NISTIR 89-4100

On the Expected Complexity of the 3-Dimensional Voronoi Diagram

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Abstract. Let S be a set of n sites chosen independently from a uniform distribution in a cube in 3-dimensional Euclidean space. In this paper, work by Bentley, Weide and Yao is extended to show that the Voronoi diagram for S has an expected $O(n)$ number of faces. A consequence of the proof of this result is that the Voronoi diagram for S can be constructed in expected $O(n)$ time. Finally, it is shown that with the exception of at most an expected $O(n^{2/3})$ number of polyhedra, each polyhedron in the Voronoi diagram for S has an expected constant number of faces.

Key words. algorithm, computational geometry, expected complexity, expected time analysis, face, facet, polyhedron, vertex, Voronoi diagram

AMS(MOS) subject classifications. 68U05

1. Introduction

Consider a set $S = \{p_1, \ldots, p_n\}$ of n sites in d-dimensional Euclidean space E^d. The *Voronoi diagram for S* is a sequence $V(p_1), \ldots, V(p_n)$ of convex polyhedra covering E^d, where for each i, $i = 1, \ldots, n$, $V(p_i)$ is the *Voronoi polyhedron of p_i relative to S*, i. e. the set of all points x in the space such that p_i is as close to x as is any other site in S.

The Voronoi diagram is an important geometrical concept that is used for solving a large number of problems in many areas. Accordingly, several algorithms have been devised and implemented for constructing it in two and higher dimensions ([1], [2], [3], [4], [5], [6], [7], [8], [12], [13], [15], [20], [21], [22], [23], [24], [26]), and many of its statistical and geometrical properties have been derived ([2], [9], [10], [11], [12], [14], [16], [17], [19], [21], [22], [25]).

In this paper, we further develop the work by Bentley, Weide and Yao [2] that relates to the expected complexity of Voronoi diagrams. Given a set S of n sites chosen independently from a uniform distribution in a $d-$dimensional hypercube, Bentley, *et al.* show that with the exception of at most an expected $O(n^{1-1/d} \cdot \log n)$ number of polyhedra, each polyhedron in the Voronoi diagram for S has an expected constant number of faces. With m defined as the largest integer less than or equal to $n^{1/d}$, i. e. the *floor* of $n^{1/d}$, Bentley, *et al.* first divide the hypercube into m^d equal-sized *cells*. Given $c > 0$ and defining $\mathrm{LG}(n)$ as the floor of $c \cdot \log n$, where log denotes the natural logarithm, Bentley, *et al.* then show that for each site p in S the expected number of faces of $V(p)$ is constant if p is not constained in any of the outermost $\mathrm{LG}(n)$ layers of cells of the hypercube. However, Bentley, *et al.* leave unclear how to compute the expected complexity of the Voronoi diagram for S due to the Voronoi polyhedra of the sites in the outermost $\mathrm{LG}(n)$ layers of cells of the hypercube.

In what follows, we extend the work by Bentley, *et al.* to show that in $3-$dimensional Euclidean space, $O(n^{2/3} \cdot (c \cdot \log n)^4)$ is an upper bound for the expected number of faces of the Voronoi diagram for S that are also faces of Voronoi polyhedra of sites in the outermost $\mathrm{LG}(n)$ layers of cells of the cube. This result and those in [2] then imply that the expected number of faces of the Voronoi diagram for the n sites is $O(n)$. Accordingly, we conjecture that in E^d, for fixed $d > 3$, similar results hold for $(d-1)-$dimensional faces or *facets*, i. e. $O(n^{1-1/d} \cdot (c \cdot \log n)^{d+1})$ is an upper bound for the expected number of facets of the Voronoi diagram for S that are also facets of Voronoi polyhedra of sites in the outermost $\mathrm{LG}(n)$ layers of cells of the hypercube, and $O(n)$ is the expected number of facets of the Voronoi diagram for the n sites. Finally, in E^d, for fixed $d \geq 2$, we show that with the exception of at most an expected $O(n^{1-1/d})$ number of sites in S, for each site p in S, the expected number of faces of $V(p)$ is constant.

2. Terminology

Let $S = \{p_1, \ldots, p_n\}$ be a set of n points in E^3 chosen independently from a uniform distribution in a cube R. In what follows, a point in E^3 will be called a *site* if and only if it belongs to S. With m defined as the floor of $n^{1/3}$, assume as in [2] that R has been divided into m^3 equal-sized cells. Given a site q, define *the* 1^{st} *layer of cells that surrounds* q as the collection of cells that contain q, and inductively, given $k \geq 1$, assume that the k^{th} layer of cells that surrounds q has been defined, and define *the* $(k+1)^{th}$ *layer of cells that surrounds* q as the collection, possibly empty, of cells that have one or more points in common with cells in the k^{th} layer, and that do not belong to the first k layers.

Let *lcell* and *vcell* represent, respectively, the length and volume of each cell.

Given numbers c, c', c'', $0 < c \leq c'$, $c'' \geq 1$, define $\mathrm{LG}(n)$ and $\mathrm{LG}'(n)$ as the floors of $c \cdot \log n$ and $c' \cdot \log n$, respectively, and assume n is large enough so that $\mathrm{LG}(n) > 2$ and $2^{3/2} \cdot c'' \cdot \mathrm{LG}'(n) \leq 2^{-1} \cdot n^{1/3}$.

Let $\hat{k}$ represent the largest integer k for which

$$2^{k/2} \cdot c'' \cdot \mathrm{LG}'(n) \leq 2^{-1} \cdot n^{1/3}.$$

It follows from the assumptions on n that $\hat{k} \geq 3$.

Set $\mathrm{LG}_0(n)$ equal to $\mathrm{LG}(n)$, and $\mathrm{LG}_k(n)$ equal to $\mathrm{LG}'(n)$ for each k, $k = 1, \ldots, \hat{k} - 2$.

Let f_i, $i = 1, \ldots, 6$, represent the facets of R, and let Π represent $\cup_{i=1}^{6} f_i$, i. e. the boundary of R.

Given a point x in E^3 and a subset W of E^3, define $\mathrm{dist}(x, W)$ as the minimum value of $||x - w||$ for w in W, where $||\cdot||$ represents the 3−dimensional Euclidean norm.

