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Submissions to a Planned Encyclopedia of Operations Research on Computational Geometry and the Voronoi/Delaunay Construct

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by

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Introduction. This report contains two submissions to a prospective Encyclopedia of Operations Research. The first submission covers an entry on "*Computational Geometry*". The second submission addresses both the Voronoi Diagram and its dual, the Delaunay triangulation, and is therefore termed the "*Voronoi/Delaunay Construct*". The latter plays such an important role in Computational Geometry and has so many potential applications in Operations Research, that a separate entry appears justified. Both submissions refer to other entries which may be included in the planned Encyclopedia. A list of such entries follows:

Chinese Postman Problem
Cluster Analysis
Complexity
Computer Graphics
Expert Systems
Facility Location
Interior Point Methods
Linear Programming
Matching
Minimum Spanning Tree
Packing
Pattern Recognition
Polyhedra
Shortest Paths
Simplex Method
Steiner Tree
Traveling Salesman Problem

Voronoi/Delaunay Construct	<	**Voronoi Diagram**
	<	**Dirichlet Tesselation**
	<	**Thiessen Polygon**
	<	**Delaunay Triangulation**

Computational Geometry is the discipline of exploring algorithms and data structures for computing geometric objects and their – often extremal – attributes. The objects are predominantly finite collections of points, flats, hyperplanes – "arrangements" –, or polyhedra, all in finite dimensions. The algorithms are typically finite, their *Complexity* playing a central role. Emphasis is on problems in low dimensions, exploiting special properties of the plane and 3-space.

A young field – its name coined in the early 1970's (e.g. A.R. Forrest 1971, for a tutorial, see R.Graham and F.Yao 1990, for a survey, see D.T.Lee and F.P.Preparata (1984) – it has since witnessed explosive growth, stimulated in part by the largely parallel development of *Computer Graphics*, *Pattern Recognition*, *Cluster Analysis*, and modern industry's reliance on Computer-Aided Design and Robotics. It plays a key role in the emerging fields of Automated Cartography and Computational Metrology.

For general texts, consult F.P. Preparata and M.I Shamos, Computational Geometry: An Introduction (1985), J. O'Rourke, Art Gallery Theorems and Algorithms (1987), H. Edelsbrunner, Algorithms in Combinatorial Geometry (1987), and P.K. Agarwal, Intersection and Decomposition Algorithms for Planar Arrangements (1991). Pertinent geometrical concepts are presented in B. Grünbaum, Convex Polytopes (1967).

There are strong connections to Operations Research, whose classical problems such as finding a *Minimum Spanning Tree*, a maximum-length *Matching*, or a *Steiner Tree* become problems in computational geometry when posed in Euclidean or related normed linear spaces. The Euclidean *Traveling Salesman Problem* remains NP-complete (C.H. Papadimitriou 1977). *Facility Location*, and *Shortest paths* in the presence of obstacles, are other examples. *Polyhedra* and their extremal properties, typical topics of computational geometry, also lie at the foundation of *Linear Programming*. Its complexity, particularly in lower dimensions, attracted early computational geometric research, heralding the achievement of linear complexity for arbitrary fixed dimension (N. Megiddo 1984, K.L. Clarkson 1986).

A fundamental problem is to determine the "convex hull" conv$S$ of a set S of n points in d-dimensional Cartesian space $\mathcal{R}^d$. This problem has a weak and a strong formulation. Its weak formulation requires only the identification of the extreme points of conv(S). In Operations Research terms, that problem is well known as (the dual of) identifying redundant constraints in a system of linear inequalities. The strong formulation requires, in addition, characterization of the facets of the polytope conv(S). For dimension $d > 3$, the optimal complexity of the strong convex hull problem in $\mathcal{R}^d$ is $O(n^{\lfloor d/2 \rfloor})$ (B. Chazelle 1991).

Early $O(n \log n)$ methods for delineating convex hulls in the plane – vertices and edges of the convex hull of a simple polygon (e.g. R.L. Graham and F.F. Yao 1983) can be found in linear time – were based on "divide-and-conquer" (F.P. Preparata and S.J. Hong 1977). In this widely used recursive strategy, a problem is divided into subproblems whose solutions, having been obtained by further subdivision, are then combined to yield the solution to the original problem. Divide-and-conquer heuristics find applications in Euclidean optimization problems such as optimum-length matching (E.M. Reingold and K.J. Supowit 1983).

