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Observations About Joined Circular Arcs

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Observations About Joined Circular Arcs

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Abstract. Smoothly joined pairs of circular arcs – termed biarcs – may serve as a device for data fitting with smooth piecewise circular curves. Geometric properties of such biarcs are investigated.

Key words: biarcs, circle pencils, cocyclic tangents, curve-fitting, piecewise circular, plane geometry

Introduction. There is interest in interpolating and approximating strings of points in the plane by piecewise-circular smooth curves because that representation lends itself readily to some computer-automated manufacturing processes. In particular, when interpolating a string of given consecutively distinct planar points, it is commonly assumed that these points are also the "knots" of the interpolating piecewise-circular curve, that is, the points at which successive circular arcs connect. If an initial direction is specified, such interpolating curves are uniquely determined by the string of points. This interpolation problem becomes overdetermined, however, if directions are prescribed at all points of the string. In that case, pairs of circular arcs joined together smoothly at some suitable intermediate point – configurations termed "biarcs" (K.M.Bolton [3]) – can be used to connect successive points with prescribed directions. Several geometric observations concerning families of biarcs will be reported in this paper.

It is unlikely that some of these observations have not been made before, particularly in the older literature, but search for a record has been unsuccessful so far. For general geometric background material, the reader may want to consult H.E.Baker [1], M.Berger [2], R.A.Johnson [4], or D.Wells [5].

1. Circular arcs and biarcs. The

(1.1) *"circular arcs"*

considered here are directed and have distinct start and endpoints. Also, they should not cover a full circle. Depending on whether they move counterclockwise or clockwise, they are, respectively, defined as the directed curves

$$A = \{(x, y) \in R^2 : \ x = \hat{x} + r\cos\varphi, \ y = \hat{y} + r\sin\varphi, \ \varphi_s \le \varphi \le \varphi_e\},$$

where $\varphi_s < \varphi_e$, and

$$A = \{(x, y) \in R^2 : \ x = \hat{x} + r\cos\varphi, \ y = \hat{y} + r\sin\varphi, \ \varphi_s \ge \varphi \ge \varphi_e\},$$

where $\varphi_s > \varphi_e$. In both cases, it is understood that $0 \leq \varphi_s < 2\pi$ and $|\varphi_e - \varphi_s| < 2\pi$. In the above definitions, the point $(\hat{x}, \hat{y})$ is the center of the arc, and $r > 0$ is its radius. At every point $(x(\varphi), y(\varphi))$, a circular arc A has a tangential direction $(-\sin\varphi, \cos\varphi)$, the term "direction" being reserved for vectors of length 1.

For each circular arc (1.1), there is a

(1.2) *"complement"* or *"complementary arc"*

that is, the arc

$$\bar{A} = \{(x,y) \in R^2 : \; x = \hat{x} + r\cos\varphi, \; y = \hat{y} + r\sin\varphi, \; \varphi_s \geq \varphi \geq \varphi_e\},$$

or the arc

$$\bar{A} = \{(x,y) \in R^2 : \; x = \hat{x} + r\cos\varphi, \; y = \hat{y} + r\sin\varphi, \; \varphi_s \leq \varphi \leq \varphi_e\},$$

respectively. In other words, the complement of a conterclockwise arc is the clockwise arc with the same start and end angles, and vice versa. The union of a circular arc and its complement covers a circle, duplicating only the start and endpoints.

In the context of piecewise circular curves, straight line segments

$$A = \{(x,y) \in R^2 : \; x = x_s + \theta(x_e - x_s), y = y_s + \theta(y_e - y_s), \; 0 \leq \theta \leq 1\},$$

with $(x_s, y_s) \neq (x_e, y_e)$, are typically considered among the circular arcs, representing the degenerate case $r = \infty$. Included among those

(1.3) *"line-degenerate arcs"*

are also arcs which consist of an entire line but exclude a straight line segment. Such an "arc" is of the form

$$A = \{(x,y) \in R^2 : \; x = x_s + \theta(x_e - x_s), y = y_s + \theta(y_e - y_s), \; 0 \geq \theta \text{ or } \theta \geq 1\},$$

may be viewed as a straight line segment passing through ∞, and will be considered the complement (1.2) of a finite straight line-degenerate arc. The tangential directions of line-degenerate arcs are given by

$$+\frac{1}{L}(x_e - x_s, y_e - y_s) \quad \text{or} \quad -\frac{1}{L}(x_e - x_s, y_e - y_s) \quad \text{with} \quad L = \sqrt{(x_e - x_s)^2 + (y_e - y_s)^2},$$

depending on whether the arc is finite or infinite. Those tangential directions are the same at each point of a line-degenerate arc.

Two arcs joined together smoothly, that is, the endpoint of the first arc is the start point of the other and the tangential directions of the two arcs coincide at that juncture, form what is called here a

(1.4) *"biarc".*

The point at which the two arcs of a biarc meet is called its

(1.5) *"knot".*

The start point of the first arc and the endpoint of the second arc will be considered the

(1.6) *"origin"* P_o and the *"destination"* P_d

of the biarc. Correspondingly, the tangential directions of the respective arcs at the origin and the destination will be referred to as the

(1.7) *"origin"* and *"destination (tangential) directions"* T_o and T_d

($\|T_o\| = \|T_d\|=1$) (Figures 1 and 2). Note also that the arcs of a biarc may be line-degenerate (1.3) (Figure 3).

The complements (1.2) of the two arcs in a biarc form again a biarc, the

(1.8) *"complement"* or *"complementary biarc"* of the original biarc. Figures 2 and 3 display instances of biarcs together with their complements. The latter are indicated by dashed lines. The heavy lines with arrows at the origin P_o and destination P_d indicate the tangential directions T_o and T_d, respectively.

It will be shown that, given any two distinct points $P_o \neq P_d$ with tangential directions T_o and T_d, biarcs exist which originate at P_o in direction T_o and terminate at P_d with direction T_d. Indeed, the following simple construction will work in most cases. Choose a circle that avoids point P_d but passes through P_o with tangent T_o. Then there exists usually a unique second circle through P_d with tangent T_d that touches the first circle at, say, point K. The two constituent arcs of a suitable biarc are then chosen in the obvious fashion from the two circles with point K as knot. This argument will be made more precise later on (see Lemma (3.4).

Since there is one degree of freedom in the choice of the first circle, the biarcs meeting the above specifications may be expected to form a 1-parameter family, as do the knots of these biarcs.

The purpose of this work is to characterize the locus of knots of such families of biarcs.

2. Families of biarcs. Let

(2.1) $\mathcal{B}(P_o, T_o, P_d, T_d)$

denote the family (=set) of biarcs from P_o to P_d, assuming the two given tangential directions T_o and T_d, respectively, at those points. Within that family, biarcs are determined uniquely by their knots. The biarcs in the

(2.2) *"complementary family"* $\bar{\mathcal{B}} = \mathcal{B}(P_o, -T_o, P_d, -T_d)$

are complementary (1.8) to the biarcs in the family (2.1), and vice versa. Any family of biarcs $\mathcal{B}$ and its complementary family $\bar{\mathcal{B}}$ have therefore the same set of knots. Of interest will be the relationship of the family (2.1) of biarcs to its

(2.3) *"opposite family"* $\mathcal{B}^* = \mathcal{B}(P_o, T_o, P_d, -T_d)$

that is, the family that differs from (2.1) only in that the destination direction is reversed. Figures 1 and 2 feature biarcs of opposite families.

The straight tangent line through the origin P_o in direction T_o, and the analogous line through the destination P_d of the biarc in direction T_d, will play a role in the subsequent geometric arguments, and will be referred to as

(2.4) *end tangents*

of the family of biarcs (2.1). Note that the complementary family (2.2) and the opposite family (2.3) both have the same end tangents as the original family.