From the assumptions on n, several nonempty subsets of R can also be defined as follows:

$$\begin{aligned} R_{-1} &\equiv \{x \in R \ : \ \mathrm{dist}(x, \Pi) \geq lcell \cdot \mathrm{LG}(n)\}. \\ R_0 &\equiv \{x \in R \ : \ lcell \cdot 2 \leq \mathrm{dist}(x, \Pi) < lcell \cdot \mathrm{LG}(n)\}. \\ R_{\hat{k}} &\equiv \{x \in R \ : \ \mathrm{dist}(x, \Pi) < lcell \cdot 2^{-\hat{k}+2}\}. \end{aligned}$$

For each k, $k = 1, \ldots, \hat{k} - 1$,

$$R_k \equiv \{x \in R \ : \ lcell \cdot 2^{-k+1} \leq \mathrm{dist}(x, \Pi) < lcell \cdot 2^{-k+2}\}.$$

For each i, k, $i = 1, \ldots, 6$, $k = 0, \ldots, \hat{k} - 2$,

$$R_k^{f_i} \equiv \{x \in R_k \ : \ \mathrm{dist}(x, f_j) \geq lcell \cdot 2^{k/2} \cdot c'' \cdot \mathrm{LG}_k(n), \ j = 1, \ldots, 6, \ j \neq i\}.$$

It follows from these definitions that the sets R_k, $k = -1, \ldots, \hat{k}$, are pair-wise disjoint nested regions of the cube R, and

$$R = \cup_{k=-1}^{\hat{k}} R_k.$$

Finally, define R_{-2}, a possibly empty subset of R, as follows:

$$R_{-2} \equiv \{x \in R \ : \ \mathrm{dist}(x, \Pi) \geq lcell \cdot (1 + c'') \cdot \mathrm{LG}(n)\}.$$

The significance of these regions as it relates to our purposes can be summarized as follows. R_{-1} is essentially that region of the cube R obtained by subtracting the outermost $\mathrm{LG}(n)$ layers of cells of R from R. From [2], the Voronoi polyhedron of a site in R_{-1} is of expected constant complexity. R_0 is essentially that region of R obtained by subtracting from the outermost $\mathrm{LG}(n)$ layers of cells of R the outermost two layers. R_k, $k = 1, \ldots, \hat{k}$, are regions of R whose union is essentially that region of R composed of the outermost two layers of cells of R, and whose thicknesses correspond to the terms of the geometric series expanded to the first $\hat{k} - 1$ terms together with the remainder. $R_k^{f_i}$, $i = 1, \ldots, 6$, $k = 0, \ldots, \hat{k} - 2$, are subsets of R_k, $k = 0, \ldots, \hat{k} - 2$, respectively, defined in such a way that due to the geometric series aspect of R_k, $k = 1, \ldots, \hat{k}$, and the position of R_0 in R, the expected complexity of the Voronoi diagram for S due to the Voronoi polyhedra of sites in these regions is linear while the expected number of sites in $\cup_{k=0}^{\hat{k}} R_k \setminus \cup_{i=1}^{6} \cup_{k=0}^{\hat{k}-2} R_k^{f_i}$ is small enough that it does not affect the linearity of the overall expected complexity of the diagram even under the worst possible circumstances (see Section 3). Finally, R_{-2} is a subset of R_{-1} defined in such a way that sites in this region are highly unlikely to have Voronoi neighbors in the outermost $\mathrm{LG}(n)$ layers of cells of R while $R_{-1} \setminus R_{-2}$ is a region of R essentially composed of $O(\mathrm{LG}(n))$ contiguous layers of cells of R.

For each facet f of R, let $H(f)$ represent the plane that contains f, and for each site q, let $T^f(q)$ represent the point in f that is the perpendicular projection of q onto f.

Given i, k, $1 \leq i \leq 6$, $0 \leq k \leq \hat{k} - 2$, and a site q in $R_k^{f_i}$, let v, v' and v'' be vertices of R in f_i for which $v' - v$ is perpendicular to $v'' - v$, and for each j, $j = 0, \ldots, 8$, define a point t_j in $H(f_i)$ by

$$t_j \equiv T^{f_i}(q) + (v' - v) \cdot \cos(j\pi/4) + (v'' - v) \cdot \sin(j\pi/4).$$

In addition, for each j, $j = 1, \ldots, 8$, let O_j be the octant in $H(f_i)$ that is the convex hull of the rays $T^{f_i}(q)\vec{t}_{j-1}$ and $T^{f_i}(q)\vec{t}_j$, and say that O_j, $j = 1, \ldots, 8$, are the *octants associated with* q. Finally, if within the first $2^{k/2} \cdot \mathrm{LG}_k(n)$ layers of cells that surround q, for each j, $j = 1, \ldots, 8$, there exists a site q_j such that $\mathrm{dist}(q_j, f_i) < lcell \cdot 2^{-k}$ and the ray $q\vec{q}_j$ intersects O_j, say that q is *octant-closed* and that q_j, $j = 1, \ldots, 8$, *render* q *octant-closed.*

Given i, k, q, v, v', v'' as above, let v''' be a vertex of R for which $v''' - v$ is perpendicular to $v' - v$ and $v'' - v$, and for each j, $j = 0, \ldots, 8$, and each m, $m = 0, \ldots, 3$, define a point

r_{jm} by

$$\begin{aligned} r_{jm} \;\; &\equiv \;\; q + ((v' - v) \cdot \cos(j\pi/4) + (v'' - v) \cdot \sin(j\pi/4)) \cdot \sin(m\pi/4) \\ &\quad + (v''' - v) \cdot \cos(m\pi/4). \end{aligned}$$

In addition, for each j, $j = 1, \ldots, 8$, and each m, $m = 1, 2, 3$, let U_{jm} be the cone that is the convex hull of the rays $q\vec{r}_{j-1,m-1}$, $q\vec{r}_{j,m-1}$, $q\vec{r}_{j-1,m}$, and $q\vec{r}_{jm}$, and say that U_{jm}, $j = 1, \ldots, 8$, $m = 1, 2, 3$, are the *cones associated with* q. Finally, if within the first $2^{k/2} \cdot \mathrm{LG}_k(n)$ layers of cells that surround q, for each j, $j = 1, \ldots, 8$, and each m, $m = 1, 2, 3$, there exists a site s_{jm}, $s_{jm} \neq q$, such that s_{jm} belongs to U_{jm}, say that q is *cone-closed* and that s_{jm}, $j = 1, \ldots, 8$, $m = 1, 2, 3$, *render* q *cone-closed.*

Given q as above, say that q is *closed* if it is octant-closed and cone-closed. As it will be shown in Section 3, Voronoi polyhedra of closed sites are of complexity acceptable for our purposes.

