Observations About Joined Circular Arcs

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Abstract. Smoothly joined pairs of circular arcs – termed biarcs – may serve as a device for data fitting with smooth piecewise circular curves. Geometric properties of such biarcs are investigated.

Key words: biarcs, circle pencils, cocyclic tangents, curve-fitting, piecewise circular, plane geometry

Introduction. There is interest in interpolating and approximating strings of points in the plane by piecewise-circular smooth curves because that representation lends itself readily to some computer-automated manufacturing processes. In particular, when interpolating a string of given consecutively distinct planar points, it is commonly assumed that these points are also the “knots” of the interpolating piecewise-circular curve, that is, the points at which successive circular arcs connect. If an initial direction is specified, such interpolating curves are uniquely determined by the string of points. This interpolation problem becomes overdetermined, however, if directions are prescribed at all points of the string. In that case, pairs of circular arcs joined together smoothly at some suitable intermediate point – configurations termed “biarcs” (K.M. Bolton [3]) – can be used to connect successive points with prescribed directions. Several geometric observations concerning families of biarcs will be reported in this paper.

It is unlikely that some of these observations have not been made before, particularly in the older literature, but search for a record has been unsuccessful so far. For general geometric background material, the reader may want to consult H.E. Baker [1], M. Berger [2], R.A. Johnson [4], or D. Wells [5].

1. Circular arcs and biarcs. The

(1.1) “circular arcs”

considered here are directed and have distinct start and endpoints. Also, they should not cover a full circle. Depending on whether they move counterclockwise or clockwise, they are, respectively, defined as the directed curves

\[ A = \{(x, y) \in R^2 : x = \hat{x} + r \cos \varphi, \ y = \hat{y} + r \sin \varphi, \ \varphi_s \leq \varphi \leq \varphi_e\}, \]

where \( \varphi_s < \varphi_e \), and

\[ A = \{(x, y) \in R^2 : x = \hat{x} + r \cos \varphi, \ y = \hat{y} + r \sin \varphi, \ \varphi_s \geq \varphi \geq \varphi_e\}, \]
where $\varphi_s > \varphi_e$. In both cases, it is understood that $0 \leq \varphi < 2\pi$ and $|\varphi_e - \varphi_s| < 2\pi$. In the above definitions, the point $(\hat{x}, \hat{y})$ is the center of the arc, and $r > 0$ is its radius. At every point $(x(\varphi), y(\varphi))$, a circular arc $A$ has a tangential direction $(-\sin \varphi, \cos \varphi)$, the term “direction” being reserved for vectors of length 1.

For each circular arc (1.1), there is a

(1.2) \textit{"complement" or \textit{"complementary arc"}}

that is, the arc

$$\tilde{A} = \{(x, y) \in \mathbb{R}^2 : x = \hat{x} + r \cos \varphi, y = \hat{y} + r \sin \varphi, \varphi_s \geq \varphi \geq \varphi_e\},$$

or the arc

$$\tilde{A} = \{(x, y) \in \mathbb{R}^2 : x = \hat{x} + r \cos \varphi, y = \hat{y} + r \sin \varphi, \varphi_s \leq \varphi \leq \varphi_e\},$$

respectively. In other words, the complement of a counterclockwise arc is the clockwise arc with the same start and end angles, and vice versa. The union of a circular arc and its complement covers a circle, duplicating only the start and endpoints.

In the context of piecewise circular curves, straight line segments

$$A = \{(x, y) \in \mathbb{R}^2 : x = x_s + \theta(x_e - x_s), y = y_s + \theta(y_e - y_s), 0 \leq \theta \leq 1\},$$

with $(x_s, y_s) \neq (x_e, y_e)$, are typically considered among the circular arcs, representing the degenerate case $r = \infty$. Included among those

(1.3) \textit{"line-degenerate arcs"}

are also arcs which consist of an entire line but exclude a straight line segment. Such an “arc” is of the form

$$A = \{(x, y) \in \mathbb{R}^2 : x = x_s + \theta(x_e - x_s), y = y_s + \theta(y_e - y_s), 0 \geq \theta \text{ or } \theta \geq 1\},$$

may be viewed as a straight line segment passing through $\infty$, and will be considered the complement (1.2) of a finite straight line-degenerate arc. The tangential directions of line-degenerate arcs are given by

$$+\frac{1}{L}(x_e - x_s, y_e - y_s) \text{ or } -\frac{1}{L}(x_e - x_s, y_e - y_s) \text{ with } L = \sqrt{(x_e - x_s)^2 + (y_e - y_s)^2},$$

depending on whether the arc is finite or infinite. Those tangential directions are the same at each point of a line-degenerate arc.

