ;\ 0\ ;\ 0) \quad , \tag{3.143}$$

$$\bar{\rho} = \bar{\rho}'^+ + \bar{\rho}' = (\vec{\rho}'^+ + \vec{\rho}'\ ; -\rho_4'^+ - \rho_4'\ ;\ 0\ ;\ 0) \quad . \tag{3.144}$$

Hence

$$\bar{\rho} = \rho^+\gamma_4 \quad . \tag{3.145}$$

The Lorentz condition is thus

$$\text{div}\ \vec{\rho} = \partial_4(\rho_4'^+ - \rho_4') = -\partial_t\rho_o \quad , \tag{3.146}$$

where

$$\rho_4 = i\ \rho_o \quad , \tag{3.147}$$

$$\rho_4^+ = -i\ \rho_o = -\rho_4 \quad , \tag{3.148}$$

which are the relations used for ω_4. We can now define $\mathscr{L}$, π and $\mathscr{H}$ for the charged real field ρ as in III.5.1a for the neutral field, equations (3.126), (3.121) and (3.127).

III.5.1c - <u>Equations of motion and the Maxwell equations</u>

For the real ω and ρ fields the Klein Gordon equations are,

$$\vec{\partial}_\mu\gamma_\mu\ \vec{\partial}_\nu\gamma_\nu\ \omega - m^2\ \omega = 0 \quad , \tag{3.149}$$

$$\vec{\partial}_\mu\gamma_\mu\ \vec{\partial}_\nu\gamma_\nu\ \rho - m^2\ \rho = 0 \quad . \tag{3.150}$$

From the result of eq.(3.120) we get successively

$$\vec{\partial}_\nu \gamma_\nu \ \omega = \begin{pmatrix} \partial_4 \vec{\omega} - \text{grad } \omega_4 \\ 0 \\ - \text{rot } \vec{\omega} \\ 0 \end{pmatrix} = \begin{pmatrix} i \vec{E} \\ 0 \\ - \vec{B} \\ 0 \end{pmatrix} , \tag{3.151}$$

$$\vec{\partial}_\mu \gamma_\mu \begin{pmatrix} i \vec{E} \\ 0 \\ - \vec{B} \\ 0 \end{pmatrix} = \begin{pmatrix} \partial_t \vec{E} - \text{rot } \vec{B} \\ - i \text{ div } \vec{E} \\ - i \text{ rot } \vec{E} - i \ \partial_t \vec{B} \\ - \text{div } \vec{B} \end{pmatrix} = m^2 \begin{pmatrix} \vec{\omega} \\ \omega_4 \\ 0 \\ 0 \end{pmatrix} . \tag{3.152}$$

In the limit of zero mass these expressions yield back the Maxwell equations. The equations (3.151) and (3.152) hold also for the ρ-field.

III.5.1d - Quantization

After quantization the conjugate fields ω and π or ρ_κ and π_κ obey the the equal time commutators

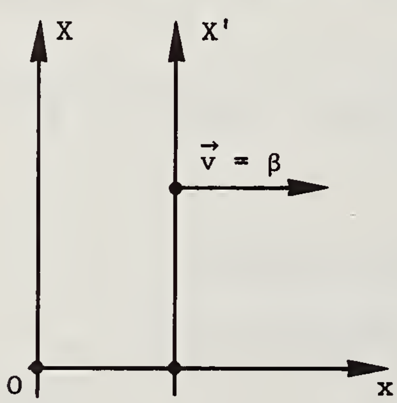
$$\left\{\omega_\alpha(x), \ \pi_{\alpha'}(x')\right\}_- = \delta_{\alpha\alpha'} \ \delta(x-x') , \qquad x_4 = x'_4 \tag{3.153}$$

$$\left\{\rho_{\kappa\alpha}(x), \ \pi_{\kappa'\alpha'}(x')\right\}_- = \delta_{\alpha\alpha'} \ \delta_{\kappa\kappa'} \ \delta(x-x') , \qquad x_4 = x'_4 \tag{3.154}$$

where α, α' are the spinor indices.

III.5.2 - Representation of the Lorentz transformation and boost

Consider the Lorentz transform from a reference system X to a system X' which is moving along Ox relatively to X with the velocity $\vec{v} = \beta$. The transformation induces the change of coordinates



$$\begin{cases} x_1 = x'_1 \cos\theta - x'_4 \sin\theta = \gamma \, x'_1 - i \, \beta\gamma x'_4 , \\ x_2 = x'_2 , \qquad x_3 = x'_3 , \\ x_4 = x'_1 \sin\theta + x_4 \cos\theta = i \, \beta\gamma x'_1 + \gamma \, x'_4 ; \end{cases}$$

or conversely

$$x'_1 = x_1 \cos\theta + x_4 \sin\theta$$

$$x'_4 = - x_1 \sin\theta + x_4 \cos\theta \tag{3.155}$$

where

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \quad , \tag{3.156}$$

$$\sin\theta = i\beta\gamma \quad , \qquad \cos\theta = \gamma \quad . \tag{3.157}$$

Let us define the infinitesimal Lorentz transformation on a four vector $A_\nu(x)$ as

$$A'_\mu(x') = \left(1 + \frac{1}{2} i \sum_{\alpha\beta} \varepsilon_{\alpha\beta} S_{\alpha\beta}\right)_{\mu\nu} A_\nu(x) \quad , \tag{3.158}$$

with the set of 4×4 matrices,

$$S_{23} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -i & \cdot \\ \cdot & i & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \qquad S_{31} = \begin{pmatrix} \cdot & \cdot & i & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ -i & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \qquad S_{12} = \begin{pmatrix} \cdot & -i & \cdot & \cdot \\ i & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$$S_{14} = \begin{pmatrix} \cdot & \cdot & \cdot & -i \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ i & \cdot & \cdot & \cdot \end{pmatrix} \qquad S_{24} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -i \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & i & \cdot & \cdot \end{pmatrix} \qquad S_{34} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -i \\ \cdot & \cdot & i & \cdot \end{pmatrix} . \tag{3.159}$$

Comparing the definition (3.155) with the infinitesimal transformation (3.158) where $\gamma \sim 1$, we get

$$\varepsilon_{14} = \sin\theta_{14} = i\,\beta_{14} \quad . \tag{3.160}$$

For a finite Lorentz transformation, setting $\vec{\beta}_{\text{infinitesimal}} = \frac{\vec{\Omega}}{n}$, we obtain

$$A'_\mu(x') = \left(1 - \frac{\vec{\Omega}}{n}\vec{S}\right)^n_{\mu\nu} A_\nu(x) = e^{-\vec{\Omega}\vec{S}} A_\nu(x) \quad . \tag{3.161}$$

Instead of the transformation from the system X to the system X', we consider now the infinitesimal boost of a 4-vector $A_\nu(x)$ in the system X to a velocity β ,

$$A'_\mu(x) = \left(1 - \frac{1}{2} i \sum_{\alpha\beta} \varepsilon_{\alpha\beta} S_{\alpha\beta}\right)_{\mu\nu} A_\nu(x) \quad , \tag{3.162}$$

and the finite boost is given by

$$\begin{aligned} A'_\mu(x) &= \left(1 + \frac{\vec{\Omega}}{n}\vec{S}\right)_{\mu\nu} A_\nu(x) \\ &= (1 - S^2_{14} + \vec{S}\sinh\vec{\Omega} + S^2_{14}\cosh\vec{\Omega})_{\mu\nu} A_\nu(x) = e^{\vec{S}\vec{\Omega}} A_\mu(x) \; . \end{aligned} \tag{3.163}$$

For example a particle of spin 1 at rest is represented by

$$\omega(x) \;=\; \mathcal{N}\, e^{-\,i\,mt}\begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ 0 \end{pmatrix} \quad , \qquad (3.164)$$

where the A_i are the spin orientation of the cartesian components. Upon boosting along Ox with velocity $\vec{\beta}$, we transform $-\,i\,mt$ into $i\,px$ in the exponent since,

$$\left\{\begin{pmatrix} 0 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 0 \end{pmatrix} + \begin{pmatrix} \cdot & \cdot & \cdot & -i \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ i & \cdot & \cdot & \cdot \end{pmatrix}\sinh\Omega + \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 0 & \cdot & \cdot \\ \cdot & \cdot & 0 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}\cosh\Omega\right\}\begin{pmatrix} 0 \\ 0 \\ 0 \\ im \end{pmatrix} = \begin{pmatrix} m\beta\gamma \\ 0 \\ 0 \\ im\gamma \end{pmatrix} = \begin{pmatrix} p \\ 0 \\ 0 \\ iE \end{pmatrix} ,$$

and noting that (3.165)

$$i\,px \;=\; i\,(p_1x_1 + p_2x_2 + p_3x_3 + p_4x_4) \;=\; i\,(0+0+0+(im)\,(it)) \quad , \qquad (3.166)$$

$$i\,\mathcal{L}(px) \;=\; i\,(m\beta\gamma x_1 + 0 + 0 + (im\gamma)\,(it)) \;=\; i\,(\vec{p}.\vec{x} - Et) \quad , \qquad (3.167)$$

since $E = m\gamma$, $\vec{p} = m\vec{\beta}\,\gamma$. One verifies that $E^2 - p^2 = m^2\,\gamma^2(1-\beta^2) = m^2$. Thus the cartesian boost along Ox on a cartesian four vector yields,

$$\begin{pmatrix} \cosh\Omega & \cdot & \cdot & -i\,\sinh\Omega \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ i\,\sinh\Omega & \cdot & \cdot & \cosh\Omega \end{pmatrix} \mathcal{N}\, e^{-\,i\,mt}\begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ 0 \end{pmatrix} = \mathcal{N}\, e^{\,i\,px}\begin{pmatrix} A_1\gamma \\ A_2 \\ A_3 \\ i\,A_1\beta\gamma \end{pmatrix} \quad . \qquad (3.168)$$

We can also verify that the Lorentz gauge $\partial_\mu A_\mu$, see Eq. (3.122), is conserved either in a Lorentz transformation, where both ∂_μ and A_μ are transformed or in a Lorentz boost, where only A_μ changes. In the latter case, for example the expression

$$\partial_\mu\, e^{-\,i\,mt}\begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ 0 \end{pmatrix} = 0 \quad , \qquad (3.169)$$

entails for the boosted vector along Ox

$$\partial_\mu\, e^{\,i\,px}\begin{pmatrix} A_1\gamma \\ A_2 \\ A_3 \\ i\,A_1\beta\gamma \end{pmatrix} = i\,p\,A_1\gamma - i\,E\,A_1\beta\gamma \equiv 0 \quad . \qquad (3.170)$$

III.5.3 - The plane wave solutions

The vector part of the real fields after quantization is of the form,

$$\vec{\rho}(\vec{r},t) = \left(\frac{1}{2\pi}\right)^{3/2} \int d^3p \sqrt{\frac{N}{2E}}\; \hat{1}^2 \Big\{ e^{i(\vec{p}\,\vec{r}-Et)} [a_{\vec{p}}^{[11]} e^{[1]} \eta^{[1]}]^{[00]} + e^{-i(\vec{p}\,\vec{r}-Et)} [\tilde{a}_{\vec{p}}^{[11]} e^{[1]} \tilde{\eta}^{[1]}]^{[00]} \Big\} \quad , \tag{3.171}$$

$$\vec{\omega}(\vec{r},t) = \left(\frac{1}{2\pi}\right)^{3/2} \int d^3p \sqrt{\frac{N}{2E}}\; \hat{1} \Big\{ e^{i(\vec{p}\,\vec{r}-Et)} [a_{\vec{p}}^{[1]} e^{[1]}]^{[0]} + e^{-i(\vec{p}\,\vec{r}-Et)} [\tilde{a}_{\vec{p}}^{[1]} e^{[1]}]^{[0]} \Big\} \quad , \tag{3.172}$$

for isospin 1 and 0 respectively. N is a normalization constant. The fourth time-like scalar component is defined by the Lorentz condition (3.134)

$$\rho_4(\vec{r},t) = -\int \partial_4 \vec{\nabla}.\vec{\rho}(r,t) = \left(\frac{1}{2\pi}\right)^{3/2} \int d^3p \sqrt{\frac{N}{2E}}\; \frac{\hat{1}^2}{E} \Big\{ e^{i(\vec{p}\,\vec{r}-Et)} [a_{\vec{p}}^{[11]}\, i\vec{p}^{[1]} \eta^{[1]}]^{[00]} + e^{-i(\vec{p}\,\vec{r}-Et)} [\tilde{a}_{\vec{p}}^{[11]}\, i\vec{p}^{[1]} \tilde{\eta}^{[1]}]^{[00]} \Big\} \quad , \tag{3.173}$$

and a similar expression for $\omega_4(\vec{r},t)$. We have here introduced the notation

$$\vec{p}^{[1]} = \sqrt{\frac{4\pi}{3}}\; p\, Y_m^{[1]}(\hat{p}) \equiv \sqrt{\frac{4\pi}{3}}\; p\, \hat{p}_m^{[1]} \quad , \tag{3.174}$$

and

$$[\vec{p}^{[1]}\, \vec{p}^{[1]}]^{[0]} = \frac{1}{\hat{1}}\, p^2 \quad . \tag{3.175}$$

We have also used the result (2.54) i.e. $\mathrm{div}[..e^{[1]}..]^{[0]} = [..\nabla^{[1]}..]^{[0]}$. The conjugate field is obtained from the equations (3.132) and (3.135), where $\vec{E}$ and E denote of course respectively the electric field and the energy, namely for isospin 1

$$-\vec{E} = \vec{\pi}(\vec{r},t) = i(\partial_4 \vec{\rho}^{\,+} + \vec{\nabla}\rho_4^+) = i\left(\frac{1}{2\pi}\right)^{3/2} \int d^3p \sqrt{\frac{N}{2E}}\; \hat{1}^2 E$$

$$\times \Big\{ e^{-i(\vec{p}\,\vec{r}-Et)} \Big([\tilde{a}_{\vec{p}}^{[11]} e^{[1]} \tilde{\eta}^{[1]}]^{[00]} - \frac{\hat{1}}{E^2} [\tilde{a}_{\vec{p}}^{[11]} \vec{p}^{[1]} \tilde{\eta}^{[1]}]^{[00]} [e^{[1]} \vec{p}^{[1]}]^{[0]} \Big) - e^{i(\vec{p}\,\vec{r}-Et)} (\mathrm{C.C.}) \Big\}$$

$$= i\left(\frac{1}{2\pi}\right)^{3/2} \int d^3p \sqrt{\frac{NE}{2}}\; \hat{I}^2 \Big\{ e^{-i(\vec{p}\,\vec{r}-Et)} \Big([\tilde{a}_{\vec{p}}^{[11]} e^{[1]} \tilde{\eta}^{[1]}]^{[00]} - \frac{1}{E^2} \sum_L \frac{\hat{L}}{\hat{1}} \Big[[\tilde{a}_{\vec{p}}^{[11]} e^{[1]} \tilde{\eta}^{[1]}]^{[L0]} [\vec{p}^{[1]} \vec{p}^{[1]}]^{[L]} \Big]^{[0]} \Big) - \mathrm{C.C.} \Big\} \quad . \tag{3.176}$$

Here we have made use of grad $[\ldots]^{[0]} = \hat{1}\, [e^{[1]} \nabla^{[1]}]^{[0]}\, [\ldots]^{[0]}$, cf. Eq. (2.50). The summation over L contains only L = 0, 2 since $\vec{p} \times \vec{p} = 0$.

$$\vec{B}^{+} = \vec{B} = \text{rot}\,\vec{\rho}(\vec{r},t) = \sqrt{2}\left(\frac{1}{2\pi}\right)^{3/2}\int d^3p\sqrt{\frac{N}{2E}}\,\hat{1}^2$$

$$\times\left\{i\,e^{-i(\vec{p}\vec{r}-Et)}\left[\tilde{a}^{[11]}_{\vec{p}}\vec{p}^{[1]}e^{[1]}\tilde{\eta}^{[1]}\right]^{[00]} - i\,e^{+i(\vec{p}\vec{r}-Et)}\left[a^{[11]}_{\vec{p}}\vec{p}^{[1]}e^{[1]}\eta^{[1]}\right]^{[00]}\right\}. \tag{3.177}$$

In this expression we have utilized the relation (cf. Eq.(2.48))

$$\vec{\nabla}\times\vec{B} = \sqrt{2}\;\hat{1}\left[\nabla^{[1]}B^{[1]}e^{[1]}\right]^{[0]}, \tag{3.178}$$

and the reordering of $a^{[1]}_{\vec{p}}$ and $\vec{p}^{[1]}$ introduces an overall sign.

The expressions of the conjugate fields of isospin 0 are similar, with a $\hat{1}$ factor missing and suppression of the isospin function $\eta^{[1]}$.

We compute now the commutation relations and the normalization.

The equal time commutator is

$$\left\{\vec{\rho}(\vec{r},t),\,\vec{\pi}(\vec{r}\,',t)\right\}_- = i\left(\frac{1}{2\pi}\right)^3\iint d^3p\,d^3p'\sqrt{\frac{NN'}{4EE'}}\;\hat{1}^4\,E$$

$$\times\left\{e^{i[(\vec{p}\vec{r}-\vec{p}\,'\vec{r}\,')-(E-E')]t}\left\{\left[a^{[11]}_{\vec{p}}e^{[1]}\eta^{[1]}\right]^{[00]},\left(\left[\tilde{a}^{[11]}_{\vec{p}\,'}e^{[1]}\tilde{\eta}^{[1]}\right]^{[00]}\right.\right.\right.$$

$$\left.\left.\left.-\frac{1}{E^2}\sum_L\frac{\hat{L}}{\hat{1}}\left[\left[\tilde{a}^{[11]}_{\vec{p}\,'}e^{[1]}\eta^{[1]}\right]^{[L0]}\left[\vec{p}^{[1]}\vec{p}^{[1]}\right]^{[L]}\right]^{[0]}\right)\right\}_- - \text{c.c.}\right\}$$

$$= i\left(\frac{1}{2\pi}\right)^3 9\int\frac{d^3p}{2E}\,NE\left(1-\frac{p^2}{3E^2}\right)\left(e^{i\vec{p}(\vec{r}-\vec{r}\,')}+e^{-i\vec{p}(\vec{r}-\vec{r}\,')}\right)$$

$$= i(2t+1)(2s+1)\;\delta(\vec{r}-\vec{r}\,')\quad. \tag{3.179}$$

We have $s=1$ and $t=1$ for the ρ-field, and $s=1$, $t=0$ for the ω-field.

Furthermore here we have set the normalization constant to be

$$N = \frac{3E^2}{3E^2-p^2} = \frac{3E^2}{2E^2+m^2}\quad. \tag{3.180}$$

The calculation of the commutator made use of the result,

$$\left\{\left[a^{[11]}_{\vec{p}}e^{[1]}\eta^{[1]}\right]^{[00]},\left[\tilde{a}^{[11]}_{\vec{p}\,'}e^{[1]}\tilde{\eta}^{[1]}\right]^{[00]}\right\}_- = \delta(\vec{p}-\vec{p}\,')\quad, \tag{3.181}$$

which follows from

$$\left\{a^{[11]}_{\tau\sigma\vec{p}},\,\tilde{a}^{[11]}_{\tau'\sigma'\vec{p}\,'}\right\}_- = \delta_{\tau\tau'}\,\delta_{\sigma\sigma'}\,\delta(\vec{p}-\vec{p}\,')\quad, \tag{3.182}$$

and from the result,

$$\left\{[a_{\vec{p}}^{[11]} e^{[1]} \eta^{[1]}]^{[00]}, \left[[\tilde{a}_{\vec{p}'}^{[11]} e^{[1]} \tilde{\eta}^{[1]}]^{[L0]} [\vec{p}^{[1]} \vec{p}^{[1]}]^{[L]}\right]^{[0]}\right\}_- = \frac{p^2}{\hat{1}} \delta(\vec{p}-\vec{p}') . \tag{3.183}$$

from the recoupling diagram of figure 3.3

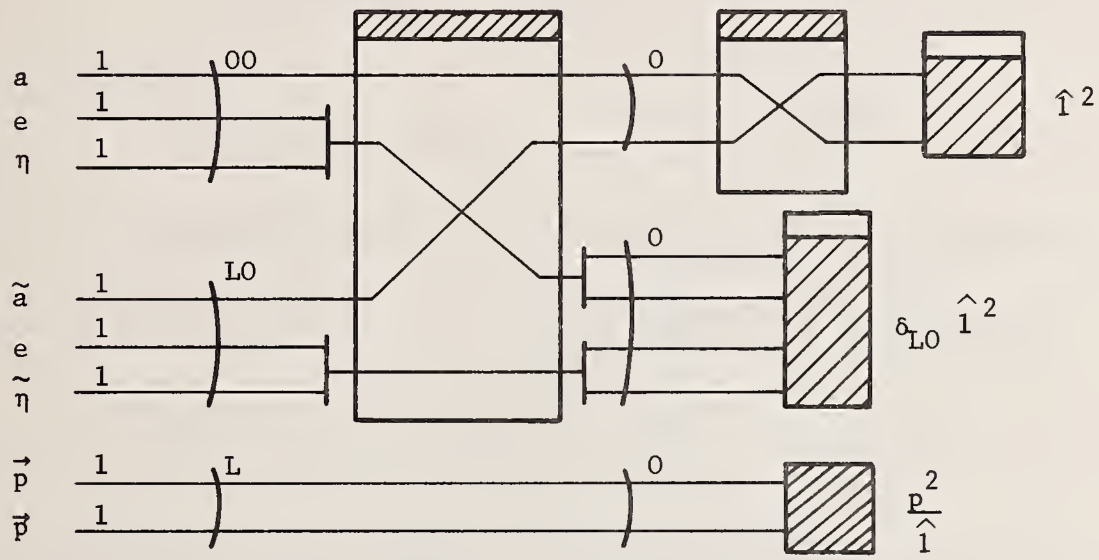


Figure 3.3.

As an exercice we can check that the various equations of motion are fulfilled by the free field solutions we have constructed. For simplification of the notation we set

$$S = \left(\frac{1}{2\pi}\right)^{3/2} \int d^3p \sqrt{\frac{N}{2E}} , \tag{3.186}$$

and we consider the fields of isospin 0. We get successively

i) $\partial_t \vec{E} - \text{rot}\,\vec{B} = m^2 \vec{\omega}$. (3.185)

$$\partial_t \vec{E} - \text{rot}\,\vec{B} = S\,\hat{1}\left\{e^{-i(\vec{p}\vec{r}-Et)}\left(E^2[\tilde{a}e]^{[0]} - \hat{1}[\tilde{a}\vec{p}]^{[0]} - 2\left[[\tilde{a}\vec{p}]^{[1]}[e\vec{p}]^{[1]}\right]^{[0]}\right) + \ldots\right\}$$

$$= S\,\hat{1}\left\{e^{-i(\vec{p}\vec{r}-Et)}\left(E^2[\tilde{a}e]^{[0]} - \sum_L\left(\frac{\hat{L}}{\hat{1}} + 2\begin{bmatrix}1&1&1\\1&1&1\\L&L&0\end{bmatrix}\right)\left[[\tilde{a}e]^{[L]}[\vec{p}\vec{p}]^{[L]}\right]^{[0]}\right) + \ldots\right\}$$

$$= S\,\hat{1}\;e^{-i(\vec{p}\vec{r}-Et)}(E^2 - p^2)[\tilde{a}e]^{[0]} = m^2\vec{\omega} \quad , \qquad (3.186)$$

where we have used the following results

$$L = 1 \qquad [\vec{p}^{[1]}\vec{p}^{[1]}]^{[1]} = 0 \quad , \qquad (3.187)$$

$$L = 2 \qquad \left(\frac{\hat{L}}{\hat{1}} + 2\begin{bmatrix}1&1&1\\1&1&1\\L&L&0\end{bmatrix}\right) = \left(\frac{\hat{2}}{\hat{1}} - 2\,\hat{2}\,\hat{1}\begin{Bmatrix}1&1&2\\1&1&1\end{Bmatrix}\right) = 0 \quad , \qquad (3.188)$$

$$L = 0 \qquad [\vec{p}^{[1]}\vec{p}^{[1]}]^{[0]} = \frac{p^2}{\hat{1}} \quad . \qquad (3.189)$$

ii) $i\,\text{div}\,\vec{E} = -\,m^2\,\omega_4$. (3.190)

$$i\,\text{div}\,\vec{E} = S\,\hat{1}\,E\left\{e^{-i(\vec{p}\vec{r}-Et)}\left([\tilde{a}(-i\vec{p})]^{[0]} - \frac{\hat{1}}{E^2}[\tilde{a}\vec{p}]^{[0]}[-i\vec{p}\,\vec{p}]^{[0]}\right) + \ldots\right\}$$

$$= -\,i\,S\,\hat{1}\,\frac{1}{E}\left\{e^{-i(\vec{p}\vec{r}-Et)}(E^2 - p^2)[\tilde{a}\vec{p}]^{[0]} + \ldots\right\} = -\,m^2\omega_4 \quad . \qquad (3.191)$$

iii) $\text{rot}\,\vec{E} = -\,\partial_t\vec{B}$. (3.192)

$$\text{rot}\,\vec{E} = S\sqrt{2}\;\hat{1}\,E\left\{e^{-i(\vec{p}\vec{r}-Et)}([\tilde{a}\,e\,\vec{p}]^{[0]}) + \ldots\right\} = -\,\partial_t\vec{B} \quad . \qquad (3.193)$$

iv) $\text{div}\,\vec{B} = 0$. (3.194)

This result comes trivially from the fact that $\vec{B}$ is a rotational field,

$$\text{div}\,\vec{B} = S\sqrt{2}\;\hat{1}\left\{e^{-i(\vec{p}\vec{r}-Et)}[\tilde{a}^{[1]}\vec{p}^{[1]}\vec{p}^{[1]}]^{[0]} + \ldots\right\} \quad , \qquad (3.195)$$

and

$$[\tilde{a}^{[1]}\vec{p}^{[1]}\vec{p}^{[1]}]^{[0]} \equiv 0 \quad . \qquad (3.196)$$

III.5.4 - Multipole expansion

The multipole expansion of the field vector potential $\vec{\rho}$ is

$$\vec{\rho}(\vec{r},t) = \sqrt{\frac{2}{\pi}} \int d^3p \sqrt{\frac{N}{2E}}\, \hat{1}^2 \sum_{\ell} (i)^{\ell}\, \hat{\ell}\, j_{\ell}(pr) \left\{ [\hat{p}^{[\ell]}\hat{r}^{[\ell]}]^{[0]} [a_{\vec{p}}^{[11]} e^{[1]} \tilde{\eta}^{[1]}]^{[00]} + (-)^{\ell} \text{C.C.} \right\}$$

$$= \sqrt{\frac{2}{\pi}} \int d^3p \sqrt{\frac{N}{2E}}\, \hat{1} \sum_{\ell J} (i)^{\ell}\, \hat{J} \left\{ \left[[a_{\vec{p}}^{[11]} \hat{p}^{[\ell]} \eta^{[1]}]^{[J0]}\, G_{\ell}^{[J]}(p\vec{r}) \right]^{[0]} + (-)^{\ell} \ldots \right\}, \tag{3.197}$$

with the notation,

$$G_{\ell}^{[J]}(p\vec{r}) = [e^{[1]}\hat{r}^{[\ell]}]^{[J]}\, j_{\ell}(pr) \quad . \tag{3.198}$$

The scalar potential ρ_4 is given by the Lorentz condition (3.134),

$$\overset{+}{\rho}_4(\vec{r},t) = -\rho_4 = \int \partial_4 \vec{\nabla}\vec{\rho}^{+}(\vec{r},t) = \sqrt{\frac{2}{\pi}} \int d^3p \sqrt{\frac{N}{2E}}\, \frac{1}{E}\, \hat{1}^2 \sum_{\ell} (-i)^{\ell}\, \hat{\ell}\, j_{\ell}(pr)$$

$$\times \left\{ [\tilde{a}^{[11]} \nabla^{[1]} \tilde{\eta}^{[1]}]^{[00]} [\hat{p}^{[\ell]}\hat{r}^{[\ell]}]^{[0]} - (-)^{\ell} \text{C.C.} \right\} \quad . \tag{3.199}$$

Thus,

$$\vec{\pi} = i(\partial_4 \vec{\rho}^{+} + \vec{\nabla}\, \overset{+}{\rho}_4) = i \sqrt{\frac{2}{\pi}} \int d^3p \sqrt{\frac{N}{2E}}\, E\, \hat{1}^2 \sum_{\ell} (-i)^{\ell}\, \hat{\ell}\, j_{\ell}(pr)$$

$$\times \left\{ [\tilde{a}^{[11]} e^{[1]} \tilde{\eta}^{[1]}]^{[00]} [\hat{p}^{[\ell]}\hat{r}^{[\ell]}]^{[0]} + \hat{1}\, [e^{[1]}\nabla^{[1]}]^{[0]} [\tilde{a}^{[11]} \nabla^{[1]} \tilde{\eta}^{[1]}]^{[00]} \times [\hat{p}^{[\ell]}\hat{r}^{[\ell]}]^{[0]} - (-)^{\ell} \ldots \right\}$$

$$= i \sqrt{\frac{2}{\pi}} \int d^3p \sqrt{\frac{N}{2E}}\, E\, \hat{1} \sum_{\ell J \lambda} (-i)^{\ell} \left(\delta_{\lambda\ell} + \frac{p^2}{E^2} \frac{\alpha_{\lambda J}\, \alpha_{J\ell}}{\hat{J}^2} \right) \hat{J}$$

$$\times \left\{ \left[[\tilde{a}^{[11]} \hat{p}^{[\ell]} \tilde{\eta}^{[1]}]^{[J0]}\, G_{\lambda}^{[J]}(p\vec{r}) \right]^{[0]} - (-)^{\ell} \text{C.C.} \right\} \quad . \tag{3.200}$$

The $\vec{\nabla}\, \overset{+}{\rho}_4$ term in $\vec{\pi}$ has been calculated with the diagram of figure 3.4 .

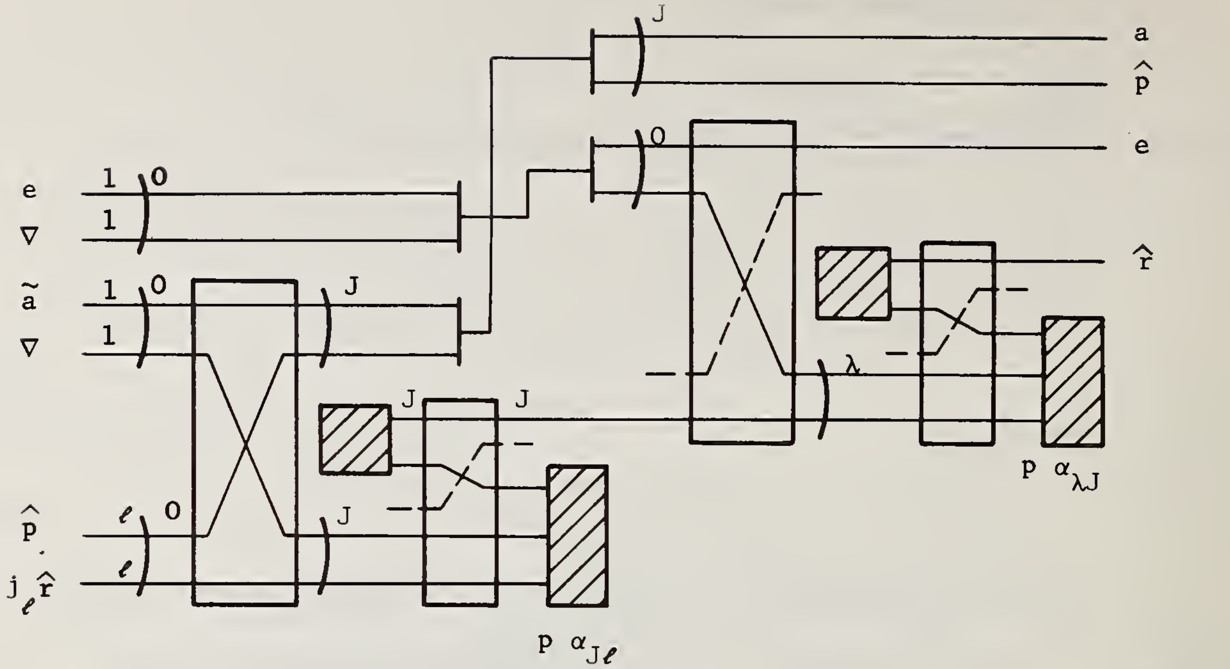


Figure 3.4

The multipole expansion of the magnetic part of the field is likewise

$$\vec{B}(\vec{r},t) = \text{rot}\,\vec{\rho}^{\,+}(\vec{r},t) = \sqrt{\frac{4}{\pi}} \int d^3p \sqrt{\frac{N}{2E}}\,\hat{1}^2 p \sum_{\ell J\lambda} (i)^{\ell}(-1)^J$$

$$\times\, \hat{J}\,\alpha_{\lambda\ell} \begin{Bmatrix}1 & 1 & 1\\ \ell & \lambda & J\end{Bmatrix} \left\{\left[\,[\tilde{a}^{[11]}_{\vec{p}}\,\hat{p}^{[\ell]}\,\tilde{\eta}^{[1]}]^{[J0]}\; G^{[J]}_{\lambda}(p\vec{r})\right]^{[0]} + (-)^{\ell}\,\text{c.c.}\right\} , \qquad (3.201)$$

from the diagram of figure 3.5.