3. The symmetric case. Before examining that problem in general, an important special case requires attention. Given the tangential directions T_o and T_d, suppose that the vector $T_o + T_d$ does not vanish and is parallel to the base line P_o, P_d:

$$T_o + T_d = \eta(P_d - P_o), \ \eta \neq 0$$

(see Figure 4). In this case, the simple construction of a specified biarc outlined in Section 1 does not work. However, there exists a single circular arc A from origin P_o to destination P_d with the prescribed tangential directions T_o and T_d. Thus any point on arc A – other than P_o or P_d – can be chosen as a knot K between two circular subarcs. Such a family of biarcs $\mathcal{B}_s$ will be called a

(3.1) *"symmetric family"*,

and the circle containing arc A will be referred to as the

(3.2) *"source circle"*

of the symmetric family $\mathcal{B}_s$. Definition (3.1) includes the family $\mathcal{B}(P_o, T_o = T, P_d, T_d = T)$, with direction T parallel to the base line P_oP_d. In that case, the arc A is line-degenerate (1.3), and the source circle becomes the base line.

The following observation highlights the special role of symmetric families.

(3.3) *Lemma: All biarcs in a symmetric family* $\mathcal{B}_s$ *are contained in its source circle.*

Proof: Let A_o and A_d be the first and second arcs of a biarc in a symmetric family, and let C_o and C_d be their respective circles. There are two circles C_1 and C_2 tangential to circle C_o and passing through P_d with tangential direction T_d. One of these two circles must be C_d. In order to distinguish which circle is in fact C_d, consider the orientations conferred on circle C_o by tangential direction T_o and on circles C_1 and C_2 by T_d. Then one of the two circles C_1 and C_2 touches circle C_o with equal tangential direction, whereas the other touches C_o with opposite tangential direction. The former circle must be the one which equals C_d, because biarcs must have equal tangential directions at their knots. Circle C_d is therefore uniquely defined by the above conditions. It is now easily seen, that the source circle also meets these conditions: it passes through P_d with tangential direction T_d, through P_o with tangential direction P_o, and thus touches circle C_o at P_o with common tangential direction. It coincides, therefore, with C_d, which implies that arc A_d is contained in the source circle. By symmetric argument, it follows that arc A_o, too, is contained in the source cycle. □

In view of Lemma (3.3), the following statement holds only in the nonsymmetric case.

(3.4) *Lemma: Given a nonsymmetric family* $\mathcal{B} = \mathcal{B}(P_o, T_o, P_d, T_d)$, *and any circle* C_o *through origin* P_o *with tangential direction* T_o, *but not meeting destination* P_d, *then there exists in* B *a unique biarc whose first arc* A_o *is contained in the circle* C_o.

Proof: Since $P_d \notin C_o$, the same argument as in the previous proof shows that there is a unique circle C_d which passes through P_d with tangential direction T_d and touches circle C_o with common tangential direction at some point K. Now $K \neq P_o$ as otherwise family $\mathcal{B}$ would be symmetric. Arc A_o can thus be chosen as proceeding in circle C_o from origin P_o to knot K. As $K \in C_o$, but $P_d \notin C_o$ by hypothesis, $K \neq P_d$, and there is a circular arc A_d from K to destination P_d within circle C_d. The two arcs A_o and A_d combine to form a biarc in family $\mathcal{B}$. The knot K of that biarc is uniquely defined by the parameters of the family $\mathcal{B}$ and the choice of circle C_o, and so is the biarc. □

By Lemma (3.3), all knots of biarcs in a symmetric family (3.1) must lie on a circle, namely the source circle, which contains the circular arc A from origin P_o to destination P_d with correct tangential directions T_o and T_d. Choosing a knot K on the source circle outside arc A will result in an

(3.5) *"overlapping biarc".*

Indeed, the first arc of such a biarc will stretch from origin P_o to knot K, passing the destination P_d along the way, while the second arc starts at knot K and passes origin P_o on the way to its endpoint P_d – all within the same circle (or straight line). The proof of the following proposition is left to the reader.

(3.6) *Proposition: Symmetric families* (3.1) *are the only families of biarcs* (2.1) *which contain* *overlapping biarcs* (3.5). *All nonoverlapping biarcs are simple curves, that is, they do not have multiple points.*

It is also seen, that if overlapping biarcs are accepted, the locus of knots of biarcs in a symmetric family (3.1) is a circle through the origin P_o and the destination P_d, with these points removed. In the following section, It will be shown that this statement holds in general.

4. The knot circle.

The the following is a necessary condition for a point to be realizable as knot of a biarc in the family of biarcs (2.1).

(4.1)*Proposition: The locus of knots of bi-arcs in the family $\mathcal{B} = \mathcal{B}(P_o, T_o, P_d, T_d)$ is contained in a unique circle, the*

"knot circle",

through the points P_o and P_d. The knot circle may degenerate to a straight line.

Proof: Two main cases are illustrated in Figures 5 and 6, respectively. A special case is examined with reference to Figure 7. Of course there are further cases that need be examined for a complete proof. In all of them, the proof proceeds in analogous fashion. Exhaustive proofs may be obtained by routine analytical procedures. In this exposition, however, arguments are based on elementary geometry are preferred because they ilustrate the underlying geometrical relationships.

Figure 5 displays a biarc in family $\mathcal{B}(P_o, T_o, P_d, T_d)$. It starts at point P_o in direction T_o, moves along an arc until it meets a second arc at knot K. That second arc terminates at the point P_d in direction T_d. The following relations between the angles indicated in Figure 5 are obvious:

$$\begin{aligned} \omega + \rho + \sigma &= \pi \\ \alpha &= 2\rho \\ \beta &= 2\sigma \\ \alpha + \beta + \gamma &= \pi. \end{aligned}$$

It follows that

$$\omega = \pi - \rho - \sigma = \pi - \frac{\alpha + \beta}{2} = \frac{\pi + \gamma}{2}.$$

The angle ω thus depends only on the angle γ, and is therefore independent of the knot position K. In other words, the line segment P_oP_d appears from any knot K under the same angle, and lies therefore on a circle through points P_o, P_d.

For the angles displayed in Figure 6, one finds the similar relationships

$$\begin{aligned} \omega + \rho + \sigma &= \pi \\ \alpha &= 2\rho \\ \beta &= 2\sigma \\ \alpha + \beta &= \gamma. \end{aligned}$$

and consequently

$$\omega = \pi - \rho - \sigma = \pi - \frac{\alpha + \beta}{2} = \pi - \frac{\gamma}{2}.$$

Again ω is constant, and the locus of the knots is therefore part of a circle passing through the points P_o and P_d.