Two arcs joined together smoothly, that is, the endpoint of the first arc is the start point of the other and the tangential directions of the two arcs coincide at that juncture, form what is called here a
"biarc".

The point at which the two arcs of a biarc meet is called its

"knot".

The start point of the first arc and the endpoint of the second arc will be considered the

"origin" $P_o$ and the "destination" $P_d$

of the biarc. Correspondingly, the tangential directions of the respective arcs at the origin and the destination will be referred to as the

"origin" and "destination (tangential) directions" $T_o$ and $T_d$

($||T_o|| = ||T_d|| = 1$) (Figures 1 and 2). Note also that the arcs of a biarc may be line-degenerate (1.3) (Figure 3).

The complements (1.2) of the two arcs in a biarc form again a biarc, the

"complement" or "complementary biarc" of the original biarc. Figures 2 and 3 display instances of biarcs together with their complements. The latter are indicated by dashed lines. The heavy lines with arrows at the origin $P_o$ and destination $P_d$ indicate the tangential directions $T_o$ and $T_d$, respectively.

It will be shown that, given any two distinct points $P_o \neq P_d$ with tangential directions $T_o$ and $T_d$, biarcs exist which originate at $P_o$ in direction $T_o$ and terminate at $P_d$ with direction $T_d$. Indeed, the following simple construction will work in most cases. Choose a circle that avoids point $P_d$ but passes through $P_o$ with tangent $T_o$. Then there exists usually a unique second circle through $P_d$ with tangent $T_d$ that touches the first circle at, say, point $K$. The two constituent arcs of a suitable biarc are then chosen in the obvious fashion from the two circles with point $K$ as knot. This argument will be made more precise later on (see Lemma (3.4).

Since there is one degree of freedom in the choice of the first circle, the biarcs meeting the above specifications may be expected to form a 1-parameter family, as do the knots of these biarcs.

The purpose of this work is to characterize the locus of knots of such families of biarcs.

2. Families of biarcs. Let

$B(P_o, T_o, P_d, T_d)$

denote the family (=set) of biarcs from $P_o$ to $P_d$, assuming the two given tangential directions $T_o$ and $T_d$, respectively, at those points. Within that family, biarcs are determined uniquely
by their knots. The biarcs in the

\[(2.2) \quad \text{"complementary family" } \tilde{B} = B(P_o, -T_o, P_d, -T_d)\]

are complementary (1.8) to the biarcs in the family (2.1), and vice versa. Any family of biarcs \(B\) and its complementary family \(\tilde{B}\) have therefore the same set of knots. Of interest will be the relationship of the family (2.1) of biarcs to its

\[(2.3) \quad \text{"opposite family" } B^* = B(P_o, T_o, P_d, -T_d)\]

that is, the family that differs from (2.1) only in that the destination direction is reversed. Figures 1 and 2 feature biarcs of opposite families.

The straight tangent line through the origin \(P_o\) in direction \(T_o\), and the analogous line through the destination \(P_d\) of the biarc in direction \(T_d\), will play a role in the subsequent geometric arguments, and will be referred to as

\[(2.4) \quad \text{end tangents}\]

of the family of biarcs (2.1). Note that the complementary family (2.2) and the opposite family (2.3) both have the same end tangents as the original family.