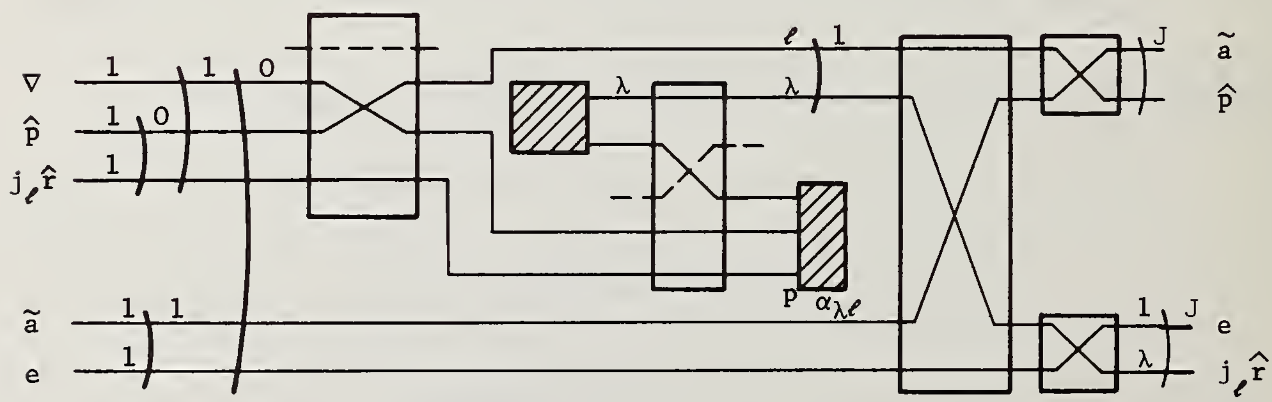


Figure 3.5

Now we can check the commutator $\{\vec{\rho}, \vec{\pi}\}_-$ on these multipole expansion forms. We immediately get a factor $\delta_{\lambda\ell}$ and we have to evaluate the commutator,

$$\left\{\left[[a_{\vec{p}}^{[11]}\hat{p}^{[\ell]}\eta^{[1]}]^{[J0]}\, G_{\ell}^{[J]}(p\vec{r})\right]^{[0]}, \left[[\tilde{a}_{\vec{p}'}^{[11]}\hat{p}'^{[\ell']}\tilde{\eta}^{[1]}]^{[J'0]}\, G_{\ell'}^{[J']}(p'\vec{r}')\right]^{[0]}\right\}_-$$

$$= \delta_{JJ'}\,\delta_{\ell\ell'}\,\delta^3(\vec{p}-\vec{p}')\,\frac{1}{\hat{\ell}}\,[\hat{r}^{[\ell]}\,\hat{r}'^{[\ell]}]^{[0]}\, j_\ell(pr)\,j_\ell(pr') \quad , \qquad (3.202)$$

from the diagram of figure 3.6.

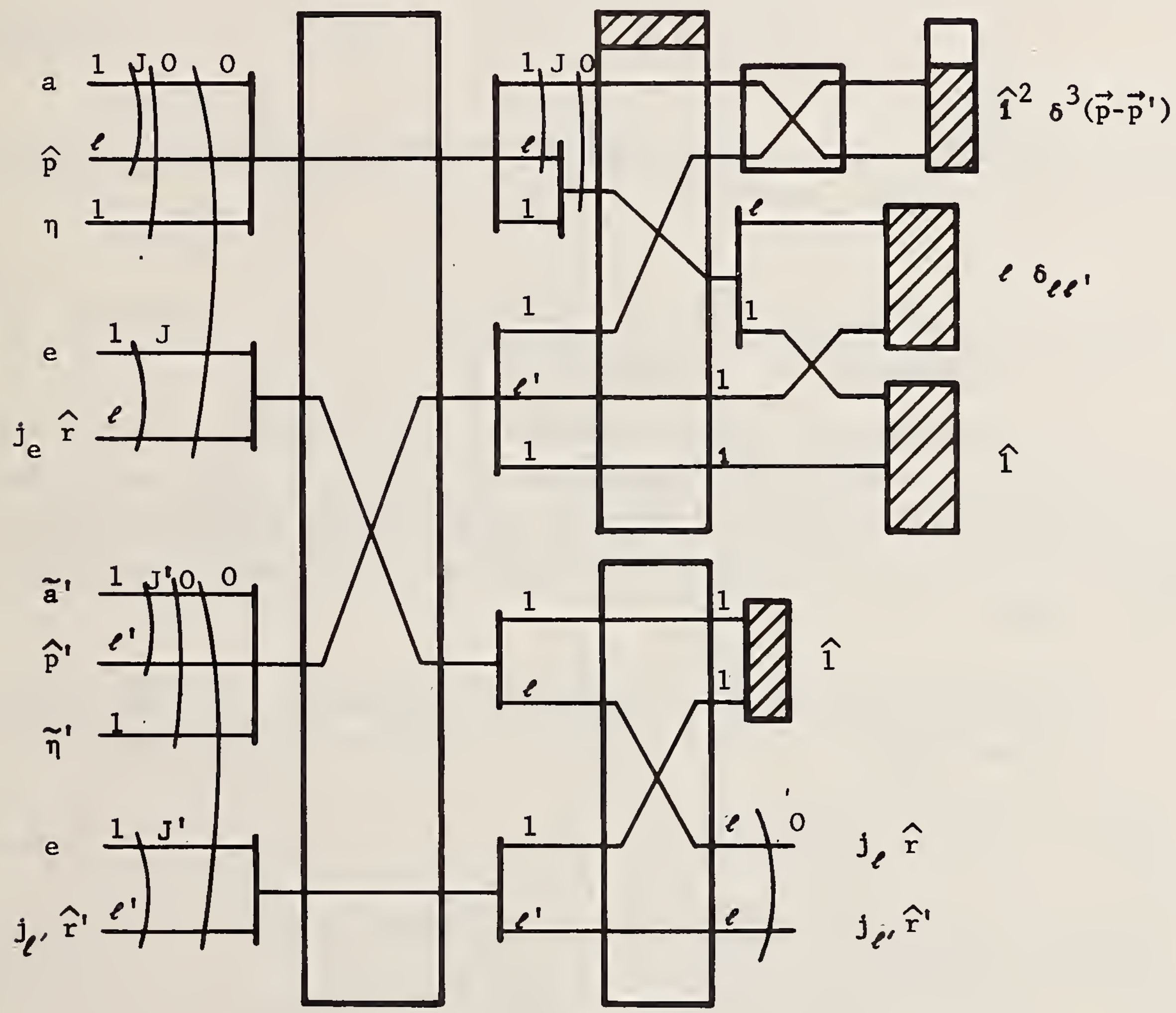


Figure 3.6

Note that after the double recoupling box the space and isospin parts are completely disentangled. Furthermore we are making use in figure 3.6 of a simplified graphical representation to simultaneously represent the couplings of $a^{[11]}$ both in angular momentum space (to $\hat{p}^{[\ell]}$) and in isospin space (to $\eta^{[1]}$). This notation will often be used in other figures.

We thus get, for the equal time commutator

$$\left\{\vec{\rho}(\vec{r},t),\vec{\pi}(\vec{r}',t)\right\}_- = i\,\frac{2}{\pi}\iint d^3p\,d^3p'\,\sqrt{\frac{NN'}{4EE'}}\,E'\,\hat{1}^2 \sum_{\ell\ell' JJ'} \left\{(i)^{\ell-\ell'}\hat{J}\hat{J}'\left(1+\frac{p^2}{E^2}\frac{\alpha_{\ell' J'}\alpha_{J'\ell'}}{\hat{J}'^2}\right)\right.$$

$$\left.\times\,\frac{1}{\hat{\ell}}\,\delta_{\ell\ell'}\delta_{JJ'}\,[\hat{r}^{[\ell]}\,\hat{r}'^{[\ell]}]^{[0]} j_\ell(pr)j_\ell(pr')\;\delta^3(\vec{p}-\vec{p}\,') + C.C.\right\}$$

$$= i\,9\,\delta^3(\vec{r}-\vec{r}\,') \equiv i(2s+1)(2t+1)\,\delta^3(\vec{r}-\vec{r}\,') \quad , \tag{3.203}$$

since we have

$$N\sum_{J=\ell\pm 1}\hat{J}^2\left(1+\frac{p^2}{E^2}\frac{\alpha_{\ell J}\alpha_{J\ell}}{\hat{J}^2}\right) = \hat{1}^2\hat{\ell}^2 \quad , \tag{3.204}$$

if we set the normalization constant equal to the value given in Eq.(3.180). For the ω field the result is similar with $t = 0$.

III.5.5 - Multipolarities and discretization

In analogy to the usual electromagnetic field expansion, we want now to define electric, magnetic and longitudinal multipoles such that each is a solution of the equation of motion with given total angular momentum and given parity. We consider here the ω field. Introduction of the isospin for the ρ field is readily made.

The magnetic, electric and longitudinal multipoles are given by the unitary transformation,

$$\underline{A}^{[J]}_{\mathcal{M}}(p) = \int d\hat{p}\;[a^{[1]}_{\vec{p}}\;\hat{p}^{[J]}]^{[J]} \quad , \tag{3.205}$$

$$\underline{A}^{[J]}_{\mathcal{E}}(p) = \int d\hat{p}\left[a^{[1]}_{\vec{p}}\left(\frac{\sqrt{J+1}}{\hat{J}}\,\hat{p}^{[J-1]} - \frac{\sqrt{J}}{\hat{J}}\,\hat{p}^{[J+1]}\right)\right]^{[J]} \quad , \tag{3.206}$$

$$\underline{A}^{[J]}_{\mathcal{L}}(p) = \int d\hat{p}\left[a^{[1]}_{\vec{p}}\left(\frac{\sqrt{J}}{\hat{J}}\,\hat{p}^{[J-1]} + \frac{\sqrt{J+1}}{\hat{J}}\,\hat{p}^{[J+1]}\right)\right]^{[J]} \quad . \tag{3.207}$$

With these definitions the field $\vec{\omega}(\vec{r},t)$

$$\vec{\omega}(\vec{r},t) = \sqrt{\frac{2}{\pi}}\int d^3p\,\sqrt{\frac{N}{2E}}\sum_{\ell J}(i)^\ell\,\hat{J}\left\{\left[[a^{[1]}_{\vec{p}}\hat{p}^{[\ell]}]^{[J]}\;G^{[J]}_\ell(p\vec{r})\right]^{[0]} + (-)^\ell C.C.\right\} , \tag{3.208}$$

can be expressed in term of its multipoles

$$\vec{\omega}(\vec{r},t) = \vec{\omega}_{\mathcal{M}} + \vec{\omega}_{\mathcal{E}} + \vec{\omega}_{\mathcal{L}} = \sqrt{\frac{2}{\pi}} \int p^2 dp \sqrt{\frac{1}{2E}} \sum_J (i)^J$$

$$\times \hat{J} \Big\{ \sqrt{N_{\mathcal{M}}} [\underline{A}_{\mathcal{M}}^{[J]}(p) G_J^{[J]}]^{[0]} - i\sqrt{N_{\mathcal{E}}} \Big[\underline{A}_{\mathcal{E}}^{[J]}(p) \Big(\frac{\sqrt{J+1}}{\hat{J}} G_{J-1}^{[J]} + \frac{\sqrt{J}}{\hat{J}} G_{J+1}^{[J]} \Big) \Big]^{[0]}$$

$$- i\sqrt{N_{\mathcal{L}}} \Big[\underline{A}_{\mathcal{L}}^{[J]}(p) \Big(\frac{\sqrt{J}}{\hat{J}} G_{J-1}^{[J]} - \frac{\sqrt{J+1}}{\hat{J}} G_{J+1}^{[J]} \Big) \Big]^{[0]} + (-)^J (C.C.) \Big\} \quad , \qquad (3.209)$$

where the normalization of the multipoles is chosen such that each multipole field by itself obeys a commutation relation with weight 1 for isospin 0 and weight 3 for isospin 1.

Utilizing the formulas (2.61) and (2.63), it can be checked that these different parts obey

$$\mathrm{div}\, \vec{\omega}_{\mathcal{M}} = 0 , \qquad \text{hence} \quad \omega_{\mathcal{M}4} = 0 \quad \text{(Lorentz)}, \qquad (3.210)$$

$$\mathrm{div}\, \vec{\omega}_{\mathcal{E}} = 0 , \qquad \text{hence} \quad \omega_{\mathcal{E}4} = 0 \quad \text{(Lorentz)}, \qquad (3.211)$$

$$\mathrm{rot}\, \vec{\omega}_{\mathcal{L}} = 0 , \qquad (3.212)$$

and

$$\mathrm{div}\, \vec{\omega}_{\mathcal{L}} = \sqrt{\frac{2}{\pi}} \int p^2 dp \sqrt{\frac{N_{\mathcal{L}}}{2E}}\, p \sum_J (i)^{J+1} \hat{J} \Big\{ [\underline{A}_{\mathcal{L}}^{[J]} G_J^{[J]}]^{[0]} + (-)^{J+1} C.C. \Big\} \qquad (3.213)$$

$$= - \partial_4 \omega_{\mathcal{L}4} \quad .$$

Hence

$$\omega_{\mathcal{L}4} = \sqrt{\frac{2}{\pi}} \int p^2 dp \sqrt{\frac{N_{\mathcal{L}}}{2E}}\, \frac{p}{E} \sum_J (i)^{J+1} \hat{J} \Big\{ [\underline{A}_{\mathcal{L}}^{[J]} G_J^{[J]}]^{[0]} + (-)^{J+1} C.C. \Big\} \quad . \qquad (3.214)$$

The condition stated above for the normalization thus yields,

$$N_{\mathcal{M}} = 1 \; , \quad N_{\mathcal{E}} = 1 \; , \quad N_{\mathcal{L}} = \frac{E^2}{m^2} \quad . \qquad (3.215)$$

We now obtain the canonical conjugate fields from the relations (3.124) and (3.125). Thus for isospin 0 the electric and the magnetic fields of the magnetic multipoles are respectively

$$\vec{\pi}_{\mathcal{M}} = - \vec{E}_{\mathcal{M}} = i \sqrt{\frac{2}{\pi}} \int p^2 dp \sqrt{\frac{N_{\mathcal{M}}}{2E}}\, E \sum_J (-i)^J \hat{J} \Big\{ [\tilde{\underline{A}}_{\mathcal{M}}^{[J]}(p) G_J^{[J]}(p\vec{r})]^{[0]} - (-)^J C.C. \Big\} , \qquad (3.216)$$

$$\vec{B}_{\mathcal{M}} = - \sqrt{\frac{2}{\pi}} \int p^2 dp \sqrt{\frac{N_{\mathcal{M}}}{2E}}\, p \sum_J (-i)^J \Big\{ [\tilde{\underline{A}}_{\mathcal{M}}^{[J]}(p) (\sqrt{J}\, G_{J+1}^{[J]} + \sqrt{J+1}\, G_{J-1}^{[J]})]^{[0]} + (-)^J C.C. \Big\} \qquad (3.217)$$

while for the electric and magnetic field of the electric multipoles we get

$$\vec{\pi}_{\mathscr{E}} = - \vec{E}_{\mathscr{E}} = i \sqrt{\frac{2}{\pi}} \int p^2 dp \sqrt{\frac{N_{\mathscr{E}}}{2E}}\, E \sum_J (-i)^{J-1} \Big\{ [\tilde{\underline{A}}^{[J]}_{\mathscr{E}} (\sqrt{J}\, G^{[J]}_{J+1} + \sqrt{J+1}\, G^{[J]}_{J-1})]^{[0]} - (-)^J \text{c.c.} \Big\} \quad (3.218)$$

$$\vec{B}_{\mathscr{E}} = - \sqrt{\frac{2}{\pi}} \int p^2 dp \sqrt{\frac{N_{\mathscr{E}}}{2E}}\, p \sum_J (-i)^{J-1}\, \hat{J} \Big\{ [\tilde{\underline{A}}^{[J]}_{\mathscr{E}} G^{[J]}_J]^{[0]} - (-)^J \text{c.c.} \Big\} \quad . \quad (3.219)$$

Finally the longitudinal part is,

$$\vec{\pi}_{\mathscr{L}} = - \vec{E}_{\mathscr{L}} = i \sqrt{\frac{2}{\pi}} \int p^2 dp \sqrt{\frac{N_{\mathscr{L}}}{2E}}\, \frac{m^2}{E} \sum_J (-i)^{J+1} \Big\{ [\tilde{A}^{[J]}_{\mathscr{L}} (\sqrt{J+1}\, G^{[J]}_{J+1} - \sqrt{J}\, G^{[J]}_{J-1})]^{[0]} + (-)^{J+1} \text{c.c.} \Big\}$$

$$\vec{B}_{\mathscr{L}} = 0 \quad . \quad (3.220)$$

It can be checked directly on these forms through a lengthy but simple calculation that these fields with the chosen normalization do fulfill the equations of motion and the commutation relations.

Finally all these fields are discretized by using the substitution,

$$\underline{A}^{[J]}(p) = \sum_\nu A^{[J]}_\nu\, f_{\nu J}(p) \quad . \quad (3.221)$$

The various multipoles of the discretized form of $\vec{\omega}(\vec{r},t)$ are thus,

$$\vec{\omega}_{\mathscr{M}}(\vec{r},t) = \sum_\nu \sum_J i^J\, \hat{J} \Big\{ [A^{[J]}_{\mathscr{M}\nu} Y^{[J]}_{1\,J}(\hat{r})]^{[0]}\, \mathscr{W}^{\mathscr{M}}_{\nu\,J\,J}(r) + (-)^J \text{c.c.} \Big\} \quad , \quad (3.222)$$

$$\vec{\omega}_{\mathscr{E}}(\vec{r},t) = \sum_\nu \sum_J i^{J-1} \Big\{ [A^{[J]}_{\mathscr{E}\nu} (\sqrt{J+1}\, Y^{[J]}_{1\,J-1}\, \mathscr{W}^{\mathscr{E}}_{\nu\,J\,J-1} + \sqrt{J}\, Y^{[J]}_{1\,J+1}\, \mathscr{W}^{\mathscr{E}}_{\nu J J+1})]^{[0]} + (-)^{J-1} \text{c.c.} \Big\} \quad (3.223)$$

$$\vec{\omega}_{\mathscr{L}}(\vec{r},t) = \sum_\nu \sum_J i^{J-1} \Big\{ [A^{[J]}_{\mathscr{L}\nu} (\sqrt{J}\, Y^{[J]}_{1\,J-1}\, \mathscr{W}^{\mathscr{L}}_{\nu\,J\,J-1} - \sqrt{J+1}\, Y^{[J]}_{1\,J+1}\, \mathscr{W}^{\mathscr{L}}_{\nu\,J\,J+1})]^{[0]} + (-)^{J-1} \text{c.c.} \Big\} \quad (3.224)$$

$$\omega_{\mathscr{L}_4}(\vec{r},t) = \sum_\nu \sum_J i^{J+1}\, \hat{J} \Big\{ [A^{[J]}_{\mathscr{L}\nu}\, \hat{r}^{[J]}]^{[0]}\, \mathcal{V}^{\mathscr{L}}_{\nu\,J}(r) + (-)^{J+1} \text{c.c.} \Big\} \quad (3.225)$$

and the conjugate fields,

$$\vec{\pi}_{\mathscr{M}} = - \vec{E}_{\mathscr{M}} = \sum_\nu \sum_J (-i)^{J-1}\, \hat{J} \Big\{ [\tilde{A}^{[J]}_{\mathscr{M}\nu}\, Y^{[J]}_{1\,J}(\hat{r})]^{[0]}\, \mathscr{X}^{\mathscr{M}}_{\nu\,J\,J}(r) + (-)^{J-1} \text{c.c.} \Big\} \quad (3.226)$$

$$\vec{B}_{\mathscr{M}} = - \sum_\nu \sum_J (-i)^J \Big\{ [\tilde{A}^{[J]}_{\mathscr{M}\nu} (\sqrt{J}\, Y^{[J]}_{1\,J+1}\, \mathscr{Y}^{\mathscr{M}}_{\nu\,J\,J+1} + \sqrt{J+1}\, Y^{[J]}_{1\,J-1}\, \mathscr{Y}^{\mathscr{M}}_{\nu\,J\,J-1})]^{[0]} + (-)^J \text{c.c.} \Big\} \quad (3.227)$$

$$\vec{\pi}_{\mathscr{E}} = -\vec{E}_{\mathscr{E}} = -\sum_{\nu}\sum_{J} (-i)^J \left\{ [\tilde{A}^{[J]}_{\mathscr{E}\nu} \left(\sqrt{J}\, Y^{[J]}_{1\,J+1}\, \mathscr{X}^{\mathscr{E}}_{\nu\,J\,J+1} + \sqrt{J+1}\, Y^{[J]}_{1\,J-1} \mathscr{X}^{\mathscr{E}}_{\nu\,J\,J-1}\right)]^{[0]} - (-)^J \text{c.c.} \right\}, \tag{3.228}$$

$$\vec{B}_{\mathscr{E}} = -\sum_{\nu}\sum_{J} (-i)^{J-1}\, \hat{J} \left\{ [\tilde{A}^{[J]}_{\mathscr{E}\nu}\, Y^{[J]}_{1\,J}]^{[0]}\, \mathscr{Y}^{\mathscr{E}}_{\nu\,J\,J} + (-)^{J-1} \text{c.c.} \right\} , \tag{3.229}$$

$$\vec{\pi}_{\mathscr{L}} = -\vec{E}_{\mathscr{L}} = \sum_{\nu}\sum_{J} (-i)^J \left\{ [\tilde{A}^{[J]}_{\mathscr{L}\nu} \left(\sqrt{J+1}\, Y^{[J]}_{1\,J+1}\, \mathscr{Z}^{\mathscr{L}}_{\nu\,J\,J+1} - \sqrt{J}\, Y^{[J]}_{1\,J-1}\, \mathscr{Z}^{\mathscr{L}}_{\nu\,J\,J-1}\right)]^{[0]} - (-)^J \text{c.c.} \right\}, \tag{3.230}$$

$$\pi_{\mathscr{L}_4} = 0 \;. \tag{3.231}$$

we have introduced the vector spherical harmonics and the functions :

$$\mathscr{W}^{\kappa}_{\nu\,J\,\lambda}(r) = \sqrt{\frac{2}{\pi}} \int p^2 dp \sqrt{\frac{N_\kappa}{2E}}\; f_{\nu J}(p)\; j_\lambda(pr) \;, \tag{3.232}$$

$$\mathscr{X}^{\kappa}_{\nu\,J\,\lambda}(r) = \sqrt{\frac{2}{\pi}} \int p^2 dp \sqrt{\frac{N_\kappa}{2E}}\; E f_{\nu J}(p)\; j_\lambda(pr) \;, \tag{3.233}$$

$$\mathscr{Y}^{\kappa}_{\nu\,J\,\lambda}(r) = \sqrt{\frac{2}{\pi}} \int p^2 dp \sqrt{\frac{N_\kappa}{2E}}\, p\, f_{\nu J}(p)\; j_\lambda(pr) \;, \tag{3.234}$$

$$\mathscr{Z}^{\mathscr{L}}_{\nu\,J\,\lambda}(r) = \sqrt{\frac{2}{\pi}} \int p^2 dp \sqrt{\frac{N_{\mathscr{L}}}{2E}}\; \frac{m^2}{E}\; f_{\nu J}(p)\; j_\lambda(pr)$$

$$= \sqrt{\frac{2}{\pi}} \int p^2 dp \sqrt{\frac{1}{2E}}\; m\; f_{\nu J}(p)\; j_\lambda(pr) \;, \tag{3.235}$$

$$\mathscr{V}^{\mathscr{L}}_{\nu\,J} = \sqrt{\frac{2}{\pi}} \int p^2 dp \sqrt{\frac{N_{\mathscr{L}}}{2E}}\; \frac{p}{E}\; f_{\nu J}(p)\; j_J(pr)$$

$$= \sqrt{\frac{2}{\pi}} \int p^2 dp \sqrt{\frac{1}{2E}}\; \frac{p}{m}\; f_{\nu J}(p)\; j_J(pr) \;. \tag{3.236}$$

It can be checked again that these discretized fields fulfill indeed the commutation relation

$$\sum_{\kappa} \left\{ \vec{\omega}_\kappa(\vec{r}),\; \vec{\pi}_\kappa(\vec{r}\,') \right\}_- = i\, 3\, \delta^3(\vec{r}-\vec{r}\,'), \qquad \kappa = \mathscr{E},\; \mathscr{M},\; \mathscr{L}, \tag{3.237}$$

where we made use of the result

$$2 \sum_{\nu,J} \mathscr{W}^{\mathscr{M}}_{\nu\,J\,J}(r)\, \mathscr{X}^{\mathscr{M}}_{\nu\,J\,J}(r') + \frac{J}{\hat{J}^2} \left(\mathscr{W}^{\mathscr{E}}_{\nu\,J\,J-1}(r)\, \mathscr{X}^{\mathscr{E}}_{\nu\,J\,J-1}(r') + \mathscr{W}^{\mathscr{L}}_{\nu\,J\,J-1}(r)\, \mathscr{Z}^{\mathscr{L}}_{\nu\,J\,J-1}(r') \right)$$

$$+ \frac{J+1}{\hat{J}^2} \left(\mathscr{W}^{\mathscr{E}}_{\nu\,J\,J+1}(r)\, \mathscr{X}^{\mathscr{E}}_{\nu\,J\,J+1}(r') + \mathscr{W}^{\mathscr{L}}_{\nu\,J\,J+1}(r)\; \mathscr{Z}^{\mathscr{L}}_{\nu\,J\,J-1}(r') \right) = 3\, \frac{\delta(r-r')}{r^2} \,. \tag{3.238}$$

For the charged field $\vec{\rho}$, we have similar expressions where the isospin functions $\eta^{[1]}$ must be coupled with the charged operators $A^{[J1]}$, and the usual $\hat{1}$ factor introduced. This yields a factor 9 in the commutator instead of 3.

III.6 - THE FREE FIELD ENERGY FOR SPIN 1 PARTICLES

The free field energy is given by equation (3.127) or more simply for the transverse case by equation (3.129). Thus the contribution of the magnetic multipoles is in the case of the ω-field

$$\int d^3r\, \mathcal{H}_{\mathcal{M}} = \frac{1}{2} : \int d^3r \left\{ E^2_{\mathcal{M}} + B^2_{\mathcal{M}} + m^2 \vec{\omega}^2_{\mathcal{M}} \right\} :$$

$$= \frac{1}{2} \int d^3r \Big\{ \Big(\sum_{\nu\nu'} \sum_J \hat{J}^2 [\tilde{A}^{[J]}_{\mathcal{M}\nu} Y^{[J]}_{1\,J}]^{[0]} [A^{[J]}_{\mathcal{M}\nu'} Y^{[J]}_{1\,J}]^{[0]} \mathfrak{X}^{\mathcal{M}}_{\nu JJ}(r)\, \mathfrak{X}^{\mathcal{M}}_{\nu' JJ}(r) + C.C. \Big)$$

$$+ \sum_{\nu\nu'} \sum_J \Big(\Big[\tilde{A}^{[J]}_{\mathcal{M}\nu} \big(\sqrt{J}\, Y^{[J]}_{1\,J+1}\, \mathcal{Y}^{\mathcal{M}}_{\nu\,J\,J+1}(r) + \sqrt{J+1}\, Y^{[J]}_{1\,J-1}\, \mathcal{Y}^{\mathcal{M}}_{\nu\,J\,J-1}(r) \big) \Big]^{[0]}$$

$$\times \Big[A^{[J]}_{\mathcal{M}\nu'} \big(\sqrt{J}\, Y^{[J]}_{1\,J+1}\, \mathcal{Y}^{\mathcal{M}}_{\nu'\,J\,J+1}(r) + \sqrt{J+1}\, Y^{[J]}_{1\,J-1}\, \mathcal{Y}^{\mathcal{M}}_{\nu'\,J\,J-1}(r) \big) \Big]^{[0]} + C.C. \Big) + m^2 \vec{\omega}^2_{\mathcal{M}}$$

$$= \sum_{\nu\nu'} \sum_J \hat{J} \Big(\int p^2 dp\, E\, f_{\nu J}(p)\, f_{\nu' J}(p) \Big) [\tilde{A}^{[J]}_{\mathcal{M}\nu}\, A^{[J]}_{\mathcal{M}\nu'}]^{[0]} \quad , \qquad (3.239)$$

with a similar expression for the electric multipoles, since there the expressions for $\vec{E}$ and $\vec{B}$ are just interchanged. For the longitudinal part we start from expression (3.127), and we get again a form similar to Eq. (3.239) with the longitudinal operators $A^{[J]}_{\mathcal{L}\nu}$.

For the ρ-fields the energy operators are identical with the isospin coupling added, namely

$$\int d^3r\, \mathcal{H}_{\mathcal{M}} = \sum_{\nu\nu'} \sum_J \hat{J}\, \hat{1} \Big(\int p^2 dp\, E\, f_{\nu J}(p) f_{\nu' J}(p) \Big) [\tilde{A}^{[J1]}_{\mathcal{M}\nu}\, A^{[J1]}_{\mathcal{M}\nu'}]^{[00]} \quad . \qquad (3.240)$$

The invariant matrix elements are easily obtained from these expressions as in the case of the pion field, see Eq. (3.35) and Eq. (3.36).

CHAPTER IV

CENTER OF MASS

IV.1 - THE CENTER OF MASS PSEUDO-HAMILTONIAN

According to the discussion of Section I.4 our treatment of the center of mass motion requires the calculation of the invariant matrix elements of the pseudo-Hamiltonian

$$\mathcal{H}_{CM} = \xi(\vec{P}^2 + \Omega^2 \vec{R}^2) \quad , \qquad (4.1)$$

which is added to the Hamiltonian of the system. The square of the center of mass momentum operator $\vec{P}^2$ will require the evaluation of one and two-body matrix elements (viz. $\vec{p}_i^2$ and $\vec{p}_i \vec{p}_j$). On the other hand the square of the center of mass coordinate $\vec{R}^2$ is a sum of many-body operators which we will transform into products of one-and two-body operators as discussed in Section I.4 , Eqs. (1.48) and (1.49).

