NISTIR 6182

Finite Precision Representation of the Conley Decomposition

Fern Y. Hunt

U.S. DEPARTMENT OF COMMERCE
Technology Administration
National Institute of Standards
and Technology
Mathematics and Computational Sciences Division
Gaithersburg, MD 20899-0001

NIST

NISTIR 6182

Finite Precision Representation of the Conley Decomposition

Fern Y. Hunt

U.S. DEPARTMENT OF COMMERCE
Technology Administration
National Institute of Standards
and Technology
Mathematics and Computational Sciences Division
Gaithersburg, MD 20899-0001

December 1998

Finite Precision Representation of the Conley Decomposition:

Fern Y. Hunt
Mathematics and Computational Sciences Division
National Institute of Standards and Technology
Gaithersburg, MD 20899

December 7, 1998

Abstract

We present a theoretical basis for a novel way of studying and representing the long time behaviour of finite dimensional maps. It is based on a finite representation of ϵ-pseudo orbits of a map by the sample paths of a suitable Markov chain based on a finite partition of phase space. The use of stationary states of the chain and the corresponding partition elements in approximating the attractors of maps and differential equations was demonstrated in [7] [3] and proved for a class of stable attracting sets in [8]. Here we explore the relationship between the communication classes of the Markov chain and basic sets of the Conley Decomposition of a dynamical system. We give sufficient conditions for the existence of a chain transitive set and show that basic sets are isolated from each other by neighborhoods associated with closed communication classes of the chain. A partition element approximation of an isolating block is introduced that is easy to express in terms of sample paths. Finally in considering the irreducibility of the chain, we show that when the map supports an SBR measure there is a unique closed communication class and the Markov chain restricted to those states is irreducible.

KEYWORDS: attractors, chain transitive sets, Conley decomposition

AMS subject classification: 58F11,58F12, 28D,28D20

1 Introduction

The relationship between the orbits of a dynamical system and the dynamic behavior of random perturbations of that system has been of interest for both practical and theoretical reasons. Various attempts at modelling the effect of roundoff in numerical computations or the effects of noise on the mechanisms modelled by the dynamical systems all involve such perturbations. The presence of stability especially stochastic stability is often interpreted as a sign of the robustness of the model to random disturbances. Our purpose here is to go in the opposite direction in some sense. Given the long-time behavior of a randomly perturbed map we wish to obtain information about the long-time behavior of the the unperturbed map. If the system is stochastically stable and has some hyperbolicity, we can expect some degree of success. Absent hyperbolicity very little is known and the correspondence between random and perturbed dynamics is quite limited in general. Nevertheless it is a useful exercise to explore the analogy between the Conley decomposition of the chain recurrent set of a homeomorphism and the well known structure theorem governing the long time behavior of Markov chain sample paths- taking advantage of the fact that they can be associated with pseudo orbits whose behavior in a statistical sense is well understood. Indeed when the map is topologically stable, the partial ordering on chain recurrent sets induced by the Conley decomposition is entirely analogous to a partial ordering in the Markov chain induced by the communication relation. Even in the cases where stability fails, this approach leads to a novel method of analyzing the dynamics of the map. In particular we can numerically approximate attractors, and identify chain transitive invariant sets by suitable identification of Markov chain stationary states (or equivalently the strong components of the the graph of the Markov chain). This means that we can approximate important elements of the long time behaviour of the dynamical system without computing long trajectories. In general the number of basic sets found from closed communication classes of the Markov chain is a lower bound on the number of basic sets in the original dynamical system. For maps with a finite number of basic sets,classes and basic sets correspond when the partition is fine enough. We conjecture that some systems with infinitely many basic sets can be well approximated too as the first example in Section 4 illustrates.

The Markov chain we discuss has been used by others notably Hsu [7], who approximated attractors and basins of attraction of maps and ordinary differential equations. Although few proofs were given, Hsu successfully demonstrated the versatility and convenience of the method. M. Dellnitz [3],[4] and co-workers applied the method we describe here and in [8] to approximate the invariant measure of the Lorenz attractor and the attractors of other maps and flows. Recently, E. Akin and W. Miller [2] identified the basic sets of a map and the basic sets in a shift space associated with a filtration based on finite partitions of the map domain. Earlier D.Ruelle and then Y. Kifer [10] discussed the Conley decomposition in terms of a Markov process based on application of the map followed by a random perturbation in a ball shaped neighborhood about the image. The sample paths interpreted as pseudo-orbits can be used to define a partial order that in certain cases characterizes attractors and basic sets. Kifer studied a finite Markov chain when the number of basic sets is finite. Each state is associated with a basic set and transitions occur when the sample path of the chain leaves one basic set and goes to another [10]. Our partition based approach has less theoretical flexibility than the general state space chains discussed by Ruelle and Kifer. However the fact that the state space is finite and that partitions are easy to generate greatly enhances their computational accessibility.