3. The symmetric case. Before examining that problem in general, an important special case requires attention. Given the tangential directions \(T_o\) and \(T_d\), suppose that the vector \(T_o + T_d\) does not vanish and is parallel to the base line \(P_o, P_d\):

\[T_o + T_d = \eta(P_d - P_o), \eta \neq 0\]

(see Figure 4). In this case, the simple construction of a specified biarc outlined in Section 1 does not work. However, there exists a single circular arc \(A\) from origin \(P_o\) to destination \(P_d\) with the prescribed tangential directions \(T_o\) and \(T_d\). Thus any point on arc \(A\) – other than \(P_o\) or \(P_d\) – can be chosen as a knot \(K\) between two circular subarcs. Such a family of biarcs \(B_s\) will be called a

\[(3.1) \quad \text{"symmetric family"},\]

and the circle containing arc \(A\) will be referred to as the

\[(3.2) \quad \text{"source circle"}\]

of the symmetric family \(B_s\). Definition (3.1) includes the family \(B(P_o, T_o = T, P_d, T_d = T)\), with direction \(T\) parallel to the base line \(P_oP_d\). In that case, the arc \(A\) is line-degenerate (1.3), and the source circle becomes the base line.

The following observation highlights the special role of symmetric families.
(3.3) Lemma: All biarcs in a symmetric family \( B \), are contained in its source circle.

Proof: Let \( A_o \) and \( A_d \) be the first and second arcs of a biarc in a symmetric family, and let \( C_o \) and \( C_d \) be their respective circles. There are two circles \( C_1 \) and \( C_2 \) tangential to circle \( C_o \) and passing through \( P_d \) with tangential direction \( T_d \). One of these two circles must be \( C_d \). In order to distinguish which circle is in fact \( C_d \), consider the orientations conferred on circle \( C_o \) by tangential direction \( T_o \) and on circles \( C_1 \) and \( C_2 \) by \( T_d \). Then one of the two circles \( C_1 \) and \( C_2 \) touches circle \( C_o \) with equal tangential direction, whereas the other touches \( C_o \) with opposite tangential direction. The former circle must be the one which equals \( C_d \), because biarcs must have equal tangential directions at their knots. Circle \( C_d \) is therefore uniquely defined by the above conditions. It is now easily seen, that the source circle also meets these conditions: it passes through \( P_d \) with tangential direction \( T_d \), through \( P_o \) with tangential direction \( T_o \), and thus touches circle \( C_o \) at \( P_o \) with common tangential direction. It coincides, therefore, with \( C_d \), which implies that arc \( A_d \) is contained in the source circle. By symmetric argument, it follows that arc \( A_o \), too, is contained in the source cycle. \( \square \)

In view of Lemma (3.3), the following statement holds only in the nonsymmetric case.

(3.4) Lemma: Given a nonsymmetric family \( B = B(P_o, T_o, P_d, T_d) \), and any circle \( C_o \) through origin \( P_o \) with tangential direction \( T_o \), but not meeting destination \( P_d \), then there exists in \( B \) a unique biarc whose first arc \( A_o \) is contained in the circle \( C_o \).

Proof: Since \( P_d \notin C_o \), the same argument as in the previous proof shows that there is a unique circle \( C_d \) which passes through \( P_d \) with tangential direction \( T_d \) and touches circle \( C_o \) with common tangential direction at some point \( K \). Now \( K \neq P_o \) as otherwise family \( B \) would be symmetric. Arc \( A_o \) can thus be chosen as proceeding in circle \( C_o \) from origin \( P_o \) to knot \( K \). As \( K \in C_o \), but \( P_d \notin C_o \) by hypothesis, \( K \neq P_d \), and there is a circular arc \( A_d \) from \( K \) to destination \( P_d \) within circle \( C_d \). The two arcs \( A_o \) and \( A_d \) combine to form a biarc in family \( B \). The knot \( K \) of that biarc is uniquely defined by the parameters of the family \( B \) and the choice of circle \( C_o \), and so is the biarc. \( \square \)

By Lemma (3.3), all knots of biarcs in a symmetric family (3.1) must lie on a circle, namely the source circle, which contains the circular arc \( A \) from origin \( P_o \) to destination \( P_d \) with correct tangential directions \( T_o \) and \( T_d \). Choosing a knot \( K \) on the source circle outside arc \( A \) will result in an

(3.5) “overlapping biarc”.

Indeed, the first arc of such a biarc will stretch from origin \( P_o \) to knot \( K \), passing the destination \( P_d \) along the way, while the second arc starts at knot \( K \) and passes origin \( P_o \) on the way to its endpoint \( P_d \) — all within the same circle (or straight line). The proof of the following proposition is left to the reader.