The following "bridge problem" is, in fact, a linear program: given two sets S_1 and S_2 of planar points separated by a line, find two points $p_1 \in S_1$ and $p_2 \in S_2$ such that the line segment $[p_1, p_2]$ is an upper edge of the convex hull $\text{conv}(S_1 \cup S_2)$, bridging the gap between the two sets. Or, through which edge does a given directed line leave the – not yet delineated – convex hull of n points in the plane? As a linear program of fixed dimension 2, the bridge problem can be solved in linear time. D.G. Kirkpatrick and R Seidel (1986) have used it along with a divide-and-conquer paradigm to devise an $O(n \log m)$ algorithm for the planar convex hull of n points, m of which are extreme.

When implementing a divide-and-conquer strategy, one typically wishes to divide a set of points $S \subset \mathcal{R}^d$ by a straight line into two parts of essentially equal cardinality, i.e., to execute a "ham-sandwich cut". This can be achieved by finding the median of, say, the first coordinates of the points in S. It is a fundamental result of the theory of algorithms, that the median of a finite set of numbers can be found in linear time. The bridge problem is equivalent to a double ham-sandwich cut of a planar set: given a first cut, find a second line quartering the set. Threeway cuts in three dimensions and results about higher dimensions are reported in (D.P. Dobkin and H. Edelsbrunner 1984).

The Euclidean "post office problem" is a prototype for a class of proximity search problems encountered, for instance, in the implementation of *Expert Systems*. "Sites" p_i of n "post offices" in $\mathcal{R}^d$ are given, and the task is to provide suitable preprocessing for efficiently identifying a post office closest to any client location.

Associated with this problem is the division of space into "postal" regions, that is, sets of locations $V_i \subset \mathcal{R}^d$ closer to "postal" site p_i than to any other site p_j. Each such region V_i around site p_i is a convex polyhedron, whose facets are determined by perpendicular bisectors, i.e. (hyper)planes or lines of equal distance from two distinct sites. Those polyhedra form a polyhedral complex covering $\mathcal{R}^d$ known as a "Voronoi diagram". For details on this particularly fruitful concept and its dual, the "Delaunay triangulation", consult the separate entry for the *Voronoi/Delaunay Construct.*

Once a Delaunay triangulation of a planar set of n sites has been established, – an $O(n \log n)$ procedure –, a pair of "nearest points" among these sites can be found in linear time. The use of Delaunay triangulations for computational geometric problems was pioneered by M.I. Shamos and D. Hoey (1975).

The problem of efficiently finding for an arbitrary query point p a Voronoi cell $V_i \ni p$ is an instance of "point location" in subdivisions. Practical algorithms for locating a given point in a subdivision of the plane generated by n line segments in time $O(\log n)$ requiring preprocessing of order $O(n \log n)$ and storage of size $O(n \log n)$ or $O(n)$, respectively, have been proposed (see F.P. Preparata 1990). For point location in planar Voronoi diagrams , H. Edelsbrunner and H.A. Maurer (1985) utilize acyclic graphs and *Packing*. For a probabilistic approach to the post office problem see K.L. Clarkson (1985).

Whether a given point lies in a certain simple polygon can be decided by an $O(n)$ process of examining the boundary intersections of an arbitrary ray emanating from the point in question. For convex polygons, an $O(n)$ preprocessing procedure permits subsequent "point inclusion" queries to be answered in $O(\log n)$ time (J.L. Bentley and W. Carruthers 1980).

Let $h_e(x)$ be the truth function expressing point inclusion in the halfplane to the left of a directed line segment e. N. Muhidinov and S. Nazirov (1978) have shown that a polygonal set can be characterized by a boolean expression of n such functions, one for each edge e of the polygonal set, where each such function occurs only once in the expression. This boolean expression transforms readily to an algebraic expression for the characteristic function of the polygon. For 3-dimensional polyhedral bodies, D. Dobkin, L. Guibas, J. Hershberger, and J. Snoeyink (1988) investigate the existence and determination of analogous "constructive solid geometry (CSG)" representations (they may require repeats of halfspace truth functions). In general, CSG representations use boolean operations to combine primitive shapes, and are at the root of some commercial CAD/CAM and display systems. For a survey of methods for representing solid objects see A.A.G. Requicha (1980).