2 Preliminary Results

Let $f : X \to X$ be continuous and nonsingular with respect to Lebesgue measure and defined on X a compact subset of $\mathcal{R}^d$. We introduce a Markov chain that can be associated with a random perturbation of the dynamical system defined by (X, f). Let $X = \bigcup_{i=1}^{n} I_i$ be a partition of X into I_i, elements of a partition Φ_n we will refer to as boxes. Each I_i is a closed set with $\ell(I_i) > 0$ and $\ell(I_i \cap I_j) = 0$ where ℓ is Lebesgue measure. Let $\mathbf{M}$ be an n x n matrix with entries $[\mathbf{M}]_{ij} = m_{ij}$.

$$m_{ij} = \ell(I_i \cap f^{-1}(I_j))/\ell(I_i) \tag{1}$$

If $I_{ij} = I_i \cap f^{-1}(I_j)$, note that $I_i = \cup_{j=1}^{n} I_{ij}$ and $\ell(I_{ij} \cap I_{ik}) = 0$ if $j \neq k$. It is not hard then to show that $\mathbf{M}$ is a stochastic matrix and thus defines a finite state Markov chain $\mathbf{MC_n}$, on the positive integers $\{1, 2, \cdots n\}$. We introduce the set valued function $\iota_n : X \to \{1, 2, \cdots n\}$ to identify the states of $\mathbf{MC_n}$ associated with points and subsets of X. Specifically if $x \in I_k \in \Phi_n$, then $\iota_n(x) = k$ when x is an interior point of Φ_n and $\iota_n(x) = \{k | x \in \partial I_k\}$ when x is a boundary point of Φ_n. The next lemma and the theorem following it will be frequently used in our discussion.

Lemma 2.1 *Let $f : X \to X$ be continuous and non-singular on X compact. Given a positive number ϵ, there exists an integer $n_0 > 0$, such that for every finite sample path of* $\mathbf{MC_n}$*, $n \geq n_0$, there is an ϵ-chain (ϵ-pseudo orbit), with points that are in the boxes of Φ_n defined by the states of the path.*

PROOF: Proceeding by induction, suppose that $x_0, x_1, \cdots x_k$ the points of an ϵ-chain have been found, corresponding to the first k states of a sample path. The points lie in the partition elements $I_{a_1}, I_{a_2}, \cdots I_{a_k}$ We suppose $x_i = f(x_{i-1}) + \zeta_i$, $|\zeta_i| < \epsilon, i = 0, \cdots k$. The existence of a sample path implies that $\ell(I_{a_k} \cap f^{-1}(I_{a_{k+1}})) \neq 0$, so there is a $u_k \in I_{a_k} \cap f^{-1}(I_{a_{k+1}})$. Now using the continuity of f, we can assume diam(Φ_n) is small enough so that $|f(u_k) - f(x_k)| < \epsilon$, by supposing $n \geq n_0$ for some sufficiently large n_0. Now $f(u_k) \in I_{k+1}$ so we set

$x_{k+1} = f(u_k)$, so $x_{k+1} = f(x_k) + \zeta_{k+1}$, where $|\zeta_{k+1}| < \epsilon$. Continuing in this way we construct an ϵ-chain for each step in the sample path. □
In fact more is true. Given an orbit of f we can find a sample path and a corresponding sequence of boxes, which contain the points of the orbit. Because of Lemma 2.1, we see that this is the reverse of the shadowing property of hyperbolic maps where ϵ-chains are approximated by orbits of f; for this reason we refer to the theorem below as the Reverse Shadowing Theorem.

Theorem 2.2 (Reverse Shadowing Theorem) *Let $\{x_k\}$ with $x_k = f^k(x_0)$ be an orbit of f starting at x_0. For any n there is a sample path of $\mathbf{MC_n}$, $(i_0, i_1, \cdots)$ such that $x_k \in I_{i_k}$ and $p_{i_{k-1},i_k} > 0$.*

PROOF: Again we proceed by induction first assuming I_{i_0} has been defined. If x_1 is an interior point of a box-call it $I(x_1)$, there is an ϵ such that $B_\epsilon(x_1) \subset I(x_1)$. By continuity of f there is a $\delta > 0$ such that $f(B_\delta(x_0)) \subset B_\epsilon(x_1)$. Thus,

$$B_\delta(x_0) \cap I(x_0) \subset I(x_0) \cap f^{-1}(I(x_1))$$

Since $B_\delta(x_0) \cap I(x_0)$ has a non-empty interior (whether x_0 is an interior or boundary point of Φ_n), $\ell(B_\delta(x_0) \cap I(x_0)) > 0$. Set $I_{i_1} = I(x_1)$. This completes the argument for x_1 an interior point. If x_1 is a boundary point of Φ_n, it belongs to several boxes. There exists at least one-call it J such that $\ell(I(x_0) \cap f^{-1}(J)) > 0$. The argument that shows this goes as follows: let $\mathcal{O}_1$ be an open set containing x_1 such that $\mathcal{O}_1 \subseteq \bigcup_{x_1 \in \partial J} \mathcal{O}_1 \cap J$. If $\delta > 0$ is chosen so that $f(B_\delta(x_0)) \subset \mathcal{O}_1$, then $B_\delta(x_0) \cap I(x_0) \subset I(x_0) \cap f^{-1}(\mathcal{O}_1)$ and $\ell(B_\delta(x_0) \cap I(x_0)) > 0$. Thus $\ell(I(x_0) \cap f^{-1}(\mathcal{O}_1)) > 0$. Now suppose $\ell(I(x_0) \cap f^{-1}(J)) = 0$, for every J, with $x_1 \in \partial J$. Since $\mathcal{O}_1 \subseteq \bigcup_{x_1 \in \partial J} \mathcal{O}_1 \cap J$ that would mean that

$$\begin{aligned} \ell(I(x_0) \cap f^{-1}(\mathcal{O}_1)) \leq \sum_{J : x_1 \in \partial J} \ell(I(x_0) \cap f^{-1}(\mathcal{O}_1 \cap J)) = \\ \sum_{J : x_1 \in \partial J} \ell(I(x_0) \cap f^{-1}(J) \cap f^{-1}(\mathcal{O}_1))) = 0 \end{aligned}$$

However this is a contradiction. Thus $\ell((I(x_0) \cap f^{-1}(J)) > 0$, for some J. Set $I_{i_1} = J$ for any such J. Note that for $x \in \partial\Phi_n$ there may be several possible J. The construction of the rest of the sample path, is done by assuming the path with boxes $I_{i_j}, j = 0, \cdots k-1$, has been constructed. Replacing x_0 with x_{k-1} and x_1 with x_k, the previous argument is repeated to obtain I_{i_k}. The desired sample path is therefore $i_j, j = 0, 1, \cdots$. □

3 Chain Recurrent Sets and Markov Chain Communication Classes

We will need the following definitions see [1], [15].