Given i, k, q as above, define $C^{f_i}(q)$ and $C(q)$ as the closed half-spaces that contain $T^{f_i}(q)$ and q, respectively, and that are determined by the plane parallel to $H(f_i)$ that contains $(T^{f_i}(q) + q)/2$. Define $S^{f_i}(q)$ as the subset of S for which a site $p \in S^{f_i}(q)$ if and only if $V(p) \cap V(q) \cap C^{f_i}(q) \neq \emptyset$, and $S(q)$ as the subset of S for which a site $p \in S(q)$ if and only if $V(p) \cap V(q) \cap C(q) \neq \emptyset$.

Finally, given sites p and q, say that p is a *Voronoi neighbor relative to* S *of* q if $V(p)$ and $V(q)$ have a facet in common.

3. Results

In this section, based on the terminology developed in Section 2, we present two theorems, the first of which, Theorem 1, is the main result of this paper.

Theorem 1. $O(n^{2/3} \cdot (c \cdot \log n)^4)$ is an upper bound for the expected number of faces of the Voronoi diagram for S that are also faces of Voronoi polyhedra of sites in $R \setminus R_{-1}$.

The proof of this theorem consists of partitioning the cube into the regions defined in Section 2 and then computing where necessary the expected number of Voronoi neighbor pairs within and between these regions. It requires some preliminary results which we present in the form of propositions. In the first two propositions it is essentially shown that Voronoi polyhedra of closed sites are of complexity acceptable for our purposes.

Proposition 1. Given i, k, $1 \leq i \leq 6$, $0 \leq k \leq \hat{k} - 2$, a site q in $R_k^{f_i}$, and octants and sites O_j, q_j, $j = 1, \ldots, 8$, such that O_j, $j = 1, \ldots, 8$, are the octants associated with q, and q_j, $j = 1, \ldots, 8$, render q octant-closed, if q' is a site such that for each j, $j = 1, \ldots, 8$, $||q' - q|| > \sqrt{2}\,||q_j' - q||$, where q_j' is the intersection of $\vec{qq_j}$ and O_j, then $q' \notin S^{f_i}(q)$.

Proof. Let q' be one such site, and define J' as the plane that perpendicularly bisects the line segment $[q', q]$, and C' as the open half-space determined by J' that contains q. We show that C' contains $V(q) \cap C^{f_i}(q)$, so that $q' \notin S^{f_i}(q)$.
Assume, without any loss of generality, that q' is in f_i, $T^{f_i}(q) \neq q_j'$, for each j, $j = 1, 2$, $T^{f_i}(q)\vec{q_1}\,' \neq T^{f_i}(q)\vec{q_2}\,'$, and q' is in the convex hull of $T^{f_i}(q)\vec{q_1}\,'$ and $T^{f_i}(q)\vec{q_2}\,'$.
Let J_1' and J_2' be the planes that are the perpendicular bisectors of the line segments $[q_1', q]$ and $[q_2', q]$, respectively. Let B be the region that is the intersection of $C^{f_i}(q)$ and the closed half-spaces determined by J_1' and J_2' that contain q. We show B is the convex hull of a region K' and a ray $\vec{u}\,'$, both of which lie in C'. Since C' is convex, and B contains $V(q) \cap C^{f_i}(q)$, the result then follows.
To this end, let H' be the plane that contains $(T^{f_i}(q) + q)/2$ and is parallel to $H(f_i)$; let H'' be the plane that contains q and is parallel to $H(f_i)$; let q'', q_1'', q_2'' be the perpendicular projections onto H'' of q', q_1', q_2', respectively; let h', h_1', h_2' be the lines that are the intersections of H' with J', J_1', J_2', respectively; and let h'', h_1'', h_2'' be the lines in H'' that perpendicularly bisect $[q'', q]$, $[q_1'', q]$, $[q_2'', q]$, respectively.
Let $\hat{q}$ be the perpendicular projection of q onto H'. Define K' as the intersection of the half-planes in H' determined by h_1' and h_2' that contain $\hat{q}$, and K'' as the intersection of the half-planes in H'' determined by h_1'' and h_2'' that contain q.
In order to show that C' contains K', we first prove that $||q'' - q|| > \sqrt{2}\,||q_j'' - q||$ for each j, $j = 1, 2$. To this end, for each j, $j = 1, 2$, we have

$$\begin{aligned} ||q'' - q||^2 + ||q' - q''||^2 &= ||q' - q||^2 \\ &> 2\,||q_j' - q||^2 \\ &= 2\,(||q_j'' - q||^2 + ||q_j' - q_j''||^2) \\ &= 2\,||q_j'' - q||^2 + 2\,||q_j' - q_j''||^2. \end{aligned}$$

But $||q' - q''||$ equals $||q'_j - q''_j||$ for each j, $j = 1, 2$, so that

$$||q'' - q||^2 > 2\,||q''_j - q||^2 + ||q'_j - q''_j||^2,$$

for each j, $j = 1, 2$, and the inequalities follow.

Since q'_1 and q'_2 belong to the contiguous octants O_1 and O_2, respectively, it follows that h'' does not intersect K''. But by similar triangles, h'', h''_1, h''_2 are the perpendicular projections onto H'' of h', h'_1, h'_2, respectively. Thus, K'' is the perpendicular projection of K' onto H'', and therefore, h' can not intersect K', which shows C' contains K'.

In order to obtain $\vec{u}\,'$, let H^* be the plane that contains q, q'_1, and q'_2; let C^* be the closed half-space determined by H^* that contains $T^{f_i}(q)$; let w' be the line that is the intersection of the planes J'_1 and J'_2; let J^* be the plane that contains q and q', and that is perpendicular to H^*; and let w'' be the perpendicular projection onto J^* of w'.

Since w' is perpendicular to H^*, so is w'', and since from the definition of q', q' is not in C^*, we must have that w'' contains a ray $\vec{u}\,''$ that lies completely in $C' \cap C^* \cap C^{f_i}(q)$. Therefore, from the definition of w'', it follows that w' must contain a ray $\vec{u}\,'$ that is also contained in $C' \cap C^* \cap C^{f_i}(q)$.

Since B is clearly the convex hull of K' and $\vec{u}\,'$, the proof is now complete.

Proposition 2. Given i, k, $1 \leq i \leq 6$, $0 \leq k \leq \hat{k} - 2$, and a site q in $R_k^{f_i}$, if q is closed then for some constant $M > 0$ independent of q, k and n, the smallest number of contiguous layers of cells that surround q and contain each Voronoi neighbor of q is bounded above by $M \cdot 2^{k/2} \cdot \mathrm{LG}_k(n)$.

Proof. Let O_j, $j = 1, \ldots, 8$, be the octants associated with q, let q_j, $j = 1, \ldots, 8$, be sites that render q octant-closed, and let s_{jm}, $j = 1, \ldots, 8$, $m = 1, 2, 3$, be sites that render q cone-closed.