Given a family of polygons, a natural generalization of point inclusion is to ask how many of those polygons include a query point. This and similar intersection-related problems are subsumed under the term "stabbing". The classical 1-dimensional stabbing problem involves n intervals. Here the "stabbing number" can be found in $O(\log n)$ time and $O(n)$ space after suitable preprocessing. Similar results hold for special classes of polygons such as rectangles (see H. Edelsbrunner 1983).

Sweep-techniques rival divide-and-conquer in popularity. "Plane-sweep" or "line-sweep", for instance, conceptually moves a vertical line from left to right over the plane, registering objects as it passes them. Plane-sweep permits to decide in $O(n \log n)$ time (optimal complexity) whether n line segments in the plane have at least one intersection (M.I. Shamos and D. Hoey, 1976).

Important special cases of the above intersection problem are testing for (self-)intersection of paths and polygons. "polygon simplicity" can be tested for in linear time by trying to "triangulate" the polygon.

"Polygon triangulation", more precisely, decomposing the interior of a simple polygon into triangles whose vertices are also vertices of the polygon, is a celebrated problem of computational geometry. In a seminal paper, M.R. Garey, D.S. Johnson, F.P. Preparata, and R.E. Tarjan (1978) proposed an $O(n \log n)$ algorithm for triangulating a simple polygon of n vertices. They used a plane sweep approach for decomposing the polygon into "monotone polygons", which can each be triangulated in linear time. A related idea is to provide a "trapezoidization" of the polygon, from which a triangulation can be obtained in linear time. In a final development, B. Chazelle (1990) introduced the concept of a "visibility map", a tree structure which might be considered a local trapezoidization of the polygon, and based on it an $O(n)$ triangulation algorithm for simple polygons. In 3-space, an analogous "tertrahedrulization" (without additional "Steiner" points for vertices) for nonconvex polyhedral bodies may not exist. Moreover, the problem of deciding such existence is NP-complete (J. Ruppert and R. Seidel 1989).

For algorithms that depend on sequential examination of objects, "bucketing" (see L. Devroye 1986) may improve performance by providing advantageous sequencing. The idea is to partition an area into a regular pattern of simple shapes such as rectangles to be traversed in a specified sequence. The problem at hand is then addressed locally within buckets or bins followed by adjustments between subsequent or neighboring buckets. Bucketing-based algorithms have provide practical solutions to Euclidean optimization problems (T. Asano, M. Edahiro, H. Imai, and M. Iri 1985) such as shortest paths, optimum-length matching, and a Euclidean version of the *Chinese Postman Problem*: minimizing the pen movement of a plotter.

The techniques of "quadtrees and octrees" (see H. Samet 1990,1990a) might be considered as hierarchical approaches to bucketing, and are often the methods of choice for image processing and spatial data analysis including surface representation.

The position of bodies and parts of bodies, relative to each other in space, determines visibility from given vantage points, shadows cast upon each other, and impediments to motion. "Hidden line" and "hidden surface" algorithms (e.g.,I.E. Sutherland, R.F. Sproull, and R.A. Shumacker 1974) are essential in computer graphics, as are procedures for shadow generation (P.Atherton, K.Weiler, and D.P. Greenberg 1978) and shading. W.R. Franklin (1980) uses bucketing techniques for an exact hidden surface algorithm.

T.Lozano-Pérez and M.A. Wesley (1979) used the concept of a "visibility graph" for planning collision-free paths: given a collection of mutually disjoint polyhedral objects, the node set of the above graph is the set of all vertices of those polyhedral objects, and two such nodes are connected if the two corresponding vertices are visible from each other.

The "piano movers problem" captures the essence of "motion planning" (J.T. Schwartz and M. Sharir 1983, 1989). Here a 2-dimensional polygonal figure, or a line segment ("ladder"), is to be moved, both translating and rotating, amidst polygonal barriers.

Geometric objects encountered in many areas such as computer-aided design (CAD) are fundamentally nonlinear (see D.P. Dobkin and D.L. Souvaine 1990). The major thrust is generation of classes of curves and surfaces with which to interpolate, approximate, or generally speaking, represent data sets and object boundaries (see R.E. Barnhill 1977, and texts R.H. Bartels, J.C. Beatty, and B.A. Barski 1987, G. Farin 1988). A classical approach – building on the concepts of "splines" and "finite elements" – has been to use piecewise polynomial functions over polyhedral tilings such as triangulations. Examples are the "TIN(=triangulated irregular network)" approach popular in terrain modeling (see M. Heller 1990), C^1 functions over triangulations (C.L. Lawson 1977), and the arduous solution of the corresponding C^2 problem (P. Alfeld and R.E. Barnhill 1984).