Definition: A set Λ is said to be chain recurrent iff for each $x \in \Lambda$ and any $\epsilon > 0$, there exists a periodic ϵ-chain of arbitrary length that contains x.

Using periodic ϵ chains one can group recurrent points into equivalence classes defined by the following relation.

Definition: $x \sim y$ iff for every $\epsilon > 0$ there is an ϵ-chain of arbitrary length joining x to y and y to x.

The equivalence classes are called chain components.

Definition: An f-invariant chain component is called a chain transitive set.

Definition: A maximal chain transitive set i.e. a set which cannot be contained in any larger chain transitive set is called a basic set.

Conley's Fundamental Theorem for homeomorphisms on a compact set says that the long time dynamics of the system occurs on the chain recurrent set $\mathcal{R}(f)$. When $\mathcal{R}(f)$ has hyperbolic structure, it can represented as the finite union of basic sets (maximal chain transitive sets). The dynamics between basic sets is gradient-like while the dynamics within a basic set itself is topologically transitive and has a degree of complexity depending on the specific structure of the stable and unstable manifolds within the set. The ϵ-chain equivalence relation is the foundation of Conley's decomposition and in the hyperbolic case we can link this to the so-called communication relation existing between states of $\mathbf{MC_n}$.

Definition: If i and j are states of the Markov chain $\mathbf{MC_n}$, then $i \rightarrow j$, if $\exists k \geq 0$, such that $m_{ij}^k > 0$. Thus state j can be reached by a sample path of length k starting from state i with positive probability.
Definition: i and j are said to be in the same communication class, that is, $i \longleftrightarrow j$, if $i \rightarrow j$ and $j \rightarrow i$.
$\longleftrightarrow$ is an equivalence relation on the states of $\mathbf{MC_n}$ and the corresponding equivalence classes are called communication classes. A communication class is maximal in the sense that if any state communicates with a state in the communication class it too belongs to the class. We will need one final definition.
Definition: A set of states C is said to closed if for $i \in C$ and $j \notin C$, implies that $m_{ij} = 0$.

A standard result in the theory of finite Markov chains states that the sample paths of the chain eventually enter closed (and therefore Markov chain recurrent) communication classes.

Statement of Results:

In addition to the previous assumptions on f we also assume:

1. For all $x \in \Lambda$ and any n, $x \in \partial\Phi_n$ implies that any neighborhood of x contains some point $p \in \Lambda$ in the interior of a box I where $x \in \partial I$

Define $\Lambda_n = \bigcup\{I \in \Phi_n : I \cap \Lambda \neq \emptyset\}$ and $I_n(x) = I(\iota_n(x)) = \cup_{j \in \iota_n(x)} I_j$.

Remark: Assumption 1 implies that the isolated points of Λ must be disjoint from $\partial\Phi_n$ for any n

Lemma 3.1 *Let 1) be satisfied and suppose Λ is a transitive invariant set. For each n, the states $\lambda_n = \{\iota_n(x)\}_{x \in \Lambda_n}$, corresponding to the set Λ_n, are contained in a single* $\mathbf{MC_n}$ *communication class.*

Proof:
Let $k_1 \in \iota_n(x)$ and $k_2 \in \iota_n(y)$. There is a transitive point $w \in int(I_{k_1})$ and integers $m_1 \geq m_2$ such that $f^{m_2}(w) \in int(I_{k_2})$ and $f^{m_1}(w) \in int(I_{k_1})$ where *int* denotes the interior of a set. Now by the Reverse Shadowing theorem there is a $\mathbf{MC_n}$ sample path from state k_1 to k_2 and back. This shows that $k_1 \leftrightarrow k_2$.
□
Remark: We will later show that there is a communciation class consisting of states that lie on sample paths that join states in λ_n

The next result about hyperbolic invariant sets assumes that f is a diffeomorphism. Recall that if $\mathcal{R}(f)$ has a hyperbolic structure then f has a finite number of hyperbolic basic sets (which are transitive). Maps with this property include those that are Axiom A. The methods of this section can be used to analyze the dynamics of such systems.

REMARK: Call the communication class discussed in Lemma 3.1 $\mathbf{c_n}$ and define the corresponding subset of X as $C_n = \bigcup\{I(\iota) : \iota \in \mathbf{c_n}\}$.

