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Distributive Lattices and Hypergraph Coloring

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The "composition ideal" is a basic notion connected with free lattices. In this paper the composition ideals of distributive lattices are characterized, and the usefulness of this characterization with respect to computation of chromatic number of hypergraphs is noted.

1. Introduction.

A *lattice* is a triple $(L, \wedge, \vee)$, where L is a set, $\wedge$ and $\vee$ are binary operations which are each idempotent, commutative, associative, and jointly satisfy the *absorption* properties:

$$\text{For } a, b \in L, a \wedge (a \vee b) = a = a \vee (a \wedge b).$$

The operation $\wedge$ is called *meet*, and $\vee$ is called *join*. We will usually denote the lattice $(L, \wedge, \vee)$ by L.

If L is a lattice then the set L is partially ordered by the relation:

$$\text{For } a, b \in L, a \leq b \text{ if } a = a \wedge b.$$

With this relation, each pair of elements have a greatest lower bound, their meet, and a least upper bound, their join.

Alternatively, a lattice can be defined as a partially ordered set in which each pair of elements have a greatest lower bound and a least upper bound. For a proof of the equivalence of these notions and further fundamental ideas of lattice theory consult the book by Birkhoff [4].

We denote by $FL(n)$ the free lattice generated by n generators, denoted $g_1, \ldots, g_n$. Given any lattice L, each element p of $FL(n)$ determines a (lattice polynomial) function mapping L^n to L: The point $(a_1, \ldots, a_n) \in L^n$ is mapped to the image of p under the unique homomorphism of $FL(n)$ to L which takes g_i to a_i for $i \in [n]$. We will sometimes denote this image by $p(a_1, \ldots, a_n)$.

A *composition ideal* in $FL(n)$ is a set $J \subseteq FL(n)$ which

(1) is an upper semi-ideal, so that if $x \in J$ and $y \geq x$ then $y \in J$, and

(2) has the property that if p and $q_1, \ldots, q_n$ are in J, then $p(q_1, \ldots, q_n) \in J$.

We may use the same definition to define the notion of a composition ideal in a free lattice satisfying some set of lattice identities. In particular, a set $J \subseteq FD(n)$ (where $FD(n)$ denotes the free distributive lattice generated by $g_1, \ldots, g_n$) is a composition ideal if it is an upper semi-ideal and its elements, when viewed as functions as above, map J^n to J.

In this paper we determine the composition ideals of $FD(n)$. It turns out that they are closely related to the notion of chromatic number of a hypergraph and we briefly examine this connection. Indeed the connection between hypergraphs and the free distributive lattices has been noted already by Benzaken in [1] and [2].

2. Terminology.

Obviously $FD(n)$ itself is a composition ideal. Also, the intersection of any collection of composition ideals is a composition ideal. It follows that for any set $S \subseteq FD(n)$ there is a smallest composition ideal which contains S. We denote this composition ideal by $E(S)$. The smallest composition ideal is

$$E(\emptyset) = \{p \in FD(n) : \text{there is } i \in [n] \text{ such that } p \geq g_i\}.$$

If L is any distributive lattice and I is an upper semi-ideal in L, then we denote by $J(L, I)$ the subset of $FD(n)$ consisting of elements $p \in FD(n)$ such that, whenever $x_1, \ldots, x_n \in I$, $p(x_1, \ldots, x_n) \in I$. Clearly $J(L, I)$ is a composition ideal.

For any positive integer $k < n$, let h_k be the element of $FD(n)$ given by

$$h_k = \bigwedge_{i \in [k+1]} (g_1 \vee g_2 \vee \cdots \vee \hat{g}_i \vee \cdots \vee g_{k+1}),$$

where the hat over g_i denotes that this term is omitted. By the distributive property, it is easily determined that

$$h_k = \bigvee_{1 \leq i < j \leq k+1} (g_i \wedge g_j).$$

The Boolean lattice of all subsets of $[k]$ under intersection and union is $\mathcal{B}_k = (\mathcal{B}_k, \cap, \cup)$.

3. Characterization of the Composition Ideals in $FD(n)$.

For $k = 1, \ldots, n-1$, let $J_k = E(\{h_k\})$. Let $J_n = E(\emptyset)$.