If the origin direction T_o equals the destination direction T_d, then the corresponding family of biarcs (see Figure 7)

$$\mathcal{B}_p = \mathcal{B}(P_o, T_o, P_d, T_d = T_o)$$

will be called a

(4.2) *"parallel family"*

of biarcs. It is readily seen, that any point on the full straight line P_oP_d with the exception of the points P_o and P_d is the knot of a biarc. To see the converse, consider a biarc in $\mathcal{B}_p$ with knot K. Let α denote the angle between the line P_oP_d and any of the two parallel end tangents (2.4) of family $\mathcal{B}_p$, and draw a straight line through K at the supplementary angle $\pi - \alpha$ to the line P_oP_d. As indicated in Figure 7, this line intersects the end tangents in points I_o and I_d. The triangles P_oKI_o and P_dKI_d are similar, and the points K, P_o, P_d are therefore collinear. □

(4.6) *Proposition: A point $K \neq P_o, P_d$ on the knot circle of a family of biarcs $\mathcal{B}(P_o, T_o, P_d, T_d)$ determines in that family a unique biarc which has K as a knot.*

Proof: In Section 3, the proposition was established for symmetric families (3.1) of biarcs. It is thus assumed that the family $\mathcal{B}$ of biarcs is not symmetric. Consider the circle (or straight line) C_o through K and P_o with tangential direction T_o. Then by Lemma (3.4), there exists a biarc in $\mathcal{B}$ with knot K'. Both K and K' belong to the knot circle as well as to C_o. Since C_o is not the knot circle – that would be the symmetric case –, the knot circle and C_o have at most two points in common, one of them being P_o. Since neither K nor K' is equal to P_o, they must share the remaining location. Thus $K = K'$. □

5. Characterizing knot circles. Consider any family $\mathcal{B} = \mathcal{B}(P_o, T_o, P_d, T_d)$ that is not symmetric (3.1) and whose opposite (2.3) family $\mathcal{B}^* = \mathcal{B}(P_o, T_o, P_d, -T_d)$ is not symmetric either. Then both $\mathcal{B}$ and $\mathcal{B}^*$ contain biarcs that are partially line-degenerate (1.3). Indeed, since the origin P_o does not lie on the end tangent through the destination P_d, there exists a circle through P_o that is tangential to both end tangents. By the nonsymmetry assumptions, that circle does not meet P_d, and by Lemma (3.4), there exists a biarc whose first arc A_o belongs to that circle, whose knot $K \neq P_d$ lies on the other end tangent, and whose second arc is, therefore, line-degenerate. Figure 8 shows such a biarc, where the second arc moves from knot K on the end tangent through the infinite point of the latter to the destination P_d.

Note that in either case the knot K is in symmetric position vis-a-vis the origin P_o, that is, it represents the mirror image of P_o with respect to an angle bisector of the the end tangents (2.4). More precisely, symmetry holds with respect to this bisector – drawn dashed in Figure 8 – whose direction is parallel to $T_o - T_d$, $\|T_o\| = \|T_d\| = 1$. This symmetry implies that the center C_k of the knot circle is to be found on the that bisector of the end tangents.

In the case of a symmetric family $\mathcal{B}$, the knot circle coincides with the source circle (3.2) of that family. The knot circle of the opposite famliy $\mathcal{B}^*$ is the circle through P_o and P_d centered at the intersection of the end tangents, if the end tangents have a unique intersection. The proof of this fact, and the examination of remaining special cases is left to the reader. With suitable interpretations of infinite points and lines, the following proposition holds.

(5.1) *Proposition: The knot circle of a family of biarcs $\mathcal{B}(P_o, T_o, P_d, T_d)$ is the unique circle or straight line, passing through origin P_o and destination P_d, whose center lies on the line bisecting the two end tangents of the family in the direction parallel to the vector $T_o - T_d$*

Figure 8 also displays the knot circle of the opposite family, showing the two knot circles to be mutually orthogonal. In what follows, this observation will be confirmed in general.

6. Circles centered on alternate angle bisectors. In this section, the two sets of circles centered, respectively, on a pair of orthogonal lines will be examined. In particular, we will establish the

(6.1) *Proposition: Consider the two families of circles centered, respectively, on two mutually orthogonal lines intersecting at a point S. Consider a pair of circles, one from each family, intersecting in two points P_1 and P_2. That pair is orthogonal if and only if the lines through S and P_1 and S and P_2 are mutually symmetric with respect to the orthogonal lines.*

Proof: Figure 9 shows two – dotted – circles centered, respectively, at point C_x on the horizontal line through the point O, and at point C_y on the vertical line through O. The two circles are orthogonal if and only if the triangle $C_x I_1 C_y$ is a right triangle. In that case, the five points C_x, O, I_1, C_y, I_2 lie on a third circle. It follows that α and ϕ are peripheral angles of that circle over the chord C_yI_1; thus $\alpha = \phi$. Analogously, $\beta = \phi$, and therefore $\alpha = \beta$. This proves the "only if" direction of the proposition.

Now suppose $\alpha = \beta$. Then the chord I_1I_3 of the circle around C_x is bisected by the horizontal line through O. The angle ρ is therefore one half of the center angle of the chord I_1I_3. On the other hand, the angle σ is a peripheral angle of that chord. Thus $\rho = \sigma$. Considering triangle OSI_2,

$$\sigma = \pi - \beta - (\frac{\pi}{2} + \epsilon) = \frac{\pi}{2} - \beta - \epsilon.$$

Since the angle at C_x of triangle C_yC_xO is given by $\pi/2 - \epsilon$,

$$\rho = (\frac{\pi}{2} - \epsilon) - \phi = \frac{\pi}{2} - \phi - \epsilon.$$

Thus $\rho = \sigma$ implies $\beta = \phi = \alpha$. The angle α at point O, and the angle ϕ at point C_x are equal angles over the same line segment I_1C_y. Therefore the four points C_x, O, I_1, C_y are cocyclic. Since triangle C_xOC_y is a right triangle, the triangle $C_xI_1C_y$ is too. □

(6.2) *Corollary: The knot circles of two opposite families* (4.2) *of biarcs are mutually orthogonal. If the two end tangents shared by those families are parallel or identical, one of the knot circles may degenerate to a straight line.*

7. The knot-tangent circle.

7. The knot-tangent circle. What can be found about the

(7.1) *"knot tangents"*

to biarcs at their knots? It turns out that many results remain true if the roles of lines and circles are interchanged.

(7.1) *Proposition: The knot tangents of a family of biarcs $\mathcal{B} = \mathcal{B}(P_o, T_o, P_d, T_d)$ are cocyclic in the sense that they are all tangential to the same*

"knot-tangent circle",

which is concentric with the knot circle (4.1) *tangential to both end tangents* (2.4).

Proof: Figure 10 illustrates a major case. A biarc in family $\mathcal{B}$ is shown, with arc A_o from origin point P_o to knot K followed by arc A_d from K to the destination point P_d. By Proposition (4.1), the knot K lies on the – heavy dotted – knot circle, centered at C_k. It is claimed that the – heavy solid – knot tangent of the biarc is also tangential to the – heavy dashed – knot-tangent circle, also centered at C_k and tangential to the end tangents, which are determined by the tangential directions T_o and T_d.

In Figure 10, C_o is the center of the circle through arc A_o. This circle intersects the knot circle at the points P_o and P_d. The – dashed – line through the centers C_o and C_k, respectively, of those two circles is a line of symmetry for both. In particular, the points P_o and K are in symmetric position, and so are the end tangent through P_o and the knot tangent through K. Since the above end tangent is also a tangent of the knot-tangent circle, whose center lies on the line of symmetry, the symmetric image of the end tangent through P_o is again a tangent to the knot-tangent circle. But that symmetric image was already seen to be the knot tangent. □

Note that in the symmetric case (3.1), the knot-tangent circle coincides with the knot circle (4.1). For a family of biarcs opposite (2.3) to a symmetric family, all knot tangents pass through a single point, the intersection of the two end tangents. In the parallel case (4.2), all knot tangents are parallel to each other.