Proposition 3.2 *Let Λ be a hyperbolic chain transitive invariant set contained in C_n. Then C_n contains the maximal chain transitive invariant set (basic set) containing Λ.*

Proof:
Given any $x \in \Lambda$, let $z \sim x$. We claim that $\iota_n(z) \subset \mathbf{c_n}$. Let $\{x_i^\epsilon\}_{i=1}^{m+l}$ be the ϵ-chain joining z to x and x to z with $x_l^\epsilon = x$ and $x_1^\epsilon = x_{m+l}^\epsilon = z$. By the Anosov Closing Lemma [15] there is a periodic orbit $\{\zeta_i^\epsilon\}_{i=1}^{m+l}$ where $\zeta_1^\epsilon = \zeta_{m+l}^\epsilon \in int(I)$ and $\zeta_l^\epsilon \in int(J)$. If x and/or z are in $\partial\Phi_n$ then I (J) is some box for which $x \in \partial I$ ($z \in \partial J$). The proof that a suitable $\epsilon > 0$ can be chosen so that $\zeta_1^\epsilon, \zeta_l^\epsilon$ are not in $\partial\Phi_n$ uses assumption 1. Again by the Reverse Shadowing Theorem there is a $\mathbf{MC_n}$ sample path through the states $\{\iota_n(\zeta_i^\epsilon)\}_{i=1}^{m+l}\}$. Thus $\iota_n(I) = \iota_n(\zeta_1^\epsilon) \longleftrightarrow \iota_n(\zeta_l^\epsilon) = \iota_n(J)$. If in particular, z is in the basic set containing Λ, then $\iota_n(J) \in \mathbf{c_n}$. so $z \in C_n$. □

The next result states that a basic class is isolated from every other chain transitive set when $\mathbf{c_n}$ is a closed communication class and n is sufficiently large.

Proposition 3.3 *Let $\mathbf{c_n}$ be a closed $\mathbf{MC_n}$ communication class and C_n, the corresponding union of boxes as defined above. Let Λ be a basic set contained in C_n and $\Gamma \neq \Lambda$ another chain transitive set. Then for n sufficiently large, $\Gamma \cap C_n = \emptyset$.*

Proof:
Let $\Gamma \neq \Lambda$ be a basic set such that $\Gamma \cap \mathbf{C_n} \neq \emptyset$ for infinitely may n. Let $\epsilon > 0$ be given and suppose $p \in \Gamma$, and $q \in \Lambda$. We may choose n large enough according to Lemma 2.1 so that we can construct a closed ϵ-chain joining x_n to q from the closed sample path joining $\iota_n(x_n)$ to $\iota_n(q)$ where $x_n \in \Gamma \cap \mathbf{C_n}$. Since Γ is a chain transitive set, there is a closed ϵ-chain joining p and x_n so combining the two chains gives one an ϵ- chain joining p and q and conversely. Since Λ is a maximal chain transitive set we must have $\Gamma \subset \Lambda$- a contradiction. There must exist an integer n_0 therefore, such that $\mathbf{C_n}$ meets no other chain transitive set for all $n \geq n_0$. □

The next series of results illustrate how the communication structure of the chain can be used to locate chain transitive sets and basic sets for f.

Lemma 3.4 *Let* $\mathbf{c}$ *be a* <u>*closed*</u> *communication class of* $\mathbf{MC_n}$. *Let* $\Im_c = \bigcup\{I(i) : i \in \mathbf{c}\}$.

- $\Im_c$ *is closed, compact and forward invariant*
- $\Lambda_c = \omega(\Im_c)$ *is closed compact and f-invariant.*

Proof :
Without loss of generality assume $\iota_n(x)$ is a single state in $\mathbf{c}$. $\Im_c$ is a finite union of closed subsets of X compact so this proves the first property. We

now show that if $x \in \Im_c$ then $f(x) \in \Im_c$. It suffices to show there is some box $I \subset I_n(f(x)) \cap \Im_c$. Since $I_n(f(x))$ contains $f(x)$ there is a one-step orbit from $I_n(x)$ to $I_n(f(x))$. By the Reverse Shadowing Theorem there is an $\mathbf{MC_n}$ sample path from $\iota_n(x)$ to a state in $\iota_n(f(x))$. Since $\mathbf{c}$ is a closed communication class $\iota_n(f(x)) \subset \mathbf{c}$ if $f(x)$ is an interior point of Φ_n. Otherwise there is $\iota \in \iota_n(f(x))$ with $\iota \in \mathbf{c}$. From the definition of $\Im_c$ it follows that $I_n(f(x)) \in \Im_c$ if $f(x)$ is $\notin \partial\Phi_n$ and $I_n(\iota) \in \Im_c$ is $f(x) \in \partial\Phi_n$. Now the existence of a compact f-invariant Λ_c follows from the forward invariance of $\Im_c$ [1] □

Given $A \subset X$ there are a set of states in $\mathbf{MC_n}$ associated with A. They are $\iota_n(A) = \{\iota_n(x) : x \in A\}$.

Theorem 3.5 *If Λ is an f-invariant set satisyfing assumption 1), $\iota_k(\Lambda)$ is contained in a single $\mathbf{MC_k}$ communication class for all sufficiently large k, then Λ is a chain transitive set and is contained in C_k.*

Proof of Theorem:
For any points x, $y \in \Lambda$, $\iota_k(x) \longleftrightarrow \iota_k(y)$ for all k sufficiently large where $\iota_k(x)$ and $\iota_k(y)$ are states in the communication class corresponding to boxes of Φ_n containing x and y respectively. Thus for any ϵ there is an ϵ-chain as in Lemma 2.1 that can be constructed from a sample path joining $\iota_k(x)$ to $\iota_k(y)$ and $\iota_k(y)$ to $\iota_k(x)$ when k is large enough. Hence $x \sim y$, and the chain transitive property of Λ is established. $\Lambda \subset \bigcup\{I(\iota) : \iota \in \mathbf{c_k}\}$ follows from the definition of $\iota_k(\Lambda)$ and the hypothesis of the theorem. □