"Bézier curves" and surfaces involve an elegant new concept: the use of "control points" to define elements of curves and surfaces, permitting intuition-guided manipulation important in CAD (see A.R. Forrest 1972). In general, polynomials are increasingly supplanted by rational functions, which suffer fewer oscillations per numbers of coefficients (see W. Tiller 1983). All these techniques culminate in "NURBS(=non-uniform rational B-splines)", which are currently recommended for curve and surface representation in most industrial applications.

In geometric calculations, round-off errors due to floating point arithmetic may cause major problems (e.g. S. Fortune and V. Milenkovic 1991). When testing, for instance, whether given points are collinear, a tolerance level "*eps*" is often specified, below which deviations from a collinearity criterion are ignored. Points p_1, p_2, p_3 and p_2, p_3, p_4, but not p_1, p_2, p_4, may thus be found collinear. Such and similar inconsistencies may cause a computation to abort. "Robust" algorithms (e.g., L.J. Guibas, D. Salesin, and J. Stolfi 1989, I. Beichl and F. Sullivan 1990) are constructed so as to avoid breakdown due to inconsistencies caused by round-off. Alternatively, various forms of "exact arithmetic" are increasingly employed (S. Fortune and C. Van Wyck 1993, C. Yap 1993). Inconsistencies occur typically whenever an inequality criterion is satisfied as an equality. An example is the degeneracy behavior of the *Simplex Method* of linear programming. Lexicographic perturbation methods can be employed to to make consistent selections of subsequent feasible bases and thus assure convergence. Similar consistent tie breaking, coupled with exact arithmetic, is the aim of the "simulation of simplicity" approach proposed by H. Edelsbrunner and E.P. Mücke (1988) in a more general computational context.

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Voronoi/Delaunay Construct. Given a finite set S of "sites" p_i located in Euclidean space $\mathcal{R}^d$, the "Voronoi polyhedron" $V(p_j)$ of site p_j is the set of all points $p \in \mathcal{R}^d$ which are at least as close to site p_j as to any other site p_i (see "post office problem" under *Computational Geometry*).

Such a Voronoi polyhedron ("Thyssen polygon", "Wigner-Seitz cell") is convex, its facets determined by perpendicular bisectors – (hyper)planes or lines of equal Euclidean distance from two distinct sites. The Voronoi polyhedra $V(p_i), p_i \in S$ cover the space $\mathcal{R}^d$ and define a polyhedral cell-complex known as a "Voronoi diagram" (G.Voronoi 1908) or "Dirichlet tesselation" (G.L.Dirichlet 1850). For a survey consult, for instance, F. Aurenhammer (1991), the text by A.Okabe, B. Boots, and K. Sugihara, Spatial tesselations: concepts and applications of Voronoi diagrams (1992), and the general texts cited in the entry *Computational Geometry.*

The cells of the dual complex are convex and, in general, simplicial. By partitioning nonsimplicial cells of the dual complex into simplices, the "Delaunay triangulation" results (Fig.1). It provides a canonical scheme for triangulating the convex hull of an arbitrary set $S \subset \mathcal{R}^d$ of sites, with these sites as vertices. The Delaunay graph provides in the plane an analog – insofar as there can be one – to sorting on the line.

For each site $p_i \in S$, the Delaunay triangulation contains an edge from p_i to each of its nearest Euclidean neighbors $q \in S$. In particular, edges in that triangulation connect all pairs of points of minimum distance in S. The 1-skeleton of the Delaunay triangulation contains the "relative neighborhood graph" (G.T.Toussaint 1979), which in turn contains a Euclidean *Minimum Spanning Tree.* The Delaunay triangulation thus provides a convenient tool for solving various proximity problems (M.I. Shamos and D. Hoey 1975). Delaunay triangulations avoid narrow triangles (see below) as much as possible, are essentially unique, and are readily determined. They are often the triangulation of choice for constructing piecewise-linear surfaces and for applications of "finite element" techniques in engineering.

