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# Spherical-Wave Characterization of Interior and Exterior Electromagnetic Sources

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# SPHERICAL-WAVE CHARACTERIZATION OF INTERIOR AND EXTERIOR ELECTROMAGNETIC SOURCES

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This report presents spherical scanning formulations for the general case where electromagnetic sources are present both inside and outside the measurement region. The fields are expanded in terms of outgoing waves (due to interior sources) and standing waves (due to exterior sources). Additional information is required to solve for the increased number of unknowns, and results are derived for the spherical wave coefficients in terms of the tangential electric and magnetic fields on a sphere. The special case of an electrically small emitter is examined in detail, and several formulations are presented for the components of electric and magnetic dipoles. The dipole formulations are intended for use in detection of a weak emitter (such as a timing device) in a noisy environment.

Key words: electric dipole; electrically small source; magnetic dipole; near-field measurement; spherical scanning; spherical wave; test volume.

## 1. INTRODUCTION

In spherical near-field antenna measurements [1] or in characterization of electromagnetic interference (EMI) sources [2], the radiated fields are determined outside a minimum sphere enclosing the sources. For radar cross section (RCS) or receiving antenna measurements [3,4], the sources are located outside the test volume. If the test volume is spherical, the interior fields can be expanded in spherical waves. In the case of either interior or exterior sources, the problem is to determine the coefficients of a spherical expansion from measurements made with a probe antenna on the surface of a sphere.

In the more general case, sources are located both inside and outside a spherical-shell, source-free region. This situation can arise in practice because of the presence of unwanted sources or scatterers located either inside the inner sphere boundary or outside the outer sphere boundary of the shell. In this case, the problem is to determine the coefficients of a spherical-wave expansion which is valid in the spherical-shell region. More information is required because the expansion includes both outgoing waves (due to the interior sources) and standing waves (due to exterior sources). The purpose of this report is to present the formulation for this more general case where both interior and exterior sources are present. For typical RCS or receiving antenna measurements (where the intentional source is outside the shell), the test volume may be electrically large, and the interior source could be due to scattering from a support structure located inside shell region. For applications to characterizing typical EMI sources or detecting weak radiators (such as timing devices for explosive detonators), the source is normally electrically small, and the spherical wave expansion for the radiated field can be reduced to electric and magnetic dipole terms [2]. When the radiator is weak, external noise sources can compete with the desired signal, and they can be modeled as the exterior sources outside the spherical shell region.

## 2. SPHERICAL-WAVE EXPANSIONS

In source-free regions with spherical boundaries, the electric and magnetic fields can be expanded in spherical waves which satisfy Maxwell's equations term by term. Various notations are available for spherical waves [5,6], but this report follows the notation in Hansen [1] because he has a number of useful mathematical results (orthogonalities, radiated power, dipole fields, etc.) already worked out. For simplicity, the special cases of interior or exterior sources only are considered in sections 2.1 and 2.2. The general case of interior and exterior fields covered in section 2.3 can be considered to be a superposition of the two special cases.

### 2.1 Interior Sources Only

This section treats the special case shown in figure 1 where the sources are confined to the interior region, $r < r_i$. In the source-free region, $r \geq r_i$, the electric field $\vec{E}$ and magnetic field $\vec{H}$ can be written as expansions [1] of outgoing waves:

$$\vec{E}(r,\theta,\phi) = \frac{k}{\sqrt{\eta}} \sum_{s=1}^{2} \sum_{n=1}^{N} \sum_{m=-n}^{n} Q_{smn}^{(3)} \vec{F}_{smn}^{(3)}(r,\theta,\phi), \tag{1}$$

$$\vec{H}(r,\theta,\phi) = -ik\sqrt{\eta} \sum_{s=1}^{2} \sum_{n=1}^{N} \sum_{m=-n}^{n} Q_{smn}^{(3)} \vec{F}_{3-s,m,n}^{(3)}(r,\theta,\phi). \tag{2}$$

The time dependence, $\exp(-i\omega t)$, is suppressed, the wavenumber $k = \omega\sqrt{\mu\epsilon}$, the intrinsic admittance $\eta = \sqrt{\epsilon/\mu}$, and $\epsilon$ and $\mu$ are the permittivity and permeability of the medium (usually free space). Standard spherical coordinates $(r,\theta,\phi)$ are used throughout.

The transverse electric (TE) spherical wave $\vec{F}_{1mn}^{(c)}$ is given by [1]

$$\vec{F}^{(c)}_{1mn}(r,\theta,\phi) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{n(n+1)}} \left( -\frac{m}{|m|} \right)^m R^{(c)}_{1n}(kr)$$
$$\left( \frac{im \overline{P^{|m|}_n}(\cos\theta)}{\sin\theta} e^{im\phi} \hat{\theta} - \frac{d\overline{P^{|m|}_n}(\cos\theta)}{d\theta} e^{im\phi} \hat{\phi} \right), \tag{3}$$

where $\overline{P^m_n}$ is the normalized associated Legendre function [1,7] and $R^{(c)}_{1n}$ is a spherical Bessel function $z^{(c)}_n$. For $c = 3$, the spherical Bessel function is

$$z^{(3)}_n(kr) = h^{(1)}_n(kr), \tag{4}$$

where $h^{(1)}_n$ is the spherical Hankel function of the first kind [8]. The transverse magnetic (TM) spherical wave $\vec{F}^{(c)}_{2mn}$ is given by [1]

$$\vec{F}^{(c)}_{2mn}(r,\theta,\phi) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{n(n+1)}} \left( -\frac{m}{|m|} \right)^m \left[ \frac{n(n+1)}{kr} z^{(c)}_n(kr) \overline{P^{|m|}_n}(\cos\theta) e^{im\phi} \hat{r} \right.$$
$$\left. + R^{(c)}_{2n}(kr) \left( \frac{d\overline{P^{|m|}_n}(\cos\theta)}{d\theta} e^{im\phi} \hat{\theta} + \frac{im \overline{P^{|m|}_n}(\cos\theta)}{\sin\theta} e^{im\phi} \hat{\phi} \right) \right], \tag{5}$$

where

$$R^{(c)}_{2n}(kr) = \frac{1}{kr} \frac{d}{d(kr)} \left[ kr\, z^{(c)}_n(kr) \right]. \tag{6}$$

The unknown coefficients $Q^{(3)}_{smn}$ remain to be determined. In theory the $n$ summation runs from 1 to $\infty$, but in practice the summation can be truncated at $N$ somewhat greater than $kr_i$.

The spherical waves $\vec{F}^{(c)}_{smn}$ are dimensionless, and the normalization is chosen so that the power $P$ radiated through the sphere of radius $r_i$ (or any larger sphere) enclosing the source is given by [1]

$$P = \frac{1}{2} \sum_{smn} |Q_{smn}^{(3)}|^2, \tag{7}$$

where $\sum_{smn}$ is shorthand for the triple sums in eqs (1) and (2). The units of $Q_{smn}^{(c)}$ are $W^{1/2}$.

Spherical near-field scanning methods for determining the coefficients $Q_{smn}^{(3)}$ are well known, including probe correction [1,3]. Probe correction is not considered in this report, and the results for $Q_{smn}^{(3)}$ will be given in terms of field components (which could be measured with dipole probes).