We now discuss an important tool for investigating the topological and dynamical properties of the Conley decomposition. An isolating block is a compact subset $N \subset X$ such that $N \cap f^{-1}(N) \cap f(N) \subset int(N)$. That is, if the image of a point in N falls on the bounday ∂N, the next iterate must fall outside of N. Isolating blocks are useful because much (but not in general all) of the information about the dynamics on or near an invariant set is retained by the isolating block that contains it. More importantly, while invariant sets can change drastically when the map is perturbed, the isolating block remains the same for small enough perturbations ([5]). Topological properties of the isolating blocks have been used to analyze dynamics of differential equations see e.g. [14]. The entrance and exit time decomposition of isolating blocks has been used both numerically and analytically to describe the dynamics and mass transport in Hamiltonian systems [11] [5]. The construction discussed below is motivated by the following proposition proved in [5].

Proposition 3.6 *If Λ is a forward invariant set, then for each $\epsilon > 0$, $N(\epsilon) = \overline{R(\Lambda, \epsilon)}$ is an isolating block where $R(\Lambda, \epsilon)$ is the union of all ϵ-chains starting and ending in Λ*

Given Λ as above, let ρ_n be the set of all sample paths beginning and ending in a state in $\lambda_n = \{\iota : \iota \in \iota_n(x), x \in \Lambda\}$. The forward invariance of Λ and the Reverse Shadowing lemma imply that for all $x \in \Lambda$ there is a state $\iota \in \iota_n(x)$ such that $\iota \in \rho_n$. Let $\mathcal{J}_n = \bigcup_{\iota \in \rho_n} I(\iota)$. Then $\mathcal{J}_n \supset \Lambda$. Now $\mathcal{J}_n$ is not an isolating block but if $x \in \mathcal{J}_n$ and $f(x) \in \partial \mathcal{J}_n$ we will show that $f^2(x) \notin int(\mathcal{J}_n)$. Thus a point on the boundary that is in the image of $\mathcal{J}_n$ is mapped outside of $int(\mathcal{J}_n)$. The image can land on the boundary of $\mathcal{J}_n$- a possibility that can't occur in an isolating block. In either case, the orbit of x cannot re-enter the interior without leaving $\mathcal{J}_n$ altogether.

Lemma 3.7 *Let $x \in \mathcal{J}_n$ and $f(x) \in \partial \mathcal{J}_n$ and let ρ_n be the set of* $\mathrm{MC_n}$ *states that lie in sample paths starting and ending in λ_n.*

1. *If $\iota \in \iota_n(x) \cap \rho_n$ and $j \in \iota_n(f(x)) \cap \rho_n^c$ then $m_{jk} = 0 \ \forall k \in \rho_n$*

2. *$f^2(x) \notin int(\mathcal{J}_n)$*

Proof of lemma: Suppose $m_{jk} > 0$ for some $k \in \rho_n$. Since there exists $x, y \in \Lambda$ such that there is an $\iota_0 \in \iota_n(x)$ with $\iota_0 \to \iota$ and a state $j_m \in \iota_n(y)$ with $k \to j_m$, we have $\iota_0 \to j \to j_m$. We conclude that $j \in \rho_n$- contradicting our hypothesis. This proves (1). Now suppose $f^2(x) \in int(\mathcal{J}_n)$. Then $\iota_n(f^2(x)) \subset \rho_n$. By the Reverse Shadowing lemma there is a sample path from j to $k \in \iota(f^2(x))$. By (1) however for such k, $m_{jk} = 0$. Thus $f^2(x) \notin int(\mathcal{J}_n)$. □
Remark: Note that when Λ is transitive, ρ_n is a communication class.

It is not hard to show that the mean entrance and exit times for ρ_n approximate the entrance and exit times of orbits up to a finite number of orbit steps $m = m(n)$ and there are well known results in the theory of Markov chains for calculating these in terms of the transition probabilities. Thus we can obtain approximate entrance and exit time decompositions for $\mathcal{J}_n$. Using Proposition 3.6 it is not hard to also show that $\mathcal{J}_n$ contains an isolating block and is contained in another. We conjecture therefore that the escape rates of invariant sets (known to be the same for all isolating blocks) can be estimated by the mean exit times of ρ_n [6].

4 Attractor-Repellor Decomposition

We conjecture that when the chain recurrent set is hyperbolic, the ordering induced by the map dynamics between basic sets is mirrored by the ordering induced on the corresponding communication classes by the communication relation. However it may be more instructive to focus on distinguished subsets of $\mathcal{R}(f)$ and examine the relationship between map and Markov chain dynamics in these cases. We therefore turn our attention to invariant decompositions of X and discuss specific instances. In the following definition assume f is a homeomorphism.

Definition: (see [1]) $\mathcal{F}$ is called an invariant decomposition if $\mathcal{F} = \{F_1, F_2, \cdots, F_n\}$, with $F_i, i = 1, \cdots n$ pairwise disjoint non-empty closed f-invariant subsets of X that cover the limit set $l[f] = \overline{(\alpha f(X) \cup \omega f(X))}$. It can be shown that $l[f] \subset \mathcal{R}(f)$.

Given an invariant set A, define $W^+(A) = \{x : \omega f(x) \subset A\}$ to be the stable set of A, and $W^-(A) = \{x : \alpha f(x) \subset A\}$ to be the unstable set of A. We will assume that A is an $l[f]$ separating subset of X.