Lemma 1. *We have the inclusions*

$$J_n \subseteq J_{n-1} \subseteq \cdots \subseteq J_1.$$

Proof. The composition ideal $J_n = E(\emptyset)$ is contained in all other composition ideals. That $J_{k+1} \subseteq J_k$ for $k \in [n-2]$ follows from the inequality $h_{k+1} \geq h_k$. $\square$

Next we present four more lemmas which will be utilized later. Lemmas 2 and 3 will be used in the proof of Theorem 2. Lemmas 4 and 5 will be used in the proof of Theorem 1.

Lemma 2. *We have the inclusion*

$$J_k \subseteq J(\mathcal{B}_k, \mathcal{B}_k \setminus \{\emptyset\}).$$

Proof. This is clear if $k = n$. For $k < n$, we need only verify that $h_k \in J(\mathcal{B}_k, \mathcal{B}_k \setminus \{\emptyset\})$. Indeed let $A_1, \ldots, A_{k+1}$ be nonempty subsets of $[k]$. Some two of them must share a common element, so that

$$\bigcup_{1 \leq i < j \leq k+1} (A_i \cap A_j)$$

is also nonempty; this set is $h_k(A_1, \ldots, A_{k+1})$. $\square$

Lemma 3. *For $k = 2, \ldots, n$, the element h_{k-1} is not in the composition ideal $J(\mathcal{B}_k, \mathcal{B}_k \setminus \{\emptyset\})$.*

Proof. For $j \in [k]$ let $A_j = [k] \setminus \{j\}$. Then $h_{k-1}(A_1, \ldots, A_k) \neq \emptyset$. $\square$

By repeatedly making use of the property of $FD(n)$ that join distributes over meet, any element of $FD(n)$ can be written as a meet of joins of the generators. Such a representation is used in the next two lemmas.

Lemma 4. *Suppose $\Lambda_1, \ldots, \Lambda_m \subseteq [n]$ and let*

$$p = \bigwedge_{j=1}^{m} \Big(\bigvee_{i \in \Lambda_j} g_i \Big) \in FD(n).$$

Suppose furthermore that each set of k or fewer Λ_j's have nonempty intersection. Then $p \in J_k$.

Proof. If $k = n$ then the assumption implies that $\cap_{i=1}^{m} \Lambda_i \neq \emptyset$. If i is an element of this intersection then $p \geq g_i$ so $p \in E(\emptyset) = J_n$. If $k < n$, we proceed by induction

on m. If $m \leq k$, the Λ_j's have an index in common, say i, so that $p \geq g_i$. Then $p \in E(\emptyset) \subseteq E(\{h_k\})$. Suppose $m \geq k+1$ and that the result holds for families of $m-1$ index sets. For $i = 1, \ldots, k+1$ let

$$p_i = \bigwedge_{j \in [m-i+1]} (\bigvee_{i \in \Lambda_j} g_i) \wedge \bigwedge_{j=m-i+3}^{m} (\bigvee_{i \in \Lambda_j} g_i).$$

By the inductive assumption, $p_i \in E(\{h_k\})$ for $i = 1, \ldots, k+1$. Also

$$h_k(p_1, \ldots, p_{k+1}) = \bigvee_{1 \leq i < j \leq k+1} (p_i \wedge p_j),$$

but each term $p_i \wedge p_j$ coincides with p, so $h_k(p_1, \ldots, p_{k+1}) = p$. It follows that $p \in E(\{h_k\}) = J_k$. □

Lemma 5. *Suppose* $\Lambda_1, \ldots, \Lambda_m \subseteq [n]$ *and let*

$$p = \bigwedge_{j=1}^{m} (\bigvee_{i \in \Lambda_j} g_i) \in FD(n).$$

Suppose further that $l \leq k \leq n$ *and* $\Lambda_1 \cap \ldots \cap \Lambda_k = \emptyset$. *Then* $J_{k-1} \subseteq E(\{p\})$.

Proof. For $i = 1, \ldots, n$, choose $j(i) \in [k]$ such that $i \notin \Lambda_{j(i)}$. Then

$$p(g_{j(1)}, \ldots, g_{j(n)}) = \bigwedge_{j=1}^{m} (\bigvee_{i \in \Lambda_j} g_{j(i)}) \leq \bigwedge_{j=1}^{k} (\bigvee_{i \in \Lambda_j} g_{j(i)}) \leq \bigwedge_{j=1}^{k} (\bigvee_{i \neq j} g_i) = h_{k-1}.$$

Therefore $h_{k-1} \in E(\{p\})$ so $J_{k-1} \subseteq E(\{p\})$. □

Theorem 1. *The composition ideals in* $FD(n)$ *are* $J_1, \ldots, J_n$.