Using arguments similar to those developed in [2], it can be shown that the existence of the sites s_{jm}, $j = 1, \ldots, 8$, $m = 1, 2, 3$, implies that for some constant $M_1 > 0$ independent of q, k and n, the smallest number of contiguous layers of cells that surround q and contain $S(q)$ is bounded above by $M_1 \cdot 2^{k/2} \cdot \mathrm{LG}_k(n)$.

We show a similar result for $S^{f_i}(q)$.

For each j, $j = 1, \ldots, 8$, $\mathrm{dist}(q_j, f_i) < lcell \cdot 2^{-k}$. Thus, by similar triangles, since q is contained in R_k so that $\mathrm{dist}(q, f_i) \geq lcell \cdot 2^{-k+1}$, we must have that for each j, $j = 1, \ldots, 8$, $||q'_j - q|| \leq 2\,||q_j - q||$, where q'_j is the intersection of $q\vec{q}_j$ and O_j.

Thus, if q' is a site such that for each j, $j = 1, \ldots, 8$, $||q' - q|| > 2\sqrt{2}\,||q_j - q||$ then for each

j, $j = 1, \ldots, 8$, $||q' - q|| > \sqrt{2}\, ||q'_j - q||$, and by Proposition 1, $q' \notin S^{f_i}(q)$.
Therefore, since for each j, $j = 1, \ldots, 8$, q_j is also contained in the first $2^{k/2} \cdot \mathrm{LG}_k(n)$ layers of cells that surround q, it follows that for some constant $M_2 > 0$ independent of q, k and n, the smallest number of contiguous layers of cells that surround q and contain $S^{f_i}(q)$ is bounded above by $M_2 \cdot 2^{k/2} \cdot \mathrm{LG}_k(n)$.
The proof of the proposition is now complete since the union of $S(q)$ and $S^{f_i}(q)$ contains each Voronoi neighbor of q.

In the next proposition it is shown that the probability that a site is not closed is very small and uniform for all sites to which the definition of a closed site applies.

Proposition 3. Given i, k, $1 \leq i \leq 6$, $0 \leq k \leq \hat{k} - 2$, and a site q in $R_k^{f_i}$, there exist positive constants M_1 and M_2 independent of q, k and n, such that the probability that q is not closed is bounded above by $M_1 \cdot \exp(-M_2 \cdot (\mathrm{LG}_k(n))^2)$, where exp is the exponential function.

Proof. Let O_j, $j = 1, \ldots, 8$, be the octants associated with q, and let U_{jm}, $j = 1, \ldots, 8$, $m = 1, 2, 3$, be the cones associated with q.
For each j, $j = 1, \ldots, 8$, define O'_j as the subset of R for which a point $p \in O'_j$ if and only if p is within the first $2^{k/2} \cdot \mathrm{LG}_k(n)$ layers of cells that surround q, $\mathrm{dist}(p, f_i) < lcell \cdot 2^{-k}$, and $\vec{qp}$ intersects O_j. In addition, for each j, $j = 1, \ldots, 8$, and each m, $m = 1, 2, 3$, define U'_{jm} as the subset of R for which a point $p \in U'_{jm}$ if and only if p is within the first $2^{k/2} \cdot \mathrm{LG}_k(n)$ layers of cells that surround q, and p is in U_{jm}.
From the definition of $R_k^{f_i}$ and since $c'' \geq 1$, the volume of $\cup_{j=1}^{8} O'_j$ is then approximately equal to

$$\begin{aligned}(2 \cdot 2^{k/2} \cdot \mathrm{LG}_k(n))^2 \cdot (2^{-k}) \cdot vcell &= (4 \cdot 2^k \cdot (\mathrm{LG}_k(n))^2) \cdot (2^{-k}) \cdot vcell \\ &= 4 \cdot (\mathrm{LG}_k(n))^2 \cdot vcell,\end{aligned}$$

so that for each j, $j = 1, \ldots, 8$, the volume of O'_j is approximately equal to

$$(1/8) \cdot 4 \cdot (\mathrm{LG}_k(n))^2 \cdot vcell = (1/2) \cdot (\mathrm{LG}_k(n))^2 \cdot vcell.$$

Thus, a positive constant M_2 exists, independent of q, k and n, such that for each j, $j = 1, \ldots, 8$, $M_2 \cdot (\mathrm{LG}_k(n))^2 \cdot vcell$ is a lower bound for the volume of O'_j.
Therefore, since for each j, $j = 1, \ldots, 8$, each m, $m = 1, 2, 3$, and each h, $h = 1, \ldots, 8$, the

volume of U'_{jm} is larger than the volume of O'_h, it follows, using arguments developed in [2], that

$$(8 + 8 \cdot 3) \cdot \exp(-M_2 \cdot (\mathrm{LG}_k(n))^2) = 32 \cdot \exp(-M_2 \cdot (\mathrm{LG}_k(n))^2)$$

is an upper bound for the probability that at least one of the sets O'_j, $j = 1, \ldots, 8$, U'_{jm}, $j = 1, \ldots, 8$, $m = 1, 2, 3$, does not contain a site.
Thus, by setting M_1 equal to 32, the proof of the proposition is then complete.

In the next four propositions it is shown that due to the geometric series aspect of R_k, $k = 1, \ldots, \hat{k}$, the position of R_0 in R, and the definitions of $R_k^{f_i}$, $i = 1, \ldots, 6$, $k = 0, \ldots, \hat{k} - 2$, Voronoi polyhedra of sites in $R_k^{f_i}$, $i = 1, \ldots, 6$, $k = 0, \ldots, \hat{k} - 2$, are of acceptable expected complexity while the expected number of sites in $\cup_{k=0}^{\hat{k}} R_k \setminus \cup_{i=1}^{6} \cup_{k=0}^{\hat{k}-2} R_k^{f_i}$ is small enough to be also acceptable for our purposes.