In order to determine the coefficients $Q_{smn}^{(3)}$ from the tangential components of the electric or magnetic field, an orthogonality relationship between the tangential components of the spherical harmonics is needed. Hansen gives such a relationship, eq (A1.69) in [1], but it is convenient to write it in a more compact form. The first step is to define tangential spherical harmonics that are independent of r:

$$\begin{aligned} \vec{F}_{1mn}(\theta,\phi) &\equiv \frac{\vec{F}_{1mn}^{(c)}(r,\theta,\phi)}{R_{1n}^{(c)}(kr)} \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{n(n+1)}} \left(-\frac{m}{|m|}\right)^m \left( \frac{im \overline{P}_n^{|m|}(\cos\theta)}{\sin\theta} e^{im\phi} \hat{\theta} - \frac{d\overline{P}_n^{|m|}(\cos\theta)}{d\theta} e^{im\phi} \hat{\phi} \right), \end{aligned} \tag{8}$$

$$\begin{aligned} \vec{F}_{2mn}(\theta,\phi) &\equiv \frac{\vec{F}_{2mn}^{(c)}(r,\theta,\phi) - \vec{F}_{2mn}^{(c)}(r,\theta,\phi)\cdot\hat{r}\hat{r}}{R_{2n}^{(c)}(kr)} \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{n(n+1)}} \left(-\frac{m}{|m|}\right)^m \left( \frac{d\overline{P}_n^{|m|}(\cos\theta)}{d\theta} e^{im\phi} \hat{\theta} + \frac{im \overline{P}_n^{|m|}(\cos\theta)}{\sin\theta} e^{im\phi} \hat{\phi} \right). \end{aligned} \tag{9}$$

Equations (8) and (9) are related by the following cross product:

$$\vec{F}_{2mn}(\theta,\phi) = \hat{r} \times \vec{F}_{1mn}(\theta,\phi). \tag{10}$$

Hansen's orthogonality relationship, eq (A1.69), can now be written as

$$\iint_{4\pi} \vec{F}_{smn}(\theta,\phi)\cdot\vec{F}_{\sigma\mu\nu}(\theta,\phi)\,d\Omega = \delta_{s\sigma}\,\delta_{m,-\mu}\,\delta_{n\nu}(-1)^m, \tag{11}$$

where the solid angle integration over $4\pi$ steradians is shorthand for

$$\iint_{4\pi}[\quad]d\Omega = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}[\quad]\sin\theta\,d\theta\,d\phi. \tag{12}$$

Consider now the case where the tangential electric field, $\vec{E}_{\tan} = E_\theta\hat{\theta} + E_\phi\hat{\phi}$, is specified (measured) over the surface of a sphere of radius $r$ ($\geq r_i$). If the dot product of a tangential spherical harmonic $\vec{F}_{\sigma,-\mu,\nu}$ with eq (1) is integrated over angle, eq (11) can be used to reduce the summation to a single term:

$$\iint_{4\pi} \vec{E}_{\tan}(r,\theta,\phi)\cdot\vec{F}_{\sigma,-\mu,\nu}(\theta,\phi)\,d\Omega = \frac{k}{\sqrt{\eta}}(-1)^\mu Q^{(3)}_{\sigma\mu\nu}R^{(3)}_{\sigma\nu}(kr). \tag{13}$$

Since $R^{(3)}_{\sigma\nu}(kr)$ is either a spherical Hankel function (for $\sigma = 1$) or its derivative (for $\sigma = 2$), it has no zeros for any real values of $kr$, and eq (13) can be solved for $Q^{(3)}_{\sigma\mu\nu}$. If the subscripts $(\sigma,\mu,\nu)$ are replaced by (s,m,n), eq (13) can be rewritten as

$$Q^{(3)}_{smn} = \frac{(-1)^m\sqrt{\eta}}{kR^{(3)}_{sn}(kr)}\iint_{4\pi} \vec{E}_{\tan}(r,\theta,\phi)\cdot\vec{F}_{s,-m,n}(\theta,\phi)\,d\Omega. \tag{14}$$

This is the desired result for the coefficients $Q^{(3)}_{smn}$ in terms of the tangential electric field, and it is consistent with Hansen's result [1]. A similar procedure can be carried out starting with eq (2) for the magnetic field to obtain the coefficients $Q^{(3)}_{smn}$ in terms of the tangential magnetic field:

$$Q^{(3)}_{smn} = \frac{i(-1)^m}{k\sqrt{\eta}R^{(3)}_{3-s,n}(kr)}\iint_{4\pi} \vec{H}_{\tan}(r,\theta,\phi)\cdot\vec{F}_{3-s,-m,n}(\theta,\phi)\,d\Omega. \tag{15}$$

The tangential electric or magnetic fields required in eqs (14) or (15) could be obtained in theory by sampling with ideal electric or magnetic dipole probes. An electrically short linear antenna is a good approximation to an electric dipole, and an electrically small loop is a good approximation to a magnetic dipole. A third idealized probe of interest is a Huygens' source [9] which responds to perpendicular components of the electric and magnetic field. A Huygens' probe can be realized approximately by a small aperture antenna, such as an open-ended waveguide [3], or by a double-gap loop [10] which simultaneously measures perpendicular components of the electric and magnetic fields. This type of idealized probe is closely related to the uniqueness vector $\vec{U}$ which is defined as [11]

$$\vec{U} \equiv \vec{E}_{\tan} - \frac{1}{\eta}\hat{n} \times \vec{H}, \tag{16}$$

where $\hat{n}$ is the unit normal directed into the region in which the fields are being expanded. For the exterior field expansion in this section, $\hat{n} = \hat{r}$. Specification of $\vec{U}$ over a closed surface uniquely determines the fields in an adjoining, source-free region (either interior or exterior). In theory, the constant $1/\eta$ can be any nonzero, real constant, but in practice a choice of $1/\eta$ $(= \sqrt{\mu/\epsilon})$ usually yields good numerical results [11]. That choice also corresponds to the response of an idealized Huygens' probe.

The spherical harmonic expansion for $\vec{U}$ is obtained by substituting eqs (1) and (2) into eq (16) and making use of the tangential spherical harmonics:

$$\vec{U}(r,\theta,\phi) = \frac{k}{\sqrt{\eta}}\sum_{smn} i^{s-1}\left[R_{1n}^{(3)}(kr) - iR_{2n}^{(3)}(kr)\right]Q_{smn}^{(3)}\vec{F}_{smn}(\theta,\phi). \tag{17}$$

The orthogonality relation in eq (11) can again be used to solve for the coefficients:

$$Q_{smn}^{(3)} = \frac{\sqrt{\eta}(-1)^m}{i^{s-1}\,k\left[R_{1n}^{(3)}(kr) - iR_{2n}^{(3)}(kr)\right]}\iint_{4\pi}\vec{U}(r,\theta,\phi)\cdot\vec{F}_{s,-m,n}(\theta,\phi)\,d\Omega. \tag{18}$$

It is useful to examine the asymptotic form of the square bracket factor in the denominator of eq (18) to confirm that it is well behaved numerically. For $kr \gg N$, the radial functions can be approximated [1]

$$R_{1n}^{(3)}(kr) \approx (-i)^{n+1}\frac{e^{ikr}}{kr} \quad\text{and}\quad R_{2n}^{(3)}(kr) \approx (-i)^{n}\frac{e^{ikr}}{kr}. \tag{19}$$

Hence for large $kr$ the terms in the square bracket of eq (18) add to give the result

$$R^{(3)}_{1n}(kr) - iR^{(3)}_{2n}(kr) \approx 2(-i)^{n+1}\frac{e^{ikr}}{kr}. \tag{20}$$

In summary, eqs (14), (15), and (17) give three methods of determining the fields outside a sphere when the sources are located inside. All three equations are well conditioned, and this is a result of uniqueness properties that either the tangential electric field, tangential magnetic field, or the uniqueness vector $\vec{U}$ is sufficient to determine the external fields.