Proof. Let $J \subseteq FD(n)$ be a composition ideal. Let k be the least positive integer such that $J \not\subseteq J_k$. Then $J \subseteq J_{k-1}$ and we need only demonstrate the reverse inclusion. Choose $p \in J \setminus J_k$. Then by Lemma 4, some k or fewer of the sets Λ_j in the canonical representation of p have empty intersection, and by Lemma 5, $h_{k-1} \in E(\{p\})$. It follows that $J_{k-1} \subseteq J$, as required. □

Theorem 2. *For* $k \in [n]$, $J(\mathcal{B}_k, \mathcal{B}_k \setminus \{\emptyset\}) = J_k$.

Proof. By lemmas 2 and 3, the smallest l such that $J_l \subseteq J(\mathcal{B}_k, \mathcal{B}_k \setminus \{\emptyset\})$ is $l = k$. By Theorem 1, we must have $J(\mathcal{B}_k, \mathcal{B}_k \setminus \{\emptyset\}) = J_k$. □

4. Hypergraph Coloring.

Following Berge [3], we define a *hypergraph* on a set S to be a collection $\mathcal{H}$ of nonempty subsets of S. A *coloring* of $\mathcal{H}$ is a function $\gamma : S \to C$, where C is a set of *colors*, such that, if $E \in \mathcal{H}$ then there are $e_1, e_2 \in E$ such that $\gamma(e_1) \neq \gamma(e_2)$. A *k-coloring* is a coloring $\gamma : S \to [k]$, where the set of colors is $[k]$. The *chromatic number* of $\mathcal{H}$ is the smallest integer k such that $\mathcal{H}$ possesses a k-coloring.

Theorem 3. *Let $\mathcal{H} = \{\Lambda_1, \ldots, \Lambda_m\}$ be a hypergraph on $[n]$. Then $\chi(\mathcal{H}) > k$ if and only if the element*

$$p = \bigvee_{j=1}^{m} (\bigwedge_{i \in \Lambda_j} g_i) \in FD(n)$$

is in $E(\{h_k\})$.

Proof. We show that $\chi(\mathcal{H}) \leq k$ if and only if $p \notin J(\mathcal{B}_k, \mathcal{B}_k \setminus \{\emptyset\})$.

Suppose that $\mathcal{H}$ admits a k-coloring $\gamma : [n] \to [k]$. Since γ is a k-coloring, for each $i \in [m]$, $\{\gamma(x) : x \in \Lambda_i\}$ has at least two (distinct) elements. For $x \in [n]$ let $A_x = \{\gamma(x)\}$. Then for each $i \in [m]$, $\cap_{x \in \Lambda_i} A_x = \emptyset$, so

$$p(A_1, \ldots, A_n) = \bigcup_{i=1}^{m} \bigcap_{x \in \Lambda_i} A_x = \emptyset.$$

It follows that $p \notin J(\mathcal{B}_k, \mathcal{B}_k \setminus \{\emptyset\})$.

Now suppose $p \notin J(\mathcal{B}_k, \mathcal{B}_k \setminus \{\emptyset\})$. Then there are nonempty sets $A_1, \ldots, A_n \subseteq [k]$ such that

$$\emptyset = p(A_1, \ldots, A_n) = \bigcup_{i=1}^{m} \bigcap_{x \in \Lambda_i} A_x.$$

For each $x \in [n]$ let $\gamma(x)$ be an element of A_x. For $i \in [m]$, $\cap_{x \in \Lambda_i} A_x = \emptyset$, so there are $x, y \in \Lambda_i$ such that $\gamma(x) \neq \gamma(y)$; i.e., γ is a k-coloring of $\mathcal{H}$. □

Theorem 3 leads to a "construction technique" for hypergraphs of chromatic number greater than k. Suppose $k \in [n-1]$ and consider the set $J_k = E(\{h_k\})$. The composition ideal J_k has the following properties:

(1) It contains the generators $g_1, \ldots, g_n$;

(2) If $p_1, \ldots, p_{k+1} \in J_k$ then $h_k(p_1, \ldots, p_{k+1}) \in J_k$; and

(3) If $p \in J_k$ and $q \geq p$ then $q \in J_k$.