Proposition 4. Given i, $1 \leq i \leq 6$, and a site q in $R_0^{f_i}$, for constants M', M_1, $M_2 > 0$ independent of q and n, the expected number of Voronoi neighbors in $R \setminus R_{-2}$ of q is bounded above by

$$M' \cdot (\mathrm{LG}(n))^3 + M_1 \cdot \exp(-M_2 \cdot (\mathrm{LG}(n))^2) \cdot O(n^{2/3} \cdot \mathrm{LG}(n)).$$

Proof. Let P_1 be the probability that q is closed, and let T_1 be the expected number of Voronoi neighbors in $R \setminus R_{-2}$ of q when q is closed. Define P_2 and T_2 similarly by replacing 'closed' with 'not closed.'
Let T be the expected number of Voronoi neighbors in $R \setminus R_{-2}$ of q, so that

$$T = P_1 \cdot T_1 + P_2 \cdot T_2.$$

We note, from Proposition 2, that if q is closed then for some constant $M > 0$ independent of q and n, the smallest number of contiguous layers of cells that surround q and contain each Voronoi neighbor of q is bounded above by $M \cdot \mathrm{LG}(n)$. Therefore, since one is the expected number of sites per cell, we must have that $T_1 \leq (2 \cdot M \cdot \mathrm{LG}(n))^3$.
In addition, from Proposition 3, we note that there exist positive constants M_1 and M_2 independent of q and n, such that $P_2 \leq M_1 \cdot \exp(-M_2 \cdot (\mathrm{LG}(n))^2)$. Thus, since $P_1 \leq 1$ and $T_2 \leq O(n^{2/3} \cdot \mathrm{LG}(n))$, it follows that

$$\begin{aligned} T &< 1 \cdot 8 \cdot M^3 \cdot (\mathrm{LG}(n))^3 + M_1 \cdot \exp(-M_2 \cdot (\mathrm{LG}(n))^2) \cdot O(n^{2/3} \cdot \mathrm{LG}(n)) \\ &= M' \cdot (\mathrm{LG}(n))^3 + M_1 \cdot \exp(-M_2 \cdot (\mathrm{LG}(n))^2) \cdot O(n^{2/3} \cdot \mathrm{LG}(n)) \end{aligned}$$

for some positive constant M' independent of q and n, which completes the proof of the proposition.

Proposition 5. Given i, k, $1 \leq i \leq 6$, $0 < k \leq \hat{k} - 2$, and a site q in $R_k^{f_i}$, for constants M, M'', M_1, $M_2 > 0$ independent of q, k and n, if $c'' \geq (2 + \sqrt{2}) \cdot M$ then the expected number of Voronoi neighbors in $\cup_{l=0}^{\hat{k}} R_l \setminus \cup_{l=0}^{k-1} R_l^{f_i}$ of q is bounded above by

$$M'' \cdot (\mathrm{LG}'(n))^2 + M_1 \cdot \exp(-M_2 \cdot (\mathrm{LG}'(n))^2) \cdot O(n^{2/3} \cdot \mathrm{LG}(n)).$$

Proof. Let P_1 be the probability that q is closed, and let T_1 be the expected number of Voronoi neighbors in $\cup_{l=0}^{\hat{k}} R_l \setminus \cup_{l=0}^{k-1} R_l^{f_i}$ of q when q is closed. Define P_2 and T_2 similarly by replacing 'closed' with 'not closed.'
Let T be the expected number of Voronoi neighbors in $\cup_{l=0}^{\hat{k}} R_l \setminus \cup_{l=0}^{k-1} R_l^{f_i}$ of q, so that

$$T = P_1 \cdot T_1 + P_2 \cdot T_2.$$

We note, from Proposition 2, that if q is closed then for some constant $M > 0$ independent of q, k and n, the smallest number of contiguous layers of cells that surround q and contain each Voronoi neighbor of q is bounded above by $M \cdot 2^{k/2} \cdot \mathrm{LG}'(n)$. Thus, if $c'' \geq (2 + \sqrt{2}) \cdot M$ and since $c' \geq c$ then

$$c'' \cdot 2^{k/2} \cdot \mathrm{LG}'(n) - M \cdot 2^{k/2} \cdot \mathrm{LG}'(n) \geq c'' \cdot 2^{(k-1)/2} \cdot \mathrm{LG}_{k-1}(n),$$

so that the intersection of these layers with $\cup_{l=0}^{\hat{k}} R_l \setminus \cup_{l=0}^{k-1} R_l^{f_i}$ is contained in $\cup_{l=k}^{\hat{k}} R_l$ and has a volume bounded above by

$$\begin{aligned}
& (2 \cdot M \cdot 2^{k/2} \cdot \mathrm{LG}'(n))^2 \cdot ((\textstyle\sum_{l=k}^{\hat{k}-1} 2^{-l+1}) + 2^{-\hat{k}+2}) \cdot vcell \\
= \; & (4 \cdot M^2 \cdot 2^k \cdot (\mathrm{LG}'(n))^2) \cdot (2^{-k+2}) \cdot vcell \\
= \; & 16 \cdot M^2 \cdot (\mathrm{LG}'(n))^2 \cdot vcell.
\end{aligned}$$

Therefore, $T_1 \leq 16 \cdot M^2 \cdot (\mathrm{LG}'(n))^2$.
That T is as desired, now follows by using arguments similar to those presented in the proof of Proposition 4.

Proposition 6. The expected number of sites in $\cup_{l=\hat{k}-1}^{\hat{k}} R_l$ is bounded above by

$$384 \cdot (c'' \cdot \mathrm{LG}'(n))^2.$$

Proof. Since $\hat{k}$ is the largest integer k' for which

$$2^{k'/2} \cdot c'' \cdot \mathrm{LG}'(n) \leq 2^{-1} \cdot n^{1/3},$$

we must have that the volume of $\cup_{l=\hat{k}-1}^{\hat{k}} R_l$ is bounded above by

$$\begin{aligned} & 6 \cdot (2 \cdot 2^{(\hat{k}+1)/2} \cdot c'' \cdot \mathrm{LG}'(n))^2 \cdot (2^{-\hat{k}+2} + 2^{-\hat{k}+2}) \cdot vcell \\ = \; & 6 \cdot (4 \cdot 2^{\hat{k}+1} \cdot (c'' \cdot \mathrm{LG}'(n))^2) \cdot (2 \cdot 2^{-\hat{k}+2}) \cdot vcell \\ = \; & 384 \cdot (c'' \cdot \mathrm{LG}'(n))^2 \cdot vcell, \end{aligned}$$

which completes the proof of the proposition.