2.2 Exterior Sources Only

This section treats the special case shown in figure 2 where the sources are confined to the exterior region, $r > r_e$. In the source-free region, $r \le r_e$, the electric and magnetic fields can be written in terms of the following expansions of standing waves that are finite at the origin:

$$\vec{E}(r,\theta,\phi) = \frac{k}{\sqrt{\eta}} \sum_{smn} Q^{(1)}_{smn} \vec{F}^{(1)}_{smn}(r,\theta,\phi), \tag{21}$$

$$\vec{H}(r,\theta,\phi) = -ik\sqrt{\eta} \sum_{smn} Q^{(1)}_{smn} \vec{F}^{(1)}_{3-s,m,n}(r,\theta,\phi). \tag{22}$$

For c = 1, the spherical Bessel function which is finite at the origin is [8]

$$z^{(1)}_n(kr) = j_n(kr). \tag{23}$$

The methods for determining the unknown coefficients $Q^{(1)}_{smn}$ follow those of the previous section. Consider first the case where the tangential electric field $\vec{E}_{\tan}$ is specified over a surface of radius $r$ ($\le r$). If a tangential spherical harmonic is dotted into eq (17), the orthogonality relationship in eq (11) can be used to obtain an expression which is similar to eq (14):

$$Q^{(1)}_{smn} = \frac{(-1)^m\sqrt{\eta}}{kR^{(1)}_{sn}(kr)} \iint_{4\pi} \vec{E}_{\tan}(r,\theta,\phi)\cdot\vec{F}_{s,-m,n}(\theta,\phi)\, d\Omega. \tag{24}$$

The only difference between eqs (14) and (24) is that the outgoing radial function $R^{(3)}_{sn}(kr)$ is replaced by the standing wave radial function $R^{(1)}_{sn}(kr)$. The derivation of eq (24) requires a division by $R_{sn}^{(1)}(kr)$ and is not valid for values of $s$, $n$, and $kr$ where $R_{sn}^{(1)}(kr) = 0$. A similar procedure can be carried out by starting with eq (22) for the tangential magnetic field, and the result for the coefficients in terms of the tangential magnetic field is

$$Q_{smn}^{(1)} = \frac{i(-1)^m}{k\sqrt{\eta}\,R_{3-s,n}^{(1)}(kr)} \iint_{4\pi} \vec{H}_{\tan}(r,\theta,\phi) \cdot \vec{F}_{3-s,-m,n}(\theta,\phi)\, d\Omega . \tag{25}$$

This equation is not valid for values of $s$, $n$, and $kr$ where $R_{3-s,n}^{(1)}(kr) = 0$. These difficulties with eqs (24) and (25) are to be expected because either the tangential electric or magnetic field by itself is not sufficient to uniquely determine the fields in an interior region [11].

If both the tangential electric and magnetic fields are specified, then the fields in an interior region are uniquely determined [12]. In such a case, eqs (24) and (25) could be used to determine the coefficients because the zeros of $R_{sn}^{(1)}(kr)$ and $R_{3-s,n}^{(1)}(kr)$ do not occur simultaneously for the same values of $s$, $n$, and $kr$. The coefficients can also be determined from the uniqueness vector $\vec{U}$ as defined by eq (16). For determining the fields in the interior region, the unit normal in eq (16) is given by $\hat{n} = -\hat{r}$, and the Huygens' probe for determining $\vec{U}$ would need to be oriented appropriately. The spherical harmonic expansion for $\vec{U}$ is obtained by substituting eqs (21) and (22) into eq (16) and making use of the tangential harmonics:

$$\vec{U}(r,\theta,\phi) = \frac{k}{\sqrt{\eta}} \sum_{smn} i^{1-s}\left[R_{1n}^{(1)}(kr) + iR_{2n}^{(1)}(kr)\right] Q_{smn}^{(1)} \vec{F}_{smn}(\theta,\phi) . \tag{26}$$

An equivalent form of eq (26) has been presented in reference [3]. The orthogonality relationship in eq (11) can again be used to solve for the coefficients:

$$Q_{smn}^{(1)} = \frac{\sqrt{\eta}(-1)^m}{i^{1-s}k\left[R_{1n}^{(1)}(kr) + iR_{2n}^{(1)}(kr)\right]} \iint_{4\pi} \vec{U}(r,\theta,\phi) \cdot \vec{F}_{s,-m,n}(\theta,\phi)\, d\Omega . \tag{27}$$

Equation (27) does not suffer from the problem encountered in eqs (24) and (25) because the denominator does not have any zeros for real values of $kr$. Practical consequences are that ideal dipole probes are not good for determining interior fields and probes that are similar to a Huygens' probe (respond to both tangential electric and magnetic fields) are preferred.

## 2.3 Interior and Exterior Sources

This section treats the most general case shown in figure 3 where sources are present in both the interior region, $r < r_i$, and the exterior region, $r > r_e$. In the source-free region,

$r_i \le r \le r_e$, the electric and magnetic fields can be written as expansions of standing waves and outgoing waves:

$$\vec{E}(r,\theta,\phi) = \frac{k}{\sqrt{\eta}} \sum_{smn} \left[ Q_{smn}^{(1)} \vec{F}_{smn}^{(1)}(r,\theta,\phi) + Q_{smn}^{(3)} \vec{F}_{smn}^{(3)}(r,\theta,\phi) \right], \tag{28}$$

$$\vec{H}(r,\theta,\phi) = -ik\sqrt{\eta} \sum_{smn} \left[ Q_{smn}^{(1)} \vec{F}_{3-s,m,n}^{(1)}(r,\theta,\phi) + Q_{smn}^{(3)} \vec{F}_{3-s,m,n}^{(3)}(r,\theta,\phi) \right]. \tag{29}$$

Equations (28) and (29) can be thought of as a superposition of eqs (1) and (21) and eqs (2) and (22). The coefficients $Q_{smn}^{(1)}$ of the standing waves characterize the exterior sources, and the coefficients $Q_{smn}^{(3)}$ of the outgoing waves characterize the interior sources.

Since twice as many unknowns need to be determined, more information is needed. If both the tangential electric and magnetic fields are specified over a sphere of radius $r$ in the source-free region, the orthogonality relationship in eq (11) can be used to derive the following simultaneous equations:

$$R_{sn}^{(1)}(kr) Q_{smn}^{(1)} + R_{sn}^{(3)}(kr) Q_{smn}^{(3)} = (-1)^m \frac{\sqrt{\eta}}{k} \iint_{4\pi} \vec{E}_{\tan}(r,\theta,\phi) \cdot \vec{F}_{s,-m,n}(\theta,\phi)\, d\Omega, \tag{30}$$

$$R_{3-s,n}^{(1)}(kr) Q_{smn}^{(1)} + R_{3-s,n}^{(3)}(kr) Q_{smn}^{(3)} = (-1)^m \frac{i}{k\sqrt{\eta}} \iint_{4\pi} \vec{H}_{\tan}(r,\theta,\phi) \cdot \vec{F}_{3-s,-m,n}(\theta,\phi)\, d\Omega. \tag{31}$$

The solution of eqs (30) and (31) can be simplified by using the Wronskian for the radial functions [1]:

$$R_{sn}^{(1)}(kr) R_{3-s,n}^{(3)}(kr) - R_{3-s,n}^{(1)}(kr) R_{sn}^{(3)}(kr) = \frac{-i(-1)^s}{(kr)^2}. \tag{32}$$