IV.2 - THE CENTER OF MASS MOMENTUM

For a system made of Bosons and Fermions the C.M. momentum $\vec{P}^2$ is given in terms of the field operators (φ and ψ respectively) by

$$\begin{aligned}\vec{P}^2 = & \sum_i \int d^3x_i \varphi^+(x_i)\, i \overleftrightarrow{\frac{\partial}{\partial t}}\, \vec{p}_i^{\,2}\, \varphi(x_i) + \sum_j \int d^3x_j \psi^+(x_j)\, \vec{p}_j^{\,2}\, \psi(x_j) \\ & + \sum_{i \neq j} \int d^3x_i d^3x_j \left[\varphi^+(x_i)\, i \overleftrightarrow{\frac{\partial}{\partial t}}\, \vec{p}_i \varphi(x_i)\right] \cdot \left[\varphi^+(x_j)\, i \overleftrightarrow{\frac{\partial}{\partial t}}\, \vec{p}_j\, \varphi(x_j)\right] \\ & + \sum_{i \neq j} \int d^3x_i d^3x_j \left[\psi^+(x_i)\, \vec{p}_i \psi(x_i)\right] \cdot \left[\psi^+(x_j)\, \vec{p}_j\, \psi(x_j)\right] \\ & + \sum_{i,j} \int d^3x_i d^3x_j \left[\varphi^+(x_i)\, i \overleftrightarrow{\frac{\partial}{\partial t}}\, \vec{p}_i \varphi(x_i)\right] \cdot \left[\psi^+(x_j)\, \vec{p}_j\, \psi(x_j)\right] \quad . \qquad (4.2)\end{aligned}$$

In that expression note that φ^+ and φ represent respectively the creation and the annihilation part of the real Boson fields, Eqs. (3.24, 3.25).

i - The quadratic terms $\vec{p}_i^{\,2}$ -

In order to evaluate the one-body invariant matrix elements of the quadratic terms (diagonal in the total angular momentum of the particle) we introduce the number operator N(p) in the p representation

$$N(p) = \int d^2\hat{p}\; a^+_{\vec{p}}\, a_{\vec{p}} \qquad .$$

For the spin 0 Boson field it is

$$N(p) = \sum_{\nu\nu'\ell} f_{\nu\ell}(p)\, f_{\nu'\ell}(p)\; \hat{1}\,\hat{\ell}\; [\tilde{A}_{\nu}^{[\ell 1]}\; A_{\nu'}^{[\ell 1]}]^{[00]} \qquad . \tag{4.3}$$

Thus in terms of fields the single particle momentum operator $\vec{p}^{\,2}$ is

$$\vec{p}^{\,2} = \sum_{\nu\nu'\ell} [\nu\ell|\vec{p}^{\,2}|\nu'\ell]\, [\tilde{\eta}^{[1]}\,|\eta^{[1]}]\, [\tilde{A}_{\nu}^{[\ell 1]}\; A_{\nu'}^{[\ell 1]}]^{[00]} \qquad , \tag{4.4}$$

where the invariant matrix element is

$$[\nu\ell|\vec{p}^{\,2}|\nu'\ell] = \hat{\ell}\int p^2 dp\; f_{\nu\ell}(p)\; p^2\; f_{\nu'\ell}(p) \qquad . \tag{4.5}$$

For the spin 1/2 Fermion field the number operator is (on similar lines as Section III.4)

$$N(p) = \frac{1}{2}\sum_{\nu\nu' j\ell} \frac{\hat{1}}{2}\,\hat{j}\; [\tilde{B}_{\nu\ell}^{[j\;1/2]}\; B_{\nu'\ell}^{[j\;1/2]}]^{[00]} \int p'^2 dp' \int r^2 dr$$

$$\times \frac{2}{\pi}\Big\{\sqrt{\frac{(E+m)(E'+m)}{EE'}}\; j_\ell(pr)\, j_\ell(p'r)\, f_{\nu\ell}(p) f_{\nu'\ell}(p')$$

$$+\sqrt{\frac{(E-m)(E'-m)}{EE'}}\; j_\lambda(pr)\, j_\lambda(p'r)\, f_{\nu\ell}(p) f_{\nu'\ell}(p')\Big\}_{\lambda=2j-\ell}$$

$$= \sum_{\nu\nu'\ell j} \frac{\hat{1}}{2}\,\hat{j}\; [\tilde{B}_{\nu\ell}^{[j\;1/2]}\; B_{\nu'\ell}^{[j\;1/2]}]^{[00]}\; f_{\nu\ell}(p)\; f_{\nu'\ell}(p) \qquad , \tag{4.6}$$

which yields

$$\vec{p}^{\,2} = \sum_{\nu\nu'\ell j} [\nu\ell j|\vec{p}^{\,2}|\nu'\ell j]\, [\tilde{\eta}^{[1/2]}\,|\eta^{[1/2]}]\, [\tilde{B}_{\nu\ell}^{[j\;1/2]}\; B_{\nu'\ell}^{[j\;1/2]}]^{[00]} \qquad ,$$

with

$$[\nu\ell j|\vec{p}^{\,2}|\nu'\ell j] = \frac{\hat{j}}{\hat{\ell}} [\nu\ell|\vec{p}^{\,2}|\nu'\ell] \quad , \tag{4.7}$$

in terms of the matrix element of Eq.(4.5).

For spin 1 Bosons with isospin t and multipolarity κ we obtain along the same line as in Eq.(3.239) for the number operator

$$N_\kappa(p) = \sum_{\nu J} f_{\nu J}(p)\, f_{\nu' J}(p)\, \hat{t}\, \hat{J}\, [\tilde{A}^{[Jt]}_{\kappa\nu}\, A^{[Jt]}_{\kappa\nu'}]^{[00]} \quad , \tag{4.8}$$

and

$$\vec{p}^{\,2}_\kappa = \sum_{\nu\nu'} [\nu J\kappa|\vec{p}^{\,2}|\nu' J\kappa]\, [\tilde{\eta}^{[t]}|\eta^{[t]}]\, [\tilde{A}^{[Jt]}_{\kappa\nu}\, A^{[Jt]}_{\kappa\nu'}]^{[00]} \quad ,$$

where for all multipolarities

$$[\nu J\kappa|\vec{p}^{\,2}|\nu' J\kappa] = [\nu J|\vec{p}^{\,2}|\nu' J] \quad , \tag{4.9}$$

of Eq.(4.5).

The radial integrals entering expression (4.5) are readily evaluated when using the harmonic oscillator basis (3.31) with parameter $\alpha = (m\,\omega)^{-1}$

$$\int p^2 dp\, p^2\, f_{\nu\ell}(p)\, f_{\nu'\ell}(p)$$

$$= - \left(\frac{C_{\nu-1,\ell}}{C_{\nu,\ell}} \frac{\nu}{\alpha^2}\, \delta_{\nu',\nu-1} + \frac{C_{\nu,\ell}}{C_{\nu+1,\ell}} \frac{\nu+1}{\alpha^2}\, \delta_{\nu',\nu+1} \right)$$

$$= - \frac{1}{\alpha^2} \left(\sqrt{\nu(\nu+\ell+1/2)}\; \delta_{\nu',\nu-1} + \sqrt{(\nu+1)(\nu+\ell+3/2)}\; \delta_{\nu',\nu+1} \right) \quad . \tag{4.10}$$

The coefficients $C_{\nu\ell}$ are defined in Eq.(3.32).

ii - The bi-linear terms $\vec{p}_i.\vec{p}_j$

We evaluate now the invariant two-body matrix elements by expressing the scalar product $\vec{p}_i.\vec{p}_j$ in terms of a product of invariant operators

$$\vec{p}_i.\vec{p}_j = \hat{1}\, [\vec{p}^{[1]}_1 \vec{p}^{[1]}_2]^{[0]} = \hat{1}\, [p^{[1]}_1 e^{[1]}]^{[0]}\, \hat{1}\, [p^{[1]}_2 e^{[1]}]^{[0]} \quad . \tag{4.11}$$

We consider first the single particle linear momentum operator $\vec{p}$ expressed in terms of the field operators.

For the spin 0 Boson case

$$\vec{p} = i \int d^3x\, \varphi^+(x)\, \frac{\overleftrightarrow{\partial}}{\partial t}\, \varphi(x)\, \hat{1}\, [p^{[1]} e^{[1]}]^{[0]}$$

$$= \sum_{\substack{\nu_1 \ell_1 \\ \nu_2 \ell_2}} [\tilde{\eta}^{[1]} | \eta^{[1]}] [\nu_1 \ell_1 | \vec{p}^{\,[1]} | \nu_2 \ell_2] [\tilde{A}_{\nu_1}^{[\ell_1 t]} e^{[1]} A_{\nu_2}^{[\ell_2 t]}]^{[00]} \quad , \qquad (4.12)$$

where

$$[\nu_1 \ell_1 | \vec{p}^{\,[1]} | \nu_2 \ell_2] = \int p^2 dp\, f_{\nu_1 \ell_1}(p)\, f_{\nu_2 \ell_2}(p)\, p \quad , \qquad (4.13)$$

as shown in figure 4.1 .

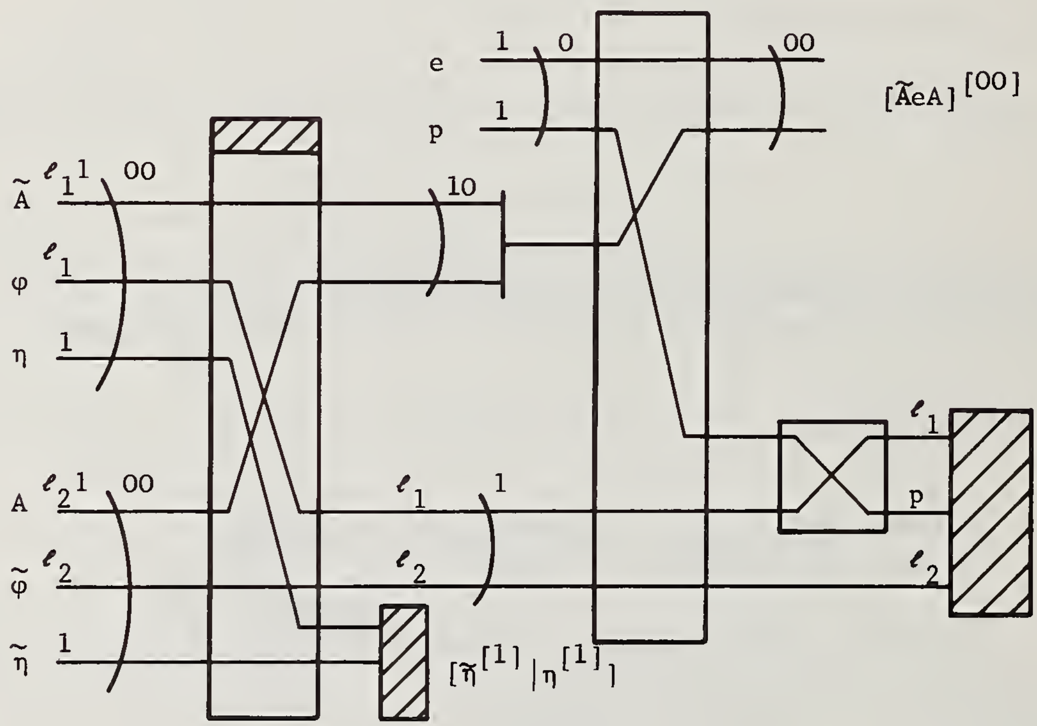


Figure 4.1

For the spin 1/2 Fermion case

$$\vec{p} = \sum_{\substack{\nu_1\ell_1 j_1 \\ \nu_2\ell_2 j_2}} [\tilde{\eta}^{[1/2]}|\eta^{[1/2]}][\nu_1\ell_1 j_1|\vec{p}^{[1]}|\nu_2\ell_2 j_2]\left[\tilde{B}^{[j_1 1/2]}_{\nu_1\ell_1}\; e^{[1]} B^{[j_2 1/2]}_{\nu_2\ell_2}\right]^{[00]} \tag{4.14}$$

where

$$[\nu_1\ell_1 j_1|\vec{p}^{[1]}|\nu_2\ell_2 j_2] = i^{\ell_2-\ell_1-1}\,(-1)^{j_2-1/2}\,\hat{j}_1\,\hat{j}_2 \times \int p^2\,dp\, f_{\nu_1\ell_1}(p)\, f_{\nu_2\ell_2}(p)$$

$$\times \frac{p}{2E}\left((-1)^{\ell_2}\begin{Bmatrix}1/2 & \ell_1 & j_1\\ 1 & j_2 & \ell_2\end{Bmatrix}\alpha_{\ell_1\ell_2}\,(E+m) + (-1)^{\lambda_2}\begin{Bmatrix}1/2 & \lambda_1 & j_1\\ 1 & j_2 & \lambda_2\end{Bmatrix}\alpha_{\lambda_1\lambda_2}\,(E-m)\right) \tag{4.15}$$

The geometry is shown on figure 4.2. The coefficients α_{ij} are defined in Eq. (2.37).

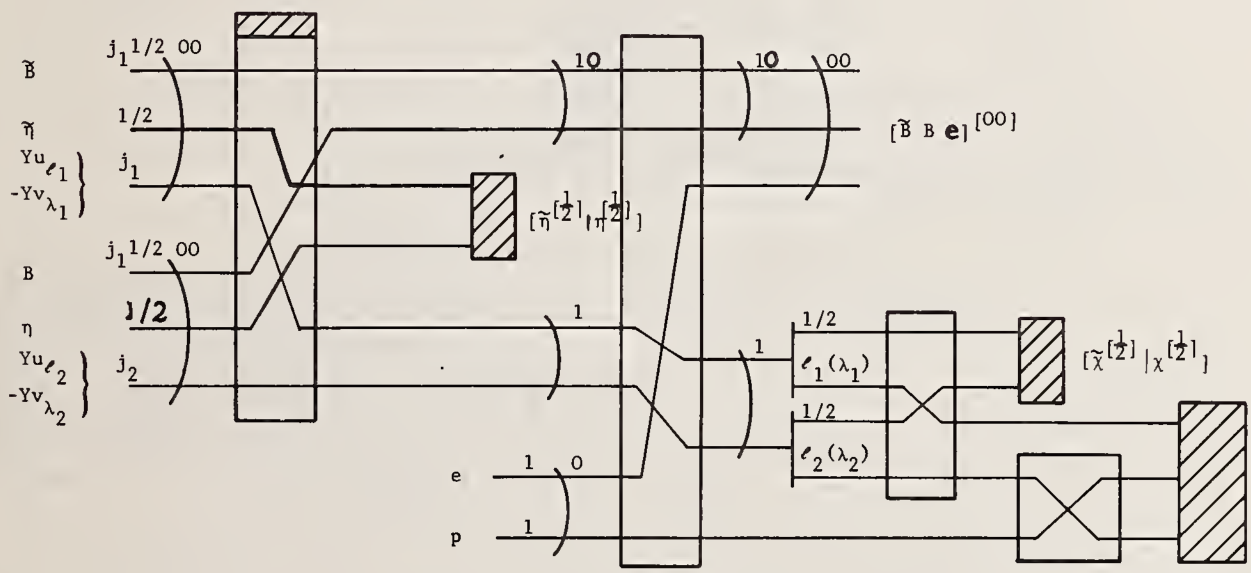


Figure 4.2

We are making use on this diagram of the same simplified graphical representation as in figure 3.6, to represent the coupling of the η's to the isospin part of the B's.

For the spin 1 Boson field with isospin t we get for the different multipolarities κ

$$\vec{P}_{(\kappa)} = \sum_{\substack{\nu_1 J_1 \\ \nu_2 J_2}} [\tilde{\eta}^{[t]}|\eta^{[t]}]\,[\nu_1 J_1 \kappa|\vec{p}^{\,[1]}|\nu_2 J_2 \kappa]\,[\tilde{A}^{[J_1 t]}_{\kappa_1 \nu}\, e^{[1]} A^{[J_2 t]}_{\kappa_2 \nu}]^{[00]} \quad , \qquad (4.16)$$

where for the magnetic multipoles

$$[\nu_1 J_1 \mathcal{M}|\vec{p}^{\,[1]}|\nu_2 J_2 \mathcal{M}] = \hat{1} \begin{bmatrix} 1 & J_1 & J_1 \\ 1 & J_2 & J_2 \\ 0 & 1 & 1 \end{bmatrix} |\alpha_{J_1 J_2}| \int p^2 dp\ f_{\nu_1 J_1}(p)\, p\, f_{\nu_2 J_2}(p) \quad , \qquad (4.17)$$

according to the diagram of figure 4.3 .

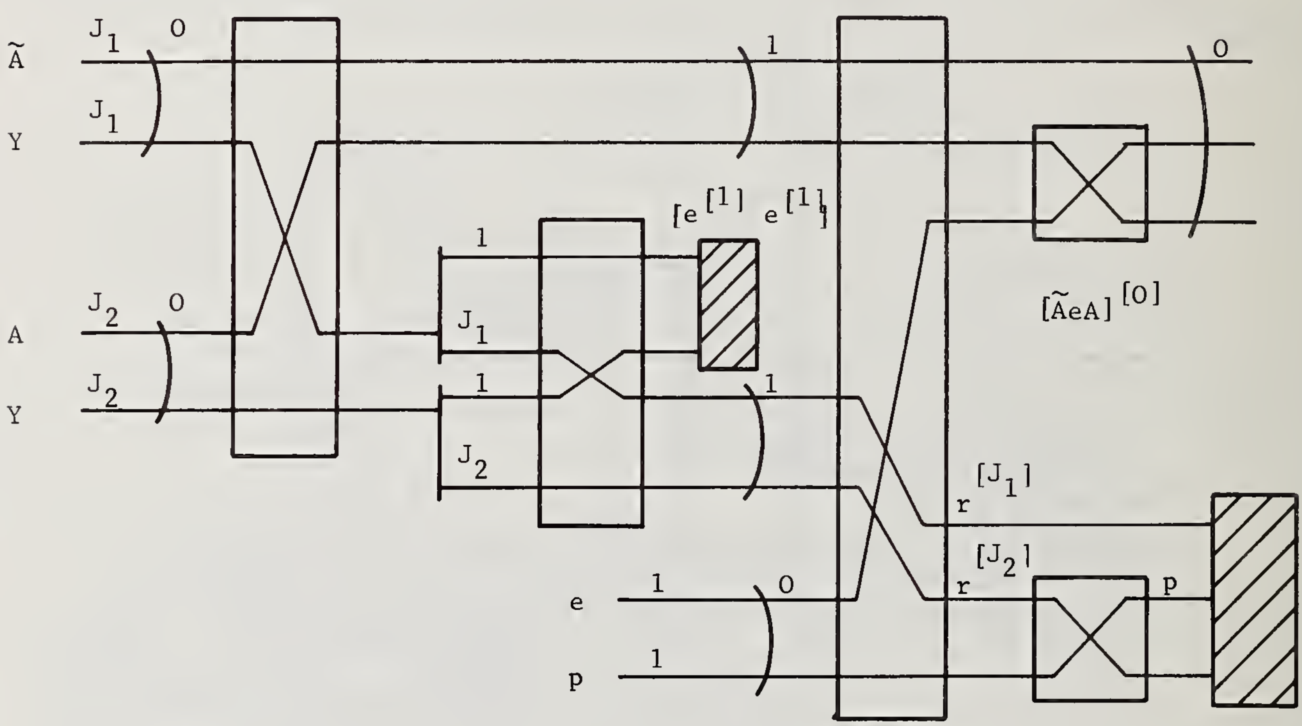


Figure 4.3

For the electric multipoles we obtain likewise

$$[\nu_1 J_1 \mathscr{E} | \vec{p}^{\,[1]} | \nu_2 J_2 \mathscr{E}] = (-i)^{J_1+1-J_2} \frac{\hat{1}}{\hat{J}_1 \hat{J}_2} \int p^2 dp\, p$$

$$\times \left\{ \sqrt{J_1(J_2+1)} \begin{bmatrix} 1 & J_1+1 & J_1 \\ 1 & J_2-1 & J_2 \\ 0 & 1 & 1 \end{bmatrix} \alpha_{J_1+1,J_2-1}\, f_{\nu_1 J_1+1}\, f_{\nu_2 J_2-1} \right.$$

$$+ \sqrt{J_1 J_2} \begin{bmatrix} 1 & J_1+1 & J_1 \\ 1 & J_2+1 & J_2 \\ 0 & 1 & 1 \end{bmatrix} \alpha_{J_1+1,J_2+1}\, f_{\nu_1 J_1+1}\, f_{\nu_2 J_2+1}$$

$$+ \sqrt{(J_1+1)(J_2+1)} \begin{bmatrix} 1 & J_1-1 & J_1 \\ 1 & J_2-1 & J_2 \\ 0 & 1 & 1 \end{bmatrix} \alpha_{J_1-1,J_2-1}\, f_{\nu_1 J_1-1}\, f_{\nu_2 J_2-1}$$

$$\left. + \sqrt{(J_1+1)J_2} \begin{bmatrix} 1 & J_1-1 & J_1 \\ 1 & J_2+1 & J_2 \\ 0 & 1 & 1 \end{bmatrix} \alpha_{J_1-1,J_2+1}\, f_{\nu_1 J_1-1}\, f_{\nu_2 J_2-1} \right\} , \qquad (4.18)$$

and for the longitudinal part

$$[\nu_1 J_1 \mathscr{L} | \vec{p}^{\,[1]} | \nu_2 J_2 \mathscr{L}] = (-i)^{J_1+1-J_2} \frac{\hat{1}}{\hat{J}_1 \hat{J}_2} \int p^2 dp\, \frac{mP}{E}$$

$$\times \left\{ \sqrt{(J_1+1)\, J_2} \begin{bmatrix} 1 & J_1+1 & J_1 \\ 1 & J_2-1 & J_2 \\ 0 & 1 & 1 \end{bmatrix} \alpha_{J_1+1,J_2-1}\, f_{\nu_1 J_1+1}\, f_{\nu_2 J_2-1} \right.$$

$$- \sqrt{(J_1+1)(J_2+1)} \begin{bmatrix} 1 & J_1+1 & J_1 \\ 1 & J_2+1 & J_2 \\ 0 & 1 & 1 \end{bmatrix} \alpha_{J_1+1,J_2+1}\, f_{\nu_1 J_1+1}\, f_{\nu_2 J_2+1}$$

$$- \sqrt{J_1 J_2} \begin{bmatrix} 1 & J_1-1 & J_1 \\ 1 & J_2-1 & J_2 \\ 0 & 1 & 1 \end{bmatrix} \alpha_{J_1-1,J_2-1}\, f_{\nu_1 J_1-1}\, f_{\nu_2 J_2-1}$$

$$\left. + \sqrt{J_1(J_2+1)} \begin{bmatrix} 1 & J_1-1 & J_1 \\ 1 & J_2+1 & J_2 \\ 0 & 1 & 1 \end{bmatrix} \alpha_{J_1-1,J_2+1}\, f_{\nu_1 J_1-1}\, f_{\nu_2 J_2+1} \right\} . \qquad (4.19)$$

Finally the two-body invariant matrix elements of the bilinear terms $\vec{p}_i . \vec{p}_j$ are obtained from standard recoupling procedures. For example in the case of spin 0 Bosons

$$[\nu_1\ell_1\nu_2\ell_2\, I\, T|\vec{P}_1\vec{P}_2|\nu_3\ell_3\nu_4\ell_4\, I\, T] = \hat{I}^2\hat{T}^2\,(-)^{\ell_1+\ell_2+I+T}$$

$$\langle|[\tilde{W}^{[IT]}[A^{[\ell_2 1]}_{\nu_2}\, A^{[\ell_1 1]}_{\nu_1}]^{[IT]}]^{[00]}\,|\vec{P}_i\cdot\vec{P}_j|\,[W^{[IT]}[\tilde{A}^{[\ell_3 1]}_{\nu_3}\,\tilde{A}^{[\ell_4 1]}_{\nu_4}]^{[IT]}]^{[00]}|\rangle$$

$$= \frac{1}{\hat{I}}\,\frac{1}{\hat{1}}\begin{bmatrix}\ell_1 & \ell_2 & I\\ \ell_3 & \ell_4 & I\\ 1 & 1 & 0\end{bmatrix}[\nu_1\ell_1|P^{[1]}|\nu_3\ell_3][\nu_2\ell_2|P^{[1]}|\nu_4\ell_4] \quad . \qquad (4.20)$$

IV.3 - THE CENTER OF MASS COORDINATE

For an N-body system of relativistic Bosons and Fermions, the center of mass coordinate term $\vec{R}^2$ in the expression (4.1) is a sum of many-body operators. In terms of the fields it is of the form (see Appendix, Eq.(A.1))

$$\vec{R}^2 = \int d^3x_1 \int d^3x_2 \ldots \int d^3x_N\, \varphi^+(x_1)\, i\frac{\overleftrightarrow{\partial}}{\partial t}\,\varphi(x_1)\,\varphi^+(x_2)\, i\frac{\overleftrightarrow{\partial}}{\partial t}\,\varphi(x_2)\ldots$$

$$\ldots\psi^+(x_N)\,\psi(x_N)\,\frac{1}{(\sum_i E_i)^2}\left\{\sum_i E_i\,\vec{x}_i^{\,2} + \sum E_iE_j\,\vec{x}_i\vec{x}_j\right\} . \qquad (4.21)$$

where the E_i's are the free field energies, $E_i = (p_i^2 + m_i^2)^{1/2}$.
The fields have been commuted (anticommuted) so as to be in corresponding pairs with same coordinate $\vec{x}_i$. No sign is introduced by this operation since anticommutation always arises between pairs of Fermion operators. The $i(\overleftrightarrow{\partial}/\partial t)$ factors operate on the immediate neighbouring Boson fields.

The non-separability character of $\vec{R}^2$ originates in the denominator $\left(\sum_{i=1}^{N} E_i\right)^2$. In order to carry out the calculation, we go through an intermediate step by employing the transformation

$$\frac{1}{(\sum_i E_i)^2} = \int_0^\infty dz_1 \int_0^\infty dz_2\; e^{-(z_1+z_2)\sum_i E_i} .$$

This way $\vec{R}^2$ is obtained from the double integration of a sum of separable operators which are functions of the variable $z = z_1 + z_2$,

$$\vec{R}^2 = \int_0^\infty dz_1 \int_0^\infty dz_2 \int d^3x_1 \int d^3x_2 \ldots \int d^3x_N$$

$$(\varphi^+(x_1)\, i \frac{\overleftrightarrow{\partial}}{\partial t}\, \varphi(x_1) e^{-zE_1})(\varphi^+(x_2)\, i \frac{\overleftrightarrow{\partial}}{\partial t}\, \varphi(x_2) e^{-zE_2}) \ldots (\psi^+(x_N)\, \psi(x_N) e^{-zE_N})$$

$$\left(\sum_{i=1}^{N} E_i^{\,2}\, \vec{x}_i^{\,2} + \sum E_i\, \vec{x}_i \,.\, E_j\, \vec{x}_j \right) \quad . \tag{4.22}$$

Thus we get, for each term of this sum, products of single particle operators which are solely functions of the energy (namely operators $\exp(-zE_i)$) and of single and two-body operators which are functions of both the energy and position coordinates (namely the operators $\exp(-zE_i)E_i^{\,2}\,\vec{x}_i^{\,2}$ and $\exp(-zE_i)E_i\vec{x}_i \,.\, \exp(-zE_j)E_j\vec{x}_j$) .

i - Energy dependent terms

For the first kind of factors which depend only upon the energy we can use the number operator N(p) of section IV.2,

$$\int d^3x\, (\varphi^+(x)\, i \frac{\overleftrightarrow{\partial}}{\partial t}\, \varphi(x)) e^{-zE} = \int p^2 dp\, N(p) e^{-zE} \quad . \tag{4.23}$$

We shall need in fact the more general expression, where $n = 0, 1, 2$

$$\int d^3x\, (\varphi^+(x)\, i \frac{\overleftrightarrow{\partial}}{\partial t}\, \varphi(x)) e^{-zE} E^n = \int p^2 dp\, N(p) e^{-zE} E^n \quad , \tag{4.24}$$

$$\int d^3x\, (\psi^+(x)\psi(x)) e^{-zE} E^n = \int p^2 dp\, N(p) e^{-zE} E^n \quad . \tag{4.25}$$

The corresponding invariant matrix elements for spin $s = 0, 1$ Bosons and spin $s = 1/2$ Fermions are given by the expressions (4.5), (4.7) and (4.9) where the factor $\vec{p}^{\,2}$ in the integrand of Eq.(4.5) is replaced by $e^{-zE}E^n$, i.e.,

$$[\nu\ell|e^{-zE}\, E^n|\nu'\ell] = \hat{\ell} \int p^2 dp\, f_{\nu\ell}(p)\, e^{-zE}\, E^n\, f_{\nu'\ell}(p) \quad . \tag{4.26}$$

ii - Mixed energy-coordinate terms

In order to evaluate the mixed energy-coordinate terms we shall first consider the simpler case of the spin 0 Boson fields. The extension of the calculational method to the other fields will be readily made thereafter.

Let $\mathscr{E}\vec{x}$ be a mixed energy coordinate operators, where $\mathscr{E}$ is a function of the energy. This mixed operator must be symmetrized since $\mathscr{E}$ and $\vec{x}$ do not necessarily commute. Thus we have to evaluate in terms of the field operators for Bosons

$$\mathscr{E}\vec{x} = i\int d^3x \frac{1}{2}\left\{\varphi^+(x)(\mathscr{E}\vec{x}+\vec{x}\mathscr{E})\pi(x) - \pi^+(x)(\mathscr{E}\vec{x}+\vec{x}\mathscr{E})\varphi(x)\right\} \quad . \qquad (4.27)$$

Each of the terms of the right hand expression is then separated into two parts by inserting the unit operator. For example

$$\int d^3x\ \varphi^+(x)\mathscr{E}\vec{x}\,\pi(x) = \int d^3x \int d^3y\ \varphi^+(x)\mathscr{E}\ \delta^3(x-y)\vec{y}\,\pi(y) \quad . \qquad (4.28)$$

For spin 0 Boson the unit operator is

$$\delta^3(x-y) = \sum_{\nu\ell} \int p_1{}^2dp_1 \int p_2{}^2dp_2 \sqrt{E_2/E_1}\ f_{\nu\ell}(p_1)\ f_{\nu\ell}(p_2)$$
$$\times\ j_\ell(p_1x)\ j_\ell(p_2y)\ \hat{\ell}\ [\hat{x}^{[\ell]}\ \hat{y}^{[\ell]}]^{[0]} \quad . \qquad (4.29)$$

It differs from the non-relativistic expression by the square root of the energies associated with the orthogonality relation (3.30) or equivalently the commutators (3.28). Of course instead of the factor $\sqrt{E_1/E_2}$ one can employ $\sqrt{E_2/E_1}$, and we use this fact in evaluating (4.30). Thus for example

$$\frac{i}{2}\int d^3x(\varphi^+(x)\mathscr{E}\vec{x}\,\pi(x) - \pi^+(x)\mathscr{E}\vec{x}\,\varphi(x) =$$

$$\frac{1}{2}\sum_{\substack{\nu_1\nu_2\\ \ell_1\ell_2}} [\nu_1\ell_1|\mathscr{E}\vec{x}^{[1]}|\nu_2\ell_2]\ [\tilde{\eta}^{[1]}|\eta^{[1]}]\ [\tilde{A}_{\nu_1}^{[\ell_1 1]}\ A_{\nu_2}^{[\ell_2 1]}\ e^{[1]}]^{[00]} \quad . \qquad (4.30)$$

In the field momentum representation the mixed energy-coordinate invariant matrix element is

$$[\nu_1\ell_1|\mathscr{E}\vec{x}^{[1]}|\nu_2\ell_2] = \int d^3x \left(\frac{2}{\pi}\right) \int p_1{}^2dp_1 \int p_2{}^2dp_2\ \frac{1}{2}\ (\sqrt{E_2/E_1} + \sqrt{E_1/E_2})$$
$$\times\ i^{\ell_2-\ell_1}\ f_{\nu_1\ell_1}(p_1)\ j_{\ell_1}(p_1x)\,[\hat{x}^{[\ell_1]}\mathscr{E}\vec{x}^{[1]}\hat{x}^{[\ell_2]}]^{[0]}\ f_{\nu_2\ell_2}(p_2)j_{\ell_2}(p_2x) \quad . \qquad (4.31)$$

After insertion of the unit operator (4.29) we obtain the separated form

$$[\nu_1\ell_1|\mathscr{E}\vec{x}^{[1]}|\nu_2\ell_2] = \sum_\nu \int d^3x \int d^3y \int p_1^2 dp_1 \int p_1'^2 dp_1 \int p_2^2 dp_2 \int p_2'^2 dp_2$$

$$\frac{1}{2}((\sqrt{E_1E_2'/E_1'E_2}) + (\sqrt{E_1'E_2/E_1E_2'}))\ i^{\ell_2-\ell_1}\ f_{\nu_1\ell_1}(p_1)j_{\ell_1}(p_1x)\mathscr{E} f_{\nu\ell_1}(p_1')j_{\ell_1}(p_1'x)$$

$$[\hat{x}^{[\ell_1]}\hat{x}^{[\ell_1]}]^{[0]} \times f_{\nu\ell_1}(p_2')j_{\ell_1}(p_2'y)[\hat{y}^{[\ell_1]}\vec{y}^{[1]}\hat{y}^{[\ell_2]}]^{[0]}\ f_{\nu_2\ell_2}(p_2)j_{\ell_2}(p_2y)$$

$$= \sum_\nu [\nu_1\ell_1|\mathscr{E}|\nu\ell_1][\nu\ell_1|\vec{y}^{[1]}|\nu_2\ell_2] \quad . \tag{4.32}$$

In this expression the matrix elements of the energy functions $\mathscr{E}$ have been given in Eq.(4.26). We are left with the calculation of the matrix elements of the coordinate operator .