8. Resulting theorem on circle pencils. The results of the previous sections yield a theorem on "circle pencils", that is, sets of circles whose equations

$$F(x, y) \equiv Q(x^2 + y^2) + Ax + By + C = 0$$

are linear combinations, with weights λ_1 and λ_2 that do not vanish simultaneously,

$$F(x, y) = \lambda_1 F_1(x, y) + \lambda_2 F_2(x, y),$$

of the equations $F_1(x,y)$ and $F_2(x,y)$, respectively, of two distinct circles. Some of the resulting equations $F(x, y) = 0$ may have complex solutions only. Also, if the above circles are not concentric, the above pencil contains a single straight line

$$F_2(x, y) - F_1(x, y) \equiv (A_2 - A_1)x + (B_2 - B_1)y + (C_2 - C_1) = 0$$

known as the "radical axis" of the circle pencil. Examples of circle pencils include the set of all circles – and the straight line – through two distinct points as well as the set of all circles orthogonal to the circles in that pencil. Of interest here are

(8.1) *"tangential circle pencils".*

Such a pencil consists of circles which are all mutually tangential at the same point and whose radical axis is therefore a common tangent to all. The end tangent through P_o and the end tangent through P_d, where P_o and P_d are the origin and destination, respectively, of the biarcs in a family (2.1) $\mathcal{B}(P_o, T_o, P_d, T_d)$, define two such tangential circle pencils, which figure in the following theorem.

(8.2) *Theorem: Given straight lines Λ_o and Λ_d in the plane. Select point $P_o \in \Lambda_o$ and point $P_d \in \Lambda_d$ such that $P_o \neq P_d$. Consider the tangential pencil C_o of circles passing through point P_o with tangent Λ_o, and the analogous tangential pencil C_d of circles tangential to line Λ_d at point P_d.*

If the lines Λ_o and Λ_d have a unique intersection S, then *"locus of tangential points",*

that is, the locus of all points at which a circle in pencil C_o touches a circle in pencil C_d, consists of the two m utually orthogonal circles intersecting each other at the points P_o and P_d, and centered, respectively, on the two angle bisectors of the lines Λ_o and Λ_d at their intersection S.

If the lines Λ_o and Λ_d are parallel or identical, then the locus of all tangential points consists of the straight line P_oP_d together with the circle through points P_o and P_d centered at the intersection of line P_oP_d with the mid-parallel Λ_m of the parallels Λ_o and Λ_d.

In the general case,

"the set of common tangents",

that is, the set of all tangents common to two tangential circles at their common point consists of the tangents to two circles. These circles are tangential to the given lines Λ_o and Λ_d, and are concentric with the above mentioned pair of mutually orthogonal circles which are the locus of all tangential points. In special cases, one of the circles described by the set of common tangents degenerates to a single – possibly infinite – point.

In this formulation, single points are considered circles of radius zero and tangential to any other circle on which the lie. For this reason, the points P_o and P_d themselves are not excluded from the above description of the locus of osculation points of two circle pencils. A similar convention leads to counting the lines Λ_o and Λ_d among the common tangents.

The above theorem was about two distinct tangential circle pencils (8.1). It might be conjectured, that the locus of tangential points of any two circle pencils consists of circles. Simple counterexamples show this not to be the case.

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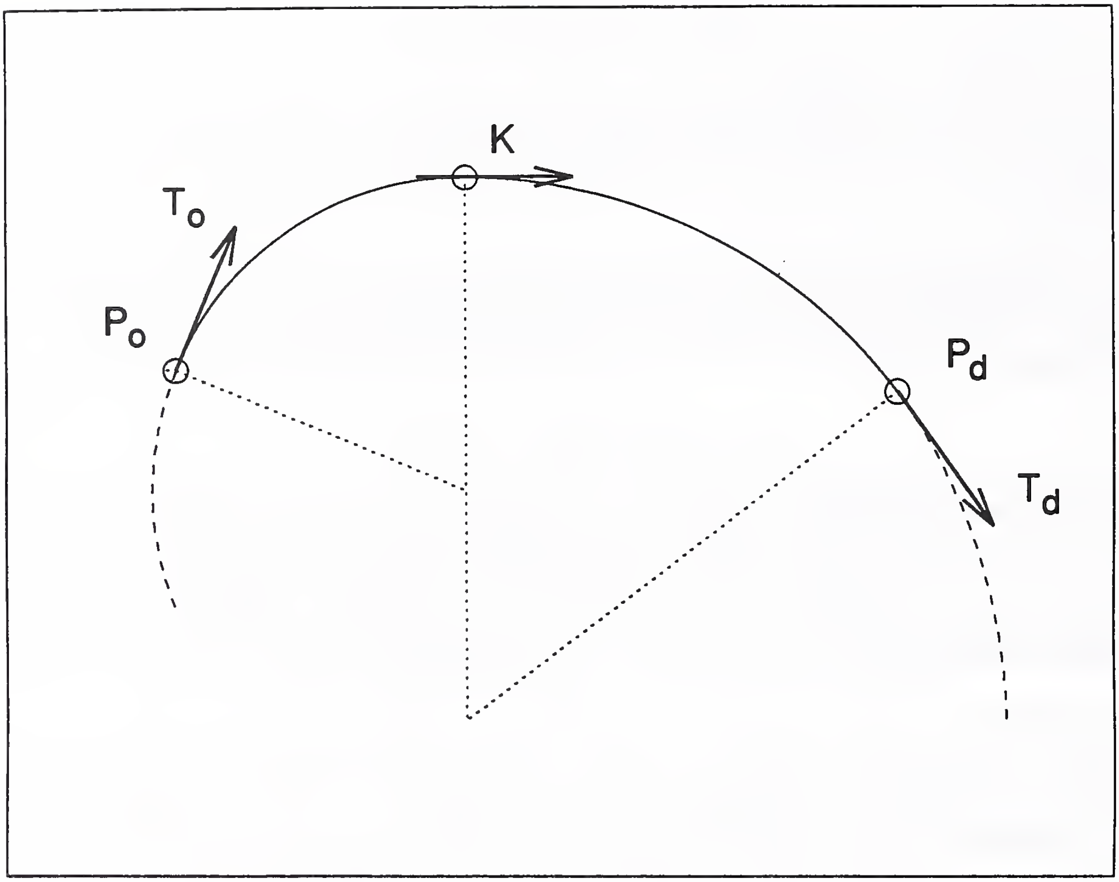

Figure 1: Biarc extending from point P_o with tangential direction T_o to point P_d with tangential direction T_d. The biarc consists of a circular arc from P_o to the knot K and a a second circular arc from K to P_d. Both circular arcs have the same tangent at knot K.

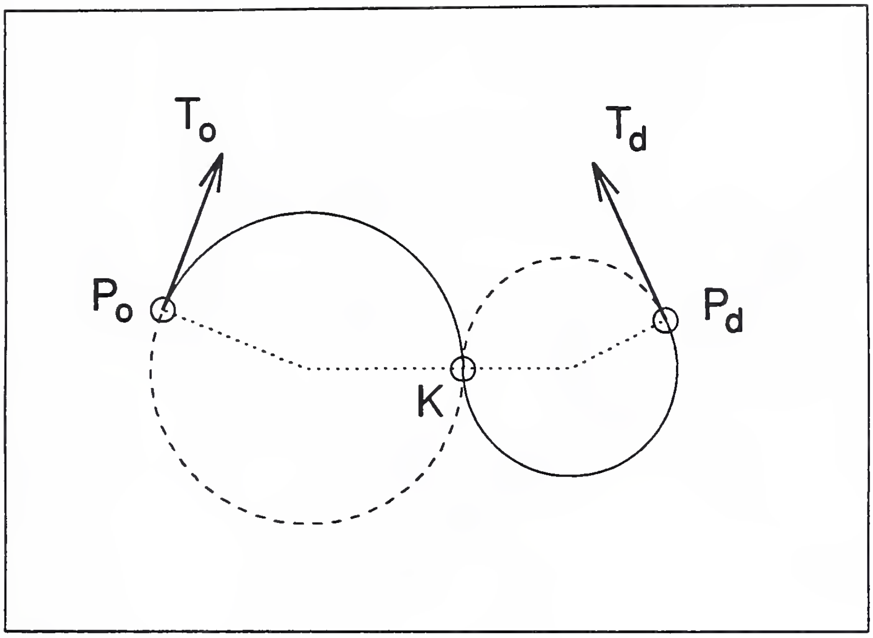

Figure 2: Origin P_o, destination P_d, and tangential direction T_o are as in Figure 1, but the tangential direction T_d at P_d has been replaced by its opposite. The biarc displayed here and the one shown in Figure 1 thus represent opposite families. The dashed curve indicates the complementary biarc.