From eqs (30) through (32), the solution for the coefficients can be written

$$Q^{(1)}_{smn} = ik\sqrt{\eta}\, r^2(-1)^{(m-s)}\left[R^{(3)}_{3-s,n}(kr)\int\!\!\int_{4\pi} \vec{E}_{\tan}(r,\theta,\phi)\cdot\vec{F}_{s,-m,n}(\theta,\phi)\,d\Omega \right.$$
$$\left. - \frac{i}{\eta}R^{(3)}_{sn}(kr)\int\!\!\int_{4\pi} \vec{H}_{\tan}(r,\theta,\phi)\cdot\vec{F}_{3-s,-m,n}(\theta,\phi)\,d\Omega\right], \tag{33}$$

$$Q^{(3)}_{smn} = -ik\sqrt{\eta}\, r^2(-1)^{(m-s)}\left[R^{(1)}_{3-s,n}(kr)\int\!\!\int_{4\pi} \vec{E}_{\tan}(r,\theta,\phi)\cdot\vec{F}_{s,-m,n}(\theta,\phi)\,d\Omega \right.$$
$$\left. - \frac{i}{\eta}R^{(1)}_{sn}(kr)\int\!\!\int_{4\pi} \vec{H}_{\tan}(r,\theta,\phi)\cdot\vec{F}_{3-s,-m,n}(\theta,\phi)\,d\Omega\right]. \tag{34}$$

The results in eqs (33) and (34) are applicable to the case where two spherical scans, one with an electric dipole probe and one with a magnetic dipole probe, have been performed at the same radius.

The solution for the coefficients can also be obtained in terms of two uniqueness vectors, $\vec{U}_1$ and $\vec{U}_3$, defined as

$$\vec{U}_1 = \vec{E}_{\tan} + \frac{1}{\eta}\hat{r}\times\vec{H} \text{ and } \vec{U}_3 = \vec{E}_{\tan} - \frac{1}{\eta}\hat{r}\times\vec{H}. \tag{35}$$

$\vec{U}_1$ is obtained by setting $\hat{n} = -\hat{r}$ and is associated with inward propagating waves. $\vec{U}_3$ is obtained by setting $\hat{n} = \hat{r}$ and is associated with outward propagating waves. From eqs (28), (29), and (35), their spherical wave expansions are

$$\vec{U}_1 = \frac{k}{\sqrt{\eta}}\sum_{smn}\left[Q^{(1)}_{smn}\vec{F}_{smn}\left(R^{(1)}_{sn}(kr) - i(-1)^sR^{(1)}_{3-s,n}(kr)\right) + Q^{(3)}_{smn}\vec{F}_{smn}\left(R^{(3)}_{sn}(kr) - i(-1)^sR^{(3)}_{3-s,n}(kr)\right)\right], \tag{36}$$

$$\vec{U}_3 = \frac{k}{\sqrt{\eta}}\sum_{smn}\left[Q^{(1)}_{smn}\vec{F}_{smn}\left(R^{(1)}_{sn}(kr) + i(-1)^sR^{(1)}_{3-s,n}(kr)\right) + Q^{(3)}_{smn}\vec{F}_{smn}\left(R^{(3)}_{sn}(kr) + i(-1)^sR^{(3)}_{3-s,n}(kr)\right)\right]. \tag{37}$$

The orthogonality relationship in eq (11) can be applied to eqs (36) and (37) to derive the following simultaneous equations for the coefficients:

$$
\begin{aligned}
&Q^{(1)}_{smn}\vec{F}_{smn}\left(R^{(1)}_{sn}(kr) - i(-1)^s R^{(1)}_{3-s,n}(kr)\right) + Q^{(3)}_{smn}\vec{F}_{smn}\left(R^{(3)}_{sn}(kr) - i(-1)^s R^{(3)}_{3-s,n}(kr)\right) \\
&\quad = \frac{(-1)^m\sqrt{\eta}}{k}\int\!\!\int_{4\pi} \vec{U}_1(r,\theta,\phi)\cdot\vec{F}_{s,-m,n}(\theta,\phi)\, d\Omega\,,
\end{aligned}
\tag{38}
$$

$$
\begin{aligned}
&Q^{(1)}_{smn}\vec{F}_{smn}\left(R^{(1)}_{sn}(kr) + i(-1)^s R^{(1)}_{3-s,n}(kr)\right) + Q^{(3)}_{smn}\vec{F}_{smn}\left(R^{(3)}_{sn}(kr) + i(-1)^s R^{(3)}_{3-s,n}(kr)\right) \\
&\quad = \frac{(-1)^m\sqrt{\eta}}{k}\int\!\!\int_{4\pi} \vec{U}_3(r,\theta,\phi)\cdot\vec{F}_{s,-m,n}(\theta,\phi)\, d\Omega\,.
\end{aligned}
\tag{39}
$$

From eqs (32), (38), and (39), the solution for the coefficients can be written

$$
\begin{aligned}
Q^{(1)}_{smn} = \frac{kr^2\sqrt{\eta}\,(-1)^m}{2}\Bigg[&\left(R^{(3)}_{sn}(kr) + i(-1)^s R^{(3)}_{3-s,n}(kr)\right)\int\!\!\int_{4\pi} \vec{U}_1(r,\theta,\phi)\cdot\vec{F}_{s,-m,n}(\theta,\phi)\, d\Omega \\
&- \left(R^{(3)}_{sn}(kr) - i(-1)^s R^{(3)}_{3-s,n}(kr)\right)\int\!\!\int_{4\pi} \vec{U}_3(r,\theta,\phi)\cdot\vec{F}_{s,-m,n}(\theta,\phi)\, d\Omega\Bigg],
\end{aligned}
\tag{40}
$$

$$
\begin{aligned}
Q^{(3)}_{smn} = \frac{kr^2\sqrt{\eta}\,(-1)^m}{2}\Bigg[&\left(R^{(1)}_{sn}(kr) - i(-1)^s R^{(1)}_{3-s,n}(kr)\right)\int\!\!\int_{4\pi} \vec{U}_1(r,\theta,\phi)\cdot\vec{F}_{s,-m,n}(\theta,\phi)\, d\Omega \\
&- \left(R^{(1)}_{sn}(kr) + i(-1)^s R^{(1)}_{3-s,n}(kr)\right)\int\!\!\int_{4\pi} \vec{U}_3(r,\theta,\phi)\cdot\vec{F}_{s,-m,n}(\theta,\phi)\, d\Omega\Bigg].
\end{aligned}
\tag{41}
$$

The results in eqs (40) and (41) are applicable to the case where two spherical scans, one with a Huygens' probe measuring $\vec{U}_1$ and one with an oppositely directed Huygens' probe measuring $\vec{U}_3$, have been performed at the same radius. For example, if an open-ended waveguide is used to approximate a Huygens' probe, $\vec{U}_1$ is measured by an outward pointing waveguide, and $\vec{U}_3$ is measured by an inward pointing waveguide.