A very important invariant decomposition is obtained when the sequence in $\mathcal{F}$ consists of an attractor A_+ and a repellor A_-. The entire phase space can then be written as [1]

$$X = A_+ \cup A_- \cup (W^+(A_+) \cap W^-(A_-))$$

It is clear the dynamics of f are that points in X are globally attracted to A_+. This is mirrored in the dynamics of the $\mathbf{MC_n}$ for n large enough. Recall that an invariant set A is asymptotically stable, if there exists an open neighborhood $U \supset A$ with $x \in U \Rightarrow \omega(x) \subset A$. As a consequence every neighborhood V of an attractor A (which by definition is asymptotically stable) contains a compact set $B \supset A$ with the property that $f(B) \subset int(B)$. Such a set is called an attractor block. We have the following lemma.

Lemma 4.1 ([8]) *Suppose f is a continuous map with attractor A. If B is an attractor block then for all sufficiently large n,*

$$B_n \subseteq f^{-1}(B_n) \tag{2}$$

where,

$$B_n = \cup\{I \in \Phi_n : I \cap B \neq \emptyset\}$$

Corrollary 1 *Let f be a non-singular and continuous map. The set of states in $\mathbf{MC_n}$ defined by the set B_n is closed under the communication relation.*

Proof:
Let B be a block and suppose $I \subset B_n$. If $J \not\subset B_n$ then $l(f^{-1}J \cap I) = 0$. To see this note that $f^{-1}(J \cap B_n) = f^{-1}J \cap f^{-1}(B_n) \supseteq f^{-1}J \cap B_n$ by the previous lemma. The latter set contains $f^{-1}J \cap I$. Now $l(J \cap B_n) = 0$ so the non-singularity of f implies that $l(f^{-1}(J \cap B_n)) = 0$. Thus when the chain is

in the state $\iota_n(I)$, it does not transtion to $\iota_n(J)$. □.

Once the chain enters the states $\iota_n(B_n)$ it cannot leave but what guarantees that the chain enters these states in the first place? Sample paths of the chain must eventually enter stationary states so a sufficient condition is that $\iota_n(B_n)$ contain these states. We can state a theorem in the case there is one closed communication class. Assume that the dynamical system has an ergodic SBR measure. It can then be shown (see Section 5) that the $\mathbf{MC_n}$ has this property for every n so in this case there is a unique stationary vector for the matrix $\mathbf{M}$ and set of stationary states. Let S_n be the union of boxes corresponding to the non-zero elements of the the stationary vector. S_n is the support of the stationary probability measure defined by the vector. If for large enough n it can be shown that $S_n \subseteq B_n$ then the chain does indeed enter the states defined by $\iota_n(B_n)$. The following preliminary result is a special case of a result proved in [13]

Proposition 4.2 ([13]) *Let f have an ergodic SBR invariant measure. If C is a union of elements (boxes) in Φ_n with $C \sqsubseteq f^{-1}(C)$ then $S_n \subseteq C$.*

where $A \sqsubseteq B$ means $\ell(A/B) = 0$

$S_n \subseteq B_n$ now follows from Lemma 4.1 and Proposition 4.2. More can be said about the relationship between Markov chain dynamics and map dynamics.

Theorem 4.3 ([8]) *Suppose μ is an ergodic SBR measure for f that has an attractor A as its support. Then for every neighborhood V of A there exists an integer n_0, such that for all $n \geq n_0$,*

$$A \subseteq S_n \subset V \tag{3}$$

The set of stationary states is $\iota_n(S_n)$. Now with the hypothesis on μ any neighborhood V of A contains an attractor block B with $S_n \subset B \subset V$ for n sufficiently large and since $\iota_n(S_n) \subset \iota_n(B_n)$, we see that the chain must enter $\iota_n(B_n)$ and never leave. The theorem tells us one more fact that is very useful and that is the attractor A can be approximated in the sense of Hausdorff metric by the set of boxes defined by the stationary states of $\mathbf{MC_n}$. Rather than using one long orbit to approximate A, the methods of graph theory and tree data structures can be used instead because the chain components of A are approximated by the communication classes of the stationary states. This is in fact equivalent to approximating the chain components by ϵ-chains starting and ending in A. We turn now to examples illustrating these ideas.

Example: Theorem 4.3 can be extended to invariant sets that are the intersection of a nested sequence of attractors. Thus attracting sets which fail to have an open basin of attraction can still be approximated by the stationary set S_n. The logistic map at the Feigenbaum parameter value which has an attracting Cantor set, is an example. Thus S_n approximates it in the Hausdorff metric. There are infinitely many periodic orbits in any neighborhood of this set so the map not only fails to be hyperbolic- it fails to have a finite number of basic sets. The set supports a unique invariant measure and the stationary vector of $\mathbf{M}$ defines a measure that converges weakly to it in the limit of n [8].