(3.6) Proposition: Symmetric families (3.1) are the only families of biarcs (2.1) which contain
overlapping biarcs (3.5). All nonoverlapping biarcs are simple curves, that is, they do not have multiple points.

It is also seen, that if overlapping biarcs are accepted, the locus of knots of biarcs in a symmetric family (3.1) is a circle through the origin \( P_0 \) and the destination \( P_d \), with these points removed. In the following section, it will be shown that this statement holds in general.

4. The knot circle. The following is a necessary condition for a point to be realizable as knot of a biarc in the family of biarcs (2.1).

(4.1) Proposition: The locus of knots of bi-arcs in the family \( B = B(P_0, T_a, P_d, T_d) \) is contained in a unique circle, the

"knot circle",

through the points \( P_0 \) and \( P_d \). The knot circle may degenerate to a straight line.

Proof: Two main cases are illustrated in Figures 5 and 6, respectively. A special case is examined with reference to Figure 7. Of course there are further cases that need be examined for a complete proof. In all of them, the proof proceeds in analogous fashion. Exhaustive proofs may be obtained by routine analytical procedures. In this exposition, however, arguments are based on elementary geometry are preferred because they illustrate the underlying geometrical relationships.

Figure 5 displays a biarc in family \( B(P_0, T_a, P_d, T_d) \). It starts at point \( P_0 \) in direction \( T_a \), moves along an arc until it meets a second arc at knot \( K \). That second arc terminates at the point \( P_d \) in direction \( T_d \). The following relations between the angles indicated in Figure 5 are obvious:

\[
\begin{align*}
\omega + \rho + \sigma &= \pi \\
\alpha &= 2\rho \\
\beta &= 2\sigma \\
\alpha + \beta + \gamma &= \pi.
\end{align*}
\]

It follows that

\[
\omega = \pi - \rho - \sigma = \pi - \frac{\alpha + \beta}{2} = \frac{\pi + \gamma}{2}.
\]

The angle \( \omega \) thus depends only on the angle \( \gamma \), and is therefore independent of the knot position \( K \). In other words, the line segment \( P_0P_d \) appears from any knot \( K \) under the same angle, and lies therefore on a circle through points \( P_0, P_d \).
For the angles displayed in Figure 6, one finds the similar relationships

\[ \omega + \rho + \sigma = \pi \]
\[ \alpha = 2\rho \]
\[ \beta = 2\sigma \]
\[ \alpha + \beta = \gamma. \]

and consequently

\[ \omega = \pi - \rho - \sigma = \pi - \frac{\alpha + \beta}{2} = \pi - \frac{\gamma}{2}. \]

Again \( \omega \) is constant, and the locus of the knots is therefore part of a circle passing through the points \( P_o \) and \( P_d \).

If the origin direction \( T_o \) equals the destination direction \( T_d \), then the corresponding family of biarcs (see Figure 7)

\[ B_p = B(P_o, T_o, P_d, T_d = T_o) \]

will be called a

(4.2) "parallel family"

of biarcs. It is readily seen, that any point on the full straight line \( P_oP_d \) with the exception of the points \( P_o \) and \( P_d \) is the knot of a biarc. To see the converse, consider a biarc in \( B_p \) with knot \( K \). Let \( \alpha \) denote the angle between the line \( P_oP_d \) and any of the two parallel end tangents (2.4) of family \( B_p \), and draw a straight line through \( K \) at the supplementary angle \( \pi - \alpha \) to the line \( P_oP_d \). As indicated in Figure 7, this line intersects the end tangents in points \( I_o \) and \( I_d \). The triangles \( P_oKI_o \) and \( P_dKI_d \) are similar, and the points \( K, P_o, P_d \) are therefore collinear. \( \square \)

(4.6) Proposition: A point \( K \neq P_o, P_d \) on the knot circle of a family of biarcs \( B(P_o, T_o, P_d, T_d) \) determines in that family a unique biarc which has \( K \) as a knot.