For the case where the scanning sphere is electrically large, approximations to eqs (40) and (41) for large $kr$ are useful for providing some physical insight. For $kr >> n$, the required radial functions can be approximated by [1,8]

$$R^{(1)}_{1n}(kr) \approx \frac{\sin\left(kr - \dfrac{n\pi}{2}\right)}{kr}, \tag{42}$$

$$R^{(1)}_{2n}(kr) \approx \frac{\cos\left(kr - \dfrac{n\pi}{2}\right)}{kr}, \tag{43}$$

$$R^{(3)}_{1n}(kr) \approx (-i)^{n+1}\,\frac{e^{ikr}}{kr}, \tag{44}$$

$$R^{(3)}_{2n} \approx (-i)^{n}\,\frac{e^{ikr}}{kr}. \tag{45}$$

From eqs (44) and (45), the coefficients of the integrals in eq (40) to order $kr^{-1}$ are

$$R^{(3)}_{sn}(kr) + i(-1)^{s}R^{(3)}_{3-s,n} \approx 2\,i^{s-1}(-1)^{n+1}\,\frac{e^{ikr}}{kr}, \tag{46}$$

$$R^{(3)}_{sn}(kr) - i(-1)^{s}R^{(3)}_{3-s,n} \approx 0. \tag{47}$$

Thus the coefficient of the $\vec{U}_3$ integral is small, and eq (40) can be approximated by

$$Q^{(1)}_{smn} \approx r\sqrt{\eta}\,(-1)^{m}(-i)^{n+1}(-i)^{n+1}e^{ikr}\int\!\!\int_{4\pi} \vec{U}_1(r,\theta,\phi)\cdot\vec{F}_{s,-m,n}(\theta,\phi)\,d\Omega\,. \tag{48}$$

The physical interpretation of eq (48) is that $Q^{(1)}_{smn}$ can be determined approximately from a single scan with a Huygens' probe that responds to $\vec{U}_1$ and not to the outgoing waves as represented by $\vec{U}_3$. However, it should be kept in mind that eq (48) is valid for $kr \gg n$, and a complete spherical scan normally requires that $n$ values up to somewhat greater than $kr$. So eq (48) will not be valid for all relevant values of $n$.

Similarly, the approximations in eqs (42) and (43) can be used to approximate the coefficients of the integrals in eq (41):

$$R_{sn}^{(1)}(kr) - i(-1)^s R_{3-s,n}^{(1)}(kr) \approx i^{2-s} \frac{e^{-i(kr - \frac{n\pi}{2})}}{kr}, \tag{49}$$

$$R_{sn}^{(1)}(kr) + i(-1)^s R_{3-s,n}^{(1)}(kr) \approx i^{s-2} \frac{e^{i(kr - \frac{n\pi}{2})}}{kr}. \tag{50}$$

With these approximations, eq (41) can be approximated by

$$\begin{aligned} Q_{smn}^{(3)} \approx \frac{r\sqrt{\eta}(-1)^m}{2} \Big[ & i^{2-s} e^{-i(kr - \frac{n\pi}{2})} \int_{4\pi}\!\!\int \vec{U}_1(r,\theta,\phi) \cdot \vec{F}_{s,-m,n}(\theta,\phi)\, d\Omega \\ & - i^{s-2} e^{i(kr - \frac{n\pi}{2})} \int_{4\pi}\!\!\int \vec{U}_3(r,\theta,\phi) \cdot \vec{F}_{s,-m,n}(\theta,\phi)\, d\Omega . \end{aligned} \tag{51}$$

This result is less advantageous than eq (48) because both the $\vec{U}_1$ and the $\vec{U}_3$ integrals contribute to the result. If the field consists primarily of outgoing waves (due to interior sources), then $\vec{U}_1$ and its associated integral will be small.

## 3. LOW-FREQUENCY EMITTERS

The case of electrically small emitters is of interest in electromagnetic interference and in detection of electronic timing devices for bomb detonators. In such cases the source that is to be characterized or detected can be represented by electric and magnetic dipole moments, and the characterization of the source involves the determination of three orthogonal components of the electric dipole moment and three orthogonal components of the magnetic dipole moment. Methods of determining these dipole moments using a three-loop device [2] or a transverse electromagnetic (TEM) cell [13] have been developed, but here the possibility of using field probes will be analyzed.

## 3.1 Electric and Magnetic Dipole Fields

Consider the geometry in figure 3 that was analyzed in section 2.3 for general sources. For electrically small sources and scan radii, the following condition holds: $kr \leq 1$. Since $r > r_i$, $kr_i$ is also small. Under these conditions, the $n$ summations can be truncated at $n = 1$, and eqs (28) and (29) reduce to

$$\vec{E}(r,\theta,\phi) = \frac{k}{\sqrt{\eta}} \sum_{s=1}^{2} \sum_{m=-1}^{1} \left[ Q_{sm1}^{(1)} \boldsymbol{F}_{sm1}^{(1)}(r,\theta,\phi) + Q_{sm1}^{(3)} \boldsymbol{F}_{sm1}^{(3)}(r,\theta,\phi) \right], \tag{52}$$

$$\vec{H}(r,\theta,\phi) = -ik\sqrt{\eta} \sum_{s=1}^{2} \sum_{m=-1}^{1} \left[ Q_{sm1}^{(1)} \boldsymbol{F}_{3-s,m,1}^{(1)}(r,\theta,\phi) + Q_{sm1}^{(3)} \boldsymbol{F}_{3-s,m,1}^{(3)}(r,\theta,\phi) \right]. \tag{53}$$

The coefficients $Q_{sm1}^{(3)}$ characterize the electrically small source of interest, and the coefficients $Q_{sm1}^{(1)}$ characterize the interference due to external sources.

The fields of an electrically small source can be considered to be those of equivalent electric and magnetic dipole moments, $\vec{d}_e$ and $\vec{d}_m$, located at the origin as indicated in figure 4. Each dipole moment can be written in terms of its rectangular components:

$$\vec{d}_e = d_{ex}\hat{x} + d_{ey}\hat{y} + d_{ez}\hat{z} \text{ and } \vec{d}_m = d_{mx}\hat{x} + d_{my}\hat{y} + d_{mz}\hat{z}. \tag{54}$$

The spherical mode coefficients can be written in terms of the dipole components as follows [1]:

$$\begin{gathered} Q_{1,-1,1}^{(3)} = \frac{-ik\sqrt{\eta}}{2\sqrt{3\pi}}(d_{mx} + i d_{mx}),\; Q_{101}^{(3)} = \frac{-ik\sqrt{\eta}}{\sqrt{6\pi}} d_{mz},\; Q_{111}^{(3)} = \frac{-ik\sqrt{\eta}}{2\sqrt{3\pi}}(-d_{mx} + i d_{mx}), \\ Q_{2,-1,1}^{(3)} = \frac{-1}{2\sqrt{3\pi}}\frac{k}{\sqrt{\eta}}(d_{ex} + i d_{ey}),\; Q_{201}^{(3)} = \frac{-1}{\sqrt{6\pi}}\frac{k}{\sqrt{\eta}} d_{ez},\; Q_{211}^{(3)} = \frac{-1}{2\sqrt{3\pi}}\frac{k}{\sqrt{\eta}}(-d_{ex} + i d_{ey}). \end{gathered} \tag{55}$$

Thus the unknowns of interest are either six mode coefficients or six dipole components. The radiated power $P$ can be written in terms of either the spherical dipole mode coefficients or the dipole components:

$$P = \frac{1}{2}\sum_{s=1}^{2}\sum_{m=-1}^{1}|Q_{sm1}^{(3)}|^2 = \frac{1}{2}\frac{k^2}{6\pi\eta}\left[|d_{ex}|^2 + |d_{ey}|^2 + |d_{ez}|^2 + \eta^2\left(|d_{mx}|^2 + |d_{my}|^2 + |d_{mz}|^2\right)\right]. \tag{56}$$

## 3.2 Dipole Components

Since the number of unknown dipole components (or outgoing spherical modes) is only six, a dense spherical scan is not needed. By selecting the locations and field components properly, the equations for the unknown dipole components and the unknown coefficients of the standing wave modes decouple to allow simple solutions. For electrically small sources, both the radial and tangential fields are useful because they are of the same order of magnitude.