Roots of a Complex Polynomial: Let P be a complex polynomial and regard it as a function $P : S^2 \mapsto S^2$ where S^2 is the Riemann sphere. We define a gradient function on S^2, regarded as the complex plane compactified with the point at infinity.

$$V_P(x) = -grad((\| P(x) \|)^2)/2 = -(DP_x)^{tr}(P(x)) \tag{4}$$

(4) defines a flow on $\mathbf{R^2}$ and has attractors or sinks at the zeroes $\{a_j\}$ of P and saddle points when distinct from the zeros of P (either hyperbolic or multipronged) at the zeroes $\{\theta_j\}$ of P', the complex derivative of P [16]. Letting $X = \mathbf{R^2} \cup \infty = S^2$, we claim that

$$X = \bigcup_j W^+(a_j) \cup \{\infty\} \cup \bigcup_j (W^-(\infty) \cap W^+(\theta_j)) \tag{5}$$

where ∞ is the point at infinity. This follows from the Poincare-Bendixson theorem and the fact that $\mathcal{L}(x) = \| P(x) \|^2$ is a Lyapunov function. To introduce the mapping in our example we let

$$T_h(x) = x + hV_P(x)$$

where h is very small. T_h is an Euler approximation of the flow defined by (4). Observing that (4) defines a gradient system we can apply a result of Stuart and Humphries [17] to assert that for h small enough T_h has the same Lyapunov function and the same fixed points as (4). Indeed finding fixed points of the map by iteration is the basis of the method developed by Hirsch and Smale. Set $A_+ = \bigcup_j \{a_j\}$, $A_- = \bigcup_j (W^-(\infty) \cap W^+(\theta_j)) \cup \bigcup_j \{\theta_j\} \cup \{\infty\}$. $\{A_+, A_-\}$ is an attractor repellor pair for T_h. Suppose there are q roots. Each root $a_i, i = 1 \cdots q$ is the only attractor in some isolating neighborhood and it supports a unique invariant measure δ_{a_i}, the point measure at a_i. Rather than a single closed communication class there is one for each a_i so that S_n consists of q sets of boxes, one for each communication class. We can apply Theorem 4.3 to each class to show that the corresponding set of boxes approximate a_i in the Hausdorff metric. Thus we have a procedure for numerically estimating the roots of a polynomial that is based on refined subdivision rather than iteration. Coupling this with a Monte Carlo [8] method for evaluating the elements of $\mathbf{M}$, makes the procedure very parallelizable and is a viable way of providing starting positions for high precision root finding algorithms based on iteration.

5 Measure Theoretic Results

All of the results with the exception of Theorem 4.3 have not depended on properties of any invariant measures an attractor might support. In this section, we will show how the requirement that μ be an SBR measure leads very naturally to the idea of an irreducible recurrent chain. A Markov chain is said to be irreducible if the entire state space is in one communication class. In our discussion we assume that the domain of convergence to μ referred to in the definition of SBR is all of X with the exception of a set of Lebesgue measure zero. We further assume that μ has no support on the boundaries of the partitions so that $\mu(\cup_n \partial \Phi_\mathbf{n}) = 0$. If μ is an f-invariant measure that is equivalent to l then we have the following result.

Theorem 5.1 ([9]) *Let μ be f-invariant, ergodic and equivalent to Lebesgue measure. Then for every n, $\mathbf{MC_n}$ is irreducible.*

There are however a lot of situations when μ is singular with respect to Lebesgue measure particularly when f is multidimensional. For example the Henon map (for certain parameter values) and its expanding cousin, the Lozi map have this property. Nevertheless, we can show such attractors can still enforce the kind of dynamics in the Ulam Markov chain that would lead to a single communication class, provided the measure is SBR. To prove this we will need to show the following;

Lemma 5.2 *Let $A \subset X$ be a Borel set with $\mu(A) > 0$. Then for all $x \in X$ there exists an integer $l > 0$ such that $P^l(\iota(x), \iota(A)) > 0$.*

$P^l(i,j)$ is the l-step transition probability for the chain.

Proof:
Given a box $I \subset \Phi_\mathbf{n}$, there is a generic point $x_0 \in I$ such that $\sigma_m(A) = 1/m \sum_{k=0}^{m-1} \delta_{f^k(x_0)}(A) > 0$ for m sufficiently large. We reason as follows. I has positive Lebesgue measure so such an x_0 must exist. If $A \subset \mathcal{F}_n$, the finite field generated by Φ_n then $\mu(A) > 0 \Longrightarrow \mu(A^o) > 0$ if $A^o \neq \emptyset$. The Portmanteau theorem applied to A^o then implies $\liminf_{n\to\infty} \sigma_m(A^o) \geq \mu(A^o)$. There exists a $\delta > 0$ so that $\sigma_m(A^o) > \delta$ and thus $\sigma_m(A) > 0$ for m sufficiently large.
If A is any Borel subset of X then using that fact that $\mu(A_n) \geq \mu(A)$, the previous argument shows that $\sigma_m(A_n) > 0$ for m sufficiently large. There is a $\mathbf{MC_n}$ sample path $\iota(x) = \iota_0, \iota_1, \cdots \iota_{m-1}$ where $p_{i_{k-1},i_k} > 0$ and such that $f^k(x_0) \in I(\iota_k)$ by the Reverse Shadowing theorem. For some l, $0 \leq l \leq m-1$, $f^l(x_0) \in A_n$. Otherwise for all such l, $\delta_{f^l(x_0)}(A_n) = 0$. This contradicts the fact that $\sigma_m(A_n) > 0$. Now $\iota_n(A) = \iota_n(A_n)$ and $P^l(\iota_n(x), \iota_n(A_n)) > p_{0,1} \cdots p_{i_{k-1},i_k} \cdots p_{i_{l-1},i_l}$ by the Chapman Kolmogorov equation. The conclusion of the theorem now follows by observing that the product is positive. □