In terms of field operators the coordinate $\vec{x}$ is,

$$\vec{x} = \int d^3x\ \varphi^+(x)\ i\ \overleftrightarrow{\frac{\partial}{\partial t}}\ \varphi(x)\vec{x} = i\int d^3x(\varphi^+(x)\vec{x}\,\pi(x) - \pi^+(x)\vec{x}\ \varphi(x)) \quad , \tag{4.33}$$

for Bosons and

$$\vec{x} = \int d^3x\ \psi^+(x)\ \psi(x)\ \vec{x} \quad , \tag{4.34}$$

for Fermions. Thus in the case of spin 0 Boson fields

$$\vec{x} = \sum_{\substack{\nu_1\nu_2\\ \ell_1\ell_2}} \frac{2}{\pi}\int p_1^2 dp_1 \int p_2^2 dp_2\ \frac{1}{2}\left(\sqrt{E_2/E_1} + \sqrt{E_1/E_2}\right)\ i^{\ell_2-\ell_1}\ f_{\nu_1\ell_1}(p_1)f_{\nu_2\ell_2}(p_2)$$

$$\times [j_{\ell_1}(p_1x)\hat{x}^{[\ell_1]}|\vec{x}^{[1]}|j_{\ell_2}(p_2x)\hat{x}^{[\ell_2]}][\tilde{\eta}^{[1]}|\eta^{[1]}]\times [\tilde{A}_{\nu_1}^{[\ell_1 1]} A_{\nu_2}^{[\ell_2 1]} e^{[1]}]^{[00]}$$

$$= \sum_{\nu_1\nu_2\ell_1\ell_2} [\nu_1\ell_1|\vec{x}^{[1]}|\nu_2\ell_2][\tilde{\eta}^{[1]}\eta^{[1]}][\tilde{A}_{\nu_1}^{[\ell_1 1]} A_{\nu_2}^{[\ell_2 1]} e^{[1]}]^{[00]} \quad , \tag{4.35}$$

where

$$i^{\ell_2-\ell_1}\ [j_{\ell_1}(p_1x)\hat{x}^{[\ell_1]}|\vec{x}^{[1]}|j_{\ell_2}(p_2x)\hat{x}^{[\ell_2]}] =$$

$$= i\ \frac{\pi}{2}\left\{(\alpha_{\ell_1\ell_2})^3\ \frac{\delta(p_1-p_2)}{p_1^3} - \alpha_{\ell_1\ell_2}\ \frac{\delta'(p_1-p_2)}{p_1p_2}\right\} \quad . \tag{4.36}$$

The invariant matrix element of $\vec{x}$ is then

$$[\nu_1\ell_1|\vec{x}^{[1]}|\nu_2\ell_2] = i\int p^2dp\Big[\frac{(\alpha_{\ell_1\ell_2})^3}{p} f_{\nu_1\ell_1}(p)\, f_{\nu_2\ell_2}(p) + \frac{\alpha_{\ell_1\ell_2}}{2}\,(f'_{\nu_1\ell_1}(p)f_{\nu_2\ell_2}(p) - f_{\nu_1\ell_1}(p)f'_{\nu_2\ell_2}(p))\Big] \quad . \tag{4.37}$$

The geometrical recouplings are given in figure 4.4.

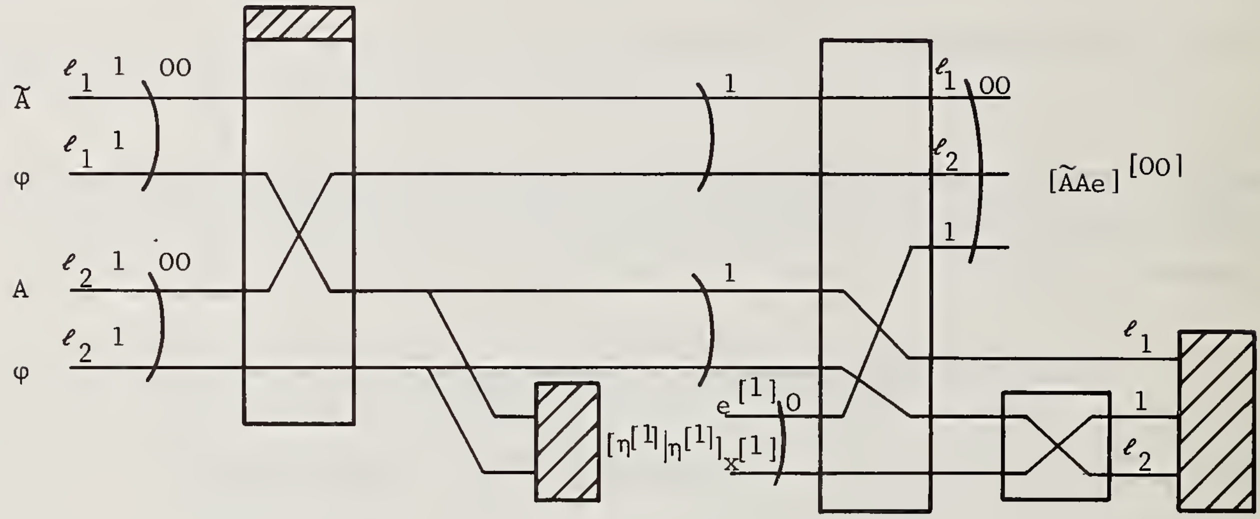


Figure 4.4

Here again as in figure 4.2 the isospin part separates simply and we are using a schematic graphical representation for the isospin coupling.

For the spin 1/2 Fermion field we get likewise

$$\vec{x} = \sum_{\substack{\nu_1\ell_1 j_1\\ \nu_2\ell_2 j_2}} [\tilde{\eta}^{[1/2]}|\eta^{[1/2]}] [\nu_1\ell_1 j_1 | \vec{x}^{[1]} | \nu_2\ell_2 j_2] [\tilde{B}^{[j_1 1/2]}_{\nu_1\ell_1} e^{[1]} B^{[j_2 1/2]}_{\nu_2\ell_2}]^{[00]} ,$$

with

$$[\nu_1\ell_1 j_1 | \vec{x}^{[1]} | \nu_2\ell_2 j_2] = i^{\ell_2-\ell_1} (-1)^{j_2-\frac{1}{2}} \hat{j}_1 \hat{j}_2$$

$$\times \Big\{ \int p^2 dp\, f_{\nu_1\ell_1}(p) f_{\nu_2\ell_2}(p) \frac{1}{p} \Big((-1)^{\ell_2} \begin{Bmatrix} 1/2 & \ell_1 & j_1 \\ 1 & j_2 & \ell_2 \end{Bmatrix} \frac{(E+m)}{2E} |\alpha_{\ell_1\ell_2}|^3$$

$$+ (-1)^{\lambda_2} \begin{Bmatrix} 1/2 & \lambda_1 & j_1 \\ 1 & j_2 & \lambda_2 \end{Bmatrix} \frac{(E-m)}{2E} |\alpha_{\lambda_1\lambda_2}|^3 \Big)$$

$$+ \int p^2 dp \frac{1}{2}\Big(f'_{\nu_1\ell_1}(p)\, f_{\nu_2\ell_2}(p) - f_{\nu_1\ell_1}(p)\, f'_{\nu_2\ell_2}(p)\Big) \Big((-)^{\ell_2} \begin{Bmatrix} 1/2 & \ell_1 & j_1 \\ 1 & j_2 & \ell_2 \end{Bmatrix} \frac{(E+m)}{2E} |\alpha_{\ell_1\ell_2}|$$

$$+ (-1)^{\lambda_2} \begin{Bmatrix} 1/2 & \lambda_1 & j_1 \\ 1 & j_2 & \lambda_2 \end{Bmatrix} \frac{(E-m)}{2E} |\alpha_{\lambda_1\lambda_2}| \Big) \Big\} . \tag{4.38}$$

For the spin 1 Boson fields with isospin t the position operator for the different multipolarities κ is

$$\vec{x}_{(\kappa)} = \sum_{\substack{\nu_1 J_1\\ \nu_2 J_2}} [\tilde{\eta}^{[t]}|\eta^{[t]}] [\nu_1 J_1 \kappa | \vec{x}^{[1]} | \nu_2 J_2 \kappa] [\tilde{A}^{[J_1 t]}_{\kappa\nu_1} e^{[1]} A^{[J_2 t]}_{\kappa\nu_2}]^{[00]} .$$

Thus the invariant matrix elements are, for the magnetic multipoles

$$[\nu_1 J_1 \mathcal{M} | \vec{x}^{[1]} | \nu_2 J_2 \mathcal{M}] = i\, \hat{1} \begin{bmatrix} 1 & J_1 & J_1 \\ 1 & J_2 & J_2 \\ 0 & 1 & 1 \end{bmatrix} \int p^2 dp\, \mathcal{F}^{\nu_1\nu_2}_{J_1,J_2}(p) , \tag{4.39}$$

with

$$\mathcal{F}^{\nu_1\nu_2}_{J_1,J_2}(p) = \frac{|\alpha_{J_1J_2}|^3}{p} f_{\nu_1J_1}(p) f_{\nu_2J_2}(p) + \frac{|\alpha_{J_1J_2}|}{2} (f'_{\nu_1J_1}(p) f_{\nu_2J_2}(p)$$

$$- f_{\nu_1J_1}(p) f'_{\nu_2J_2}(p)) , \tag{4.40}$$

for the electric multipoles

$$[\nu_1 J_1' \mathscr{E} | \vec{x}^{[1]} | \nu_2 J_2 \mathscr{E}] = (-i)^{J_1 - J_2} \frac{\hat{1}}{\hat{J}_1 \hat{J}_2} \int p^2 dp$$

$$\times \Bigg[\sqrt{J_1(J_2+1)} \begin{bmatrix} 1 & J_1+1 & J_1 \\ 1 & J_2-1 & J_2 \\ 0 & 1 & 1 \end{bmatrix} \mathscr{F}^{\nu_1 \nu_2}_{J_1+1, J_2-1}(p) + \sqrt{J_1 J_2} \begin{bmatrix} 0 & J_1+1 & 1 \\ 1 & J_2+1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \mathscr{F}^{\nu_1 \nu_2}_{J_1+1, J_2+1}(p)$$

$$+ \sqrt{(J_1+1)(J_2+1)} \begin{bmatrix} 1 & J_1-1 & J_1 \\ 1 & J_2-1 & J_2 \\ 0 & 1 & 1 \end{bmatrix} \mathscr{F}^{\nu_1 \nu_2}_{J_1-1, J_2-1}(p)$$

$$+ \sqrt{(J_1+1)(J_2)} \begin{bmatrix} 1 & J_1-1 & J_1 \\ 1 & J_2+1 & J_2 \\ 0 & 1 & 1 \end{bmatrix} \mathscr{F}^{\nu_1 \nu_2}_{J_1-1, J_2+1}(p) \Bigg] , \qquad (4.41)$$

and for the longitudinal multipoles

$$[\nu_1 J_1 \mathscr{L} | \vec{x}^{[1]} | \nu_2 J_2 \mathscr{L}] = (-i)^{J_1 - J_2} \frac{\hat{1}}{\hat{J}_1 \hat{J}_2} \int p^2 dp \ \frac{mP}{E}$$

$$\times \Bigg[\sqrt{(J_1+1)J_2} \begin{bmatrix} 1 & J_1+1 & J_1 \\ 1 & J_2-1 & J_2 \\ 0 & 1 & 1 \end{bmatrix} \mathscr{F}^{\nu_1 \nu_2}_{J_1+1, J_2-1}(p) -$$

$$- \sqrt{(J_1+1)(J_2+1)} \begin{bmatrix} 1 & J_1+1 & J_1 \\ 1 & J_2+1 & J_2 \\ 0 & 1 & 1 \end{bmatrix} \mathscr{F}^{\nu_1 \nu_2}_{J_1+1, J_2+1}(p)$$

$$- \sqrt{J_1 J_2} \begin{bmatrix} 1 & J_1-1 & J_1 \\ 1 & J_2-1 & J_2 \\ 0 & 1 & 1 \end{bmatrix} \mathscr{F}^{\nu_1 \nu_2}_{J_1-1, J_2-1}(p) + \sqrt{J_1(J_2+1)} \begin{bmatrix} 1 & J_1-1 & J_1 \\ 1 & J_2+1 & J_2 \\ 0 & 1 & 1 \end{bmatrix} \mathscr{F}^{\nu_1 \nu_2}_{J_1-1, J_2+1}(p) \Bigg] . \qquad (4.42)$$

We can now write straightforwardly the mixed energy-coordinate factors of the invariant matrix elements of the C.M. squared coordinate operator $\vec{R}^2$, Eq.(4.22). For example, we get for the quadratic operator $E^2 e^{-zE} \vec{x}^2$, spin 0 case

$$[\nu_1 \ell_1 | E^2 e^{-zE} \vec{x}^2 | \nu_2 \ell_2] = \sum_{\nu} [\nu_1 \ell_1 | E^2 e^{-zE} | \nu \ell_1] [\nu \ell_1 | \vec{x}^2 | \nu_2 \ell_1] \delta_{\ell_1 \ell_2} . \qquad (4.43)$$

Here the quadratic operator $\vec{x}^2$ is again expressed in terms of the linear ones by introducing the relativistic unit operator of Eq.(4.29)

$$[\nu_1\ell|\vec{x}^{\,2}\ |\nu_2\ell] = \int d^3y\ [\nu_1\ell|\vec{x}^{\,[1]}\delta^3(x-y)\,\vec{y}^{\,[1]}|\nu_2\ell]$$

$$= \sum_{\mu\lambda} \frac{1}{\hat{\ell}}\ [\nu_1\ell|\vec{x}^{\,[1]}|\mu\lambda]\,[\mu\lambda|\vec{x}^{\,[1]}|\nu_2\ell] \quad . \qquad (4.44)$$

Likewise the invariant matrix elements of the bilinear terms are

$$[\nu_1\ell_1\nu_2\ell_2\, I\, T|E_i e^{-zE_i}\vec{x}_i \cdot E_j e^{-zE_j}\vec{x}_j|\nu_3\ell_3\nu_4\ell_4\, I\, T]$$

$$= \frac{1}{\hat{I}\,\hat{1}} \begin{bmatrix} \ell_1 & \ell_2 & I \\ \ell_3 & \ell_4 & I \\ 1 & 1 & 0 \end{bmatrix} [\nu_1\ell_1|E_i e^{-zE_i}\,\vec{x}^{\,[1]}|\nu_3\ell_3]\,[\nu_2\ell_2\ E_j e^{-zE_j}\,\vec{x}^{\,[1]}|\nu_4\ell_4] \quad . \qquad (4.45)$$

Finally we sketch on figure 4.5 the angular coupling diagram of one of the term of the invariant many-body matrix element of $\vec{R}^2$. We have chosen the case of identical particles in a same shell j,t. The calculation involves now the well-known technique of angular momentum coupling for many-particle systems of given statistics. On figure 4.5 the $\mathscr{E}$'s denote the energy factors and we have adopted an obvious graphical representation for the single particle CFP's .

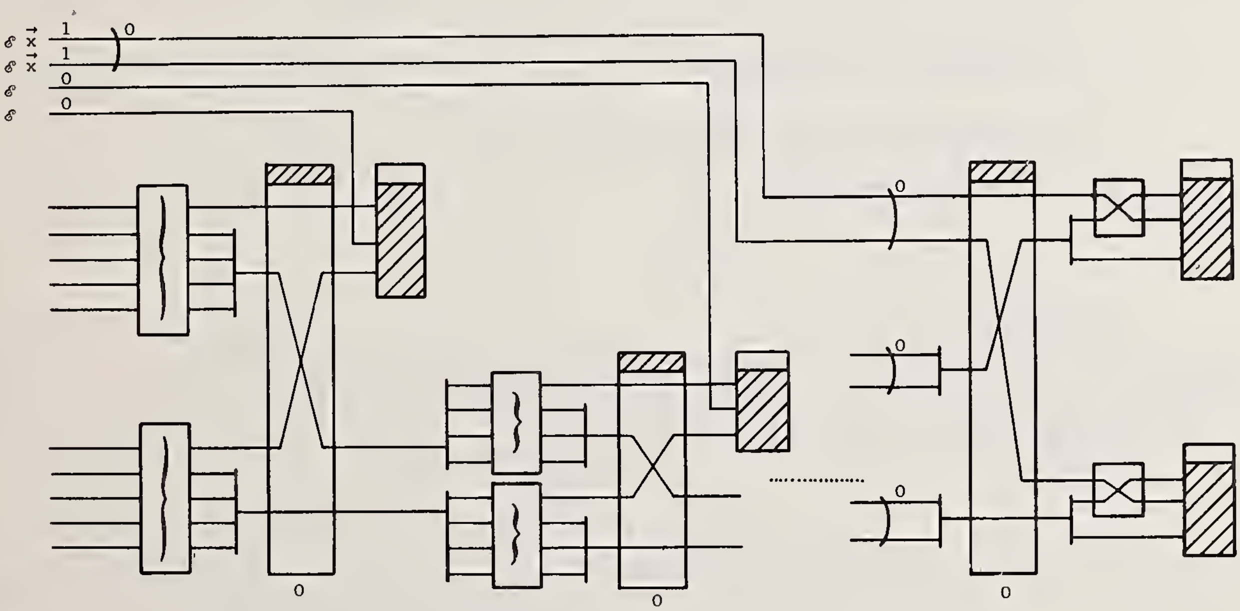


Figure 4.5

CHAPTER V

THE INTERACTION HAMILTONIAN

V.1 - THE PION-NUCLEON INTERACTION

We shall consider successively the pseudoscalar and the pseudo vector couplings,

$$\mathscr{L}_{PS} = i\, G_{PS(\pi NN)} : \sum_{\substack{\kappa\kappa'\mu \\ \alpha\beta}} \bar{\psi}_{\kappa\alpha} (\gamma_5)_{\alpha\beta} (\tau_\mu)_{\kappa\kappa'} \psi_{\kappa'\beta} \Phi_\mu : \quad , \quad . \tag{5.1}$$

and

$$\mathscr{L}_{PV} = -i\, G_{PV(\pi NN)} : \sum_{\substack{\kappa\kappa'\mu \\ \alpha\beta}} \bar{\psi}_{\kappa\alpha} (\gamma_\lambda\gamma_5)_{\alpha\beta} (\tau_\mu)_{\kappa\kappa'} \psi_{\kappa'\beta} \partial_\lambda \Phi_\mu : \quad , \tag{5.2}$$

where α,β are spinor indices, κ,κ' isospin indices and μ denotes the components in 3-dimensional isospin space of the pion field. As before we use the notation $[\quad]^{[00]}$ for double coupling in orbital and isospin spaces.

V.1.1 - Pseudoscalar coupling

The pseudoscalar interaction Hamiltonian is

$$- \int d^3r\, \mathscr{L}_{PS} = + i\, G_{PS(\pi NN)} \frac{1}{\sqrt{8}} \sum_{\substack{\nu\nu'\nu'' \\ \ell\ell'\ell'' \\ j'j''}} i^{\ell+\ell'-\ell''}\, \hat{\ell}\, \hat{j}'\, \hat{j}''\, \hat{1}\, \hat{\tfrac{1}{2}} \int d^3r$$

$$\Bigg[\tilde{B}^{[j''1/2]}_{\nu''\ell''} \begin{pmatrix} \tilde{Y}^{[j'']}_{1/2\ \ell''}(\hat{r})\ u_{\nu''\ell''}(r) \\ \\ \tilde{Y}^{[j'']}_{1/2\ \lambda''}(\hat{r})\ v_{\nu''\lambda''}(r) \end{pmatrix} \tilde{\eta}^{[1/2} \Bigg]^{[00]} \Big\{ \Big[A^{[\ell\ 1]}_{\nu} \hat{r}^{[\ell]}_{\ } \tau^{[1]} \Big]^{[00]}$$

$$+ (-)^{\ell} \Big[\tilde{A}^{[\ell\ 1]}_{\nu} \hat{r}^{[\ell]} \tilde{\tau}^{[1]} \Big]^{[00]} \Big\} \Bigg[B^{[j'1/2]}_{\nu'\ell'} \begin{pmatrix} -Y^{[j']}_{1/2\ \lambda'}(\hat{r})\ v_{\nu'\lambda'}(r) \\ \\ Y^{[j']}_{1/2\ \ell'}(\hat{r})\ u_{\nu'\ell'}(r) \end{pmatrix} \eta^{[1/2]} \Bigg]^{[00]} g_{\nu\ell}(r) \quad . \tag{5.3}$$

We have performed here the scalar product $\vec{\tau}.\vec{\Phi}$, hence the replacement of the isospin vectors $\eta^{[1]}$ by $\tau^{[1]}$ in the boson field. The τ matrices being hermitian we have

$$\tilde{\tau}^{[1]} \equiv \tau^{[1]} \quad . \tag{5.4}$$

In order to evaluate the spin and isospin summation and the 3-dimensional space integral we have the following recoupling diagram of figure 5.1

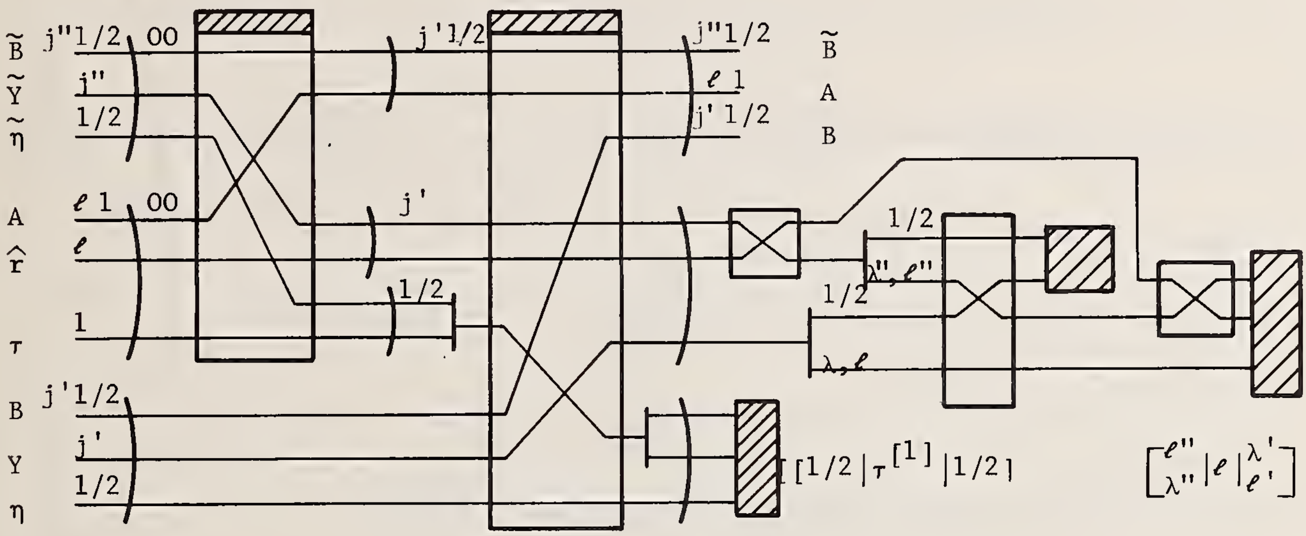


Figure 5.1

We have introduced again double recoupling coefficients which are the product of recoupling coefficients in j and t spaces, as defined in Chapter 2, figure 2.4.

We finally obtain for the pseudoscalar interaction Hamiltonian

$$\mathcal{H}_{PS(\pi NN)} = - i \frac{\sqrt{3}}{2} G_{PS(\pi NN)} \sum_{\substack{\nu\nu'\nu'' \\ \ell\ell'\ell'' \\ j'j''}} i^{-\ell''+\ell'+\ell} \hat{j}' \hat{j}'' (-)^{j'+\lambda'+1/2}$$

$$\times \left\{ \begin{Bmatrix} \ell'' & j'' & 1/2 \\ j' & \lambda' & \ell \end{Bmatrix} [\ell''|\lambda'|\ell] \int r^2 dr\, g_{\nu\ell}(r) u_{\nu''\ell''}(r) v_{\nu'\ell'\lambda'}(r) + \begin{Bmatrix} \lambda'' & j'' & 1/2 \\ j' & \ell' & \ell \end{Bmatrix} [\lambda''|\ell'|\ell] \right.$$

$$\left. \times \int r^2 dr\, g_{\nu\ell}(r) u_{\nu'\ell'}(r) v_{\nu''\ell''\lambda''}(r) \right\} \left[\tilde{B}^{[j''1/2]}_{\nu''\ell''} (A^{[\ell\ 1]}_{\nu} + (-)^{\ell} \tilde{A}^{[\ell\ 1]}_{\nu}) B^{[j'1/2]}_{\nu'\ell'} \right]^{[00]} . \tag{5.5}$$

V.1.2 - Pseudo-vector coupling

The pseudo vector coupling interaction yields two terms one space-like, one time-like :

$$\mathscr{H}_{PV(\pi NN)} = - \int d^3r \, \mathscr{L}_{PV(\pi NN)} = i \, G_{PV(\pi NN)} \int d^3r$$

$$\times : \left\{ \psi \left[\begin{pmatrix} 0 & -\partial_4 \\ \partial_4 & 0 \end{pmatrix} + \hat{1} \begin{pmatrix} i[\sigma^{[1]}\nabla^{[1]}]^{[0]} & 0 \\ 0 & -i[\sigma^{[1]}\nabla^{[1]}]^{[0]} \end{pmatrix} \right] (\vec{\tau}.\vec{\Phi}) \psi \right\} : . \tag{5.6}$$

We tackle the time-part first, where (Eq. (3.5))

$$\partial_4 \Phi = - i \, \pi^+ \quad . \tag{5.7}$$

The recoupling diagram is essentially similar to the pseudoscalar one and we get

$$\mathscr{H}_{PV(\pi NN)} (\text{time-like}) = \frac{\sqrt{3}}{2} G_{PV(\pi NN)} \sum_{\substack{\nu\nu'\nu'' \\ \ell\ell'\ell'' \\ j'j''}} i^{-\ell''+\ell'+1} \, \hat{j}' \, \hat{j}'' \, (-)^{j'+\lambda'+1/2}$$

$$\times \left\{ \begin{Bmatrix} \ell'' & j'' & 1/2 \\ j' & \lambda' & \ell \end{Bmatrix} [\ell|\ell''|\lambda'] \int r^2 dr \, h_{\nu\ell}(r) u_{\nu''\ell''}(r) v_{\nu'\ell'\lambda'}(r) \right.$$

$$\left. - \begin{Bmatrix} \lambda'' & j'' & 1/2 \\ j' & \ell' & \ell \end{Bmatrix} [\ell|\lambda''|\ell'] \int r^2 dr \, h_{\nu\ell}(r) u_{\nu'\ell'}(r) v_{\nu''\ell''\lambda''}(r) \right\}$$

$$\times \left[\tilde{B}^{[j''1/2]}_{\nu''\ell''} (A^{[\ell \; 1]}_{\nu} - (-)^{\ell} \, \tilde{A}^{[\ell \; 1]}_{\nu}) \, B^{[j'1/2]}_{\nu'\ell'} \right]^{[00]} \quad . \tag{5.8}$$

The (-) sign in front of $\tilde{A}$ comes from the time derivative.

The space-like part has a geometry given by the recoupling diagram of figure 5.2. (Here we are using again a simplified notation for the coupling of B's both in angular and isospin space, see the remark at the bottom of Fig. 3.6.)

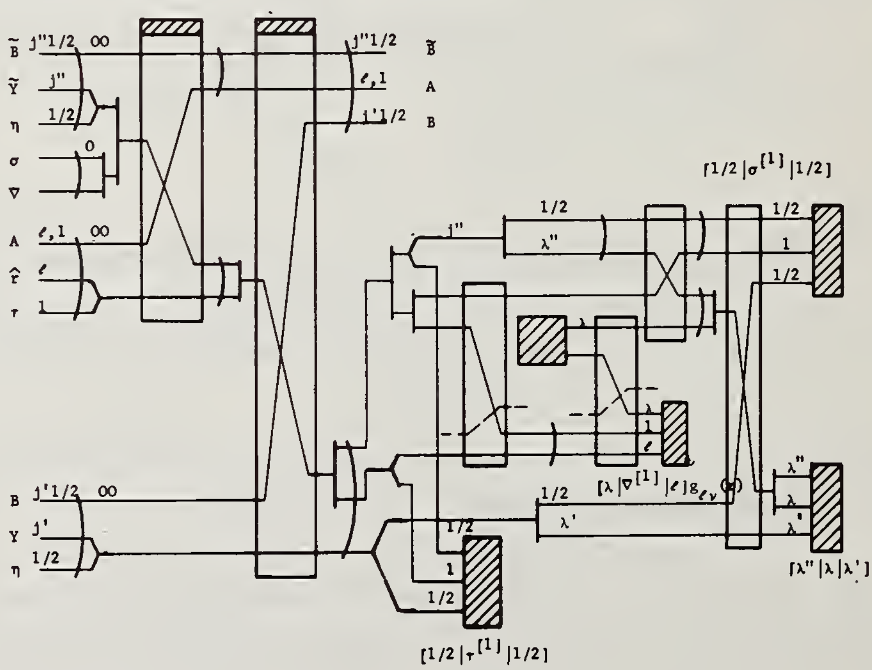


Figure 5.2

The space-like part is thus

$$\mathcal{H}_{PV(\pi NN)}(\text{space-like}) = \frac{3}{\sqrt{2}} G_{PV(\pi NN)} \sum_{\substack{\nu\nu'\nu'' \\ \ell\ell'\ell'' \\ j'j''}} \sum_{\lambda\lambda'\lambda''} i^{-\ell''+\ell'+\ell} \hat{j}' \hat{j}'' \begin{Bmatrix} 1/2 & \lambda'' & j'' \\ 1 & \lambda & \ell \\ 1/2 & \lambda' & j' \end{Bmatrix}$$

$$\times[\lambda''|\lambda|\lambda'] \Big\{ \int r^2 dr \, [\lambda|\nabla^{[1]}|\ell] \, g_{\nu\ell}(r) u_{\nu''\ell''}(r) u_{\nu'\ell'}(r) \, \delta_{\lambda''\ell''} \, \delta_{\lambda'\ell'}$$

$$+ \int r^2 dr \, [\lambda|\nabla^{[1]}|\ell] \, g_{\nu\ell}(r) v_{\nu''\ell''\lambda''}(r) v_{\nu'\ell'\lambda'}(r) \, \delta_{\lambda'',2j''-\ell''} \, \delta_{\lambda',2j'-\ell'} \Big\}$$

$$\times \left[\tilde{B}^{[j''1/2]}_{\nu''\ell''} (A^{[\ell\ 1]}_{\nu} + (-)^{\ell} \tilde{A}^{[\ell\ 1]}_{\nu}) \, B^{[j'1/2]}_{\nu'\ell'} \right]^{[00]} . \tag{5.9}$$

In this expression the gradient matrix element is given by, see Eq.(2.34)

$$[\ell-1|\nabla^{[1]}|\ell] \, g_{\nu\ell}(r) = \sqrt{\ell} \left(\frac{\partial}{\partial r} + \frac{\ell+1}{r}\right) g_{\nu\ell}(r)$$

$$= \sqrt{\frac{2}{\pi}} \int \frac{p^2 dp}{\sqrt{E}} \, p \sqrt{\ell} \, j_{\ell-1}(pr) \, f_{\nu\ell}(p) \quad , \tag{5.10}$$

$$[\ell+1|\nabla^{[1]}|\ell] \, g_{\nu\ell}(r) = \sqrt{\ell+1} \left(\frac{\partial}{\partial r} - \frac{\ell}{r}\right) g_{\nu\ell}(r)$$

$$= - \sqrt{\frac{2}{\pi}} \int \frac{p^2 dp}{\sqrt{E}} \, p \sqrt{\ell+1} \, j_{\ell+1}(pr) \, f_{\nu\ell}(p) \quad . \tag{5.11}$$

V.1.3 - Pion-nucleon invariant matrix elements

We evaluate now the matrix elements of the two processes of figure 5.3

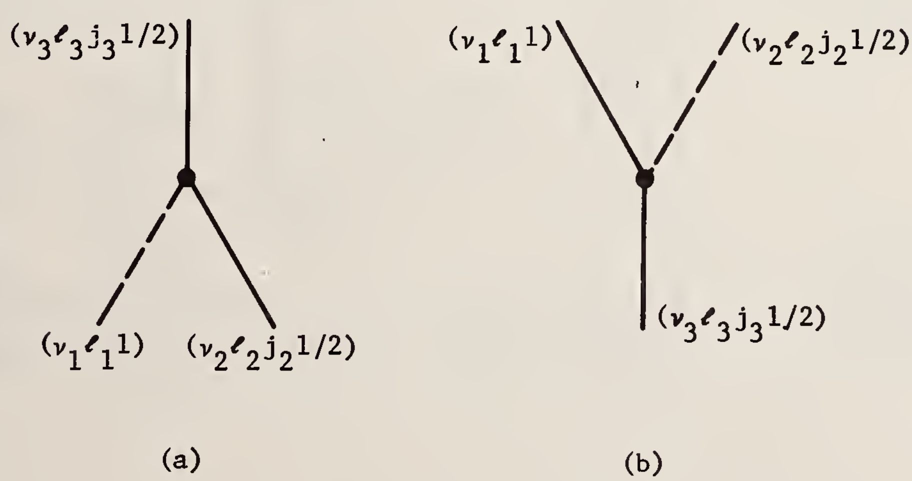


Figure 5.3

The states are here defined with amplitude vectors W,

$$|\nu_3\ell_3 j_3 1/2\rangle_f = \hat{j}_3\, \hat{\tfrac{1}{2}} \left[W_f^{[j_3 1/2]} \tilde{B}^{[j_3 1/2]}_{\nu_3\ell_3} \right]^{[00]} |0\rangle \quad , \tag{5.12}$$

$$|(\nu_1\nu_2\ell_1\ell_2 j_2) j_3 1/2\rangle_i = \hat{j}_3\, \hat{\tfrac{1}{2}} \left[W_i^{[j_3 1/2]} \left[\tilde{A}^{[\ell_1 1]}_{\nu_1} \tilde{B}^{[j_2 1/2]}_{\nu_2\ell_2} \right]^{[j_3 1/2]} \right]^{[00]} |0\rangle , \tag{5.13}$$

and processes(a) and (b) correspond respectively to the contractions (the + (-) sign is associated to the space (time) part),

$${}_f\langle \nu_3\ell_3 j_3 1/2 | [\tilde{B}^{[j''1/2]}_{\nu''\ell''} (A^{[\ell\, 1]}_{\nu} \pm (-)^{\ell} \tilde{A}^{[\ell\, 1]}_{\nu}) B^{[j'1/2]}_{\nu'\ell'}]^{[00]} | (\nu_1\nu_2\ell_1\ell_2 j_2) j_3 1/2\rangle_i$$

$$= [\tilde{W}_f^{[j_3 1/2]} \; W_i^{[j_3 1/2]}]\, \delta_{\nu\nu_1}\delta_{\nu'\nu_2}\delta_{\nu''\nu_3}\delta_{\ell\ell_1}\delta_{\ell'\ell_2}\delta_{\ell''\ell_3}\delta_{j'j_2}\delta_{j''j_3} \quad , \tag{5.14}$$

and

$${}_f\langle (\nu_1\nu_2\ell_1\ell_2 j_2) j_3 1/2 | [\tilde{B}^{[j''1/2]}_{\nu''\ell''} (A^{[\ell\, 1]}_{\nu} \pm (-)^{\ell} A^{[\ell\, 1]}_{\nu}) B^{[j'1/2]}_{\nu'\ell'}]^{[00]} | \nu_3\ell_3 j_3 1/2\rangle_i$$

$$= \pm (-)^{j_2+j_3} [\tilde{W}_f^{[j_3 1/2]} \; W_i^{[j_3 1/2]}]\, \delta_{\nu\nu_1}\delta_{\nu'\nu_3}\delta_{\nu''\nu_2}\delta_{\ell\ell_1}\delta_{\ell'\ell_3}\delta_{\ell''\ell_2}\delta_{j'j_3}\delta_{j''j_2} \tag{5.15}$$

according to the following recoupling diagrams, of figure 5.4 and figure 5.5,where we are using for the contraction symbols of figure 2.12 a double notation as in figure 2.4 or 2.5b to include isospin.