### 3.2.1 Solution in Terms of Radial Fields

The radial components of the electric and magnetic fields can be obtained by taking the dot product of eqs (52) and (53):

$$E_r(r,\theta,\phi) = \frac{k}{\sqrt{\eta}}\sum_{m=-1}^{1}\left[Q_{2m1}^{(1)}\vec{F}_{2m1}^{(1)}(r,\theta,\phi)\cdot\hat{r} + Q_{2m1}^{(3)}\vec{F}_{2m1}^{(3)}(r,\theta,\phi)\cdot\hat{r}\right], \tag{57}$$

$$H_r(r,\theta,\phi) = -ik\sqrt{\eta}\sum_{m=-1}^{1}\left[Q_{1m1}^{(1)}\vec{F}_{2m1}^{(1)}(r,\theta,\phi)\cdot\hat{r} + Q_{1m1}^{(3)}\vec{F}_{2m1}^{(3)}(r,\theta,\phi)\cdot\hat{r}\right]. \tag{58}$$

Equations (57) and (58) are decoupled because eq (57) includes only electric (TM) modes and eq (58) includes only magnetic (TE) modes. A simple method for the solution of eqs (57) and (58) makes use of the values of the spherical functions on the positive $z$ axis ($\theta = 0$):

$$\vec{F}_{201}^{(1)}(r,0,\phi)\cdot\hat{r} = \frac{\sqrt{6}}{2\sqrt{\pi}}\frac{j_1(kr)}{kr},\quad \vec{F}_{201}^{(3)}(r,0,\phi)\cdot\hat{r} = \frac{\sqrt{6}}{2\sqrt{\pi}}\frac{h_1^{(1)}(kr)}{kr},$$
$$\vec{F}_{2,-1,1}^{(1)}(r,0,\phi)\cdot\hat{r} = \vec{F}_{2,-1,1}^{(3)}(r,0,\phi)\cdot\hat{r} = \vec{F}_{211}^{(1)}(r,0,\phi)\cdot\hat{r} = \vec{F}_{211}^{(3)}(r,0,\phi)\cdot\hat{r} = 0. \tag{59}$$

Since four of the six functions are zero, both eqs (57) and (58) can be reduced to three two-by-two equations.

Consider first the radial electric field in eq (57). Only the $z$-directed electric dipole produces a radial electric field on the $z$ axis as shown in figure 5. If the radial electric field is measured at two radii, $r_1$ and $r_2$, then eq (57) yields the following two equations in two unknowns:

$$\frac{k}{\sqrt{\eta}}\frac{\sqrt{6}}{2\sqrt{\pi}}\frac{j_1(kr_1)}{kr_1}Q_{201}^{(1)} + \frac{k}{\sqrt{\eta}}\frac{\sqrt{6}}{2\sqrt{\pi}}\frac{h_1^{(1)}(kr_1)}{kr_1}Q_{201}^{(3)} = E_r(r_1,0,\phi), \tag{60}$$

$$\frac{k}{\sqrt{\eta}}\frac{\sqrt{6}}{2\sqrt{\pi}}\frac{j_1(kr_2)}{kr_2}Q_{201}^{(1)} + \frac{k}{\sqrt{\eta}}\frac{\sqrt{6}}{2\sqrt{\pi}}\frac{h_1^{(1)}(kr_2)}{kr_2}Q_{201}^{(3)} = E_r(r_2,0,\phi), \tag{61}$$

Simultaneous solution of eqs (60) and (61) yields

$$Q_{201}^{(1)} = \frac{k\sqrt{6}}{2\Delta\sqrt{\eta}\sqrt{\pi}}\left[\frac{h_1^{(1)}(kr_2)}{kr_2}E_r(r_1,0,\phi) - \frac{h_1^{(1)}(kr_1)}{kr_1}E_r(r_2,0,\phi)\right], \tag{62}$$

$$Q_{201}^{(3)} = \frac{k\sqrt{6}}{2\Delta\sqrt{\eta}\sqrt{\pi}}\left[\frac{j_1(kr_1)}{kr_1}E_r(r_2,0,\phi) - \frac{j_1(kr_2)}{kr_2}E_r(r_1,0,\phi)\right], \tag{63}$$

$$\text{where} \quad \Delta = \frac{3k^2}{2\pi\eta}\left[\frac{j_1(kr_1)}{kr_1}\frac{h_1^{(1)}(kr_2)}{kr_2} - \frac{j_1(kr_2)}{kr_2}\frac{h_1^{(1)}(kr_1)}{kr_1}\right]. \tag{64}$$

$Q_{201}^{(1)}$ represents the external sources and is not of direct interest except to indicate the strength of the interfering fields. $Q_{201}^{(3)}$ is proportional to the $z$-directed dipole moment, and from eqs (55) and (63) $d_{ez}$ is given by

$$d_{ez} = \frac{3}{\Delta}\left[\frac{j_1(kr_2)}{kr_2}E_r(r_1,0,\phi) - \frac{j_1(kr_1)}{kr_1}E_r(r_2,0,\phi)\right]. \tag{65}$$

Similarly, $d_{ex}$ can be determined from two measurements of the radial electric field on the $x$ axis, and $d_{ey}$ can be determined from two measurements of the radial electric field on the $y$ axis:

$$d_{ex} = \frac{3}{\Delta}\left[\frac{j_1(kr_2)}{kr_2}E_r(r_1,\frac{\pi}{2},0) - \frac{j_1(kr_1)}{kr_1}E_r(r_2,\frac{\pi}{2},0)\right], \quad (66)$$

$$d_{ey} = \frac{3}{\Delta}\left[\frac{j_1(kr_2)}{kr_2}E_r(r_1,\frac{\pi}{2},\frac{\pi}{2}) - \frac{j_1(kr_1)}{kr_1}E_r(r_2,\frac{\pi}{2},\frac{\pi}{2})\right]. \quad (67)$$

The determination of the magnetic dipole components follows a similar method starting with eq (58) for the radial magnetic field. From two measurements of the radial magnetic field on the $z$ axis, the following two coefficients can be determined:

$$Q_{101}^{(1)} = \frac{ik\sqrt{6}}{2\sqrt{\pi}\,\eta^{3/2}\Delta}\left[\frac{h_1^{(1)}(kr_2)}{kr_2}H_r(r_1,0,\phi) - \frac{h_1^{(1)}(kr_1)}{kr_1}H_r(r_2,0,\phi)\right], \quad (68)$$

$$Q_{101}^{(3)} = \frac{ik\sqrt{6}}{2\sqrt{\pi}\,\eta^{3/2}\Delta}\left[\frac{j_1(kr_1)}{kr_1}H_r(r_2,0,\phi) - \frac{j_1(kr_2)}{kr_2}H_r(r_1,0,\phi)\right], \quad (69)$$

where $Q_{101}^{(1)}$ represents the external sources, and $Q_{101}^{(3)}$ is proportional the $z$-directed magnetic dipole component $d_{mz}$. From eqs (55) and (69), $d_{mz}$ is given by

$$d_{mz} = \frac{3}{\eta^2\Delta}\left[\frac{j_1(kr_2)}{kr_2}H_r(r_1,0,\phi) - \frac{j_1(kr_1)}{kr_1}H_r(r_2,0,\phi)\right]. \quad (70)$$