Proposition 7. The expected number of sites in $R_0 \setminus \cup_{i=1}^{6} R_0^{f_i}$ is bounded above by

$$12 \cdot n^{1/3} \cdot (c'' \cdot \mathrm{LG}(n))^2,$$

and in $\cup_{l=1}^{\hat{k}-2} R_l \setminus \cup_{i=1}^{6}(\cup_{l=1}^{\hat{k}-2} R_l^{f_i})$ by

$$(1 + \sqrt{2}) \cdot 48 \cdot n^{1/3} \cdot c'' \cdot \mathrm{LG}'(n).$$

Proof. From the definitions, the volume of $R_0 \setminus \cup_{i=1}^{6} R_0^{f_i}$ is bounded above by

$$12 \cdot n^{1/3} \cdot (c'' \cdot \mathrm{LG}(n))^2 \cdot vcell,$$

and that of $\cup_{l=1}^{\hat{k}-2} R_l \setminus \cup_{i=1}^{6}(\cup_{l=1}^{\hat{k}-2} R_l^{f_i})$ by

$$\begin{aligned} & \textstyle\sum_{l=1}^{\hat{k}-2} 2 \cdot 12 \cdot n^{1/3} \cdot 2^{l/2} \cdot c'' \cdot \mathrm{LG}'(n) \cdot 2^{-l+1} \cdot vcell \\ = \; & \textstyle\sum_{l=1}^{\hat{k}-2} 48 \cdot n^{1/3} \cdot 2^{-l/2} \cdot c'' \cdot \mathrm{LG}'(n) \cdot vcell \\ \leq \; & (1 + \sqrt{2}) \cdot 48 \cdot n^{1/3} \cdot c'' \cdot \mathrm{LG}'(n) \cdot vcell. \end{aligned}$$

The proposition now follows.

Proof of Theorem 1. It suffices to prove the theorem for the 2−dimensional faces or facets, since applications of the Euler formula to each of the Voronoi polyhedra of the sites in $R \setminus R_{-1}$ produces the desired result for the 0− and 1−dimensional faces. As mentioned above, the proof consists of computing where necessary the expected number of Voronoi neighbor pairs within and between the regions $R_k^{f_i}$, $i = 1, \ldots, 6$, $k = 0, \ldots, \hat{k} - 2$, $\cup_{k=0}^{\hat{k}} R_k \setminus \cup_{i=1}^{6} \cup_{k=0}^{\hat{k}-2} R_k^{f_i}$,

$R_{-1} \setminus R_{-2}$ and R_{-2}.
To this end, let p be a site in R_{-1}.
From [2], since for each site q in $\cup_{k=1}^{\hat{k}} R_k$, $\mathrm{dist}(q, R_{-1}) \geq lcell \cdot (\mathrm{LG}(n) - 2)$, we must have that constants M_1' and $M_2' > 0$ exist independent of n and p, such that the probability that p has Voronoi neighbors in $\cup_{k=1}^{\hat{k}} R_k$ is bounded above by

$$M_1' \cdot \exp(-M_2' \cdot (\mathrm{LG}(n))^3).$$

Therefore, the expected number of facets of the Voronoi diagram for S that are shared by Voronoi polyhedra of sites in R_{-1} with Voronoi polyhedra of sites in $\cup_{k=1}^{\hat{k}} R_k$ is bounded above by

$$O(n^{2/3}) \cdot n \cdot M_1' \cdot \exp(-M_2' \cdot (\mathrm{LG}(n))^3).$$

Similarly, positive constants M_1'' and M_2'' exist independent of n, such that the number of facets of the Voronoi diagram for S that are shared by Voronoi polyhedra of sites in R_{-2} with Voronoi polyhedra of sites in R_0 is bounded above by

$$O(n^{2/3} \cdot \mathrm{LG}(n)) \cdot n \cdot M_1'' \cdot \exp(-M_2'' \cdot (\mathrm{LG}(n))^3).$$

For each i, $i = 1, \ldots, 6$, define $R_{-1}^{f_i}$, a possibly empty subset of $R_{-1} \setminus R_{-2}$, as follows:

$$R_{-1}^{f_i} \equiv \{x \in R_{-1} \setminus R_{-2} \ : \ \mathrm{dist}(x, f_j) \geq lcell \cdot (1 + c'') \cdot \mathrm{LG}(n), \ j = 1, \ldots, 6, \ j \neq i\}.$$

Given i, $1 \leq i \leq 6$, let p be a site in $R_{-1}^{f_i}$.
Again, from [2], since for each site q in $R_0 \setminus \cup_{j=1}^{6} R_0^{f_j}$, $\mathrm{dist}(q, R_{-1}^{f_i}) \geq lcell \cdot \mathrm{LG}(n)$, we must have that constants M_1''' and $M_2''' > 0$ exist independent of n and p, such that the probability that p has Voronoi neighbors in $R_0 \setminus \cup_{j=1}^{6} R_0^{f_j}$ is bounded above by

$$M_1''' \cdot \exp(-M_2''' \cdot (\mathrm{LG}(n))^3).$$

In addition, as in the proof of Proposition 7, it can be shown that the expected number of sites in $(R_{-1} \setminus R_{-2}) \setminus \cup_{j=1}^{6} R_{-1}^{f_j}$ is bounded above by $12 \cdot n^{1/3} \cdot (c'' \cdot \mathrm{LG}(n))^2$. Therefore, from proposition 7, the expected number of facets of the Voronoi diagram for S that are shared by Voronoi polyhedra of sites in $R_{-1} \setminus R_{-2}$ with Voronoi polyhedra of sites in $R_0 \setminus \cup_{j=1}^{6} R_0^{f_j}$ is bounded above by

$$12 \cdot n^{1/3} \cdot (c'' \cdot \mathrm{LG}(n))^2 \cdot O(n^{2/3} \cdot \mathrm{LG}(n)) \cdot M_1''' \cdot \exp(-M_2''' \cdot (\mathrm{LG}(n))^3) + (12 \cdot n^{1/3} \cdot (c'' \cdot \mathrm{LG}(n))^2)^2.$$

Thus, from Propositions 4, 5, 6 and 7, constants M', M'', M_1, M_2, $c'' > 0$ exist such that the expected number of facets of the Voronoi diagram for S that are also facets of Voronoi polyhedra of sites in $R \setminus R_{-1}$ is bounded above by