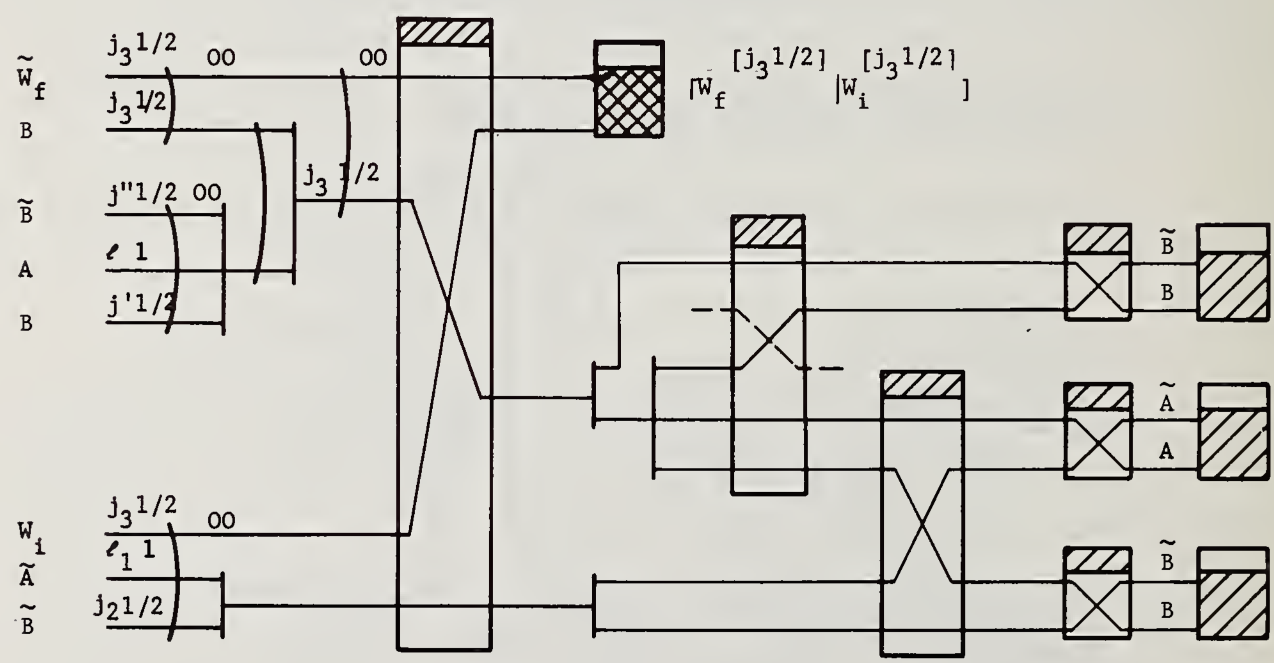


Figure 5.4

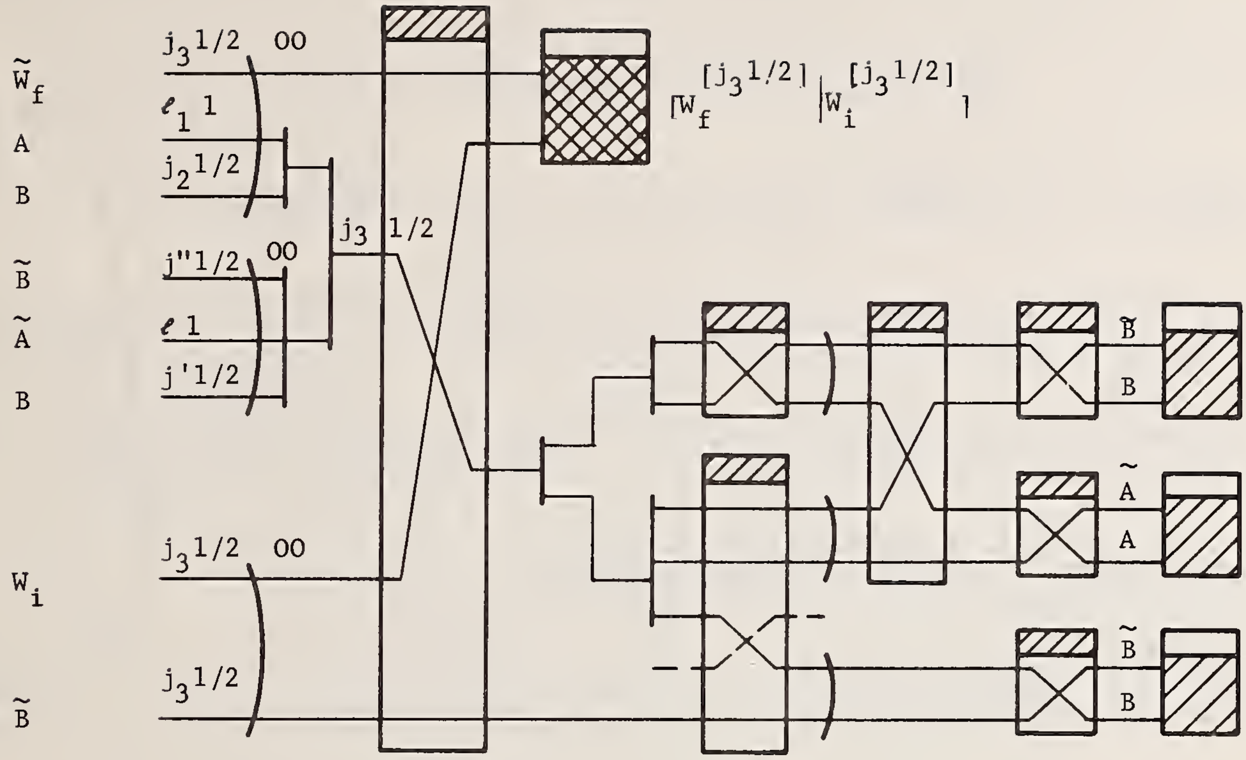


Figure 5.5

We finally get for the pion-nucleon matrix element of process (a) for example the expression (5.16). It is of course hermitic and one may check simply from the average value (5.15) that the matrix element (b) is the complex conjugate of (a).

$$[\nu_3\ell_3 j_3 1/2 |\mathcal{H}_{(\pi NN)}| (\nu_1\nu_2\ell_1\ell_2 j_2) j_3 1/2] = \hat{j}_2\hat{j}_3 \, (i)^{-\ell_3+\ell_2+\ell_1}$$

$$\times(-1)^{j_2+\lambda_2+1/2} \Big\{ \frac{\sqrt{3}}{2} G_{PS(\pi NN)} \Big(\begin{Bmatrix} \ell_3 & j_3 & 1/2 \\ j_2 & \lambda_2 & \ell_1 \end{Bmatrix} [\ell_3|\lambda_2|\ell_1] \int r^2 dr \, g_{\nu_1\ell_1} u_{\nu_3\ell_3} v_{\nu_2\ell_2\lambda_2}$$

$$+ \begin{Bmatrix} \lambda_3 & j_3 & 1/2 \\ j_2 & \ell_2 & \ell_1 \end{Bmatrix} [\lambda_3|\ell_2|\ell_1] \int r^2 dr \, g_{\nu_1\ell_1} u_{\nu_2\ell_2} v_{\nu_3\ell_3\lambda_3} \Big) + \frac{\sqrt{3}}{2} G_{PV(\pi NN)}$$

$$\times \Big(\begin{Bmatrix} \ell_3 & j_3 & 1/2 \\ j_2 & \ell_2 & \ell_1 \end{Bmatrix} [\ell_3|\lambda_2|\ell_1] \int r^2 dr \, h_{\nu_1\ell_1} u_{\nu_3\ell_3} v_{\nu_2\ell_2\lambda_2} - \begin{Bmatrix} \lambda_3 & j_3 & 1/2 \\ j_2 & \ell_2 & \ell_1 \end{Bmatrix} [\lambda_3|\ell_2|\ell_1]$$

$$\times \int r^2 dr \, h_{\nu_1\ell_1} u_{\nu_2\ell_2} v_{\nu_3\ell_3\lambda_3} \Big) + (-1)^{j_2+\lambda_2+1/2} \frac{3}{\sqrt{2}} G_{PV(\pi NN)} \sum_\lambda \Big(\begin{bmatrix} 1/2 & \ell_3 & j_3 \\ 1 & \lambda & \ell_1 \\ 1/2 & \ell_2 & j_2 \end{bmatrix}$$

$$\times [\ell_3|\lambda|\ell_2] \int r^2 dr \, [\lambda|\nabla^{[1]}|\ell_1] \, g_{\nu_1\ell_1} u_{\nu_3\ell_3} u_{\nu_2\ell_2}$$

$$+ \begin{bmatrix} 1/2 & \lambda_3 & j_3 \\ 1 & \lambda & \ell_1 \\ 1/2 & \lambda_2 & j_2 \end{bmatrix} [\lambda_3|\lambda|\lambda_2] \int r^2 dr \, [\lambda|\nabla^{[1]}|\ell_1] \, g_{\nu_1\ell_1} v_{\nu_3\ell_3\lambda_3} v_{\nu_2\ell_2\lambda_2} \Big) \Big\} \quad . \qquad (5.16)$$

V.1.4 - Momentum conservation

In these expansions the integration over r brings in the momentum conservation requirements. Thus we have, see ref. [14]

$$\int r^2 dr \begin{Bmatrix} g_{\nu_1\ell_1}(r) \\ h_{\nu_1\ell_1}(r) \end{Bmatrix} u_{\nu_2\ell_2}(r) v_{\nu_3\ell_3\lambda_3}(r) = \int p_1^2 dp_1 \int p_2^2 dp_2 \int p_3^2 dp_2 \left(\frac{2}{\pi}\right)^{3/2}$$

$$\times \Delta^{\ell_1\ell_2\lambda_3}_{p_1p_2p_3} \begin{Bmatrix} 1/\sqrt{E_1} \\ \sqrt{E_1} \end{Bmatrix} \sqrt{\frac{(E_2+m)(E_3-m)}{E_2E_3}} \, f_{\nu_1\ell_1}(p_1) \, f_{\nu_2\ell_2}(p_2) \, f_{\nu_3\ell_3}(p_3) \, , \qquad (5.17)$$

where

$$\Delta^{\ell_1\ell_2\ell_3}_{p_1p_2p_3} = \int r^2 dr \, j_{\ell_1}(p_1 r) \, j_{\ell_2}(p_2 r) \, j_{\ell_3}(p_3 r)$$

$$= \frac{\pi}{4} \frac{\Delta(p_1p_2p_3)}{p_1p_2p_3} \frac{i^{-\ell_1-\ell_2-\ell_3}}{\begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix}} \sum_m (-)^m \left[\frac{(\ell_1-m)!\,(\ell_2+m)!}{(\ell_1+m)!\,(\ell_2-m)!}\right]^{1/2}$$

$$\times \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m & -m & 0 \end{pmatrix} P^m_{\ell_1}(\cos\theta_{13}) \, P^{-m}_{\ell_2}(\cos\theta_{23}) \quad , \qquad (5.18)$$

with

$$\Delta(p_1\ p_2\ p_3) = \begin{cases} 1 & \text{when } p_1p_2p_3 \text{ form a non degenerate triangle}, \\ 1/2 & \text{when } p_1p_2p_3 \text{ form a degenerate triangle}, \\ 0 & \text{when } p_1p_2p_3 \text{ do not form a triangle}, \end{cases} \tag{5.19}$$

$$\cos\theta_{23} = \frac{1}{2}\,\frac{p_1^2 - p_2^2 - p_3^2}{p_2p_3} \quad , \qquad \cos\theta_{13} = \frac{1}{2}\,\frac{p_2^2 - p_1^2 - p_3^2}{p_1p_3} \quad . \tag{5.20}$$

Likewise the integrals with a gradient are explicitly given by,

$$\int r^2\, dr \begin{Bmatrix} [\ell_1+1|\nabla^{[1]}|\ell_1] \\ [\ell_1-1|\nabla^{[1]}|\ell_1] \end{Bmatrix} g_{\nu_1\ell_1}(r)u_{\nu_2\ell_2}(r)u_{\nu_3\ell_3}(r) = \int p_1^2dp_1 \int p_2^2dp_2 \int p_3^2dp_3 \left(\frac{2}{\pi}\right)^{3/2}$$

$$\times \begin{Bmatrix} -\sqrt{\ell_1+1}\ p_1\ \Delta_{p_1\ p_2\ p_3}^{\ell_1+1\ \ell_2\ \ell_3} \\ \sqrt{\ell_1}\ \ p_1\ \Delta_{p_1\ p_2\ p_3}^{\ell_1-1\ \ell_2\ \ell_3} \end{Bmatrix} \sqrt{\frac{(E_2+m)(E_3+m)}{E_1E_2E_3}}\ f_{\nu_1\ell_1}(p_1)f_{\nu_2\ell_2}(p_2)f_{\nu_3\ell_3}(p_3)\ , \tag{5.21}$$

and

$$\int r^2 dr \begin{Bmatrix} [\ell_1+1|\nabla^{[1]}|\ell_1] \\ [\ell_1-1|\nabla^{[1]}|\ell_1] \end{Bmatrix} g_{\nu_1\ell_1}(r)v_{\nu_2\ell_2\lambda_2}(r)v_{\nu_3\ell_3\lambda_3}(r) = \int p_1^2dp_1 \int p_2^2dp_2 \int p_3^2dp_3 \left(\frac{2}{\pi}\right)^{3/2}$$

$$\times \begin{Bmatrix} -\sqrt{\ell_1+1}\ p_1\ \Delta_{p_1\ p_2\ p_3}^{\ell_1+1\ \lambda_2\ \lambda_3} \\ \sqrt{\ell_1}\ \ p_1\ \Delta_{p_1\ p_2\ p_3}^{\ell_1-1\ \lambda_2\ \lambda_3} \end{Bmatrix} \sqrt{\frac{(E_2-m)(E_3-m)}{E_1E_2E_3}}\ f_{\nu_1\ell_1}(p_1)f_{\nu_2\ell_2}(p_2)f_{\nu_3\ell_3}(p_3)\ . \tag{5.22}$$

V.2 - THE PION-PION INTERACTION

V.2.1 - The π^4 interaction

The Φ^4 operator without overall angular coupling is

$$\mathscr{H}_{(4\pi)} = \frac{G_{(4\pi)}}{4} \int r^2 dr\ g_\alpha(r)g_\beta(r)g_\gamma(r)g_\delta(r)\ i^{\alpha+\beta+\gamma+\delta}\ \hat{\alpha}\ \hat{\beta}\ \hat{\gamma}\ \hat{\delta}\ (\hat{1})^4$$

$$\times \int d\hat{r}\ \{\ \}_\alpha \{\ \}_\beta \{\ \}_\gamma \{\ \}_\delta \quad , \tag{5.23}$$

where

$$g_\alpha = g_{\nu_\alpha \ell_\alpha}(r)\ ; \quad i^\alpha = i^{\ell_\alpha}\ ; \quad \hat{\alpha} = \hat{\ell}_\alpha\ ; \tag{5.24}$$

$$\{\ \}_\alpha = \left[A^{[\ell_\alpha\, 1]}_{\nu_\alpha}\ \hat{r}^{[\ell_\alpha]}\eta^{[1]}\right]^{[00]} + (-)^{\ell_\alpha}\left[\tilde{A}^{[\ell_\alpha\, 1]}_{\nu_\alpha}\ \hat{r}^{[\ell_\alpha]}\tilde{\eta}^{[1]}\right]^{[00]} \ . \qquad (5.25)$$

We rewrite this operator in the various couplings approriate to the possible processes.

i) First the four pion annihilation term is given in figure 5.6 .

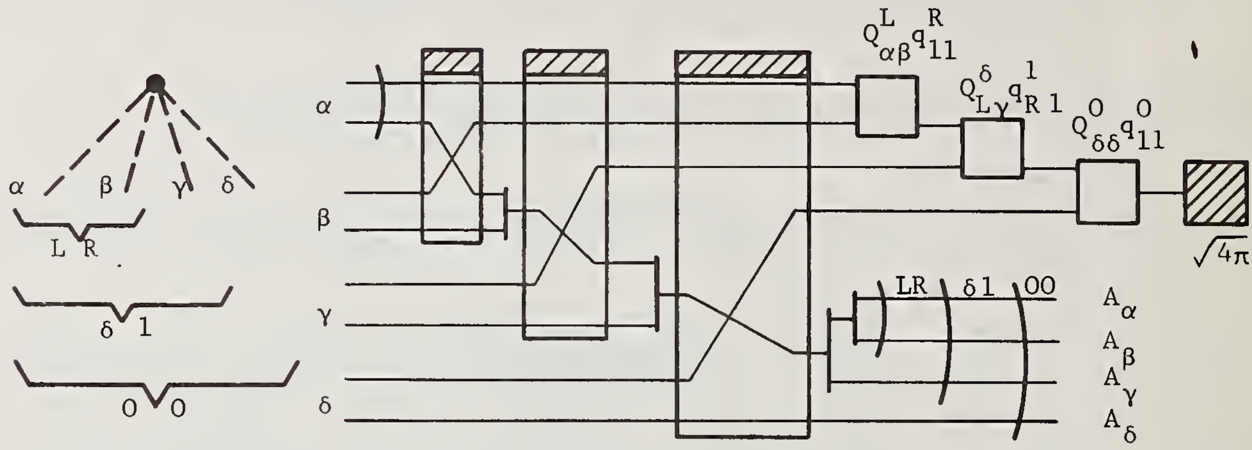


Figure 5.6

$$\mathcal{H}_{(4\pi)} = \frac{G_{(4\pi)}}{4}\sum_{L}\ \sum_{R=0,2} I(\alpha\beta\gamma\delta)\, i^{\alpha+\beta+\gamma+\delta}\, Q^L_{\alpha\beta}\, Q^\delta_{L\gamma}\, Q^0_{\delta\delta}\sqrt{4\pi}$$

$$\times\ q^R_{11}\, q^1_{R1}\, q^0_{11}\left[\left[\left[A^{[\alpha 1]}\, A^{[\beta 1]}\right]^{[LR]} A^{[\gamma 1]}\right]^{[\delta 1]} A^{[\delta 1]}\right]^{[00]} , \qquad (5.26)$$

where $I(\alpha\beta\gamma\delta) = \int r^2 dr\ g_\alpha(r) g_\beta(r) g_\gamma(r) g_\delta(r)$.

In this expression we have $R = 0,2$ because with a point interaction, the relative two pion states are all even (symmetric under interchange of r). We have introduced the notation

$$q^L_{\alpha\beta} = \sqrt{4\pi}\, Q^L_{\alpha\beta} = \hat{\alpha}\,\hat{\beta}\begin{pmatrix}\alpha & \beta & L\\ 0 & 0 & 0\end{pmatrix}(-)^{1/2(\alpha+\beta+L)} \ .$$

where the coefficient $Q^L_{\alpha\beta}$ is defined in Eq. (2.41) and the corresponding square symbol appearing in figure 5.6 is defined in figure 2.8 .

We note in Eq.(5.25) the selection rules resulting from the symmetry in isospin space which corresponds to the symmetry in space imposed by a point interaction.

We now give the various other terms of the pion-pion interaction operator, as shown in figure 5.7.

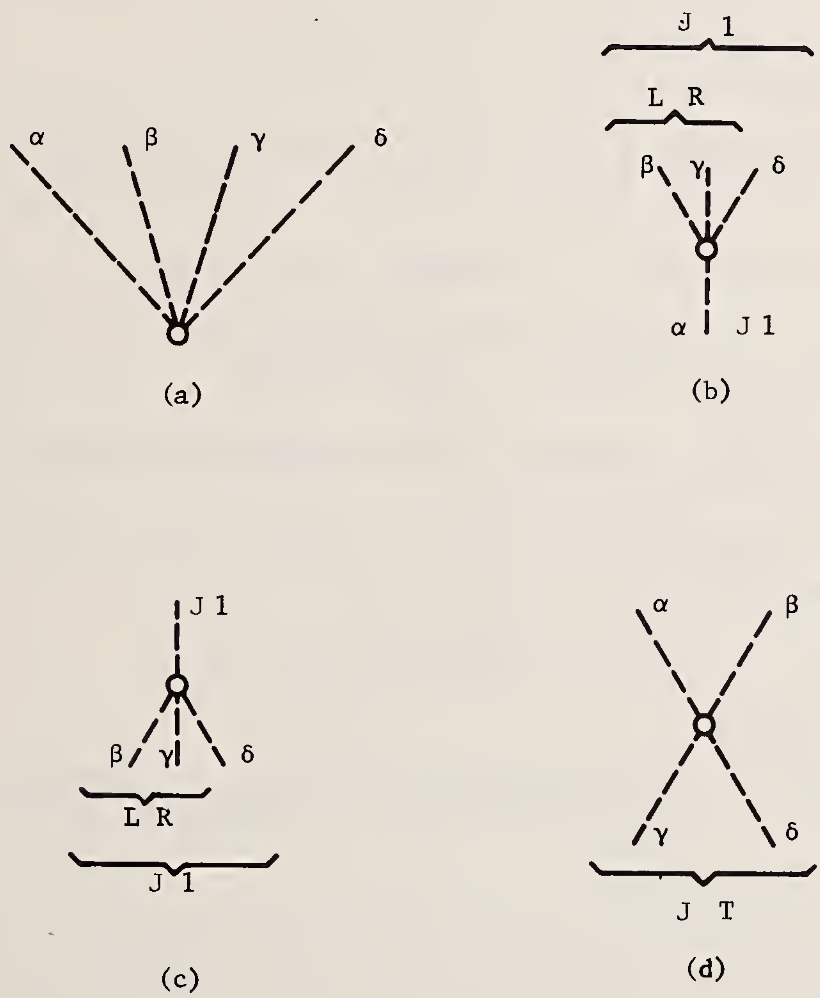


Figure 5.7

ii) The creation of four pions, figure 5.8a

$$\mathcal{H}_{(4\pi)} = \frac{G_{(4\pi)}}{4} \sum_{L} \sum_{R=0,2} I(\alpha\beta\gamma\delta)\, i^{-(\alpha+\beta+\gamma+\delta)}\, Q^{L}_{\alpha\beta}\, Q^{L}_{L\gamma}$$

$$\times\, Q^{0}_{\delta\delta}\, \sqrt{4\pi}\, q^{R}_{11}\, q^{1}_{R1}\, q^{0}_{11} \left[[[\tilde{A}^{[\alpha 1]}\tilde{A}^{[\beta 1]}]^{[LR]}\tilde{A}^{[\gamma 1]}]^{[\delta 1]}\tilde{A}^{[\delta 1]}\right]^{[00]} \quad . \qquad (5.27)$$

iii) Scattering with one creation, figure 5.8b

$$\mathcal{H}_{(4\pi)} = \frac{G_{(4\pi)}}{4} \sum_{L} \sum_{R=0,2} I(J\beta\gamma\delta)\, i^{(I-\beta-\gamma-\delta)}\, Q^{L}_{\beta\gamma}\, Q^{J}_{L\delta}$$

$$\times\, Q^{0}_{JJ}\, \sqrt{4\pi}\, q^{R}_{11}\, q^{1}_{R1}\, q^{0}_{11} \left[[[\tilde{A}^{[\beta 1]}\tilde{A}^{[\gamma 1]}]^{[LR]}\tilde{A}^{[\delta 1]}]^{[J1]}A^{[J1]}\right]^{[00]} \quad . \qquad (5.28)$$

iv) Scattering with one absorption, figure 5.8c

$$\mathcal{H}_{(4\pi)} = \frac{G_{(4\pi)}}{4} \sum_{L} \sum_{R=0,2} I(J\beta\gamma\delta)\, i^{(-J+\beta+\gamma+\delta)}\, Q^{L}_{\beta\gamma}\, Q^{J}_{L\delta}$$

$$\times\, Q^{0}_{JJ}\, \sqrt{4\pi}\, q^{R}_{11}\, q^{1}_{R1}\, q^{0}_{11} \left[\tilde{A}^{[J1]}[[A^{[\beta 1]}A^{[\gamma 1]}]^{[LR]}A^{[\delta 1]}]^{[J1]}\right]^{[00]} \quad . \qquad (5.29)$$

v) Pion-pion scattering, figure 5.8d

$$\mathcal{H}_{(4\pi)} = \frac{G_{(4\pi)}}{4} \sum_{L} \sum_{R=0,2} I(\alpha\beta\gamma\delta)\, i^{(-\alpha-\beta+\gamma+\delta)}\, Q^{J}_{\alpha\beta}\, Q^{J}_{\gamma\delta}$$

$$\times\, Q^{0}_{JJ}\, \sqrt{4\pi}\, q^{T}_{11}\, q^{T}_{11}\, q^{0}_{TT} \left[[\tilde{A}^{[\alpha 1]}\tilde{A}^{[\beta 1]}]^{[JT]}[A^{[\gamma 1]}A^{[\delta 1]}]^{[JT]}\right]^{[00]} \quad . \qquad (5.30)$$

V.2.2 - Momentum conservation

In order to calculate the integral I of Eqs. (5.23)-(5.30),

$$\int r^2 dr\, g_{\nu_1\ell_1}(r) g_{\nu_2\ell_2}(r) g_{\nu_3\ell_3}(r) g_{\nu_4\ell_4}(r) =$$

$$\frac{4}{\pi}2\int p_1^2 dp_1 \int p_2^2 dp_2 \int p_3^2 dp_3 \int p_4^2 dp_4\, \Box^{\ell_1\ell_2\ell_3\ell_4}_{p_1p_2p_3p_4} \frac{1}{(E_1E_2E_3E_4)^{1/2}}$$

$$\times f_{\nu_1\ell_1}(p_1) f_{\nu_2\ell_2}(p_2) f_{\nu_3\ell_3}(p_3) f_{\nu_4\ell_4}(p_4) \ , \tag{5.31}$$

we have to evaluate the discontinuous function which represents the momentum conservation law

$$\Box^{\ell_1\ell_2\ell_3\ell_4}_{p_1p_2p_3p_4} = \int r^2 dr\, j_{\ell_1}(p_1 r) j_{\ell_2}(p_2 r) j_{\ell_3}(p_3 r) j_{\ell_4}(p_4 r) \quad . \tag{5.32}$$

This is done by applying twice the definition (5.18)

$$\int r^2 dr\, j_\lambda(pr) j_{\ell_1}(p_1 r) j_{\ell_2}(p_2 r) = \int r^2 dr \int q^2 dq\, N(p_1 p_2 p) j_\lambda(pr) j_\lambda(qr) \quad . \tag{5.33}$$

Hence

$$\frac{\pi}{2} N(p_1\ p_2\ p) = \Delta^{\lambda\,\ell_1\,\ell_2}_{p\ p_1\ p_2} \ , \tag{5.34}$$

and

$$\Box^{\ell_1\ell_2\ell_3\ell_4}_{p_1p_2p_3p_4} = \int \frac{2}{\pi} p^2 dp\ \Delta^{\lambda\,\ell_1\,\ell_2}_{p\ p_1 p\ 2}\ \Delta^{\lambda\,\ell_3\,\ell_4}_{p\ p_3\ p_4} \quad . \tag{5.35}$$

Here λ must obey the triangular rules (λ,ℓ_1,ℓ_2) and (λ,ℓ_3,ℓ_4) since otherwise (5.18) is indeterminate and useless. Of course, only such cases will in fact arise as all triangularities are guaranteed by the angular integrations.

V.2.3 - Pion-pion invariant matrix elements

We give now a few relevant matrix elements of the Φ^4 interaction, as examples.

i) The 4-pion annihilation operator yields a particular simple result for the case of the s^4 configuration. Here the unique state vector is of the form, see section II.5.2.3

$$|s^4\rangle = \sum_{R=0,2} C_R\Big[[\tilde{A}^{[01]}\tilde{A}^{[01]}]^{[OR]} [\tilde{A}^{[01]}\tilde{A}^{[01]}]^{[OR]}\Big]^{[00]} |0\rangle \quad . \tag{5.36}$$

The coefficients C_R are given by the condition that the state $|s^4\rangle$ be of even T in all recoupled pairs, namely

$$\left(C_0\begin{bmatrix}1&1&0\\1&1&0\\K&K&0\end{bmatrix} + C_2\begin{bmatrix}1&1&2\\1&1&2\\K&K&0\end{bmatrix}\right)\left[[\tilde{A}^{[01]}\tilde{A}^{[01]}]^{[0K]}[\tilde{A}^{[01]}\tilde{A}^{[01]}]^{[0K]}\right]^{[00]}|0\rangle = 0 \;, \qquad (5.37)$$

if $K = 1$. This condition yields together with normalization

$$C_0 = \frac{\sqrt{5}}{3} \;, \qquad C_2 = \frac{2}{3} \;. \qquad (5.38)$$

We have used the value

$$\begin{bmatrix}1&1&2\\1&1&2\\1&1&0\end{bmatrix} = \frac{\hat{2}}{\hat{1}\;2} = \frac{1}{2}\sqrt{\frac{5}{3}} \;. \qquad (5.39)$$

Then

$$[0|\mathcal{H}_{4\pi}|s^4] = \frac{G^{(4\pi)}}{4}\, I(0\,0\,0\,0)\sum_{R=0,2} C_R\, q_{11}^{R}\, q_{R1}^{1}\, q_{11}^{0}\; 8\sum_{K=0,2}\left\{\delta_{KR}+\frac{\hat{R}}{\hat{K}\,\hat{1}}\right\}, \qquad (5.40)$$

from the recoupling diagram, in isospin space, of the average value

$$\sum_{KR} C_R\langle 0|[[AA]^{[K]}[AA]^{[K]}]^{[0]}\;[[\widetilde{AA}]^{[R]}[\widetilde{AA}]^{[R]}]^{[0]}|0\rangle \;, \qquad (5.41)$$

which is given in figure 5.8 .