Suppose λ_n is the set of states corresponding to the set Λ_n; i.e. $\lambda_n = \{\iota_n(x)\}_{x \in \Lambda_n}$. For each $k \in \lambda_n$, the corresponding box $I(k)$ has positive μ measure since it contains a point in the support. By Lemma 5.2 $\iota \to k$ for all states $\iota \in \mathbf{MC}_n$. Let us define the set of states reachable by λ_n as

$$\gamma_n = \{j : \exists i \in \lambda_n, i \to j\}$$

Lemma 5.3 *γ_n is a closed communication class*

Proof: The communication property follows from the definition of γ_n, the previous remark and Lemma 5.2. If $j \notin \gamma_n$ and for some $r \in \gamma_n$, and $m_{rj} > 0$, then for $i \in \lambda_n$, $i \to r \to j$ or $i \to j$. This contradicts the fact that $j \notin \gamma_n$. Thus γ_n is closed □

Since every state of the chain must eventually enter λ_n, γ_n is the only closed communication class of $\mathbf{MC_n}$.

Theorem 5.4 *If μ is a SBR measure, then $\mathbf{MC_n}$ restricted to γ_n is irreducible for every n. All other states are transient.*

We turn now to a description of S_n, the support of the stationary measure of the chain. It depends on the fact that all states enter a single finite irreducible subchain and that such a chain is Markov chain recurrent. Irreducibility and recurrence in the theory of finite Markov chains is a special case of a condition known as Harris recurrence. The following is an adaptation to finite state spaces of a theorem of Meyn and Tweedie for Harris recurrent chains on general state spaces.

Theorem 5.5 ([12]) *Let π_n be the unique stationary measure of the chain $\mathbf{MC_n}$, described in the previous theorem. Then π_n is equivalent (as a measure on the finite field $\mathcal{F}_n$) to*

$$\nu_n(\cdot) = \sum_{\iota} \overline{\mu}(\iota)\mathbf{K_n}(\iota, \cdot) \tag{6}$$

$$\text{where } \mathbf{K_n}(\iota, \cdot) = \textstyle\sum_{k=0}^{\infty} P^k(\iota, \cdot)/2^{-(k+1)}$$
$$\text{and } \overline{\mu}(\iota) = \mu(I(\iota))$$

Theorem 5.4 tells us that π_n is indeed unique and its support is in fact γ_n. To see this observe that γ_n is the support of ν_n. Equivalence of π_n and ν_n means they have the same support.

The picture we have then is this. Given an invariant set Λ supporting an f-invariant SBR measure we have proved that all states in $\mathbf{MC_n}$ eventually enter a closed communication class containing states whose corresponding partition boxes are at once a cover and a finite set approximation to Λ. The communication class itself is the support of a unique stationary measure that can be expressed (up to equivalence) in terms of μ. If Λ is an attracting set with some stability the set approximation is a good one. For example suppose Λ is the intersection of a nested sequence of asymptotically stable sets then for any neighborhood U of Λ the set $\Lambda \subset S_n \subset U$ for n sufficiently large.

References

[1] E. Akin, *The general topology of dynamical systems*, American Mathematical Society, Providence, 1993

[2] E. Akin, W. Miller, *Invariant Measures for Set-Valued Mappings* Transactions of the AMS (to appear 1998)

[3] M. Dellnitz, O. Junge, *On the Approximation of Complicated Dynamical Behavior* SIAM J. Num. Anal. to appear 1998

[4] M. Dellnitz, A. Hohmann, O. Junge, M. Rumpf, *Exploring invariant sets and invariant measures*, CHAOS, 7, no.2 221-228, 1997

[5] R. W. Easton, *Isolating blocks and epsilon chains for maps*, Physica D, Vol. 39, no. 1, 95-110, 1989

[6] R. W. Easton, *Transport of phase space volume near isolated invariant sets*, J. Dynamics Differential Equations 5, no. 3 529-536 1993

[7] C. S. Hsu, *Cell to Cell Mapping: a method of global analysis for nonlinear systems*, New York, Springer Verlag, 1987

[8] F. Y. Hunt, *Unique ergodicity and the approximation of attractors and their invariant measures using Ulam's method*, Nonlinearity, 11, 307-317, 1998

[9] F. Y. Hunt, W. M. Miller, *On the approximation of invariant measures*, J.Stat.Phys., 66, 535-548, 1992

[10] Y. Kifer, *Random Perturbations of Dynamical Systems*, Birkhauser, Boston, 1988

[11] R. W. Easton, J. D. Meiss, *Exit times and transport for symplectic twist maps*, Chaos 4, no. 2, 153-165, 1993

[12] S. Meyn, R. Tweedie, *Markov chains and stochastic stability*, Communications and Control Engineering Series, Springer Verlag London Ltd., London, 1993

[13] W. M. Miller, *Stability and approximation of invariant measures for a class of nonexpanding transofrmations*, Nonlinear Analysis, Theory Methods and Applications, 23, 1013-1025, 1994

[14] R. Mischaikow, M. Mrozek, *Chaos in the Lorenz equations: a computer assisted proof. II. Details*, Math. Comp. 67, no. 223, 1023-1046, 1998

[15] C. Robinson, *Dynamical Systems: Stability, Symbolic Dynamics, and Chaos*, Boca Raton, CRC Press 1995

[16] M. Shub, D. Tischler, R. Williams, *The Newtonian graph of a complex polynomial*, SIAM J. Math. Anal. 19, 246-256, 1988

[17] A. M. Stuart, A. R. Humphries, *Dynamical Systems and Numerical Analysis*, New York, Cambridge University Press, 1996