$$\begin{aligned}
& O(n^{2/3} \cdot \mathrm{LG}(n)) \cdot (M' \cdot (\mathrm{LG}(n))^3 + M_1 \cdot \exp(-M_2 \cdot (\mathrm{LG}(n))^2) \cdot O(n^{2/3} \cdot \mathrm{LG}(n))) + \\
& O(n^{2/3}) \cdot (M'' \cdot (\mathrm{LG}'(n))^2 + M_1 \cdot \exp(-M_2 \cdot (\mathrm{LG}'(n))^2) \cdot O(n^{2/3} \cdot \mathrm{LG}(n))) + \\
& (384 \cdot (c'' \cdot \mathrm{LG}'(n))^2 + 12 \cdot n^{1/3} \cdot (c'' \cdot \mathrm{LG}(n))^2 + (1+\sqrt{2}) \cdot 48 \cdot n^{1/3} \cdot c'' \cdot \mathrm{LG}'(n))^2 + \\
& O(n^{2/3}) \cdot n \cdot M_1' \cdot \exp(-M_2' \cdot (\mathrm{LG}(n))^3) + \\
& O(n^{2/3}) \cdot \mathrm{LG}(n) \cdot n \cdot M_1'' \cdot \exp(-M_2'' \cdot (\mathrm{LG}(n))^3) + \\
& 12 \cdot n^{1/3} \cdot (c'' \cdot \mathrm{LG}(n))^2 \cdot O(n^{2/3} \cdot \mathrm{LG}(n)) \cdot M_1''' \cdot \exp(-M_2''' \cdot (\mathrm{LG}(n))^3) + \\
& (12 \cdot n^{1/3} \cdot (c'' \cdot \mathrm{LG}(n))^2)^2 \\
= \; & M \cdot n^{2/3} \cdot (\mathrm{LG}(n))^4 \\
= \; & M \cdot n^{2/3} \cdot (c \cdot \log n)^4,
\end{aligned}$$

where M is a function of n, c and c' that decreases for fixed c and c', $0 < c \leq c'$. This completes the proof of the theorem.

The following corollary is a direct consequence of results in [2] and Theorem 1.

Corollary 1. $O(n)$ is the expected number of faces of the Voronoi diagram for S.

Proof. From [2] there exist positive constants M_1' and M_2' independent of n such that

$$O(1) + n \cdot M_1' \cdot \exp(-M_2' \cdot (\mathrm{LG}(n))^3)$$

is the expected number of faces of the Voronoi diagram for S that are also faces of the Voronoi polyhedron of any given site in R_{-1}. Thus, from Theorem 1, the expected number of faces of the Voronoi diagram for S is

$$O(n) \cdot (O(1) + n \cdot M_1' \cdot \exp(-M_2' \cdot (c \cdot \log n)^3) + O(n^{2/3} \cdot (c \cdot \log n)^4) = M \cdot n,$$

where M is a function of n, c and c' that decreases for fixed c and c', $0 < c \leq c'$.

The geometrical nature of the proofs of Theorem 1 and Corollary 1, and the fact that $O(n^2)$ is the maximum number of facets that the Voronoi diagram for a set of n sites in E^d,

$d \geq 3$, can have ([9], [16], [17], [19]), suggest the following conjecture. Here, it is assumed that S is a set of n sites in E^d, $d > 3$, chosen independently from a uniform distribution in a $d-$dimensional hypercube R, and that R has been divided into m^d equal-sized cells, where m is the floor of $n^{1/d}$.

Conjecture. For fixed d, $O(n^{1-1/d} \cdot (c \cdot \log n)^{d+1})$ is an upper bound for the expected number of facets or $(d-1)-$dimensional faces of the Voronoi diagram for S that are also facets of Voronoi polyhedra of sites in the outermost $\mathrm{LG}(n)$ layers of cells of R. Consequently, $O(n)$ is the expected number of facets of the Voronoi diagram for S.

The following remark relates to the expected number of faces of the convex hull of S.

Remark. From [2] there exist positive constants M_1' and M_2' independent of n such that the probability that the Voronoi polyhedron of any site in R_{-1} is unbounded is bounded above by

$$M_1' \cdot \exp(-M_2' \cdot (\mathrm{LG}(n))^3).$$

From Proposition 3 and the definition of a closed site there exist positive constants M_1 and M_2 independent of n such that the probability that the Voronoi polyhedron of any site in $\cup_{i=1}^{6} \cup_{k=0}^{k-2} R_k^{f_i}$ is unbounded is bounded above by

$$M_1 \cdot \exp(-M_2 \cdot (\mathrm{LG}(n))^2).$$

Thus, from Propositions 6 and 7, and the Euler formula, the expected number of faces of the convex hull of S is bounded above by

$$\begin{aligned}
& n \cdot M_1' \cdot \exp(-M_2' \cdot (\mathrm{LG}(n))^3) + O(n^{2/3} \cdot \mathrm{LG}(n)) \cdot M_1 \cdot \exp(-M_2 \cdot (\mathrm{LG}(n))^2) + \\
& 384 \cdot (c'' \cdot \mathrm{LG}'(n))^2 + 12 \cdot n^{1/3} \cdot (c'' \cdot \mathrm{LG}(n))^2 + (1 + \sqrt{2}) \cdot 48 \cdot n^{1/3} \cdot c'' \cdot \mathrm{LG}'(n) \\
= \; & O(n^{1/3} \cdot (\log n)^2).
\end{aligned}$$

The next result is also of related interest, although its proof does not depend on any of the results obtained thus far in this paper. Here, it is assumed that S and R are as in the conjecture above and that $d \geq 2$.

Theorem 2. For fixed d, with the exception of at most an expected $O(n^{1-1/d})$ number of sites in S, the Voronoi polyhedron of each site in S can be constructed in expected constant time.

Proof. Let k be the nonnegative integer for which $4k^2 < d \leq 4(k+1)^2$.
Let R' denote the hypercube obtained by surrounding R with $\mathrm{LG}(n) + k + 1$ layers of cells of the type into which R has been divided, and let S' denote the set of sites that are the centroids of cells in the $\mathrm{LG}(n) + k + 1$ new layers.
Given a site q in $S \cup S'$, using arguments developed in [2], it can be shown that the Voronoi polyhedron relative to $S \cup S'$ of q can be constructed in expected constant time if q is not contained in any of the outermost $\mathrm{LG}(n)$ layers of cells of R'.
Let S'_0 denote the set of sites in S' contained in the first $k+1$ layers of cells in R' that surround R, and let S_0 be the set of sites in S that are Voronoi neighbors relative to $S \cup S'$ of sites in S'.
It follows from the geometry of the sets S and S' that the sites in S'_0 are the only sites in S' that can be Voronoi neighbors relative to $S \cup S'$ of sites in S. Thus, since each of the first $k+1$ layers of cells in R' that surround R contains an expected $O(n^{1-1/d})$ number of sites, the expected number of sites in S_0 must be

$$(k+1) \cdot O(n^{1-1/d}) \cdot O(1) = O(n^{1-1/d}).$$

Finally, since for each q in $S \setminus S_0$, the Voronoi polyhedron relative to $S \cup S'$ of q equals the Voronoi polyhedron relative to S of q, the proof of the theorem is complete.

We note that in the proof of Theorem 2, in order to use the arguments developed in [2], the $d-$dimensional hypercube R is surrounded by $\mathrm{LG}(n) + k + 1$ additional layers of cells, where k is the nonnegative integer for which $4k^2 < d \leq 4(k+1)^2$. However, as hinted by the proof, $k+1$ layers would have sufficed.