Similarly, $d_{mx}$ and $d_{my}$ can each be determined from two measurements of the radial magnetic field:

$$d_{mx} = \frac{3}{\eta^2\Delta}\left[\frac{j_1(kr_2)}{kr_2}H_r(r_1,\frac{\pi}{2},0) - \frac{j_1(kr_1)}{kr_1}H_r(r_2,\frac{\pi}{2},0)\right], \quad (71)$$

$$d_{my} = \frac{3}{\eta^2\Delta}\left[\frac{j_1(kr_2)}{kr_2}H_r(r_1,\frac{\pi}{2},\frac{\pi}{2}) - \frac{j_1(kr_1)}{kr_1}H_r(r_2,\frac{\pi}{2},\frac{\pi}{2})\right]. \quad (72)$$

For $kr \ll 1$, the spherical Bessel functions can be replaced by their small-argument approximations [8]:

$$j_1(kr) \approx \frac{kr}{3},\ R_{21}^{(1)}(kr) \approx \frac{2}{3},\quad \text{and} \quad h_1^{(1)}(kr) \approx \frac{-i}{(kr)^2}. \tag{73}$$

For small $kr_1$ and $kr_2$, eq (73) can be used to reduce the dipole moment expressions to

$$d_{ex} \approx \frac{E_r(r_1, \frac{\pi}{2}, 0) - E_r(r_2, \frac{\pi}{2}, 0)}{\frac{i}{2\pi\eta k}\left(\frac{1}{r_1^3} - \frac{1}{r_2^3}\right)}, \tag{74}$$

$$d_{ey} \approx \frac{E_r(r_1, \frac{\pi}{2}, \frac{\pi}{2}) - E_r(r_2, \frac{\pi}{2}, \frac{\pi}{2})}{\frac{i}{2\pi\eta k}\left(\frac{1}{r_1^3} - \frac{1}{r_2^3}\right)}, \tag{75}$$

$$d_{ez} \approx \frac{E_r(r_1, 0, \phi) - E_r(r_2, 0, \phi)}{\frac{i}{2\pi\eta k}\left(\frac{1}{r_1^3} - \frac{1}{r_2^3}\right)}, \tag{76}$$

$$d_{mx} \approx \frac{H_r(r_2, \frac{\pi}{2}, 0) - H_r(r_1, \frac{\pi}{2}, 0)}{\frac{i\eta}{2\pi k}\left(\frac{1}{r_1^3} - \frac{1}{r_2^3}\right)}, \tag{77}$$

$$d_{my} \approx \frac{H_r(r_2, \frac{\pi}{2}, \frac{\pi}{2}) - H_r(r_1, \frac{\pi}{2}, \frac{\pi}{2})}{\frac{i\eta}{2\pi k}\left(\frac{1}{r_1^3} - \frac{1}{r_2^3}\right)}, \tag{78}$$

$$d_{mz} \approx \frac{H_r(r_2, 0, \phi) - H_r(r_1, 0, \phi)}{\frac{i\eta}{2\pi k}\left(\frac{1}{r_1^3} - \frac{1}{r_2^3}\right)}. \tag{79}$$

To illustrate the immunity of this dipole formulation to interfering external sources, consider a plane-wave field (from a distant external source) propagating through the source-free region where the radial electric and magnetic fields are being measured. (A single plane wave is not necessarily the best example of an interference field, but it is easy to analyze.) For propagation in the positive $x$ direction and electric field polarization in the $y$ direction, the electric field $\boldsymbol{E}_p$ and the magnetic field $\boldsymbol{H}_p$ are given by:

$$\vec{E}_p = \hat{y} E_0 e^{ikx} \quad \text{and} \quad \vec{H}_p = \hat{z} \eta E_0 e^{ikx}. \tag{80}$$

If these plane-wave fields are substituted into the dipole moment solutions in eqs (74) through (79), all of the results are zero: $d_{ex} = d_{ey} = d_{ez} = d_{mx} = d_{my} = d_{mz} = 0$. The form of eqs (74) through (79) is essentially that of a gradiometer antenna which responds to the gradient (or difference) of the field rather than the field itself. This property allows rejection of interference distant sources and simultaneous detection of nearby sources.

### 3.2.2 Solution in Terms of Tangential Fields

An advantage of using the tangential (transverse) fields is that measurements of the electric and magnetic fields taken at a single radius r are sufficient to determine the 12 unknown coefficients in eqs (52) and (53). By using various symmetries, it is possible to determine two dipole moments (out of six) at a time. For example, consider the determination of $d_{ey}$ and $d_{mz}$ from two electric-field and two magnetic-field measurements as shown in figure 6. From eqs (3), (5), (52), and (53), the relevant field components can be written:

$$E_\phi(r,\frac{\pi}{2},0) = \frac{\sqrt{6}\,k}{4\sqrt{\pi}\sqrt{\eta}}\Bigg[j_1(kr)\,Q^{(1)}_{101} + h_1^{(1)}(kr)\,Q^{(3)}_{101} - \frac{i}{\sqrt{2}}R^{(1)}_{21}(kr)\Big(Q^{(1)}_{2,-1,1} + Q^{(1)}_{211}\Big) - \frac{i}{\sqrt{2}}R^{(3)}_{21}(kr)\Big(Q^{(3)}_{2,-1,1} + Q^{(3)}_{211}\Big)\Bigg], \tag{81}$$

$$E_\phi(r,\frac{\pi}{2},\pi) = \frac{\sqrt{6}\,k}{4\sqrt{\pi}\sqrt{\eta}}\Bigg[j_1(kr)\,Q^{(1)}_{101} + h_1^{(1)}(kr)\,Q^{(3)}_{101} + \frac{i}{\sqrt{2}}R^{(1)}_{21}(kr)\Big(Q^{(1)}_{2,-1,1} + Q^{(1)}_{211}\Big) + \frac{i}{\sqrt{2}}R^{(3)}_{21}(kr)\Big(Q^{(3)}_{2,-1,1} + Q^{(3)}_{211}\Big)\Bigg], \tag{82}$$

$$H_\theta(r,\frac{\pi}{2},0) = \frac{i\sqrt{6}\,k\sqrt{\eta}}{4\sqrt{\pi}}\Bigg[R^{(1)}_{21}(kr)\,Q^{(1)}_{101} + R^{(3)}_{21}(kr)\,Q^{(3)}_{101} + \frac{i}{\sqrt{2}}j_1(kr)\Big(Q^{(1)}_{2,-1,1} + Q^{(1)}_{211}\Big) + \frac{i}{\sqrt{2}}h_1^{(1)}(kr)\Big(Q^{(3)}_{2,-1,1} + Q^{(3)}_{211}\Big)\Bigg], \tag{83}$$

$$H_\theta(r,\frac{\pi}{2},\pi) = \frac{i\sqrt{6}\,k\sqrt{\eta}}{4\sqrt{\pi}}\Bigg[R^{(1)}_{21}(kr)\,Q^{(1)}_{101} + R^{(3)}_{21}(kr)\,Q^{(3)}_{101} - \frac{i}{\sqrt{2}}j_1(kr)\Big(Q^{(1)}_{2,-1,1} + Q^{(1)}_{211}\Big) - \frac{i}{\sqrt{2}}h_{1_{(1)}}(kr)\Big(Q^{(1)}_{2,-1,1} + Q^{(1)}_{211}\Big)\Bigg]. \tag{84}$$