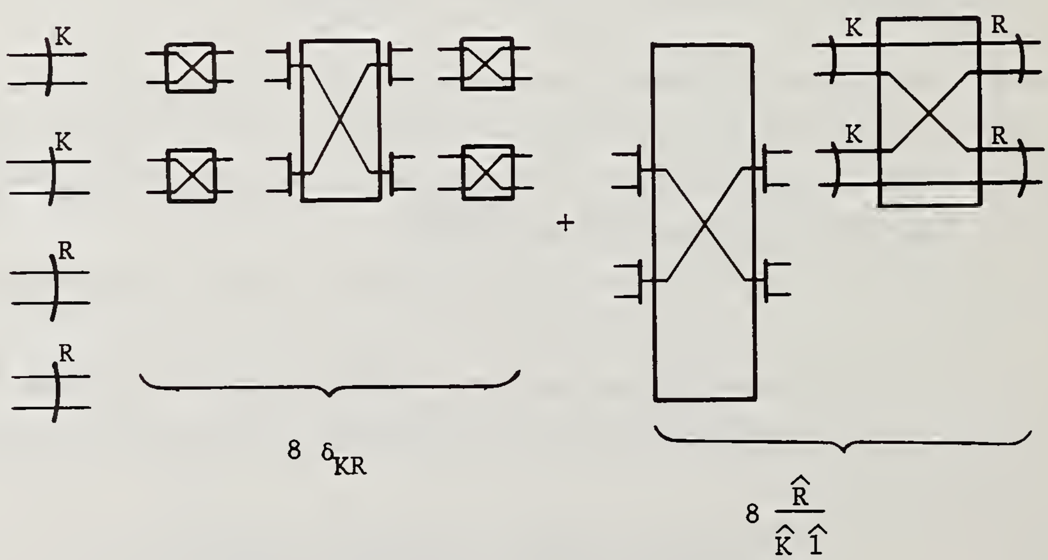


Figure 5.8

ii) In the case of the pion-pion scattering, Eq.(5.30) we get for a 2 pion system (JT)

$$\langle f|\mathcal{H}_{(4\pi)}|i\rangle = \frac{G_{(4\pi)}}{4} \sum_{\alpha\beta\gamma\delta} I(\alpha\beta\gamma\delta)\, i^{(-\alpha-\beta+\gamma+\delta)}\, Q^{J}_{\alpha\beta}\, Q^{J}_{\gamma\delta}\, Q^{0}_{JJ}\, \sqrt{4\pi}\, q^{T}_{11}\, q^{T}_{11}\, q^{0}_{TT}$$

$$\frac{1}{((1+\delta_{k\ell})\quad(1+\delta_{ij}))^{1/2}}\, \hat{J}^2\hat{T}^2 \langle 0|[\tilde{W}_f^{[JT]}[A^{[\ell]}A^{[k]}]^{[JT]}$$

$$\times\Big[[\tilde{A}^{[\alpha 1]}\tilde{A}^{[\beta 1]}]^{[JT]}\,[A^{[\gamma 1]}A^{[\delta 1]}]^{[JT]}\Big]^{[00]}\, W_i^{[JT]}[\tilde{A}^{[i]}\tilde{A}^{[j]}]^{[JT]}|0\rangle$$

$$= \frac{G_{(4\pi)}}{4}\, I(i\,j\,k\,\ell)\, i^{(-i-j+k+\ell)}\, Q^{J}_{ij}\, Q^{J}_{k\ell}\, Q^{0}_{JJ}\, \sqrt{4\pi}\, q^{T}_{11}\, q^{T}_{11}\, q^{0}_{TT}$$

$$\times \frac{1}{((1+\delta_{k\ell})(1+\delta_{ij}))^{1/2}}\; 4\, \hat{J}\, \hat{T}\, [\tilde{W}_f^{[JT]}\, W_i^{[JT]}] \,. \tag{5.42}$$

In this case the invariant matrix element is :

$$[(k\ell)JT|\mathcal{H}_{(4\pi)}|(ij)JT] = G_{(4\pi)}\, I(i\,j\,k\,\ell)\, i^{(-i-j+k+\ell)} \left(\frac{4\pi}{(1+\delta_{ij})(1+\delta_{k\ell})}\right)^{1/2}$$

$$\times\; q^{T}_{11}\, q^{T}_{11}\, q^{0}_{TT}\, Q^{J}_{ij}\, Q^{J}_{k\ell}\, Q^{0}_{JJ}\, \hat{J}\, \hat{T} \,. \tag{5.43}$$

V.3 - THE NUCLEON-SPIN 1 MESON INTERACTION

V.3.1 - The ωNN interaction

Let us first consider the absorption or emission of an ω meson by the nucleon. The adopted interaction is

$$\mathcal{L}_{\omega NN} = G_{\omega NN}\, i\, \bar{\psi}\, \gamma_\mu \psi\, \omega_\mu = G_{\omega NN} \left\{ i[\bar{\psi}\, \gamma^{[1]}\, \psi\, \bar{\omega}^{[1]}]^{[0]} + i\, \bar{\psi}\, \gamma_4\, \psi\, \omega_4 \right\} , \tag{5.44}$$

where the second(time-like)part comes only from the longitudinal field $\omega_{4\mathcal{L}}$. This form is hermitic since : $\omega_4^+ = -\omega_4$; $\vec{\omega}^+ = \vec{\omega}$; $\gamma_x\gamma_4 = -\gamma_4\gamma_x$; $\gamma_4^2 = 1$ and

$$(\psi^+\gamma_4\gamma_\mu\, \psi\, \omega_\mu)^+ = \omega_\mu^+\, \bar{\psi}\, \gamma_4\gamma_\mu\gamma_4\, \psi = -\, \bar{\psi}\, \gamma_\mu\, \psi\, \omega_\mu \,. \tag{5.45}$$

The space-like part of the energy requires the calculation of

$$\mathcal{H}_{\omega NN} = - \int d^3r\, G_{\omega NN}\, i \left[\psi^+ \begin{pmatrix} 0 & -i\sigma^{[1]} \\ -i\sigma^{[1]} & 0 \end{pmatrix} \psi\, \omega^{[1]}\right]^{[0]} , \tag{5.46}$$

or more explicitly for the magnetic multipole contribution of (3.222)

$$\mathcal{H}_{\omega NN} = - G_{\omega NN} \frac{1}{2} \sum_{\substack{\nu\, J \\ \nu_i\, j_i\, \ell_i \\ \nu_f\, j_f\, \ell_f}} i^{(1-\ell_f+\ell_i+J)}\, \hat{J}\, \hat{\tfrac{1}{2}}^2\, \hat{j}_i \hat{j}_f \int d^3\vec{r}\, \mathcal{W}^{\mathcal{M}}_{\nu JJ}(r)$$

$$\times \left\{ \left[\tilde{B}^{[j_f 1/2]}_{\nu_f \ell_f}\, \tilde{\eta}^{[1/2]} \begin{pmatrix} \tilde{Y}^{[j_f]}_{1/2\ell_f}(\hat{r})\, u_{\nu_f \ell_f}(r) \\ -\tilde{Y}^{[j_f]}_{1/2\lambda_f}(\hat{r})\, v_{\nu_f \ell_f \lambda_f}(r) \end{pmatrix} \right]^{[00]} \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \right.$$

$$\left. \times \left[\left(A^{[J]}_{\mathcal{M}\nu} + (-)^J \tilde{A}^{[J]}_{\mathcal{M}\nu}\right) [\sigma^{[1]} \hat{r}^{[J]}]^{[J]} \right]^{[0]} \left[B^{[j_i 1/2]}_{\nu_i \ell_i}\, \eta^{[1/2]} \begin{pmatrix} Y^{[j_i]}_{1/2\ell_i}(\hat{r})\, u_{\nu_i \ell_i}(r) \\ -Y^{[j_i]}_{1/2\lambda_i}(\hat{r})\, v_{\nu_i \ell_i \lambda_i}(r) \end{pmatrix} \right]^{[00]} \right\} . \tag{5.47}$$

We see that the geometry for this contribution as well as those of the electric and space-like longitudinal parts is of the form of figure 5.9

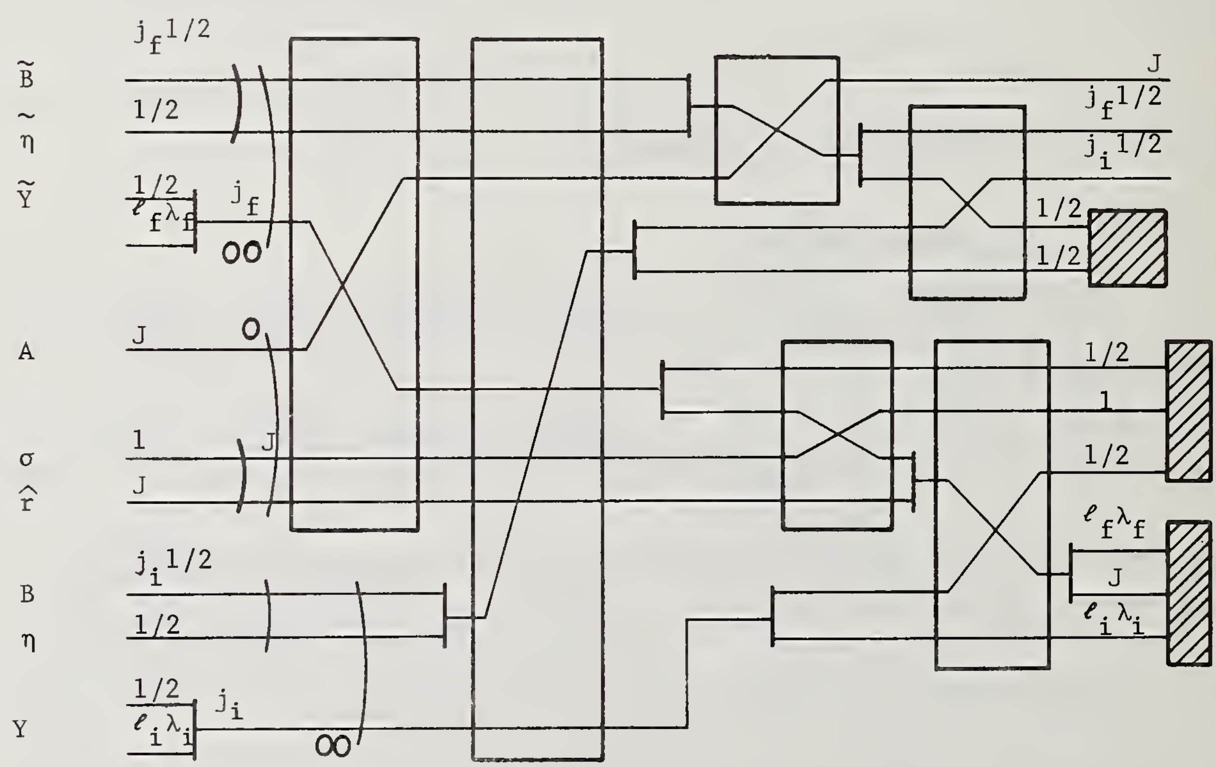


Figure 5.9

Thus let us define the quantity $S_\kappa(K)$ for multipole K, $\kappa = \mathcal{M},\mathcal{E},\mathcal{L}$

$$S_\kappa(K) = \sqrt{\frac{3}{2}} \sum_{\substack{\nu\ J \\ \nu_i\ j_i\ \ell_i \\ \nu_f\ j_f\ \ell_f}} i^{(-\ell_f+\ell_i+J+1)} \frac{\hat{j}_i}{\hat{J}} \hat{\frac{1}{2}} (-)^{j_f+J-j_i}$$

$$\times \left\{ \begin{bmatrix} 1/2 & \ell_f & j_f \\ 1 & K & J \\ 1/2 & \lambda_i & j_i \end{bmatrix} \frac{[\ell_f|K|\lambda_i]}{\hat{\lambda}_i} \int r^2 dr\, \mathcal{W}^\kappa_{\nu JK}\, u_{\nu_f \ell_f}\, v_{\nu_i \ell_i \lambda_i} \right.$$

$$\left. + \begin{bmatrix} 1/2 & \lambda_f & j_f \\ 1 & K & J \\ 1/2 & \ell_i & j_i \end{bmatrix} \frac{[\lambda_f|K|\ell_i]}{\hat{\ell}_i} \int r^2 dr\, \mathcal{W}^\kappa_{\nu JK}\, v_{\nu_f \ell_f \lambda_f}\, u_{\nu_i \ell_i} \right\} . \qquad (5.48)$$

Then the interaction with the magnetic ω field (3.222) is

$$\mathcal{H}^{\mathcal{M}}_{\omega NN} = G_{\omega NN} \sum \hat{J}\, S_{\mathcal{M}}(J) \left[(A^{[J]}_{\mathcal{M}\nu} + (-)^J \tilde{A}^{[J]}_{\mathcal{M}\nu})\, \tilde{B}^{[j_f 1/2]}_{\nu_f \ell_f}\, B^{[j_i 1/2]}_{\nu_i \ell_i} \right]^{[00]} . \qquad (5.49)$$

With the electric one, Eq.(3.223), it is

$$\mathcal{H}^{\mathcal{E}}_{\omega NN} = - i\, G_{\omega NN} \sum \left\{ \sqrt{J+1}\, S_{\mathcal{E}}(J-1) + \sqrt{J}\, S_{\mathcal{E}}(J+1) \right\} \left[(A^{[J]}_{\mathcal{E}\nu} + (-)^{J+1} \tilde{A}^{[J]}_{\mathcal{E}\nu}) \right.$$

$$\left. \times\ \tilde{B}^{[j_f 1/2]}_{\nu_f \ell_f}\, B^{[j_i 1/2]}_{\nu_i \ell_i} \right]^{[00]} , \qquad (5.50)$$

and with the longitudinal space-like part $\vec{\omega}_{\mathcal{L}}$, Eq.(3.224) it is

$$\underset{(\text{space})}{\mathcal{H}^{\mathcal{L}}_{\omega NN}} = - i\, G_{\omega NN} \sum \left\{ \sqrt{J}\, S(J-1) - \sqrt{J+1}\, S(J+1) \right\} \left[(A^{[J]'}_{\mathcal{L}\nu} + (-)^{J+1} \tilde{A}^{[J]}_{\mathcal{L}\nu}) \right.$$

$$\left. \times\ \tilde{B}^{[j_f 1/2]}_{\nu_f \ell_f}\, B^{[j_i 1/2]}_{\nu_i \ell_i} \right]^{[00]} . \qquad (5.51)$$

We finally have the time-like longitudinal contribution, (which corresponds to a spin 0 transfer), Eq.(3.225)

$$\underset{(\text{time})}{\mathcal{H}^{\mathcal{L}}_{\omega NN}} = - G_{\omega NN} \int d^3\vec{r}\, i\, \bar{\psi}\, \gamma_4\, \psi\, \omega_4 = - G_{\omega NN} \int d^3 r\, i\, \psi^+\, \psi\, \omega_4$$

$$= - G_{\omega NN} \frac{i}{2} \sum_{\substack{\nu\ J \\ \nu_i\ \ell_i\ j_i \\ \nu_f\ \ell_f\ j_f}} i^{(-\ell_f+\ell_i+J+1)}\, \hat{\frac{1}{2}}^2\, \hat{J}\, \hat{j}_i \hat{j}_f \int d^3 r\, \mathcal{V}^{\mathcal{L}}_{\nu J}(r)$$

$$\times\Big[\tilde{B}^{[j_f 1/2]}_{\nu_f \ell_f}\,\tilde{\eta}^{[1/2]}\begin{pmatrix}\tilde{Y}^{[j_f]}_{1/2\ell_f} & u_{\nu_f\ell_f}\\ -\tilde{Y}^{[j_f]}_{1/2\lambda_f} & v_{\nu_f\ell_f\lambda_f}\end{pmatrix}\Big]^{[00]}\Big[B^{[j_i 1/2]}_{\nu_i\ell_i}\,\eta^{[1/2]}\begin{pmatrix}Y^{[j_i]}_{1/2\ell_i} & u_{\nu_i\ell_i}\\ -Y^{[j_f]}_{1/2\lambda_i} & v_{\nu_i\ell_i\lambda_i}\end{pmatrix}\Big]^{[00]}$$

$$\times\Big\{[A^{[J]}_{\mathscr{L}\nu}\hat{r}^{[J]}]^{[0]} + (-)^{J+1}\ \text{c.c.}\Big\} \quad , \qquad (5.52)$$

the geometry of which is given in figure 5.10 .

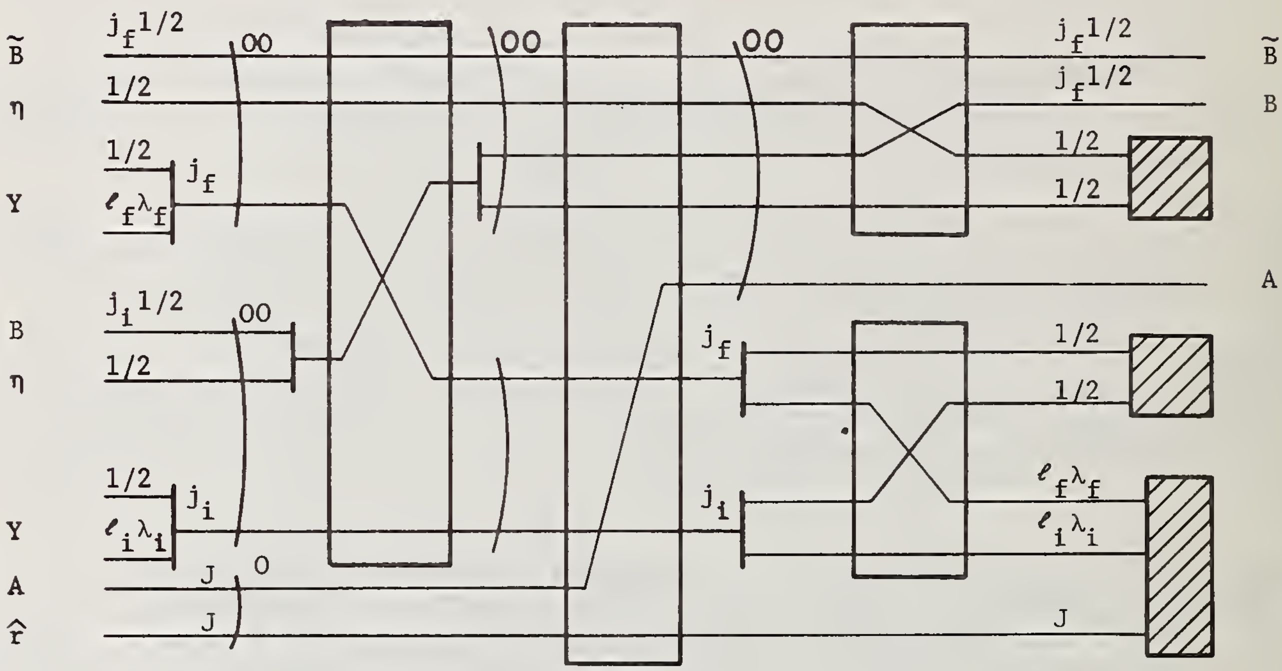


Figure 5.10

Hence,

$$\mathscr{H}^{\mathscr{L}}_{\omega NN\,(\text{time})} = -\,G_{\omega NN}\sum i^{(-\ell_f+\ell_i+J)}\left\{\begin{bmatrix}1/2 & \ell_f & j_f\\ 1/2 & \ell_i & j_i\\ 0 & J & J\end{bmatrix}[\ell_f|\ell_i|J]\int r^2 dr\, v^{\mathscr{L}}_{\nu J}\, u_{\nu_f\ell_f}\, u_{\nu_i\ell_i}\right.$$

$$\left. + \begin{bmatrix}1/2 & \lambda_f & j_f\\ 1/2 & \lambda_i & j_i\\ 0 & J & J\end{bmatrix}[\lambda_f|\lambda_i|J]\int r^2 dr\, v^{\mathscr{L}}_{\nu J}\, v_{\nu_f\ell_f\lambda_f}\, v_{\nu_i\ell_i\lambda_i}\right\}$$

$$\times\Big[(A^{[J]}_{\mathscr{L}\nu} - (-)^J\,\tilde{A}^{[J]}_{\mathscr{L}\nu})\,\tilde{B}^{[j_f 1/2]}_{\nu_f\ell_f}\,B^{[j_i 1/2]}_{\nu_i\ell_i}\Big]^{[00]} \quad . \qquad (5.53)$$

V.3.2 - The ρNN interaction

For the ρ field the expressions are similar except for an isospin contribution which is given in figure 5.11 ,

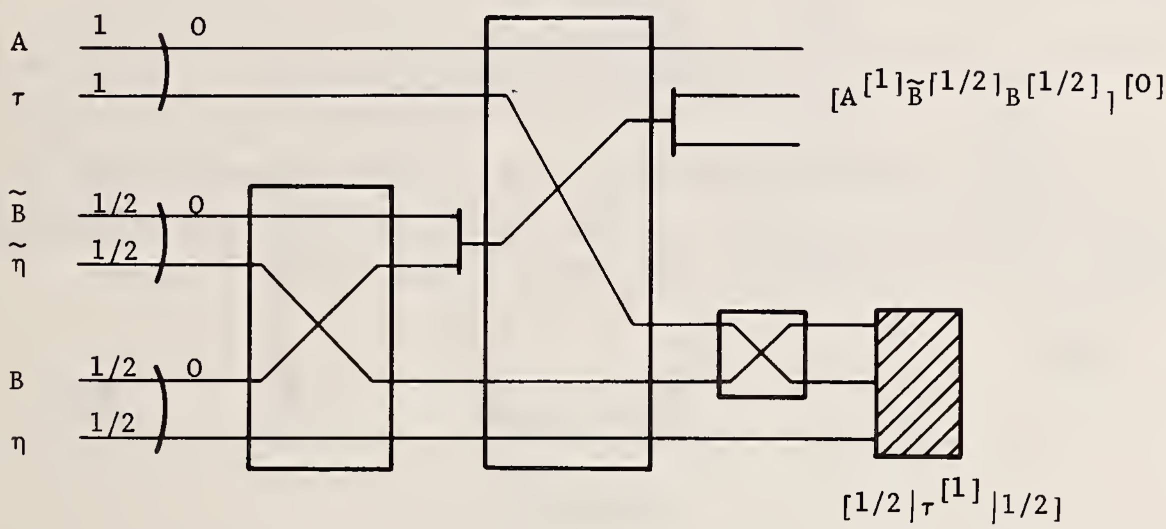
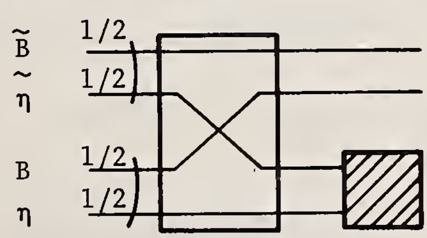


Figure 5.11

where as usual the isospin function $\eta^{[1]}$ in the ρ field is replaced by the isospin transition operator $\tau^{[1]}$. This diagram yields a factor $-i/(\frac{\hat{1}}{2})$. In order to use with it the previous expressions for the ωNN interaction, one must furthermore multiply the latter by $\hat{1}$ for isospin coupling in the ρ field expansion and by $\frac{\hat{1}}{2}$ to cancel the previous isospin contribution which is given by figure 5.12.



$$= \frac{[\tilde{B}B]^{[0]}}{\sqrt{2}}$$

Figure 5.12

We finally get

$$\mathcal{H}_{\rho NN} = - i \sqrt{3}\, \mathcal{H}_{\omega NN} \frac{G_{\rho NN}}{G_{\omega NN}} \quad . \tag{5.54}$$

V.4 - THE SPIN 0 - SPIN 1 MESON INTERACTION

V.4.1 - The $\rho\pi\pi$ interaction

The ρ meson must interact with two pions in order to conserve $\mathscr{G}$ parity. The simplest form for this interaction is,

$$\mathscr{L}_{\rho\pi\pi} = - G_{\rho\pi\pi}\, \rho_{\mu}(\vec{\tau})\,\partial_{\mu}\, \Phi^2$$

$$= - G_{\rho\pi\pi} \left\{ \rho_{\mu}\, \hat{1}\, [e^{[1]}\nabla^{[1]}]^{[0]}\, \Phi^2 + \rho_4(\vec{\tau})\,\partial_4\, \Phi^2 \right\} . \qquad (5.55)$$

Here $\rho_{\mu}(\vec{\tau})$ is the ρ field with substitution of its isospin wave function $\eta^{[1]}$ by the operator $\tau^{[1]}$. The gradient must be a symmetrized operator in the two pion coordinates. The various possible processes are shown in figure 5.13.

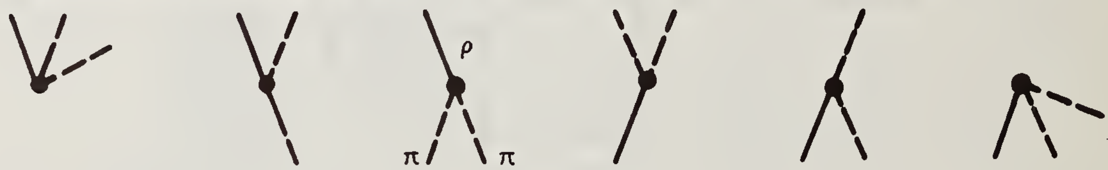


Figure 5.13

They give the same geometry since they yield invariant triple products in the operators. For example using the result,

$$\hat{1}\, [e^{[1]}\nabla^{[1]}]^{[0]}\, [A^{[J]} Y^{[J]}_{1\,L}]^{[0]} = [A^{[J]} Y^{[J]}_{\nabla L}]^{[0]} , \qquad (5.56)$$

where

$$Y^{[J]}_{\nabla\, L} = [\nabla^{[1]} \hat{r}^{[L]}]^{[J]} , \qquad (5.57)$$

we get for the various multipoles of the ρ field expressions the structure of which is of the form $(x = \mathscr{E}, \mathscr{M}, \mathscr{L})$

$$\mathscr{H}_{\rho\pi\pi} = - \int d^3 r\, \mathscr{L}_{\rho\pi\pi} = \frac{1}{2} \sum_{\substack{\nu\ J \\ \nu_1\ \ell_1 \\ \nu_2\ \ell_2}} i^{(J+\ell_1+\ell_2)}\, \hat{\ell}_1\, \hat{\ell}_2\, \hat{1}^3$$

$$\times \int d^3 r\, \mathscr{W}^{x}_{\nu JL}(r)\, g_{\nu_1 \ell_1}(r)\, g_{\nu_2 \ell_2}(r) \left\{ \left(\left[A^{[J1]}_{x\nu}\, Y^{[J]}_{\nabla L}\, \tau^{[1]} \right]^{[00]} + \ldots \right) \right.$$

$$\left. \times \left(\left[C^{[\ell_1 1]}_{\nu_1}\, \hat{r}^{[\ell_1]}\, \eta^{[1]} \right]^{[00]} + (-)^{\ell_1} \mathrm{c.c.} \right) \left(\left[C^{[\ell_2 1]}_{\nu_2}\, \hat{r}^{[\ell_2]}\, \eta^{[1]} \right]^{[00]} + (-)^{\ell_2} \mathrm{c.c.} \right) \right\} , \qquad (5.58)$$

with recoupling diagrams of the type given in figure 5.14.

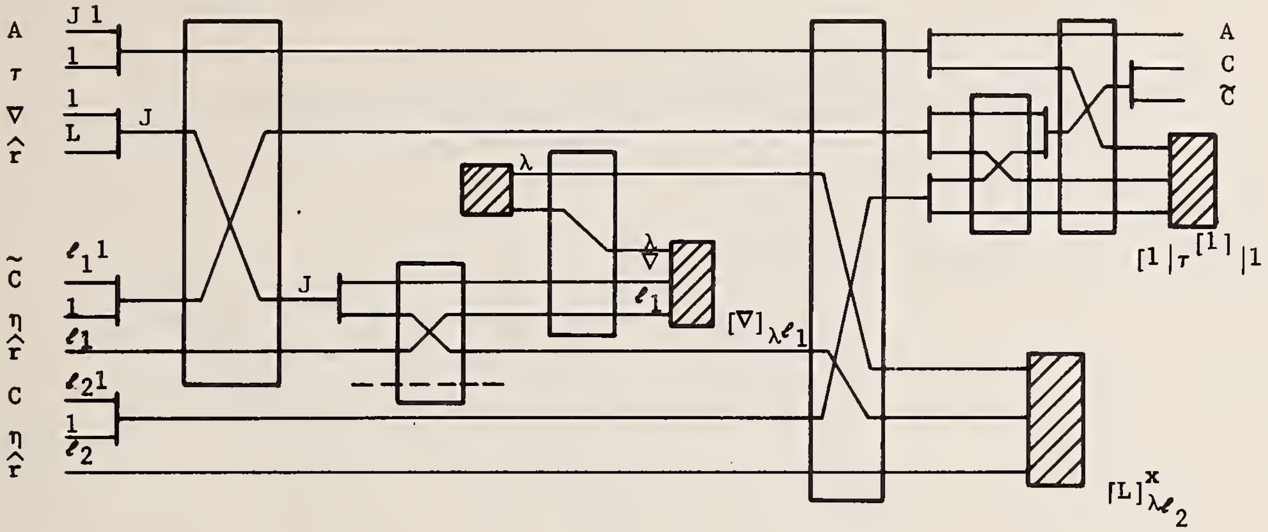


Figure 5.14

Here we have made a change of notation to distinguish the ρ and π creation operators : we use $\tilde{C}$ for the ρ mesons. Furthermore product of invariant matrix elements $[\nabla]_{\lambda\ell_1}$ and $[L]^x_{\lambda\ell_2}$ is given in detail by

$$[\nabla]_{\lambda\ell_1} [L]^x_{\lambda\ell_2} = [\lambda|\nabla|\ell_1] [\lambda|\dot{L}|\ell_2] \left(\frac{2}{\pi}\right)^{3/2}$$

$$\times \int p^2 dp \; p_1^2 dp_1 \; p_2^2 dp_2 \sqrt{\frac{N_x}{2E}} \sqrt{\frac{1}{E_1}} \sqrt{\frac{1}{E_2}} \; f_{\nu_1\ell_1}(p_1) f_{\nu_2\ell_2}(p_2) f_{\nu J}(p) \; \Delta^{J \; \ell_1 \; \ell_2}_{p \; p_1 \; p_2} \; . \tag{5.59}$$

Finally we note the identity

$$[1|\tau^{[1]}|1] = [1/2|\tau^{[1]}|1/2] \quad , \tag{5.60}$$

since

$$[J|S^{[1]}|J] = i\,\hat{J}\sqrt{J(J+1)} \qquad \text{and} \qquad S = |J|\,\sigma \; . \tag{5.61}$$