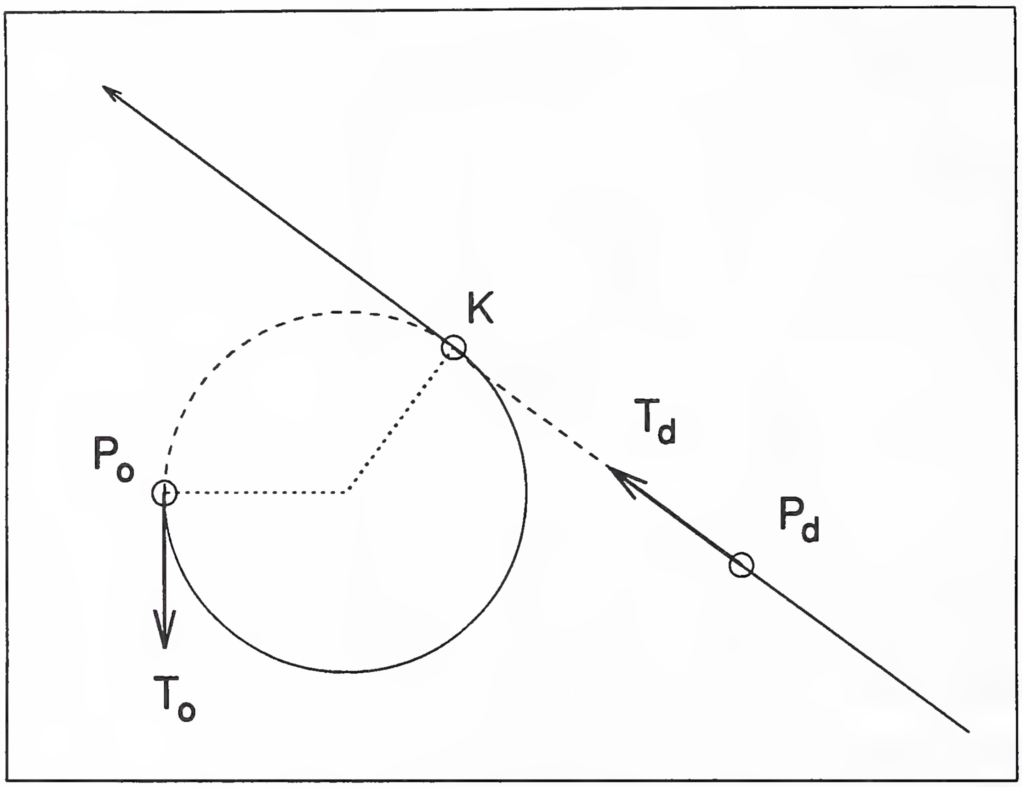

Figure 3: Example of a biarc containing a line-degenerate arc. The biarc follows a proper circular arc from origin P_o to knot K, and continues straight in direction T_d to ∞, passes through ∞, and returns within the same straight line of direction T_d to destination P_d. The dashed lines indicate the complementary biarc.

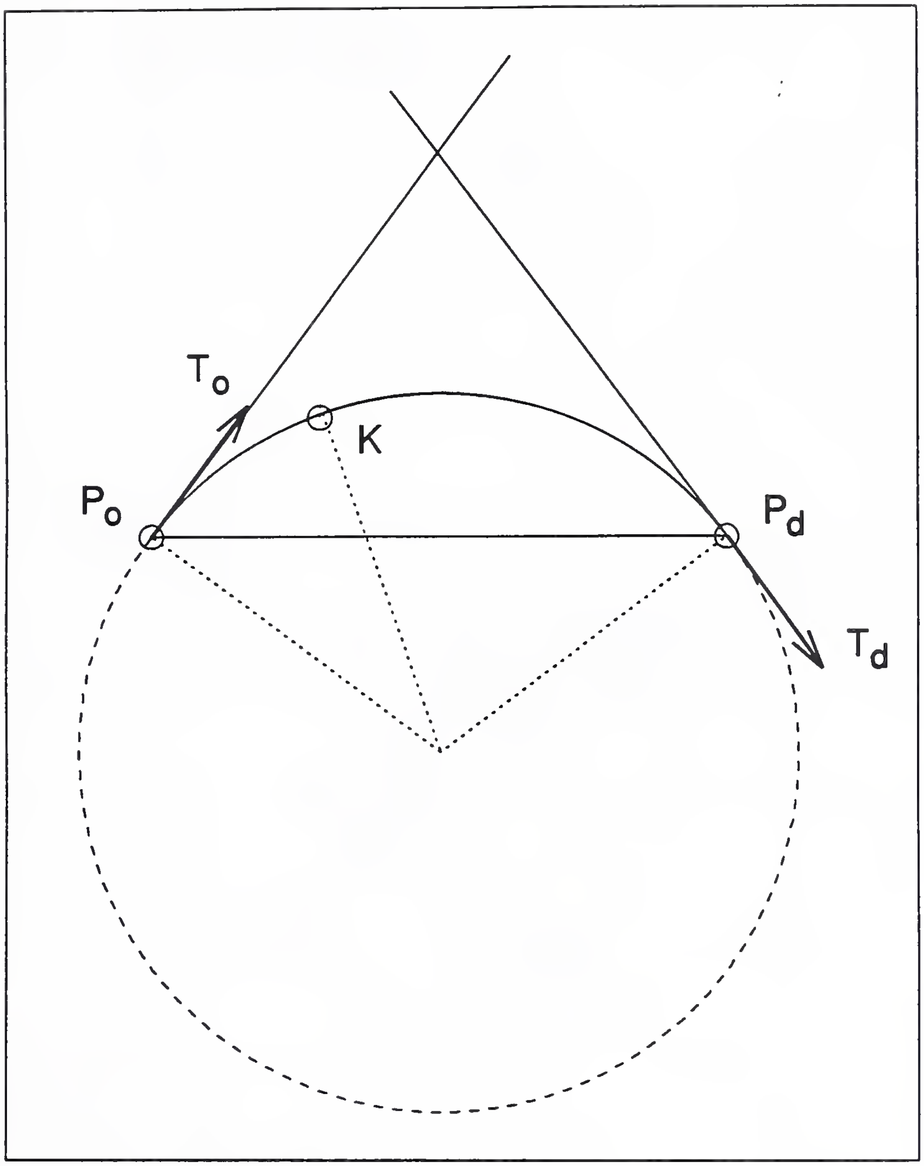

Figure 4: Biarc in the symmetric case. The two circular arcs connecting at K belong to the same circle.