The final result of this paper is a corollary to Theorem 2 that shows the existence of relatively simple 2$-$ and 3$-$dimensional algorithms of good expected complexity for constructing Voronoi diagrams. It simplifies results in [3].

Corollary 2. For $d = 2$ and $d = 3$, the Voronoi diagram for S can be constructed in expected $O(n)$ and $O(n^{4/3})$ time, respectively.

Proof. Let S_0 and S_0' be as defined in the proof of Theorem 2, and let S_1 be the set of sites in $S \setminus S_0$ that are Voronoi neighbors relative to $S \cup S_0'$ (equivalently, relative to S) of at least one site in S_0.
According to Theorem 2 and its proof, the expected number of sites in S_0 is $O(n^{1-1/d})$, and the expected number of Voronoi neighbors relative to $S \cup S_0'$ of a site in S_0 is $O(1)$. Thus, the expected number of sites in $S_0 \cup S_1$ is

$$O(n^{1-1/d}) + O(n^{1-1/d}) \cdot O(1) = O(n^{1-1/d}).$$

For $d = 2$, the Voronoi polygons of sites in S_0 can be obtained by applying to $S_0 \cup S_1$ an $O(k \cdot \log k)$ algorithm, e. g. Shamos' [21]. Thus, for $d = 2$, the Voronoi diagram for S can be constructed in

$$O(n) \cdot O(1) + O(n^{1/2} \cdot \log(n^{1/2})) = O(n)$$

expected time.
For $d = 3$, the Voronoi polyhedra of sites in S_0 can be obtained by applying to $S_0 \cup S_1$ an $O(k^2)$ algorithm, e. g. Bowyer's [4] (proven to be $O(k^2)$ in [3]). Thus, for $d = 3$, the Voronoi diagram for S can be constructed in

$$O(n) \cdot O(1) + O((n^{2/3})^2) = O(n^{4/3})$$

expected time.
The proof of the corollary is now complete.

4. Summary

Let S be a set of n sites chosen independently from a uniform distribution in a $d-$dimensional hypercube R, and assume R has been divided into m^d equal-sized cells, where m is the floor of $n^{1/d}$. In addition, let c and c' be positive numbers, and define LG(n) as the floor of $c \cdot \log n$, where log denotes the natural logarithm. Influenced by Bentley, Weide and Yao's work [2], we have shown that if d equals 3 then $M \cdot n^{1-1/d} \cdot (c \cdot \log n)^{d+1}$ is an upper bound for the expected number of facets of the Voronoi diagram for S that are also facets of Voronoi polyhedra of sites in the outermost LG(n) layers of cells of R, where M is a function of n, c and c' that decreases for fixed c and c', $0 < c \leq c'$. Subsequently, from this result and results in [2], we have shown that $O(n)$ is an upper bound for the expected number of facets of the Voronoi diagram for S. Accordingly, we have conjectured that similar results hold for fixed $d > 3$, and from the Euler formula have concluded that for $d = 3$, the same results hold for the 0− and 1−dimensional faces of the Voronoi diagram for S. Actually, without explicitly stating it, we have established the existence of an expected $O(n)$ algorithm for constructing Voronoi diagrams in three dimensions. To see this, we note that for each site in the outermost LG(n) layers of cells of R, we have implicitly shown the feasibility of obtaining a subset of S that contains all of the Voronoi neighbors of the site, in such a way that the expected time involved in obtaining all such subsets for all such sites is bounded above by $M \cdot n^{2/3} \cdot (c \cdot \log n)^4$, where M is a function of n, c and c' that decreases for fixed c and c', $0 < c \leq c'$. Thus, since the intersection of k half-spaces in 3−dimensional space can be found in time $O(k \cdot \log k)$ [18], a computation can be carried out to show that the Voronoi polyhedra of the sites in the outermost LG(n) layers of R can be found in at most

$$O(n^{2/3} \cdot (c \cdot \log n)^4) \cdot \log(O(n^{2/3} \cdot (c \cdot \log n)^4)) = M' \cdot n^{2/3} \cdot (c \cdot \log n)^4 \cdot \log n$$

expected time, where M' is a function of n, c and c' that decreases for fixed c and c', $0 < c \leq c'$. This observation, together with results in [2], then shows the existence of the algorithm. Finally, independently of the results described above, we have shown that for fixed $d \geq 2$, with the exception of at most an expected $O(n^{1-1/d})$ number of polyhedra, each polyhedron in the Voronoi diagram for S can be constructed in expected constant time. This result and its proof, together with results in [2], have allowed us to show the existence of 'simple' expected $O(n)$ and $O(n^{4/3})$ methods for constructing Voronoi diagrams in two and three dimensions, respectively.

Acknowledgments

The author wishes to express his gratitude to Professor James F. Lawrence for his criticism and suggestions on the paper. He also wishes to thank Dr. Christoph Witzgall for his encouragement.

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NBS-114A (REV. 2-8C)

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4. TITLE AND SUBTITLE

On the Expected Complexity of the 3-Dimensional Voronoi Diagram

5. AUTHOR(S)

Javier Bernal

6. PERFORMING ORGANIZATION *(If joint or other than NBS, see instructions)*

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11. ABSTRACT *(A 200-word or less factual summary of most significant information. If document includes a significant bibliography or literature survey, mention it here)*

Given n sites chosen independently from a uniform distribution in a cube in 3-dimensional Euclidean space, Bentley, Weide and Yao have shown that with the exception of at most an expected $O(n^{2/3} \cdot \log n)$ number of polyhedra, each polyhedron in the Voronoi diagram for the n sites has an expected constant number of vertices. In this paper, their work is extended to show that $O(n^{2/3} \cdot (\log n)^4)$ is an upper bound for the expected number of distinct points that are vertices of the expected $O(n^{2/3} \cdot \log n)$ polyhedra not considered by Bentley, et al. A consequence of this result is that the Voronoi diagram for the n sites has an expected $O(n)$ number of vertices. Finally, it is shown that with the exception of at most an expected $O(n^{2/3})$ number of polyhedra, each polyhedron in the Voronoi diagram for the n sites has an expected constant number of vertices.

12. KEY WORDS *(Six to twelve entries; alphabetical order; capitalize only proper names; and separate key words by semicolons)*

computational geometry; expected complexity; expected time analysis; face; facet; polyhedron; vertex; Voronoi diagram

13. AVAILABILITY

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14. NO. OF PRINTED PAGES

22

15. Price

$9.95