Thus we define the symmetrized function (with $s_{12} = 1/\sqrt{2}$ if $\ell_1\nu_1 \neq \ell_2\nu_2$ or $s_{12} = 1$ if $\ell_1\nu_1 = \ell_2\nu_2$),

$$T_x(L) = s_{12} \left(\frac{3}{2}\right)^{1/2} \sum_{\lambda} i^{(J+\ell_1+\ell_2-1)} \frac{1}{\hat{J}\hat{\lambda}} \left\{ \begin{bmatrix} 1 & L & J \\ \ell_1 & 0 & \ell_1 \\ \lambda & L & \ell_2 \end{bmatrix} [\nabla]_{\lambda\ell_1} [L]^x_{\lambda\ell_2} + \begin{bmatrix} 1 & L & J \\ \ell_2 & 0 & \ell_2 \\ \lambda & L & \ell_1 \end{bmatrix} [\nabla]_{\lambda\ell_2} [L]^x_{\lambda\ell_1} \right\} \text{ if } \nu_1\ell_1 \neq \nu_2\ell_2 , \quad (5.62)$$

and the various contributions are,

$$\mathcal{H}^{\mathcal{M}}_{\rho\pi\pi} = \sum_{\substack{J\ \nu \\ \ell_1\ \nu_1 \\ \ell_2\ \nu_2}} G_{\rho\pi\pi} \hat{J}\, T_{\mathcal{M}}(J) \left[(A^{[J1]}_{\mathcal{M}\nu} + (-)^J \tilde{A}^{[J1]}_{\mathcal{M}\nu})(C^{[\ell_1 1]}_{\nu_1} + (-)^{\ell_1} \tilde{C}^{[\ell_1 1]}_{\nu_1}) \times (C^{[\ell_2 1]}_{\nu_2} + (-)^{\ell_2} \tilde{C}^{[\ell_2 1]}_{\nu_2}) \right]^{[00]} , \quad (5.63)$$

$$\mathcal{H}^{\mathcal{E}}_{\rho\pi\pi} = \sum_{\substack{J\ \nu \\ \ell_1\ \nu_1 \\ \ell_2\ \nu_2}} -i\, G_{\rho\pi\pi} (\sqrt{J+1}\, T_{\mathcal{E}}(J-1) + \sqrt{J}\, T_{\mathcal{E}}(J+1)) \times \left[(A^{[J1]}_{\mathcal{E}\nu} - (-)^J \tilde{A}^{[J1]}_{\mathcal{E}\nu})(C^{[\ell_1 1]}_{\nu_1} + (-)^{\ell_1} \tilde{C}^{[\ell_1 1]}_{\nu_1}) \times (C^{[\ell_2 1]}_{\nu_2} + (-)^{\ell_2} \tilde{C}^{[\ell_2 1]}_{\nu_2}) \right]^{[00]} , \quad (5.64)$$

$$\underset{(\text{space})}{\mathcal{H}^{\mathcal{L}}_{\rho\pi\pi}} = \sum_{\substack{J\ \nu \\ \ell_1\ \nu_1 \\ \ell_2\ \nu_2}} -i\, G_{\rho\pi\pi} (\sqrt{J}\, T_{\mathcal{L}}(J-1) - \sqrt{J+1}\, T_{\mathcal{L}}(J+1)) \times \left[(A^{[J1]}_{\mathcal{L}\nu} - (-)^J \tilde{A}^{[J1]}_{\mathcal{L}\nu})(C^{[\ell_1 1]}_{\nu_1} + (-)^{\ell_1} \tilde{C}^{[\ell_1 1]}_{\nu_1}) \times (C^{[\ell_2 1]}_{\nu_2} + (-)^{\ell_2} \tilde{C}^{[\ell_2 1]}_{\nu_2}) \right]^{[00]} . \quad (5.65)$$

The time-like part of the longitudinal ρ field yields

$$\underset{(\text{time})}{\mathcal{H}^{\mathcal{L}}_{\rho\pi\pi}} = \int d^3\vec{r}\, G_{\rho\pi\pi}\, \rho_4\, \partial_4\, \vec{\Phi}^2 = G_{\rho\pi\pi} \frac{1}{2} \sum_{\substack{\nu\ J \\ \nu_1\ \ell_1 \\ \nu_2\ \ell_2}} i^{(J+\ell_1+\ell_2+1)}\, \hat{J}\, \hat{\ell}_1\, \hat{\ell}_2\, \hat{1}^3\, s_{12}$$

$$\times \Big\{ \left([A^{[J1]}_{\mathcal{L}\nu} \hat{r}^{[J]}{}_\tau{}^{[1]}]^{[00]} + (-)^{J+1} \text{c.c.} \right) \left([C^{[\ell_1 1]}_{\nu_1} \hat{r}^{[\ell_1]}\, \eta^{[1]}]^{[00]} + (\mp)(-)^{\ell_1} \text{c.c.} \right) \times \left([C^{[\ell_2 1]}_{\nu_2} \hat{r}^{[\ell_2]}\, \eta^{[1]}]^{[00]} + (\pm)(-)^{\ell_2} \text{c.c.} \right) \Big\}\; P\begin{pmatrix} E_1 \\ E_2 \end{pmatrix} , \quad (5.66)$$

with a sum over the two lines if $\nu_1\ell_1 \neq \nu_2\ell_2$ for antisymmetrization purposes. The factor $P\binom{E_1}{E_2}$ is

$$P\begin{pmatrix}E_1\\E_2\end{pmatrix} = \int p^2 dp\; p_1^2 dp_1\; p_2^2 dp_2 \begin{pmatrix}E_1\\E_2\end{pmatrix} \sqrt{\frac{N_{\mathscr{L}}}{2E}}\sqrt{\frac{1}{E_1}}\sqrt{\frac{1}{E_2}}\, f_{\nu J}(p) f_{\nu_1\ell_1}(p_1) \times f_{\nu_2\ell_2}(p_2)\, \Delta^{J\;\ell_1\;\ell_2}_{p\;p_1\;p_2} \quad . \qquad (5.67)$$

After integration over angles we get

$$\mathscr{H}^{\mathscr{L}}_{\substack{\rho\pi\pi\\(\text{time})}} = G_{\rho\pi\pi}\left(\frac{3}{2}\right)^{1/2} \sum_{\substack{\nu\;J\\ \nu_1\;\ell_1\\ \nu_2\;\ell_2}} i^{(J+\ell_1+\ell_2)}\, s_{12}\,[J|\ell_1|\ell_2]\, P\begin{pmatrix}E_1\\E_2\end{pmatrix}$$

$$\times\Big[(A^{[J1]}_{\mathscr{L}\nu} - (-)^J\,\tilde{A}^{[J1]}_{\mathscr{L}\nu})\,(C^{[\ell_1 1]}_{\nu_1} + (\mp)\,(-)^{\ell_1}\,\tilde{C}^{[\ell_1 1]}_{\nu_1}) \times (C^{[\ell_2 1]}_{\nu_2} + (\pm)\,(-)^{\ell_2}\,\tilde{C}^{[\ell_2 1]}_{\nu_2})\Big]^{[00]} \quad . \qquad (5.68)$$

V. 4.2 - The $\omega\pi^3$ interaction

The ω can interact with at least three pions for $\mathscr{G}$ parity conservation and we adopt the following form for the interaction, where we have performed the scalar product $\vec{\Phi}.\vec{\tau}$ in isospin space (hence the replacement of the isospin vectors $\eta^{[1]}$ by $\tau^{[1]}$ in one of the boson field as explained in Eq.(5.3))

$$\mathscr{L}_{\omega\pi^3} = -\,G_{\omega\pi^3}\,\omega_\mu\,\partial_\mu\,\Phi(\tau)\,\Phi^2$$

$$= -\,G_{\omega\pi^3}\left\{\omega_x\,\partial_x\,\Phi(\tau)\,\Phi^2 + \omega_4\,\partial_4\,\Phi(\tau)\,\Phi^2\right\} \quad , \qquad (5.69)$$

where the derivative must be symmetrized between the three pion coordinates. The evaluation of the energy goes along the same lines as above. We first note that the isospin part yields an invariant triple product of the operators in isospin space according to figure 5.15.

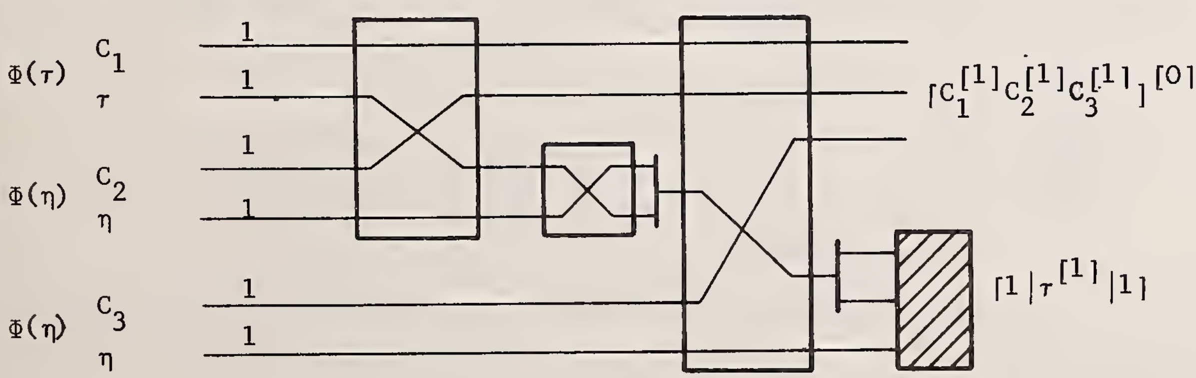


Figure 5.15

while the geometry of the angular momentum part is of the form shown on figure 5.16.

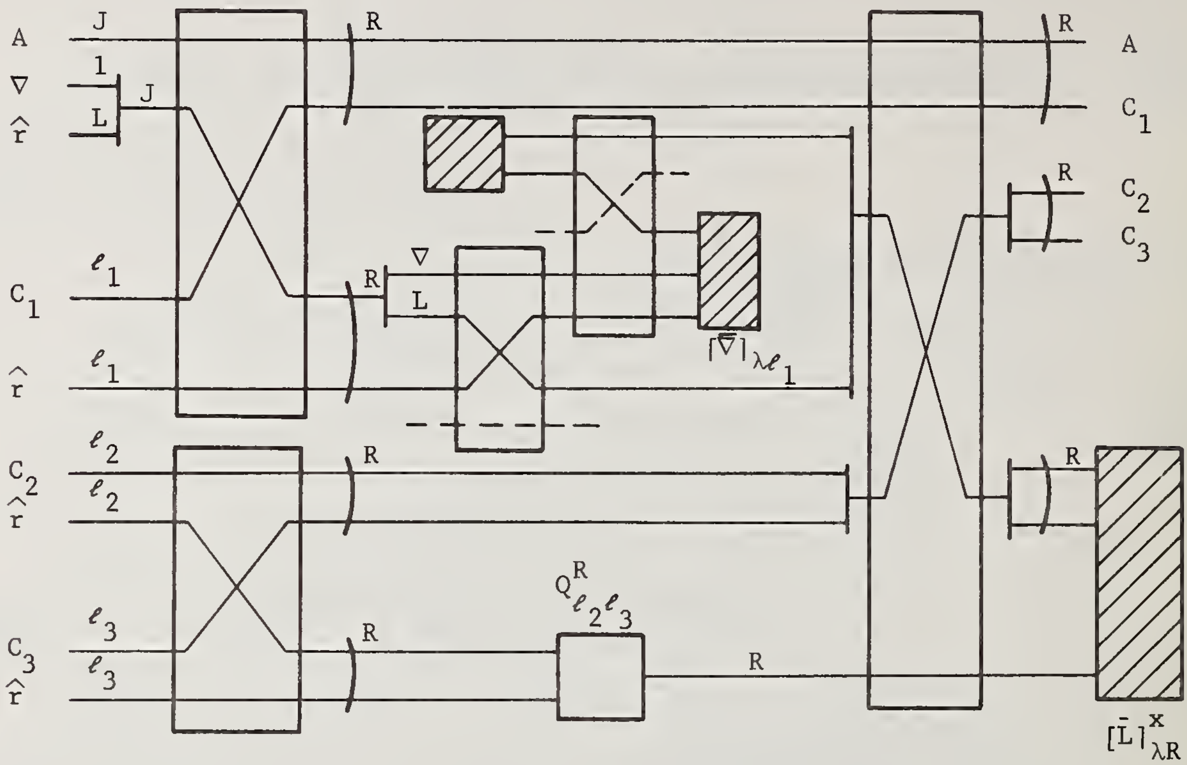


Figure 5.16

Hence we define the function,

$$V_x^{123}(LR) = -\frac{\sqrt{3}}{2}\, i^{(J+\ell_1+\ell_2+\ell_3+1)} \sum_{\lambda} \frac{1}{\hat{J}\,\hat{\lambda}} \begin{bmatrix} 1 & L & J \\ \ell_1 & 0 & \ell_1 \\ \lambda & L & R \end{bmatrix} Q^R_{\ell_2\ell_3}\, [\bar{\nabla}]_{\lambda\ell_1}\, [\bar{L}]^x_{\lambda R} \quad , \qquad (5.70)$$

where

$$[\bar{\nabla}]_{\lambda\ell_1}\, [\bar{L}]^x_{\lambda R} = [\lambda|\nabla|\ell_1]\,[\lambda|L|R] \int p^2 dp \;\; p_1^2 dp_1 \; p_2^2 dp_2 \; p_3^2 dp_3$$

$$\times \frac{1}{\sqrt{E_1E_2E_3}} \sqrt{\frac{N_x}{2E}}\; f_{\nu J}(p) f_{\nu_1\ell_1}(p_1) f_{\nu_2\ell_2}(p_2) f_{\nu_3\ell_3}(p_3)\; \square^{J\;\ell_1\;\ell_2\;\ell_3}_{p\;p_1\;p_2\;p_3} \quad , \qquad (5.71)$$

and the symmetrized sum

$$V_x(LR) = \sum_{\mathcal{P}(ijk)} s_{ijk}\; V_x^{ijk}(L) \quad , \qquad (5.72)$$

where s_{ijk} is the proper symmetrization weight . Finally we can define the interaction energy operators in a way similar to the ωNN interaction :

$$\mathcal{H}^{\mathcal{M}}_{\omega\pi^3} = G_{\omega\pi^3} \sum_{\substack{\nu \quad J \\ \nu_1\ \nu_2\ \nu_3 \\ \ell_1\ \ell_2\ \ell_3}} \sum_R \hat{J}\, V_{\mathcal{M}}(J,R)$$

$$\times \Big\{ \Big[\Big[(A^{[J0]}_{\mathcal{M}\nu} + (-)^J \tilde{A}^{[J0]}_{\mathcal{M}\nu}) (C^{[\ell_1 1]}_{\nu_1} + (-)^{\ell_1} \tilde{C}^{[\ell_1 1]}_{\nu_1}) \Big]^{[R1]}$$

$$\times \Big[(C^{[\ell_2 1]}_{\nu_2} + (-)^{\ell_2} \tilde{C}^{[\ell_2 1]}_{\nu_2}) (C^{[\ell_3 1]}_{\nu_3} + (-)^{\ell_3} \tilde{C}^{[\ell_3 1]}_{\nu_3}) \Big]^{[R1]} \Big]^{[00]} \Big\} , \qquad (5.73)$$

$$\mathcal{H}^{\mathcal{E}}_{\omega\pi^3} = - i\, G_{\omega\pi^3} \sum \Big\{ \sqrt{J+1}\, V_{\mathcal{E}}(J-1,R) + \sqrt{J}\, V_{\mathcal{E}}(J+1,R) \Big\}$$

$$\times \Big\{ \Big[\Big[(A^{[J0]}_{\mathcal{E}\nu} + (-)^{J+1} \tilde{A}^{[J0]}_{\mathcal{E}\nu}) (\ldots\ldots\ldots \Big\} , \qquad (5.74)$$

$$\underset{\text{(space)}}{\mathcal{H}^{\mathcal{L}}_{\omega\pi^3}} = - i\, G_{\omega\pi^3} \sum \Big\{ \sqrt{J}\, V_{\mathcal{L}}(J-1,R) - \sqrt{J+1}\, V_{\mathcal{L}}(J+1,R) \Big\}$$

$$\times \Big\{ \Big[\Big[(A^{[J0]}_{\mathcal{L}\nu} + (-)^{J+1} \tilde{A}^{[J0]}_{\mathcal{L}\nu}) (\ldots\ldots\ldots \Big\} . \qquad (5.75)$$

Finally the longitudinal time-like contribution is

$$\underset{\text{(time)}}{\mathcal{H}^{\mathcal{L}}_{\omega\pi^3}} = G_{\omega\pi^3} \int d^3r\ \omega_4\ \partial_4\ \Phi(\tau)\Phi^2 . \qquad (5.76)$$

Here the isospin summation is the same as above while the geometry in orbital space is given straightforwardly by the diagram of figure 5.17

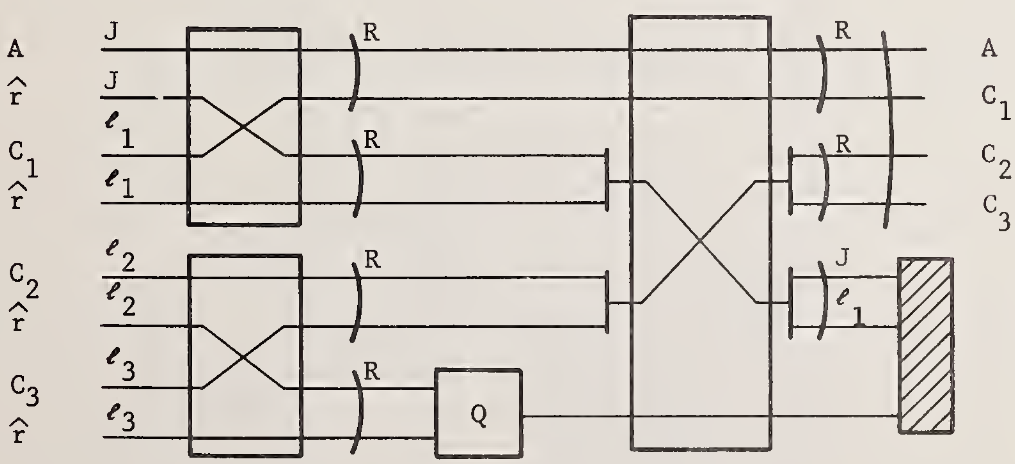


Figure 5.17

As above we define a function,

$$W^{123}_{\mathscr{L}}(R) = \sqrt{\frac{3}{2}}\, i^{(J+\ell_1+\ell_2+\ell_3)}\, Q^{R}_{\ell_2\ell_3}[J|\ell_1|R] \quad , \tag{5.77}$$

and the integral ($E_i = E_1, E_2, E_3$)

$$P(E_i) = \int p^2dp\, p_1^2dp_1\, p_2^2dp_2\, p_3^2dp_3\, \frac{1}{\sqrt{E_1E_2E_3}}\,(E_i)\,\sqrt{\frac{1}{2E}}\,\frac{p}{m}$$

$$\times\, f_{\nu J}(p) f_{\nu_1\ell_1}(p_1) f_{\nu_2\ell_2}(p_2) f_{\nu_3\ell_3}(p_3)\, \square^{J\,\ell_1\,\ell_2\,\ell_3}_{p\,p_1\,p_2\,p_3} \quad . \tag{5.78}$$

We get, with proper symmetrization,

$$\underset{(\text{time})}{\mathscr{H}^{\mathscr{L}}_{\omega\pi^3}} = G_{\omega\pi^3} \sum_{\substack{\nu \\ \nu_1\nu_2\nu_3 \\ \ell_1\ell_2\ell_3}} \sum_J \sum_R \sum_{\substack{ijk \\ (\text{even permutation} \\ \text{of } 123)}} s_{ijk}\, W^{ijk}_{\mathscr{L}}(R)\, P(E_i)$$

$$\times [[(A^{[J0]}_{\mathscr{L}\nu} - (-)^J \tilde{A}^{[J0]}_{\mathscr{L}\nu})(C^{[\ell_i 1]} - (-)^{\ell_i}\tilde{C}^{[\ell_i 1]}]^{[R1]}$$

$$\times [(C^{[\ell_j 1]} + (-)^{\ell_j}\tilde{C}^{[\ell_j 1]})(C^{[\ell_k 1]} + (-)^{\ell_k}\tilde{C}^{[\ell_k 1]})]^{[R1]}]^{[00]} \quad . \tag{5.79}$$

CHAPTER VI

MODELS OF HADRONS

VI.1 - THE HAMILTONIAN MATRIX

As an example, for a system of nucleons and pions the pseudo Hamiltonian (1.50) in terms of the field operators could be

$$\mathcal{H} = H_o + \mathcal{H}_{NN\pi} + \alpha\Phi^4 + \ldots + \frac{1}{2}\xi(P^2 + \Omega^2 R^2) \tag{6.1}$$

where H_o are the free field Hamiltonians of Chapter III and where we have introduced the artificial kinematical center of mass Hamiltonian (Chapter IV) in order to extract the intrinsic states. With a configuration space limited to four pions and one nucleon the energy matrix looks for example as shown on figure 6.1 .

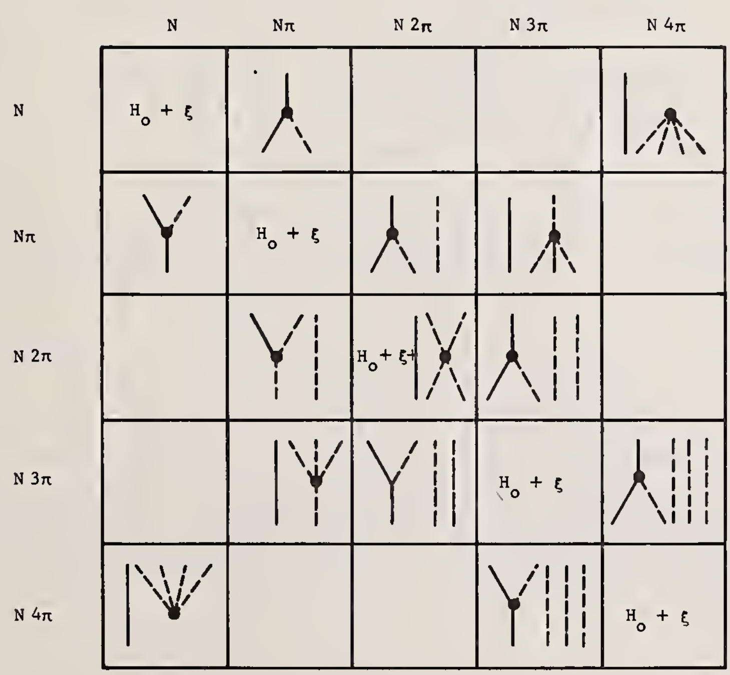


Figure 6.1

VI.2 - THE PARTONS

By now it is a generally accepted idea that hadrons are composite systems, the constituents of which have been called partons by Feynman. In this language our fields represent the partons. We will want to test for example a model based on the following partons :

	B	T	S	P	G
N	1	1/2	1/2	+	
Δ	1	3/2	3/2	+	
π	0	1	0	-	-
ρ	0	1	1	-	+
ω	0	0	1	-	-
etc					

We limit this list to the fields with quantum numbers corresponding to those of the first few lowest physical particles. For completeness we have included the Δ, although the spin 3/2 field is not treated in this work.

In the calculation the parton masses are parameters to be adjusted for fitting the physical masses. The parton configurational energies are used as a criterium to truncate the functional space. These energies are evaluated by taking N in the 0s state as the reference energy, and roughly the energy scale of the harmonic oscillator basis to be $\hbar\omega = 2m_\pi \approx 280$ MeV, which yields a level spacing for the unperturbed Hamiltonian (which contains p^2 only) of the order of magnitude of the pion mass. This way Δ in the 0s state is at an energy of about 2 effective quanta and the π, ρ, ω at energies of 1, 4, 4 effective quanta respectively. These values are utilized below for the truncation.

The total list of the distributions $(j_1^{n_1}\ j_2^{n_2} \ldots)_J$ (in the sense of occupied orbits) for the pion cloud up to 6 effective quanta with n_1 particles in state j_1 etc... and total angular momentum J is given in Table VI.1 first column n denotes the number of effective quanta , i.e. the harmonic oscillator total principal quantum number N of the configuration plus the corresponding number of pion masses.

n	π	2π	3π	4π	5π	6π
1	$(Os)_0$					
2	$(Op)_1$	$(Os^2)_0$				
3	$(Od)_2$ $(1s)_0$	$(OsOp)_1$	$(Os^3)_0$			
4	$(Of)_3$ $(1p)_1$	$(Os1s)_0$ $(OsOd)_2$ $(Op^2)_{0,2}$	$(Os^2Op)_1$	$(Os^4)_0$		
5	$(Og)_4$ $(1d)_2$ $(2s)_0$	$(OsOf)_3$ $(Os1p)_1$ $(OpOd)_{3,2,1}$ $(Op1s)_1$	$(Os^21s)_0$ $(Os^2Od)_2$ $(OsOp^2)_{0,2}$	$(Os^3Op)_1$	$(Os^5)_0$	
6	$(Oh)_5$ $(1f)_3$ $(2p)_1$	$(OsOg)_4$ $(Os1d)_2$ $(Os2s)_0$ $(OpOf)_{4,3,2}$ $(Op1p)_{0,1,2}$ $(Od^2)_{0,2,4}$ $(1s^2)_0$ $(Od1s)_2$	$(Os^2Of)_3$ $(Os^21p)_1$ $(OsOpOd)_{3,2,1}$ $(Op^3)_{1,2,3}$ $(Os1sOp)_1$	$(Os^31s)_0$ $(Os^3Od)_2$ $(Os^2Op^2)_{0,2}$	$(Os^4Op)_1$	$(Os^6)_0$

Table VI.1

VI.3 - THE NUCLEON CONFIGURATIONS

Thus for the $(\frac{1}{2}\ \frac{1}{2})$ system with B = 1 we get up to a truncation energy of 1 GeV above the nucleon mass the configurations of Table VI.2 : N, N + π, N + 2π ..., N + ρ, N + ρ + π ..., Δ + π, Δ + 2π etc ... .

N	+ π	+ 2π	+ 3π	+4π	+5π	+6π
(0s)	(0p) (1p) (2p)	$(0s^2)$ (0s1s),$(0p^2)$ (0s2s),(0p1p) $(1s^2)$, $(0d^2)$	$(0s^2 0p)$ $(0s^2 1p)$ (0s0p0d) $(0p^3)$,(0s1s0p)	$(0s^4)$ $(0s^3 1s)$ $(0s^2 0p^2)$	$(0s^4 0p)$	$(0s^6)$
(0p)	(0s) (1s),(0d) (1d),(2s)	(0s0p), (0s1p) (0p0d), (0p1s)	$(0s^3)$, $(0s^2 1s)$ $(0s 0p^2)$	$(0s^3 0p)$	$(0s^5)$	
(1s),(0d)	(0p) (0f),(1p)	(0s0p), (0s1s) (0s0d), $(0p^2)$	$(0s^2 0p)$	$(0s^4)$		
(1p),(0f)	(0s) (0d),(1s)	(0s0p)	$(0s^3)$			
(2s) (1d)	(0p)					

N	+ ρ	+ ρ,π
(0s) (0p)	(0p) (0s)	(0s,0s)

Δ	+ π	+ 2 π	+ 3 π	+ ρ
(0s)	(0p) (1p)	$(0p^2)$, (0s0d)	$(0s^2 0p)$	(0s)
(0p)	(0s) (0d)	(0s0p)	$(0s^3)$	
(1s) (0d)	(0p)	$(0s)^2$		
(1p)	(0s)			

N
0s
1s
2s
3s

Table VI.2

We have included the Δ, which couples strongly to the one pion system ($g_{\Delta N \pi} \sim 4 g_{N N \pi}$) and the ρ meson which couples extremely strongly to the pion system ($g_{\rho\pi\pi} \sim 5 g_{N\pi\pi}$).

VI.4 - THE DEUTERON CONFIGURATIONS

For the deuteron case we consider separately the configurations where the two nucleon orbitals yield an even or odd space symmetry respectively. For the even case all the listed 2N configurations can contribute without any pion except (OsOg). For the odd case at least one pion must be present. We also give between parenthesis the number of quanta for the NN and the nπ configurations. The configurations are limited to six $\hbar\omega$ and by the parity and angular momentum requirements. The even space symmetry configurations are given in Table VI.3 and the odd space ones in Table VI.4. If the ω and ρ are added we get in addition the Table VI.5 .

2N	+ π (2)	+ π (4)	+ π (6)	+ 2π (2)	+ 2π (4)	+ 2π (6)	+ 3π (4)	+ 3π (6)	+ 4π (4)	+ 4π (6)
(0) (Os^{2})	(Op)	(1p) (Of)	(2p)	(Os^{2})	(Os1s) (OsOd) (Op^{2})	(Os2s) (Os1d) (1sOd) (Od^{2}) (1s^{2}) (Op1p)	(Os^{2}Op)	(Os^{2}1p) (Op^{3})	(Os^{4})	(Os^{3}1s) (Os^{3}Od) (Os^{2}Op^{2})
(2) (OsOd)	(Op)	(1p) (Of)		(Os^{2})	(Os1s) (OsOd) (Op^{2})		(Os^{2}Op)		(Os^{4})	
(2) (Op^{2})	(Op)	(1p) (Of)		(Os^{2})	(Os1s) (OsOd) (Op^{2})		(Os^{2}Op)		(Os^{4})	
(4) (Os1d)	(Op)			(Os^{2})						
(4) (OsOg)	(Op)			(Os^{2})						
(4) (Op1p)	(Op)			(Os^{2})						
(4) (OpOf)	(Op)			(Os^{2})						
(4) (Od^{2})	(Op)			(Os^{2})						
(4) (Od1s)	(Op)			(Os^{2})						
(4) (1s^{2})	(Op)			(Os^{2})						
(6) (Os2d)										
(6) (Op2p)										
(6) (Op1f)										
(6) (Od2s)										
(6) (OdOg)										
(6) (1s2s)										
(6) (Of^{2})										
(6) (Of1p)										
(6) (1p^{2})										

2N	+ 5π (6)	+ 6π (6)
(0) (Os^{2})	(Os^{4}Op)	(Os^{6})

Table VI.3

2N	+ π (1)	(3)	(5)	+ 2π (3)	(5)	+ 3π (3)	(5)
(1) (0s0p)	(0s)	(1s) (0d)	(2s) (1d)	(0s0p)	(0s1p) (0p1s) (0p0d)	$(0s^3)$	$(0s^2 1s)$ $(0s0p^2)$
(3) (0s1p)	(0s)	(1s) (0d)		(0s0p)		$(0s^3)$	
(3) (0s0f)	(0s)	(1s) (0d)		(0s0p)		$(0s^3)$	
(3) (0p0d)	(0s)	(1s) (0d)		(0s0p)		$(0s^3)$	
(3) (0p1s)	(0s)	(1s) (0d)		(0s0p)		$(0s^3)$	
(5) (0s2p)	(0s)						
(5) (0p1d)	(0s)						
(5) (0p2s)	(0s)						
(5) (0d1p)	(0s)						
(5) (0d0f)	(0s)						
(5) (1s1p)	(0s)						

2N	+ 4π (5)	+ 5π (5)
(1) (0s0p)	$(0s^3 0p)$	$(0s^5)$

Table VI.4

2N	+ ρ	+ ρπ	+ ω	+ ωπ
$(0s^2)$	(0p)	(0s0s)	(0p)	(0s0s)
(0s0p)	(0s)		(0s)	

Table VI.5

VI.5 - THE PION CONFIGURATIONS

For the pion system we get likewise the configurations of Table VI.6 for a cut-off energy of about 1 GeV with our chosen oscillator parameter.

π	(0s)	(1s)	(2s)	(3s)	
3π	$(0s^3)$ (0s1s2s)	$(0s^2 1s)$ $(0p^2 0d)$	$(0s^2 2s)$ $(0s0p^2)$	$(0d^2 0s)$ (0s0p1p)	$(0s1s^2)$
5π	$(0s^5)$	$(0s^4 1s)$			
$\pi\rho$	(0s0p)	(0p0s)			

Table VI.6

Thus we see on these various examples that the truncated spaces up to six pion masses (~ 1 GeV) are of moderate size. The complications introduced by the symmetrization problem are limited essentially to the s^n case, see section II.5.2 . The only redundant configurational set which appears in the above tables is the very simple p^3 case. Of course the effect of the cut-off energy on the low energy properties of the systems (form factors, electromagnetic moments etc...) has to be studied numerically.

APPENDIX

RELATIVISTIC KINEMATICS OF THE CENTER OF MASS

We demonstrate in this appendix the result of Eq.(1.45), namely the C.M. coordinate $\vec{R}$ for a system of relativistic particles of energies E_i all considered at an equal time is given by[8]

$$\vec{R} = \frac{\sum_i E_i \vec{x}_i}{\sum_k E_k} \quad . \tag{A.1}$$

The discussion shall be carried out by considering successively the case of a single particle of mass M which decays into two others of masses m_1 and m_2, then the case of two distinct particles of coordinates $(x_1 t_1)$ and $(x_2 t_2)$.

In the first case the C.M. trajectory is given by the trajectory of the initial particle M and we simply must compute the geometrical relations between that trajectory and the trajectories of the daughter particles.