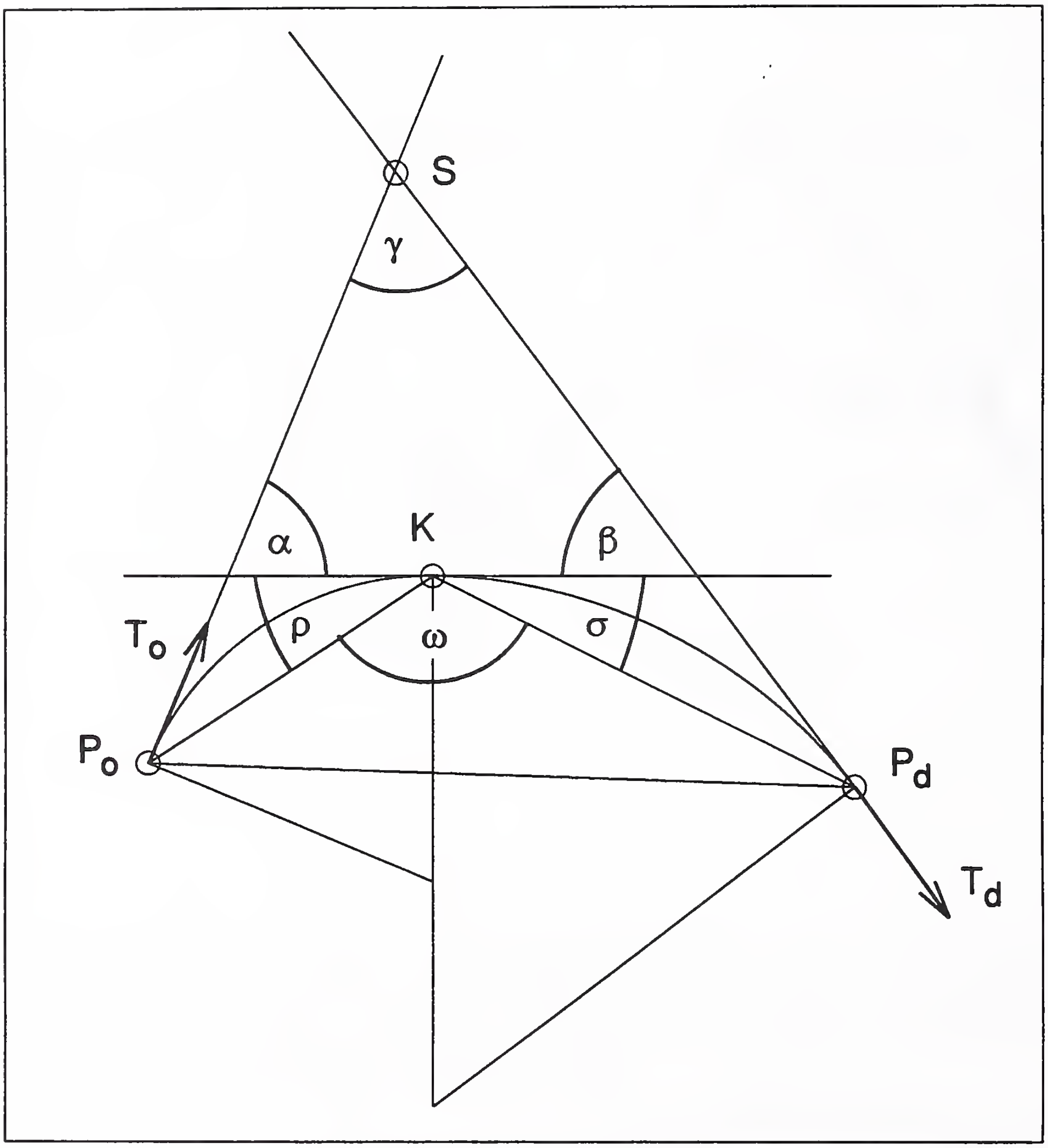

Figure 5: Illustration of first main case in the proof of Proposition (4.1).

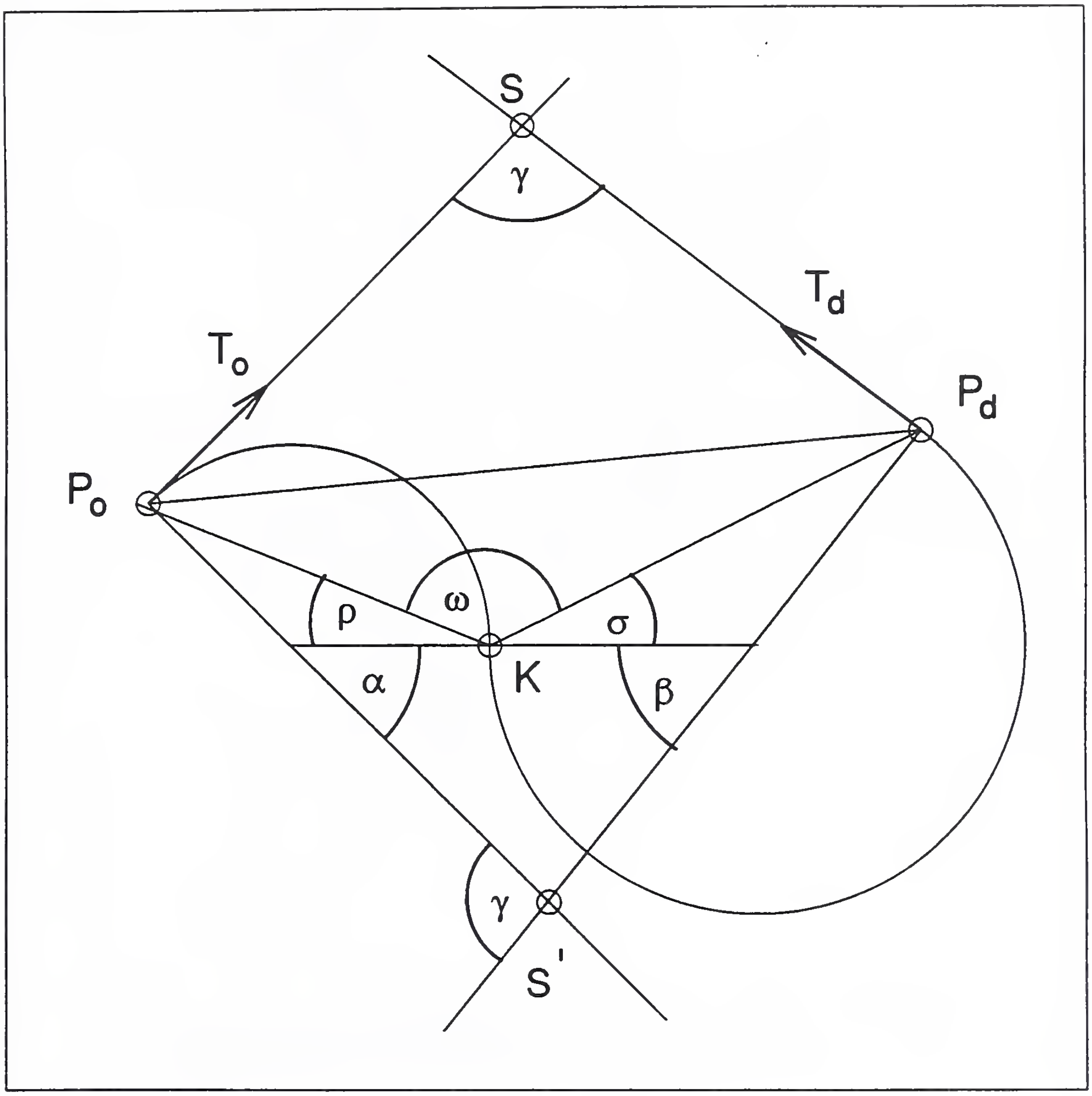

Figure 6: Illustration of second main case in the proof of Proposition (4.1).