Delaunay triangulations are characterized by the "empty sphere criterion": the circumsphere of a simplex in a Delaunay triangulation does not contain any of the triangulation vertices in its interior (B.Delaunay 1934). This criterion determines a triangulation uniquely in the absence of "degeneracy", that is, the occurrence of several simplices sharing a circumsphere.

In two dimensions, the empty circle criterion is equivalent to the requirement that the ascending sequence of angles, formed by selecting a smallest interior angle from each triangle in the triangulation, lexicographically maximizes the corresponding sequences for all triangulations of the same vertex set ("equiangularity"). The requirement that the sequence of *all* interior angles be lexicographically maximum is, in the presence of degeneracy, stronger, and can therefore serve in some instances as a tie-breaker in the presence of degeneracy.

The Delaunay triangulation of a set $S \subset \mathcal{R}^d$ of n sites can be obtained as a projection of the face lattice of the convex hull of n suitable points in $\mathcal{R}^{d+1}$. Those points can be chosen on a sphere – "stereographic projection" – or on a rotational paraboloid whose axis is perpendicular to the space of the triangulation. This implies that the Voronoi/Delaunay problem in d dimensions is computationally subsumed under the strong formulation of the convex hull problem (see *Computational Geometry*) in $d + 1$ dimensions.

In order to check whether a given triangulation satisfies the empty sphere criterion, it is not necessary to verify that criterion for each simplex by scanning all sites which are not vertices of the simplex: only pairs of facet-adjacent simplices whose union is convex need to be examined as to whether anyone of the two vertices not in the common facet might lie in the interior of its opposite circumsphere. This corresponds to establishing convexity of a (hyper)surface by examining the angles at which adjacent facets are joined. In two dimensions, the above criterion reduces to checking each strictly convex quadrangle formed by edge-adjacent triangles as to whether the correct diagonal of the quadrangle belongs to the triangulation (C.L. Lawson 1977). Based on this observation, several simple and efficient methods such as the "insertion method" swap diagonals in quadrangles. Alternatively, "divide-and-conquer" as well as "plane sweep" techniques (see *Computational Geometry*) yield $O(n \log n)$ algorithms for planar Delaunay triangulation of n sites. For determination of Voronoi diagrams in linear expected time see J.L. Bentley, B.W. Weide, and A.C. Yao (1980) and R.A. Dwyer (1991).

In many applications, it is desirable to construct a planar triangulation with some prescribed edges while preserving the advantages – avoiding unnecessarily narrow triangles, essential uniqueness – of the Delaunay approach. In that case, the empty circle criterion can be generalized by testing for potential inclusion only those sites whose "visibility" from any point of the triangle is not blocked by a prescribed edge. This generalized empty circle criterion defines a "constrained Delaunay triangulation", which is unique except for sites on the peripheries of empty circles (L. De Floriani and E. Puppo 1988).

A second important generalization of the Voronoi diagram is the "power diagram" (see F.Aurenhammer 1987) or "radical Voronoi diagram" (B.J. Gellatly and J.L. Finney 1982). Here sites may be enlarged to spheres of positive radius. The intersection, real or imaginary, of two spheres lies on and defines the "radical" (hyper)plane of that pair. These (hyper)planes then play the role of the perpendicular bisectors in the classical Voronoi diagram. The radical Voronoi diagram of site spheres of radius $r_i \geq 0$ centered at locations $p_i \in \mathcal{R}^d$, respectively, can be obtained by intersecting the classical Voronoi diagram for the sites (p_i, r_i) in $d + 1$ dimensions with the original d-dimensional space. Radical Voronoi diagrams are used in crystallography in order to account for differences in atomic radii.

The corresponding "radical Delaunay triangulation" in $\mathcal{R}^d$ satisfies a modified empty spheres criterion. For each simplex, consider the unique sphere K which is orthogonal to the $d+1$ site spheres. Then no other site sphere is "orthogonally interior" to sphere K, that is, is shifted towards the center of K from a position orthogonal to K.

There are numerous other generalizations of the Voronoi/Delaunay construct. Alternatives to the Euclidean norm, as well as general sets instead of single point sites, are considered. There are "order-k", "furthest point", "weighed", "discrete", and "abstract" Voronoi diagrams. Voronoi constructs based on Euclidean metric are instances of cell-complexes derived from "arrangements" of hyperplanes. Data structures, algorithms and combinatorial results concerning such cell-complexes in general are presented by H. Edelsbrunner, J. O'Rourke, and R. Seidel (1986).