In a given reference system (S), see figure A.1, considering only the projections on the (xt) plane and assuming that the trajectory of the initial particle goes through the origin, the CM trajectory is

$$x = V t \quad , \tag{A.2}$$

with

$$V = P_x / E \quad , \tag{A.3}$$

$$E = \sqrt{P_x^2 + P_y^2 + P_z^2 + M^2} \quad . \tag{A.4}$$

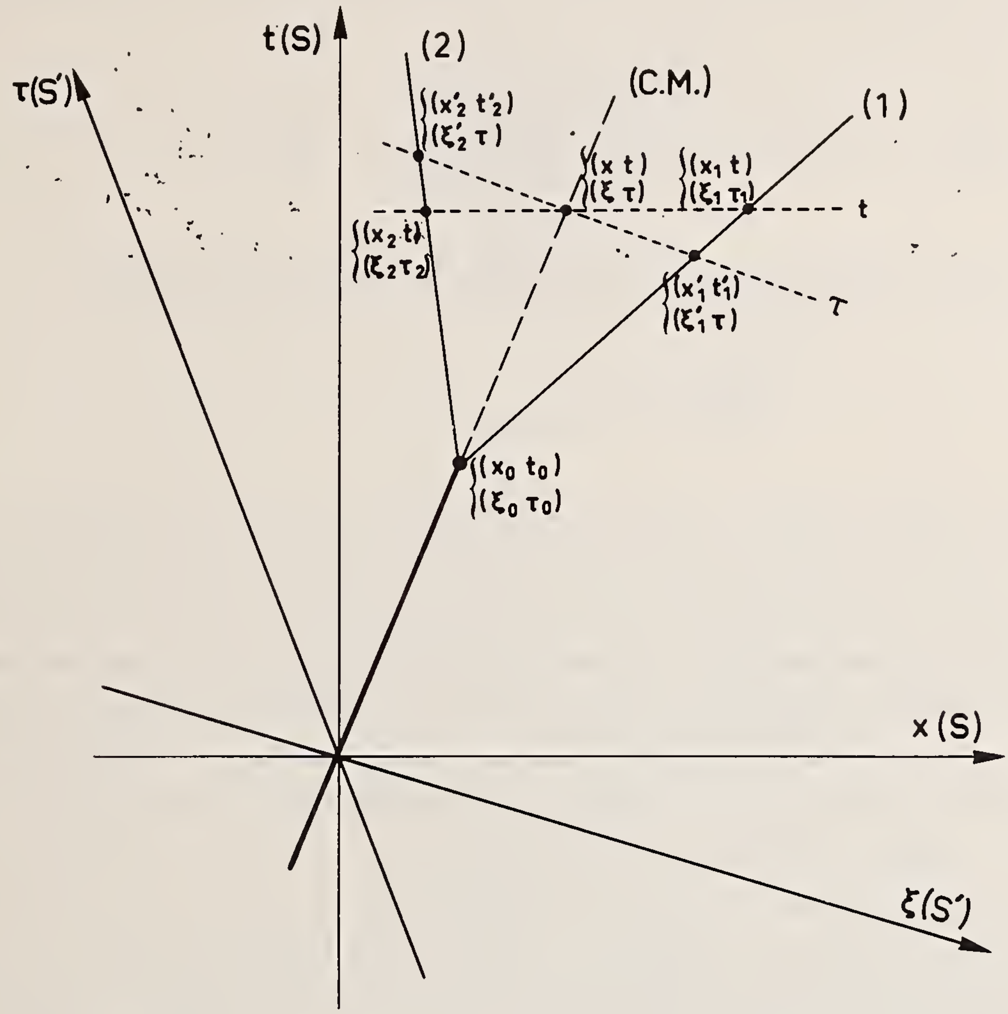


Figure A.1

At the point (x_o, t_o) the particle M decays into the two daughter particles m_1 and m_2 the positions of which at time t are

$$x_1 = x_o + v_1(t - T_o) \quad , \tag{A.5}$$

$$x_2 = x_o + v_2(t - T_o) \quad , \tag{A.6}$$

with

$$v_1 = \frac{p_{1_x}}{E_1} \quad , \quad E_1 = \sqrt{p_{1_x}^2 + p_{1_y}^2 + p_{1_z}^2 + m_1^2} \quad ; \tag{A.7}$$

$$v_2 = \frac{p_{2_x}}{E_2} \quad , \quad E_2 = \sqrt{p_{2_x}^2 + p_{2_y}^2 + p_{2_z}^2 + m_2^2} \quad ; \tag{A.8}$$

and

$$E_1 + E_2 = E \quad , \quad \vec{p}_1 + \vec{p}_2 = \vec{P} \quad . \tag{A.9}$$

Hence at the equal time t, the ratio of the distances to the C.M. coordinate x given in Eq.(A.2), is

$$\frac{\Delta x_1}{\Delta x_2} = \frac{x_1 - x}{x_2 - x} = - \frac{\frac{p_{1_x}}{E_1} - \frac{p_{1_x} + p_{2_x}}{E_1 + E_2}}{\frac{p_{2_x}}{E_2} - \frac{p_{1_x} + p_{2_x}}{E_1 + E_2}} = - \frac{E_2}{E_1} , \qquad \text{(A.10)}$$

which for the non relativistic limit goes into the usual relation

$$\frac{\Delta x_1}{\Delta x_2} = - \frac{m_2}{m_1} . \qquad \text{(A.11)}$$

Now in some other system S' the CM and the two daughter particles have coordinates ξ, ξ_1 and ξ_2 corresponding to x, x_1 and x_2 with now however non equal times τ, τ_1 and τ_2. Nevertheless these three world points being connected by the above physical relations still retain the character that $(\xi\ \tau)$ is the position of the CM for the particles m_1 and m_2 at positions $(\xi_1\tau_1)$ and $(\xi_2\tau_2)$ moving now with velocities v_1' and v_2'. On the other hand the point $(\xi\ \tau)$ is also the CM of the particles m_1 and m_2 at the equal time τ in S', i.e., at the positions $(\xi_1'\tau)$ and $(\xi_2'\tau)$ as shown on figure A.1. Conversely going back to the initial system (S), the world point (xt) (corresponding to $(\xi\ \tau)$ in (S')) is also the CM of the particles m_1 and m_2 at $(x_1't_1')$ and $(x_1't_2')$ (corresponding to $(\xi_1\tau_1)$ and $(\xi_2\tau_2)$ in (S') respectively). Thus we see that the C.M. in relativity is associated with a whole family of particle positions with space-like separations, i.e., all the positions obtained at the crossing points of any straight line containing (xt) with the particle trajectories. To each of these straight line corresponds a Lorentz transform into a reference system in which the three points have equal times.

We now turn the problem around : given two particles with arbitrary 4-momenta $(\vec{p}_1,E_1)$ and $(\vec{p}_2,E_2)$ in (S), find the center of mass if the particles have the positions $(\vec{r}_1t_1)$ and $(\vec{r}_2t_2)$. The C.M. energy and momentum are again given by Eq.(A.9).

Restricting ourselves again to the x-t plane one can find the crossing of the world lines by drawing lines through the points (x_1t_1) and (x_2t_2) with slopes (see figure A.2)

$$v_{1_x} = \frac{p_{1_x}}{E_1} , \qquad v_{2_x} = \frac{p_{2_x}}{E_2} . \qquad \text{(A.12)}$$

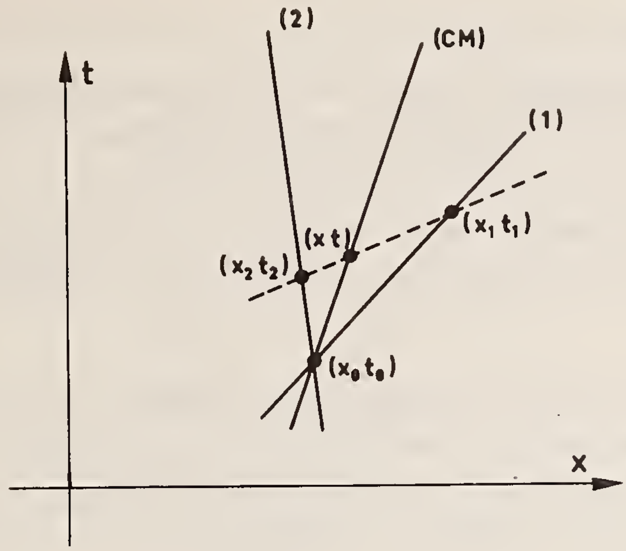


Figure A.2

Let us call the crossing point $(x_o t_o)$. In contrast to the above example of the decay of a particle, here the point $(x_o t_o)$ has no physical meaning, since in general the world lines 1 and 2 do not cross, i.e. the projections of the world lines on the x-t, y-t and z-t planes cross at different times. We now can construct the trajectory of the CM as it goes through the point $(x_o t_o)$ with the slope

$$V = \frac{P_x}{E} \tag{A.13}$$

and finally, the CM corresponding to the given points $(x_1 t_1)$ is given by the crossing of the CM trajectory with the line connecting the two points. After some algebra we find, considering the x-t plane, for the CM coordinates

$$x = \frac{(x_1 - x_2)(E_1 x_1 + E_2 x_2) - (t_1 - t_2)(p_{1_x} x_1 + p_{2_x} x_2)}{(x_1 - x_2)E - (t_1 - t_2)P_x} , \tag{A.14}$$

$$t = \frac{(x_1 - x_2)(E_1 t_1 + E_2 t_2) - (t_1 - t_2)(p_{1_x} t_1 + p_{2_x} t_2)}{(x_1 - x_2)E - (t_1 - t_2)P_x} . \tag{A.15}$$

For the y-t plane, we have

$$y = \frac{(y_1 - y_2)(E_1 y_1 + E_2 y_2) - (t_1 - t_2)(p_{1_y} y_1 + p_{2_y} y_2)}{(y_1 - y_2)E - (t_1 - t_2)P_y} , \tag{A.16}$$

$$t = \frac{(y_1 - y_2)(E_1 t_1 + E_2 t_2) - (t_1 - t_2)(p_{1_y} t_1 + p_{2_y} t_2)}{(y_1 - y_2)E - (t_1 - t_2)P_y} , \tag{A.17}$$

and, for the z-t plane

$$z = \frac{(z_1 - z_2)(E_1 z_1 + E_2 z_2) - (t_1 - t_2)(p_{1_z} z_1 + p_{2_z} z_2)}{(z_1 - z_2)E - (t_1 - t_2)P_z} \quad , \tag{A.18}$$

$$t' = \frac{(z_1 - z_2)(E_1 t_1 + E_2 t_2) - (t_1 - t_2)(p_{1_z} t_1 + p_{2_z} t_2)}{(z_1 - z_2)E - (t_1 - t_2)P_z} \quad . \tag{A.19}$$

Of course, one could express these formulae in terms of the trajectories of the particles. We shall, however, not go into further detail since we will need only the case $t_1 = t_2$, for which we immediately obtain the Eq. (1.45). We only note that the CM coordinates are not given by the ansatz

$$\vec{R} = \frac{1}{2}(\vec{r}_1 + \vec{r}_2) \quad , \qquad T = \frac{1}{2}(t_1 + t_2) \quad , \tag{A.20}$$

which is used in the Bethe-Salpeter equation. Finally, there holds

$$\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right) x = 1 \quad , \tag{A.21}$$

which can be verified by direct calculation from (A.14). Thus, formally, the commutation relations $[P_x, x] = i \ldots$ are fulfilled.

This procedure for finding the CM coordinate can be continued to more particles, for example in analogy to the Jacobi coordinates of non-relativistic kinematics. Thus one first defines the CM for particles 1 and 2, x_{12}, t_{12}, by (A.14), (A.15). Now one adds the third particle by replacing in (A.14), (A.15) $x_1 \to x_{12}$, $x_2 \to x_3$, $t_1 \to t_{12}$, $t_2 \to t_3$. This way one obtains $x_{12,3}$, $t_{12,3}$ and so on. As to be expected, the general expressions are cumbersome. However in the present work we are interested only in the equal time case $(t_1 = t_2 = t_3 = \ldots)$ for which the expression for the CM coordinate is simply

$$x_{12,3}(t_1 = t_2 = t_3) = \frac{E_{12} x_{12} + E_3 x_3}{E_{12} + E_3} = \frac{(E_1 + E_2)\left(\frac{E_1 x_1 + E_2 x_2}{E_1 + E_2}\right) + E_3 x_3}{E_1 + E_2 + E_3}$$

$$= \frac{E_1 x_1 + E_2 x_2 + E_3 x_3}{E_1 + E_2 + E_3} \quad , \tag{A.22}$$

as asserted in (1.45).

Finally we only point out that the definition of the non equal-time C.M. coordinate used above in the case of space-like seprations can be analytically continued to yield the definition of the CM coordinate for time like separations, i.e., for slopes of the line connecting $(x_1 t_1)$ and $(x_2 t_2)$, figure A.3, larger than 45^o. This case arises in covariant treatments of many body systems, e.g., in the Bethe-Salpeter equation. Therefore the CM of two points $(x_1 t_1)$ and $(x_2 t_2)$ is the intersection of the straight line connecting these points with the trajectory of the CM defined in (S).

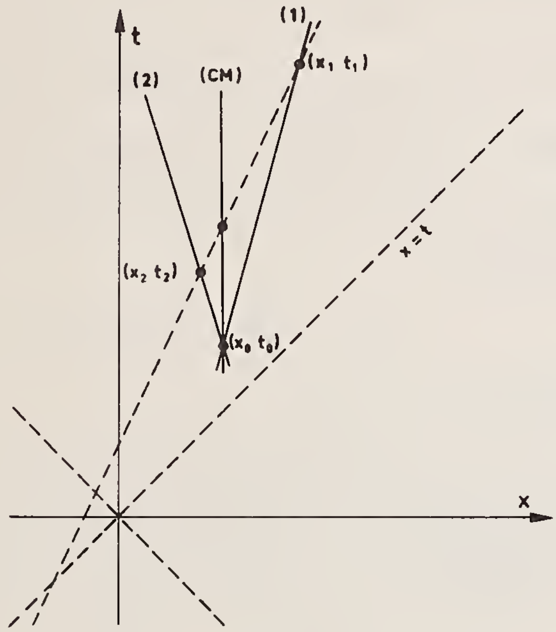


Figure A.3

An example of such points is the pair $(x_1 t_1)$, $(x_2 t_2)$ of figure A.3. In fact, by a Lorentz transform one can always achieve that a pair of points with a time-like separation is parallel to the time-axis. An observer in that coordinate system then will note that first the particle 2, then the CM, then the particle 1 fly by his observation post. This way the straight line criterion acquires a physical meaning. With this definition we have a uniform description for the center of mass of a system of two particles.

For completeness we list the following relations. To re-write the familiar CM-separation

$$e^{i(\vec{p}_1 \vec{r}_1 + \vec{p}_2 \vec{r}_2)} = e^{i(\vec{P}\,\vec{R} + \vec{p}\,\vec{r})} \tag{A.23}$$

in relativistic kinematics we have, in the equal time case

$$\vec{P} = \vec{p}_1 + \vec{p}_2 \tag{A.24}$$

$$\vec{p} = \left(\frac{\vec{p}_1}{E_1} - \frac{\vec{p}_2}{E_2}\right) \frac{E_1 E_2}{E_1 + E_2} \qquad (A.25)$$

$$\vec{R} = \frac{E_1 \vec{r}_1 + E_2 \vec{r}_2}{E_1 + E_2} \qquad (A.26)$$

$$\vec{r} = \vec{r}_1 - \vec{r}_2 \qquad (A.27)$$

and the inverse formulae

$$\vec{p}_1 = \vec{P} \frac{E_1}{E_1 + E_2} + \vec{p} \qquad (A.28)$$

$$\vec{p}_2 = \vec{P} \frac{E_2}{E_1 + E_2} - \vec{p} \qquad (A.29)$$

$$\vec{r}_1 = \vec{R} + \vec{r} \frac{E_2}{E_1 + E_2} \qquad (A.30)$$

$$\vec{r}_2 = \vec{R} - \vec{r} \frac{E_1}{E_1 + E_2} \qquad (A.31)$$

They all arise from the non-relativistic formulae by the replacement $M_i \rightarrow E_i$.

B. BOOSTED STATES

When computing, say, the electron scattering form factors one needs the wave functions of the system at momenta $\vec{P}_i$ and $\vec{P}_f = \vec{P}_i + \vec{q}$, which may correspond to relativistic velocities. To that end one will have to boost the solutions of the composite physical particle. With these boosted functions one will obtain a result, correct within the model, i.e., no relativistic or recoil corrections will have to be applied. In the procedure outlined in this monograph, the wave functions Ψ_n of the system are obtained as configuration mixtures in a discretized basis (cf. Eq. 1.29, or 1.30, together with 1.32 and 1.35) in the form

$$\Psi_n(r_i,t) = e^{-iE_n t} \sum_\nu X_\nu^{(n)} \psi_\nu(\vec{r}_i) \qquad (B.1)$$

The amplitudes $X_\nu^{(n)}$ are obtained upon diagonalization of the secular matrix (1.37). We have here changed the notation in writing $\psi_\nu(\vec{r}_i)$ instead of $|r\rangle$ of (1.37). According to (1.55) the CM motion of each configuration $\psi_\nu(\vec{r}_i)$ can be separated off using

$$\psi_\nu(\vec{r}_i) = \chi_\nu(\vec{r}_i)\, \varphi_o(\vec{R}_\nu) \quad , \qquad (B.2)$$

$$\varphi_o(R_\nu) = e^{-(\vec{R}_\nu^{\,2}/2R_o^{\,2})} \qquad (B.3)$$

The CM coordinate $\vec{R}_\nu$ is a function of the coordinates $\vec{r}_i$ and is given by (1.45). Since the number of particles is different in the different configurations ψ_ν the explicit form of $\vec{R}_\nu$ depends on the index ν. Owing to (1.55) the functions $\chi_\nu(\vec{r}_i)$ in fact depend only on relative coordinates, which, however, are not specified explicitly.

Now to describe a system in free motion having momentum $\vec{P}$, it is best to introduce a moving coordinate system:

$$\vec{r}_i = \vec{R} + \vec{\xi}_i \qquad (B.4)$$

Then we have, beginning with the non-relativistic case

$$\psi_n(\vec{P};\vec{R},\vec{\xi}_i,t) = e^{i(\vec{P}\cdot\vec{R}-Et)} \sum_\nu X_\nu^{(n)} \chi_\nu(\vec{P},\vec{\xi}_i) \tag{B.5}$$

$$\chi_\nu(\vec{P},\vec{\xi}_i) = e^{(R_\xi^2/2R_o^2)} \psi_n(\vec{P},\vec{\xi}_i) \tag{B.6}$$

Here $\vec{R}_\xi$ is given by (1.45) upon the replacement $\vec{r}_i \to \vec{\xi}_i$ (recall, the $\vec{\xi}_i$ are not explicitly relative coordinates!) and,

$$E = (M_n^2 + P^2)^{1/2} \tag{B.7}$$

with M_n from Eq. 1.58.

The physical meaning of (B.4) through (B.7) is: the wave function of the system is translated to the point $\vec{R}$ from the origin by (B.4). (The point $\vec{\xi}_i = 0$ in (B.5),(B.6) corresponds to the point $\vec{r}_i = 0$ in (B.1), (B.2)). The CM wave function is divided out by (B.6). A new CM wave function is supplied by the plane wave in (B.5). The energy of the new system is now given by (B.7). Note that for non-relativistic velocities $\chi_\nu(\vec{P},\vec{\xi}_i)$ in fact is independent of $\vec{P}$. For relativistic velocities, however, the coordinates $\vec{\xi}_i$ undergo a Lorentz contraction this way the functions χ_ν acquire a dependence on $\vec{P}$. Otherwise the form of (B.5) and (B.6) remains unchanged. Because of the Lorentz contraction of the $\vec{\xi}_i$ the spherical multipole functions $j_\ell(kr)\hat{r}_n^{[\ell]}$ in the basis functions ψ_n are replaced by spheroidal functions. (Recall that a sphere in the moving frame becomes an ellipsoid in the lab frame.) The spheroidal functions can be expanded in terms of spherical multipole functions.[15] As can be seen from the tables in Ref. 15, this expansion converges rapidly for not too highly relativistic velocities. Inserting this expansion into the basis functions given in Chapter III, remembering to boost the spin of the spin carrying fields one can write down the boosted functions ψ_n in a straightforward, if perhaps tedious manner. Somewhat more involved is the boost of the factors $\exp(R_\xi^2/2R_o^2)$ appearing in (B.6). This way we have

$$\chi_n(\vec{x}_i) \rightarrow \sum_{\lambda} \left[b_n^{[\lambda]}(\vec{p}) \chi_n^{[\lambda]}(\vec{\xi}) \right]^{[0]} \tag{B.8}$$

We here shall not derive the explicit form of the boost vectors $b^{[\lambda]}(\vec{p})$. We only note that they depend on the structure of the boosted system.

C. SCATTERING STATES

The diagonalization procedure of this monograph in principle yields solutions which approach the exact solutions of the model specified by the chosen Langrangian, upon enlargement of the set of basis states, i.e., upon letting the truncation energy E_{max} grow, and taking the limit $\Omega^2 \to 0$ in the CM pseudo-Hamiltonian (4.1). Therefore the solutions having an energy which lies above the threshold for particle emission in fact directly describe scattering states. This is in contrast to R-matrix type theories in which the problem is solved in the "inside" region, together with suitable artificial boundary conditions. A scattering state then must be expanded in terms of the "inside" solutions. In the present treatment, on the other hand, one still has to analyse the solutions to extract the parameters of the S-matrix. This is similar to the description of the continuum in a shell-model framework.[16] We shall discuss this procedure first for the case of a single open channel, and we use the system $B = 1$, $J^{\pi} = 1/2^{+}$, i.e., a nucleon, as an example.

We expect the following spectrum: a single bound state, i.e., the state of the free nucleon, and after a gap corresponding approximately to the mass of π-meson a discrete spectrum of states which correspond to the p-wave π-meson-nucleon scattering continuum. Above the 2-π threshold a new set of states would appear which correspond to the 3-body system π-π-N. As long as no photons are included in the set of basis states no radiative pion capture can take place and below the 2π threshold the meaning of the states is unambiguous.

Let us now consider specifically a state, say Ψ, obtained by diagonalization of the Hamiltonian. From the energy eigenvalue one derives by means of (1.56), (1.58) the intrinsic energy of the system. Using the previously obtained rest mases of the nucleon and the pion one can derive the discrete relative momentum of the scattering state, and one can verify that Ψ belongs to the single channel region. Thus this solution must have the following character: at small $|\vec{x}_i|$ the solution describes a complicated compound system; at $|\vec{x}_i| > R_N + R_{\pi}$, i.e., in the "asymptotic" region, the system must break apart into a pion and a nucleon. Here R_N and R_π are the nucleon and pion radii.

We now discuss the form of the solution at this asymptotic separation. It can be constructed as follows. Consider the states Ψ_N and Ψ_π of the nucleon and π-meson, boosted to $\vec{p}$ and $-\vec{p}$ respectively. According to (B.8) the boosted intrinsic parts of the configurations are given by

$$\chi_{\nu N}(\vec{p},\vec{\xi}_i) = \sum_k \left[b_N^{[k]}(\vec{p})\chi_{\nu N}^{[k]}(\vec{\xi}_i)\right]^{[0]} \qquad (C.1)$$

$$\chi_{\mu\pi}(\vec{p},\vec{\xi}_j') = \sum_k \left[b_\pi^{[\ell]}(-\vec{p})\ \chi_{\mu\pi}^{[\ell]}(\vec{\xi}_j')\right]^{[0]} \qquad (C.2)$$

Here and below $\vec{p}$ and $\vec{r}$ are the relative momentum and coordinate, respectively. They are obtained by putting in (A.24) through (A.31) $\vec{r}_1 \equiv \vec{r}_N$ = nucleon CM position and $\vec{r}_2 \equiv \vec{r}_\pi$ = pion CM position, etc., and taking $\vec{R} = \vec{P} = 0$ since here we are considering the scattering state in the intrinsic, i.e., the CM system. Finally, the $\vec{\xi}_i$ are measured from $\vec{r}_N$ and the $\vec{\xi}_j'$ from $\vec{r}_\pi$. The relative motion is provided by the expansion of a plane wave into multipoles in which in the radial part $j_\lambda(pr)$ is replaced by

$$F_{\lambda j}(pr) = \cos\delta_{\lambda j}\ j_\lambda(pr) - \sin\delta_{\lambda j}\ n_\lambda(pr) \qquad (C.3)$$

$\delta_{\lambda j}$ are the phase shifts for a given λ, j scattering state. A particular λ, j system then can be extracted by angular projection.

Thus, a general intrinsic scattering state (the CM wave function is still absent) is of the form

With the abbreviation, in analogy to, say, (3.14)

$$B_{k\ell}^{[j]}(p) = \int d^2\hat{p} \left[\left[b_N^{[k]}(\vec{p})\, b_\pi^{[\ell]}(-\vec{p}) \right]^{[s]} \hat{p}^{[\lambda]} \right]^{[j]} \tag{C.5}$$

We have

$$\Psi \simeq 4\pi \sum_\lambda i^\lambda\, \hat{j}\, F_{\lambda j}(pr) \sum_{k\ell} \left[B_{k\ell}^{[j]}(p) \left[\chi_{\nu N}^{[k]}(\vec{\xi}_i) \chi_{\mu\pi}^{[\ell]}(\vec{\xi}\,'_j) \right]^{[s]} \hat{r}^{[\lambda]} \right]^{[0]} \tag{C.6}$$

and thus the intrinsic configurational scattering state of a given multipolarity is of the form

$$\chi_{\lambda j}^{(\nu,\mu)} = \mathcal{N}\, \hat{j}\, F_{\lambda j}(pr) \sum_{k\ell} \left[B_{k\ell}^{[j]}(p) \left[\chi_{\nu N}^{[k]}(\vec{\xi}_i) \chi_{\mu\pi}^{[\ell]}(\vec{\xi}\,'_j) \right]^{[s]} \hat{r}^{[\lambda]} \right]^{[0]} \tag{C.7}$$

where $\mathcal{N}$ is a normalization constant. Replacing now the coordinates $\vec{\xi}_i, \vec{\xi}\,'_j$, by means of (A.3), (A.32), (B.3) by the parton laboratory coordinates, equation (C.7) is already almost in the form (B.2). To achieve that form one must rewrite the multipoles making up the nucleon system and the pion system: they are still written about the points $\vec{r}_N$ for the nucleon and $\vec{r}_\pi$ for the pion. They can be rewritten about the center of mass point by means of the translation operator.[17] The state which one obtains in the diagonalization of the Hamiltonian, say Ψ_n, thus is (implicitly) identical to the state, say Ψ_S, which one obtains from the rewritten equation (C.7) when multiplying each configuration (C.7) with the CM functions $\varphi_o(\vec{R}_{\nu\mu})$, multiplying with the amplitudes $X_N^{(\nu)}$ and $X_\pi^{(\mu)}$ (cf. Eq. 1.37) of the solutions for the nucleon and pion, respectively, and summing over the configurations:

$$\Psi_S = \sum_{\nu,\mu} X_N^{(\nu)} X_\pi^{(\mu)} \chi_{\lambda j}^{(\nu,\mu)}(\vec{r}_i, \vec{r}\,'_j)\, \varphi_o(\vec{R}_{\nu\mu}) \tag{C.8}$$

One then obtains $\cos\delta_{\lambda j}$ and $\sin\delta_{\lambda j}$ by computing $\Psi_s(r)$ at two values of r, say r_α and r_β, which are suitable roots of

$$j_\lambda(pr_\alpha) = 0 \tag{C.9}$$

$$n_\lambda(pr_\beta) = 0 \tag{C.10}$$

and computing the overlaps of $\langle\Psi_n|\Psi_s(r_\alpha)\rangle$ and $\langle\Psi_n|\Psi_s(r_\beta)\rangle$. The ratio of these two overlaps equals $-\cot\delta_{\lambda j}$; cf.(C.3). (Of course, in our example $j = 1/2$, $\lambda = 1$.) We recall here that this is true only for non-relativistic CM motion. The relative motion between the scattering particles, however, is allowed to be relativistic. Note, that for non-relativistic relative motion no Lorentz contraction takes place and the sum over k, ℓ in Eq. (C.7) collapses into a single term ($k = 1/2$, $\ell = 0$ for the considered nucleon-pion system).

We shall not discuss here the analysis of Ψ_n in the case of several open channels since no new problems arise in that case. Finally, of course, the scattering states are correct as far as the CM motion is concerned. No recoil or relativistic corrections have to be made.

D. THE PHYSICAL VACUUM

The lowest eigenstate, say $|V\rangle$, obtained in diagonalizing the pseudo-Hamiltonian for a system having the quantum numbers of the vacuum is the (not manifestly covariant) physical vacuum state of the model field theory. All energies have to be measured from the expectation value of the Hamiltonian for this state:

$$E_V = \langle V|H|V\rangle \tag{D.1}$$

This has been incorporated in (1.56), Chapter I.

The vacuum state arises as a consequence of the matrix elements of the kind of Fig. 5.7(a) and its time reversed form, or of the similar matrix elements of Fig. 5.13, i.e., of those matrix elements which connect the "ground configuration" $|0\rangle$ with a configuration containing partons. (These matrix elements thus are proportional to the $\vec{P} = 0$ component of the parton configuration.) This situation is familiar in the non-relativistic shell model in which the ground state of a many-body system is not equal to the ground configuration.

Owing to the finite size of the model Hilbert space it is always possible to construct a unitary operator which allows to represent the eigenstates on the basis of the physical vacuum state, $|V\rangle$, instead of on the basis of the ground configuration $|0\rangle$. We now list the principal characteristics of $|V\rangle$:

$$|V\rangle = v^+|0\rangle \tag{D.2}$$

$$v|0\rangle = 0 \tag{D.3}$$

$$\langle 0|vv^+|0\rangle = 1 \tag{D.4}$$

$$\mathcal{H}v^+|0\rangle = \epsilon_V v^+|0\rangle \tag{D.5}$$

$$\mathcal{H}v^+v^+|0\rangle \neq \lambda v^+v^+|0\rangle \tag{D.6}$$

Thus, for an eigen - state vector, say $|S\rangle = s^+|0\rangle$, we have

$$|S\rangle = s^+|0\rangle \equiv \sigma^+|V\rangle \tag{D.7}$$

This defines implicitly the physical state vector σ^+. Note that an explicit definition is not easy since only $\langle 0|[v,v^+]_-|0\rangle = 1$ while $[v,v^+]_-$ is not a c-number. At any rate, the physical state vector σ^+ shows which parton configurations in excess of the physical vacuum make up the physical particle. (Note, that in non-relativistic physics the definition of an operator anologous to (D.7) is possible, and, in fact, particularly in nuclear physics might be an interesting quantity to investigate.)

In the model of Chapter VI, $|V\rangle$ will have the configurations: $|0\rangle$, $|\pi^2\rangle$, $|\pi^4\rangle$, $|\pi^6\rangle$, $|\pi\omega\rangle$, $|\pi^2\rho\rangle$. At higher truncation energy more configurations will participate. If a neutral scalar meson field (σ-model) is included, $|V\rangle$ would contain likewise terms of the form $|\sigma\rangle$, $|\sigma^2\rangle$, $|\sigma^3\rangle$, etc. In writing out the configuration mixture of $|V\rangle$ one must supply a particular creation operator, v_o^+, to generate the amplitude of the ground configuration $|0\rangle$ in $|V\rangle$, in order to achieve (D.3) and (D.4). Thus one must write

$$v^+ = Av_o^+ + Ba^+a^+ + Ca^+a^+a^+a^+ + \ldots \tag{D.8}$$

where v_o^+ and v_o obey the usual Boson commutation relations

$$[v_o, v_o^+] = 1 \tag{D.9}$$

Equation (D.8) above has been written symbolically, omitting the integrations over momenta and summations over spin and isospin, etc.

In time-dependent perturbation theory the structure of the physical vacuum can be eliminated from explicit treatment by omitting the disconnected graphs and by ignoring the Pauli principle within the diagrams, and by dividing the S-matrix elements by the vacuum expectation value of the S-matrix. (See, e.g., Ref. 5, Chapter 7.2.)

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16. ABSTRACT *(A 200-word or less factual summary of most significant information. If document includes a significant bibliography or literature survey, mention it here.)*

The principles and the mathematical details of a fully relativistic nuclear theory are given. Since the concept of nuclear forces is a strictly non-relativistic construct, it must be abandoned and the forces must be replaced explicitly by their physical origin, i.e., by the interaction between nucleons and mesons. Thus, in this monograph the description of a nucleus has been formulated as a problem of relativistic quantum field theory which is solved by nuclear physics methods. To wit: The physics is described by specifying a Lagrangian which is a functional of the constituent fields (= of the parton fields). The solutions for the physical systems then are obtained in a time-independent treatment as expansions in the parton fields: both particles and nuclei are composite systems, made up of parton configurations, which define a representation of the Hamiltonian (associated with the specified Lagrangian). The Hamiltonian is truncated by omitting all configurations having a diagonal element exceeding that of the lowest configuration by a pre-determined value, E_{max}, and is diagonalized. All formulae needed to carry out this program are derived and given in full detail for spin 0, 1/2, and 1 parton fields for PS, PV, and φ^4 interactions. Particular attention is devoted to the center-of-mass position coordinate which in relativistic kinematics is a non-separable many-body operator. Finally, the configurations up to E_{max} = 1 GeV are listed for the nucleon, the deuteron, and the pion.

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