Proof: In Section 3, the proposition was established for symmetric families (3.1) of biarcs. It is thus assumed that the family \( B \) of biarcs is not symmetric. Consider the circle (or straight line) \( C_o \) through \( K \) and \( P_o \) with tangential direction \( T_o \). Then by Lemma (3.4), there exists a biarc in \( B \) with knot \( K' \). Both \( K \) and \( K' \) belong to the knot circle as well as to \( C_o \). Since \( C_o \) is not the knot circle – that would be the symmetric case –, the knot circle and \( C_o \) have at most two points in common, one of them being \( P_o \). Since neither \( K \) nor \( K' \) is equal to \( P_o \), they must share the remaining location. Thus \( K = K' \). \( \square \)
5. Characterizing knot circles. Consider any family \( B = B(P_0, T_o, P_d, T_d) \) that is not symmetric (3.1) and whose opposite (2.3) family \( B^* = B(P_0, T_o, P_d, -T_d) \) is not symmetric either. Then both \( B \) and \( B^* \) contain biarcs that are partially line-degenerate (1.3). Indeed, since the origin \( P_o \) does not lie on the end tangent through the destination \( P_d \), there exists a circle through \( P_o \) that is tangential to both end tangents. By the nonsymmetry assumptions, that circle does not meet \( P_d \), and by Lemma (3.4), there exists a biarc whose first arc \( A_o \) belongs to that circle, whose knot \( K \neq P_d \) lies on the other end tangent, and whose second arc is, therefore, line-degenerate. Figure 8 shows such a biarc, where the second arc moves from knot \( K \) on the end tangent through the infinite point of the latter to the destination \( P_d \).

Note that in either case the knot \( K \) is in symmetric position vis-a-vis the origin \( P_o \), that is, it represents the mirror image of \( P_o \) with respect to an angle bisector of the the end tangents (2.4). More precisely, symmetry holds with respect to this bisector—drawn dashed in Figure 8—which direction is parallel to \( T_o - T_d, ||T_o|| = ||T_d|| = 1 \). This symmetry implies that the center \( C_k \) of the knot circle is to be found on the that bisector of the end tangents.

In the case of a symmetric family \( B \), the knot circle coincides with the source circle (3.2) of that family. The knot circle of the opposite family \( B^* \) is the circle through \( P_o \) and \( P_d \) centered at the intersection of the end tangents, if the end tangents have a unique intersection. The proof of this fact, and the examination of remaining special cases is left to the reader. With suitable interpretations of infinite points and lines, the following proposition holds.

(5.1) Proposition: The knot circle of a family of biarcs \( B(P_0, T_o, P_d, T_d) \) is the unique circle or straight line, passing through origin \( P_o \) and destination \( P_d \), whose center lies on the line bisecting the two end tangents of the family in the direction parallel to the vector \( T_o - T_d \).

Figure 8 also displays the knot circle of the opposite family, showing the two knot circles to be mutually orthogonal. In what follows, this observation will be confirmed in general.

6. Circles centered on alternate angle bisectors. In this section, the two sets of circles centered, respectively, on a pair of orthogonal lines will be examined. In particular, we will establish the

(6.1) Proposition: Consider the two families of circles centered, respectively, on two mutually orthogonal lines intersecting at a point \( S \). Consider a pair of circles, one from each family, intersecting in two points \( P_1 \) and \( P_2 \). That pair is orthogonal if and only if the lines through \( S \) and \( P_1 \) and \( S \) and \( P_2 \) are mutually symmetric with respect to the orthogonal lines.

Proof: Figure 9 shows two—dotted—circles centered, respectively, at point \( C_x \) on the horizontal line through the point \( O \), and at point \( C_y \) on the vertical line through \( O \). The two circles are orthogonal if and only if the triangle \( C_x C_y I_i I_j \) is a right triangle. In that case, the five points \( C_x, O, I_1, C_y, I_2 \) lie on a third circle. It follows that \( \alpha \) and \( \phi \) are peripheral
angles of that circle over the chord \( C_y I_1 \); thus \( \alpha = \phi \). Analogously, \( \beta = \phi \), and therefore \( \alpha = \beta \). This proves the “only if” direction of the proposition.

Now suppose \( \alpha = \beta \). Then the chord \( I_1 I_3 \) of the circle around \( C_x \) is bisected by the horizontal line through \( O \). The angle \( \rho \) is therefore one half of the center angle of the chord \( I_1 I_3 \). On the other hand, the angle \( \sigma \) is a peripheral angle of that chord. Thus \( \rho = \sigma \).