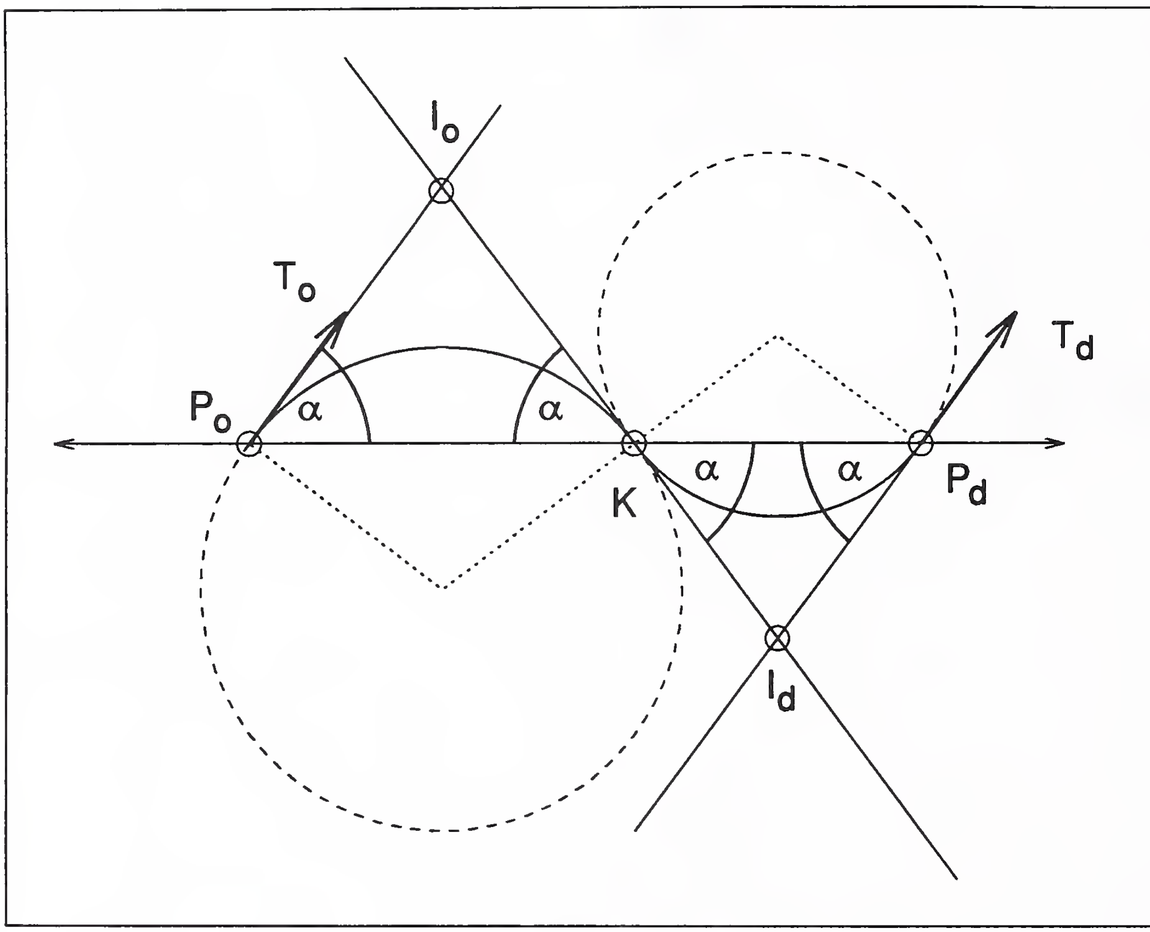

Figure 7: Illustration of Proposition (4.1) in the case of a parallel family (4.2).

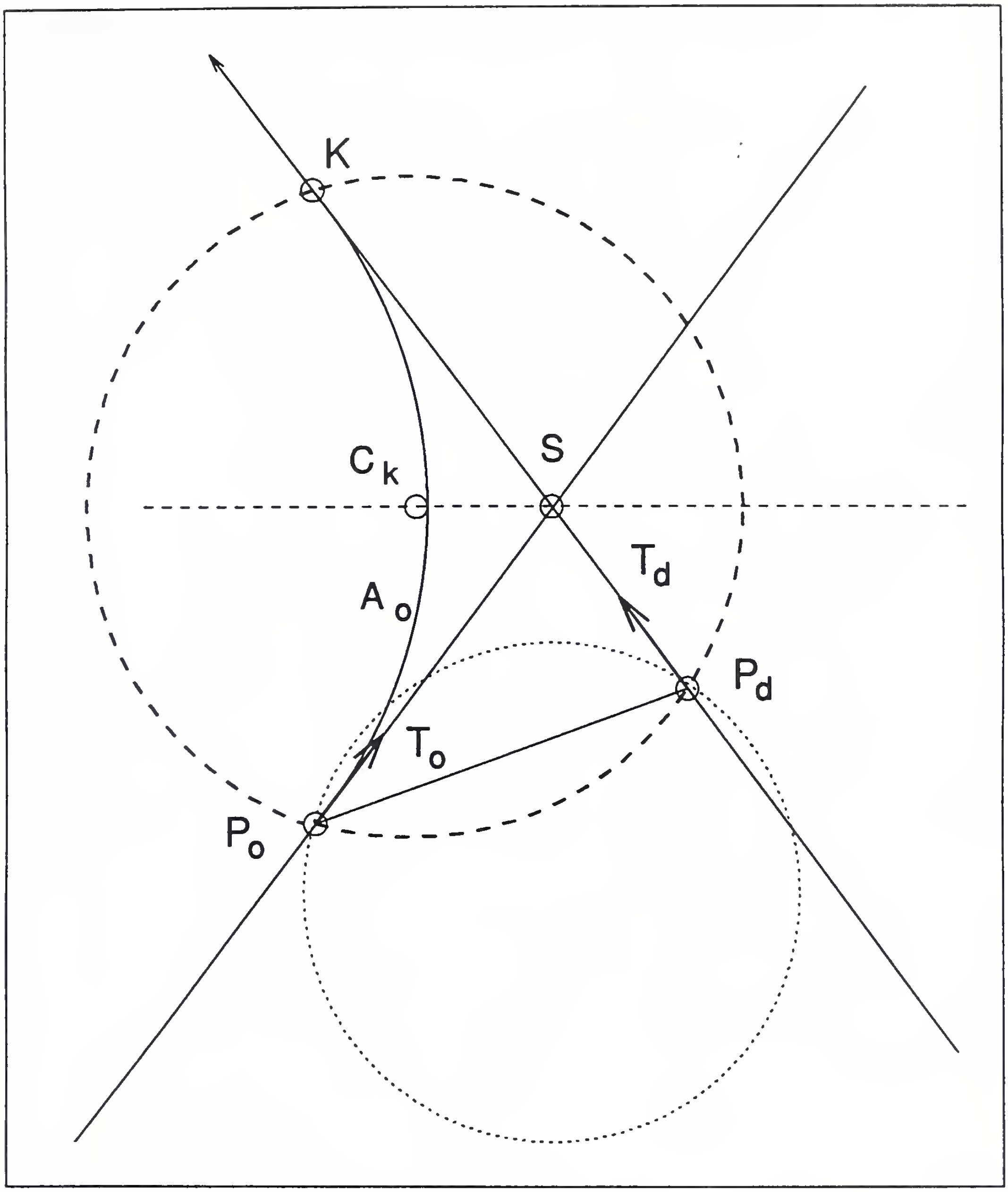

Figure 8: Characterization of the knot circle. A knot K is found by proceding from P_o along a circle tangential to both tangent lines and then following a line-degenerate arc. The knot circle is P_oP_dK. Dots indicate the knot circle of the opposite family.