If one lets $\tilde{J}_k$ denote the set of elements of $FD(n)$ which can be built up starting from $g_1, \ldots, g_n$ using h_k and composition of functions then it is easily seen that $J_k =$

$\{q \in FD(n) :$ there exists $p \in \tilde{J}_k$ for which $q \geq p\}$. Elements of $\tilde{J}_k$ can be conveniently described by utilizing $(k+1)$-ary trees having leaves labelled with the generators $g_1, \ldots, g_n$.

Let T be a tree having a root r with the property that each node of T has either $k+1$ branches or no branches. Those with no branches are called *leaves* of T. Assume that the root r is not the sole node of T. Let each leaf of T be labelled with an element of $[n]$.

We now associate with each node x of T an element of $\tilde{J}_k$ in such a way that the following hold:

(1) If x is a leaf then the corresponding element of $\tilde{J}_k$ is the generator g_i, where x is labelled with i;

(2) If x is not a leaf then the corresponding element of $\tilde{J}_k$ is $h_k(p_1, \ldots, p_{k+1})$, where the $p_1, \ldots$, and p_{k+1} are the elements of $\tilde{J}_k$ corresponding to the root nodes of the branches from x.

This correspondence can obviously be built up by starting at the leaves of T and working toward the root, and it is uniquely determined by the properties above.

Finally, we associate T (with its labelling) with the element $p(T)$ of $\tilde{J}_k$ which is associated with the root r of T.

It is clear that any element p of $\tilde{J}_k$ corresponds in this way to some labelled $(k+1)$-ary tree T: $p = p(T)$. By earlier comments, the elements of J_k are the elements q of $FD(n)$ such that there exists some such tree T with $q \geq p(T)$.

We state this now entirely in terms of hypergraphs.

If T is a $(k+1)$-ary labelled tree as above then we say that a set $S \subseteq [n]$ *turns on* a node x of T if

(1) x is a leaf of T and it is labelled with an element of S, or

(2) x is not a leaf, and at least two of the root nodes of the branches at x are turned on by S. Let $\mathcal{H}_T$ denote the hypergraph consisting of subsets of $[n]$ which turn on the root of T.

Theorem 4. *The chromatic number of $\mathcal{H}_T$ is at least $k+1$. If $\mathcal{H}$ is any hypergraph of $[n]$ such that $\chi(\mathcal{H}) > k$ then there is a labelled $(k+1)$-ary tree T such that each edge of $\mathcal{H}_T$ contains some edge of $\mathcal{H}$.*

Proof. Let $\mathcal{H}_T = \{\Lambda_1, \ldots, \Lambda_m\}$. Then

$$p(T) = \bigvee_{j=1}^{m} (\bigwedge_{i \in \Lambda_j} g_i) \in E(\{h_k\})$$

so, by Theorem 3, $\chi(\mathcal{H}_T) > k$.

If $\mathcal{H} = \{\Gamma_1, \ldots, \Gamma_l\}$ is a hypergraph on $[n]$ such that $\chi(\mathcal{H}) > k$ then by Theorem 3 the element

$$p = \bigvee_{j=1}^{l} \left(\bigwedge_{i \in \Gamma_j} g_i \right)$$

is in $E(\{h_k\})$, so there is a labelled $(k+1)$-ary tree T such that $p \geq p(T)$. It is easily seen that this inequality is equivalent to the assertion that each edge of $\mathcal{H}_T$ contains an edge of $\mathcal{H}$. $\square$

References.

[1] C. Benzaken, Critical hypergraphs for the weak chromatic number, *Journal of Combinatorial Theory, B*, **29** (1980), 328-338.

[2] C. Benzaken, Hypergraphs critiques pour le nombre chromatique et conjecture de Lovász. In *Combinatorics '79, Annals of Discrete Mathematics*, **8** (1980), 91-100.

[3] C. Berge, *Graphs and Hypergraphs*, North Holland, 1973.

[4] G. Birkhoff, *Lattice Theory*, American Mathematical Society, 1940.