Considering triangle \( OSI_2 \),

\[
\sigma = \pi - \beta - \left( \frac{\pi}{2} + \epsilon \right) = \frac{\pi}{2} - \beta - \epsilon.
\]

Since the angle at \( C_x \) of triangle \( C_y C_x O \) is given by \( \pi/2 - \epsilon \),

\[
\rho = \left( \frac{\pi}{2} - \epsilon \right) - \phi = \frac{\pi}{2} - \phi - \epsilon.
\]

Thus \( \rho = \sigma \) implies \( \beta = \phi = \alpha \). The angle \( \alpha \) at point \( O \), and the angle \( \phi \) at point \( C_x \) are equal angles over the same line segment \( I_1 C_y \). Therefore the four points \( C_x, O, I_1, C_y \) are cocyclic. Since triangle \( C_x O C_y \) is a right triangle, the triangle \( C_x I_1 C_y \) is too. \( \Box \)

(6.2) Corollary: The knot circles of two opposite families (4.2) of biarcs are mutually orthogonal. If the two end tangents shared by those families are parallel or identical, one of the knot circles may degenerate to a straight line.

7. The knot-tangent circle. What can be found about the

(7.1) “knot tangents”

to biarcs at their knots? It turns out that many results remain true if the roles of lines and circles are interchanged.

(7.1) Proposition: The knot tangents of a family of biarcs \( B = B(P_0, T_0, P_d, T_d) \) are cocyclic in the sense that they are all tangential to the same

“knot-tangent circle”,

which is concentric with the knot circle (4.1) tangential to both end tangents (2.4).

Proof: Figure 10 illustrates a major case. A biarc in family \( B \) is shown, with arc \( A_o \) from origin point \( P_o \) to knot \( K \) followed by arc \( A_d \) from \( K \) to the destination point \( P_d \). By Proposition (4.1), the knot \( K \) lies on the – heavy dotted – knot circle, centered at \( C_k \). It is claimed that the – heavy solid – knot tangent of the biarc is also tangential to the – heavy dashed – knot-tangent circle, also centered at \( C_k \) and tangential to the end tangents, which are determined by the tangential directions \( T_o \) and \( T_d \).

In Figure 10, \( C_o \) is the center of the circle through arc \( A_o \). This circle intersects the knot circle at the points \( P_o \) and \( P_d \). The – dashed – line through the centers \( C_o \) and \( C_k \),
respectively, of those two circles is a line of symmetry for both. In particular, the points $P_0$ and $K$ are in symmetric position, and so are the end tangent through $P_0$ and the knot tangent through $K$. Since the above end tangent is also a tangent of the knot-tangent circle, whose center lies on the line of symmetry, the symmetric image of the end tangent through $P_0$ is again a tangent to the knot-tangent circle. But that symmetric image was already seen to be the knot tangent. □

Note that in the symmetric case (3.1), the knot-tangent circle coincides with the knot circle (4.1). For a family of biarcs opposite (2.3) to a symmetric family, all knot tangents pass through a single point, the intersection of the two end tangents. In the parallel case (4.2), all knot tangents are parallel to each other.

8. Resulting theorem on circle pencils. The results of the previous sections yield a theorem on “circle pencils”, that is, sets of circles whose equations

$$F(x, y) \equiv Q(x^2 + y^2) + Ax + By + C = 0$$

are linear combinations, with weights $\lambda_1$ and $\lambda_2$ that do not vanish simultaneously,

$$F(x, y) = \lambda_1 F_1(x, y) + \lambda_2 F_2(x, y),$$

of the equations $F_1(x, y)$ and $F_2(x, y)$, respectively, of two distinct circles. Some of the resulting equations $F(x, y) = 0$ may have complex solutions only. Also, if the above circles are not concentric, the above pencil contains a single straight line

$$F_2(x, y) - F_1(x, y) \equiv (A_2 - A_1)x + (B_2 - B_1)y + (C_2 - C_1) = 0$$

known as the “radical axis” of the circle pencil. Examples of circle pencils include the set of all circles – and the straight line – through two distinct points as well as the set of all circles orthogonal to the circles in that pencil. Of interest here are

(8.1) “tangential circle pencils”.