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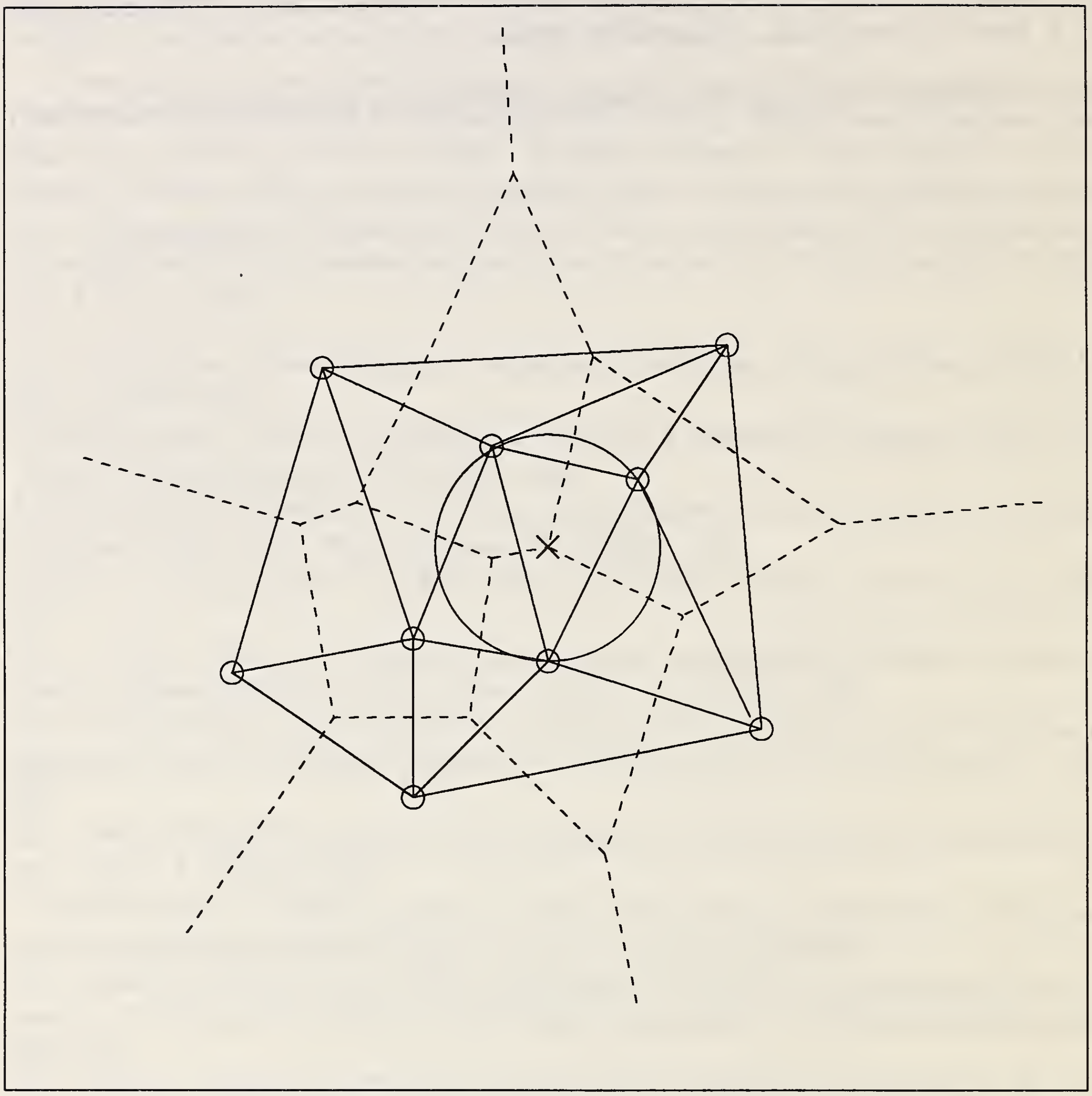

Figure 1: Planar Voronoi diagram (dashed lines) and Delaunay triangulation of nine sites. The circle around one of the Delaunay triangles illustrates the "empty circle criterion".