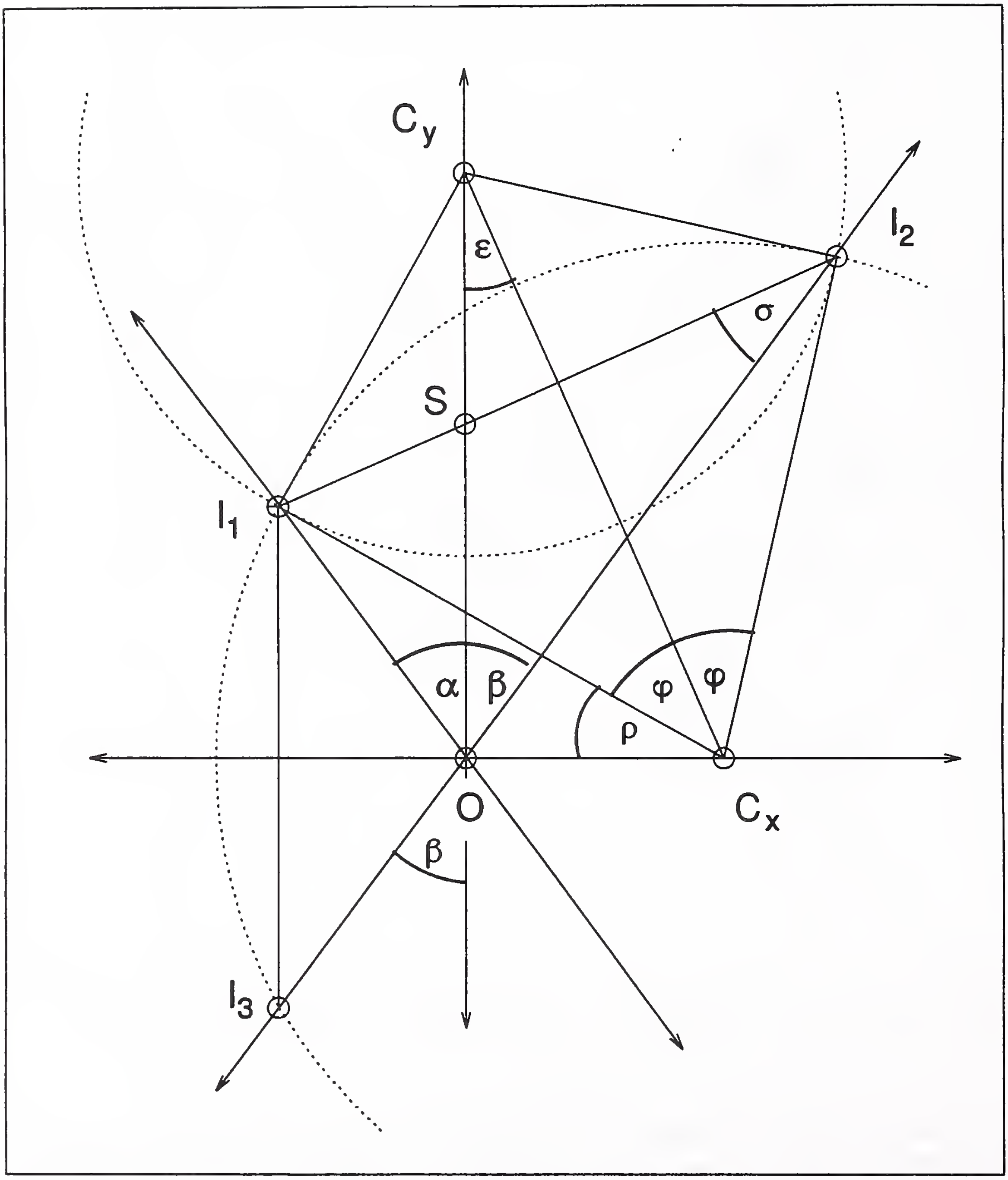

Figure 9: Illustration of proof of Proposition (6.1).

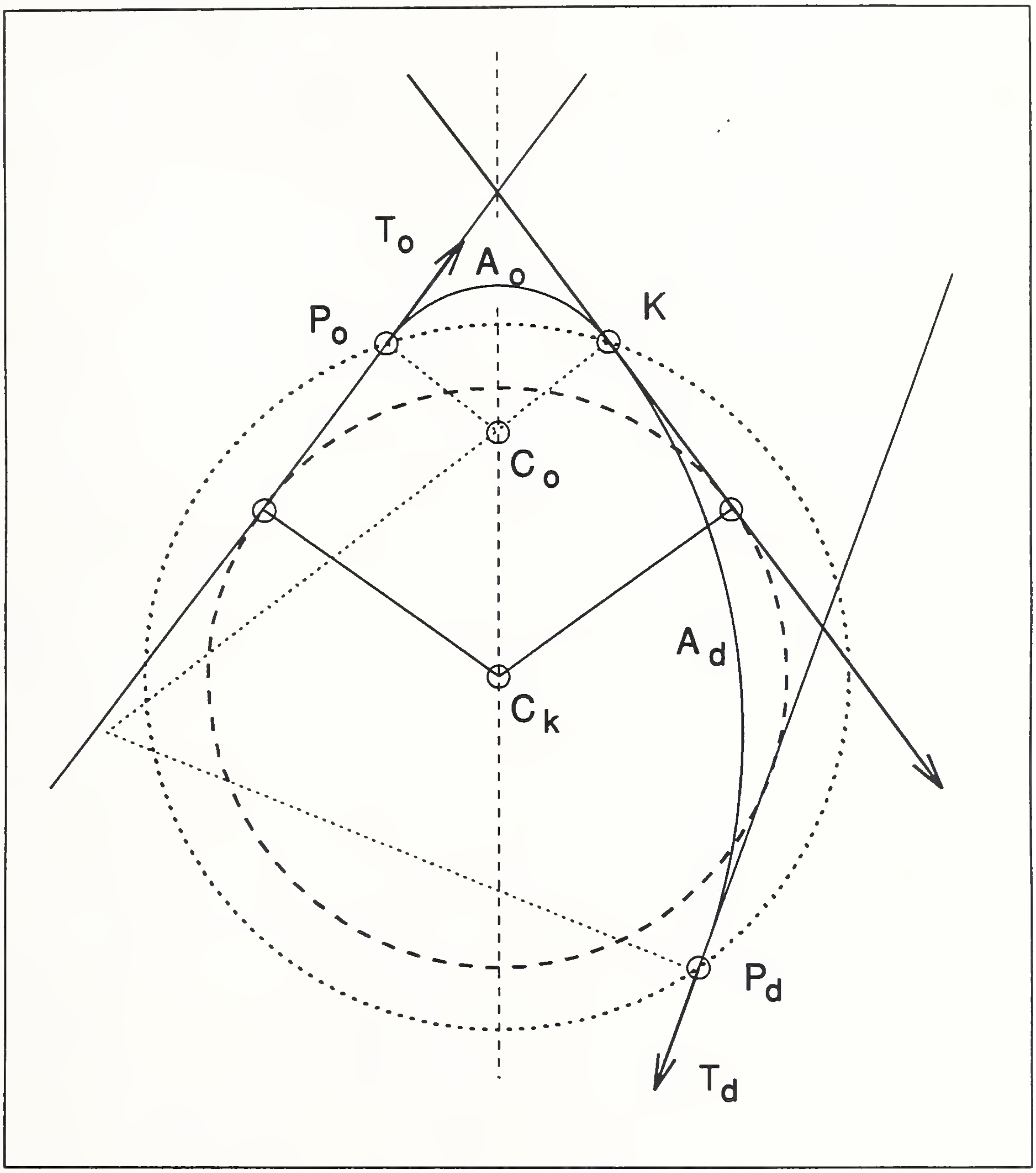

Figure 10: The knot-tangent circle – heavily dashed – and the knot circle – dotted – are concentric.