Such a pencil consists of circles which are all mutually tangential at the same point and whose radical axis is therefore a common tangent to all. The end tangent through $P_0$ and the end tangent through $P_d$, where $P_0$ and $P_d$ are the origin and destination, respectively, of the biarcs in a family (2.1) $B(P_0, T_0, P_d, T_d)$, define two such tangential circle pencils, which figure in the following theorem.

(8.2) Theorem: Given straight lines $\Lambda_o$ and $\Lambda_d$ in the plane. Select point $P_o \in \Lambda_o$ and point $P_d \in \Lambda_d$ such that $P_o \neq P_d$. Consider the tangential pencil $\mathcal{C}_o$ of circles passing through point $P_o$ with tangent $\Lambda_o$, and the analogous tangential pencil $\mathcal{C}_d$ of circles tangential to line $\Lambda_d$ at point $P_d$.

If the lines $\Lambda_o$ and $\Lambda_d$ have a unique intersection $S$, then
"locus of tangential points",

that is, the locus of all points at which a circle in pencil \( C_e \) touches a circle in pencil \( C_d \), consists of the two mutually orthogonal circles intersecting each other at the points \( P_o \) and \( P_d \), and centered, respectively, on the two angle bisectors of the lines \( \Lambda_o \) and \( \Lambda_d \) at their intersection \( S \).

If the lines \( \Lambda_o \) and \( \Lambda_d \) are parallel or identical, then the locus of all tangential points consists of the straight line \( P_o P_d \) together with the circle through points \( P_o \) and \( P_d \) centered at the intersection of line \( P_o P_d \) with the mid-parallel \( \Lambda_m \) of the parallels \( \Lambda_o \) and \( \Lambda_d \).

In the general case,

"the set of common tangents",

that is, the set of all tangents common to two tangential circles at their common point consists of the tangents to two circles. These circles are tangential to the given lines \( \Lambda_o \) and \( \Lambda_d \), and are concentric with the above mentioned pair of mutually orthogonal circles which are the locus of all tangential points. In special cases, one of the circles described by the set of common tangents degenerates to a single -- possibly infinite -- point.

In this formulation, single points are considered circles of radius zero and tangential to any other circle on which the lie. For this reason, the points \( P_o \) and \( P_d \) themselves are not excluded from the above description of the locus of osculation points of two circle pencils. A similar convention leads to counting the lines \( \Lambda_o \) and \( \Lambda_d \) among the common tangents.

The above theorem was about two distinct tangential circle pencils (8.1). It might be conjectured, that the locus of tangential points of any two circle pencils consists of circles. Simple counterexamples show this not to be the case.


Figure 1: Biarc extending from point $P_o$ with tangential direction $T_o$ to point $P_d$ with tangential direction $T_d$. The biarc consists of a circular arc from $P_o$ to the knot $K$ and a second circular arc from $K$ to $P_d$. Both circular arcs have the same tangent at knot $K$. 
Figure 2: Origin $P_o$, destination $P_d$, and tangential direction $T_o$ are as in Figure 1, but the tangential direction $T_d$ at $P_d$ has been replaced by its opposite. The biarc displayed here and the one shown in Figure 1 thus represent opposite families. The dashed curve indicates the complementary biarc.
Figure 3: Example of a biarc containing a line-degenerate arc. The biarc follows a proper circular arc from origin $P_o$ to knot $K$, and continues straight in direction $T_d$ to $\infty$, passes through $\infty$, and returns within the same straight line of direction $T_d$ to destination $P_d$. The dashed lines indicate the complementary biarc.
Figure 4: Biarc in the symmetric case. The two circular arcs connecting at $K$ belong to the same circle.
Figure 5: Illustration of first main case in the proof of Proposition (4.1).
Figure 6: Illustration of second main case in the proof of Proposition (4.1).
Figure 7: Illustration of Proposition (4.1) in the case of a parallel family (4.2).
Figure 8: Characterization of the knot circle. A knot $K$ is found by proceeding from $P_o$ along a circle tangential to both tangent lines and then following a line-degenerate arc. The knot circle is $P_oP_dK$. Dots indicate the knot circle of the opposite family.
Figure 9: Illustration of proof of Proposition (6.1).
Figure 10: The knot-tangent circle – heavily dashed – and the knot circle – dotted – are concentric.