The solutions for the coefficients of the outgoing waves are obtained by taking sums and differences of the preceding equations in much the same as is done with the three-loop sensor [2]. Using the Wronskian in eq (32), the results are

$$Q^{(3)}_{101} = -\frac{2\sqrt{\pi}\,k\,r^2}{\sqrt{6}\sqrt{\eta}}\Bigg[j_1(kr)\left(H_\theta(r,\frac{\pi}{2},0) + H_\theta(r,\frac{\pi}{2},\pi)\right) - i\,\eta R^{(1)}_{21}(kr)\left(E_\phi(r,\frac{\pi}{2},0) + E_\phi(r,\frac{\pi}{2},\pi)\right)\Bigg], \tag{85}$$

$$Q^{(3)}_{2,-1,1} + Q^{(3)}_{211} = \frac{2\sqrt{\pi}\,k\,r^2}{\sqrt{3}\sqrt{\eta}}\left[\eta j_1(kr)\left(E_\phi(r,\frac{\pi}{2},0) - E_\phi(r,\frac{\pi}{2},\pi)\right)\right.$$
$$\left. - iR^{(1)}_{21}(kr)\left(H_\theta(r,\frac{\pi}{2},0) - H_\theta(r,\frac{\pi}{2},\pi)\right)\right]. \tag{86}$$

Similar solutions could be obtained for the coefficients of the standing waves (due to external sources). The dipole moments can be written in terms of the outgoing wave coefficients by using eq (55):

$$d_{mz} = \frac{\sqrt{6\pi}}{-ik\sqrt{\eta}} Q^{(3)}_{101} \text{ and } d_{ey} = \frac{-i2\pi r^2}{\eta}\left(Q^{(3)}_{2,-1,1} + Q^{(3)}_{211}\right). \tag{87}$$

From eqs (85) through (87), the dipole moments can be written in terms of the measured fields:

$$d_{mz} = \frac{-i2\pi r^2}{\eta}\left[j_1(kr)\left(H_\theta(r,\frac{\pi}{2},0) + H_\theta(r,\frac{\pi}{2},\pi)\right)\right.$$
$$\left. - i\eta R^{(1)}_{21}(kr)\left(E_\phi(r,\frac{\pi}{2},0) + E_\phi(r,\frac{\pi}{2},\pi)\right)\right], \tag{88}$$

$$d_{ey} = i2\pi r^2\left[\eta j_1(kr)\left(E_\phi(r,\frac{\pi}{2},0) - E_\phi(r,\frac{\pi}{2},\pi)\right)\right.$$
$$\left. - iR^{(1)}_{21}(kr)\left(H_\theta(r,\frac{\pi}{2},0) - H_\theta(r,\frac{\pi}{2},\pi)\right)\right]. \tag{89}$$

The four remaining dipole components can be obtained from eqs (88) and (89) by rotation:

$$d_{my} = \frac{-i2\pi r^2}{\eta}\left[j_1(kr)\left(-H_\theta(r,0,\frac{\pi}{2}) + H_\theta(r,\pi,\frac{\pi}{2})\right)\right.$$
$$\left. - i\eta R^{(1)}_{21}(kr)\left(-E_\phi(r,0,\frac{\pi}{2}) + E_\phi(r,\pi,\frac{\pi}{2})\right)\right], \tag{90}$$

$$d_{ex} = i2\pi r^2\left[\eta j_1(kr)\left(-E_\phi(r,0,\frac{\pi}{2}) - E_\phi(r,\pi,\frac{\pi}{2})\right)\right.$$
$$\left.- iR_{21}^{(1)}(kr)\left(-H_\theta(r,0,\frac{\pi}{2}) - H_\theta(r,\pi,\frac{\pi}{2})\right)\right], \tag{91}$$

$$d_{mx} = \frac{-i2\pi r^2}{\eta}\left[j_1(kr)\left(H_\phi(r,\frac{\pi}{2},\frac{\pi}{2}) - H_\phi(r,\frac{\pi}{2},\frac{3\pi}{2})\right)\right.$$
$$\left.- i\eta R_{21}^{(1)}(kr)\left(-E_\theta(r,\frac{\pi}{2},\frac{\pi}{2}) + E_\theta(r,\frac{\pi}{2},\frac{3\pi}{2})\right)\right], \tag{92}$$

$$d_{ez} = i2\pi r^2\left[\eta j_1(kr)\left(-E_\theta(r,\frac{\pi}{2},\frac{\pi}{2}) - E_\theta(r,\frac{\pi}{2},\frac{3\pi}{2})\right)\right.$$
$$\left.- iR_{21}^{(1)}(kr)\left(H_\phi(r,\frac{\pi}{2},\frac{\pi}{2}) + H_\theta(r,\frac{\pi}{2},\frac{3\pi}{2})\right)\right]. \tag{93}$$

For the simpler case where there are no external sources, the determination of the dipole moments does not require both electric and magnetic fields. For example, the expressions for the dipole moments in terms of electric field values are given in the Appendix.

For small values of $kr$, the radial functions in eqs (88) through (93) can be replaced by their small-argument approximations given in eq (73). Then the dipole expressions reduce to

$$d_{mz} \approx \frac{-i2\pi r^2}{3\eta}\left[kr\left(H_\theta(r,\frac{\pi}{2},0) + H_\theta(r,\frac{\pi}{2},\pi)\right) - i2\eta\left(E_\phi(r,\frac{\pi}{2},0) + E_\phi(r,\frac{\pi}{2},\pi)\right)\right], \tag{94}$$

$$d_{ey} \approx \frac{i2\pi r^2}{3}\left[\eta\, kr\left(E_\phi(r,\frac{\pi}{2},0) - E_\phi(r,\frac{\pi}{2},\pi)\right) - 2\left(H_\theta(r,\frac{\pi}{2},0) - H_\theta(r,\frac{\pi}{2},\pi)\right)\right], \tag{95}$$

$$d_{my} \approx \frac{-i2\pi r^2}{3\eta}\left[kr\left(-H_\theta(r,0,\frac{\pi}{2}) + H_\theta(r,\pi,\frac{\pi}{2})\right) - i2\eta\left(-E_\phi(r,0,\frac{\pi}{2}) + E_\phi(r,\pi,\frac{\pi}{2})\right)\right], \tag{96}$$

$$d_{ex} \approx \frac{i2\pi r^2}{3}\left[\eta kr\left(-E_\phi(r,0,\frac{\pi}{2}) - E_\phi(r,\pi,\frac{\pi}{2})\right) - 2\left(-H_\theta(r,0,\frac{\pi}{2}) - H_\theta(r,\pi,\frac{\pi}{2})\right)\right], \tag{97}$$

$$d_{mx} \approx \frac{-i2\pi r^2}{3\eta}\left[kr\left(H_\phi(r,\frac{\pi}{2},\frac{\pi}{2}) - H_\phi(r,\frac{\pi}{2},\frac{3\pi}{2})\right) - i2\eta\left(-E_\theta(r,\frac{\pi}{2},\frac{\pi}{2}) + E_\theta(r,\frac{\pi}{2},\frac{3\pi}{2})\right)\right], \tag{98}$$

$$d_{ez} \approx \frac{i2\pi r^2}{3}\left[\eta kr\left(-E_\theta(r,\frac{\pi}{2},\frac{\pi}{2}) - E_\theta(r,\frac{\pi}{2},\frac{3\pi}{2})\right) - 2\left(H_\phi(r,\frac{\pi}{2},\frac{\pi}{2}) + H_\phi(r,\frac{\pi}{2},\frac{3\pi}{2})\right)\right]. \tag{99}$$

To illustrate the immunity of this dipole formulation to external sources, consider again the plane-wave incident field (from a distant external source) as given by eq (80). If eq (80) is substituted into eqs (94) through (99), the following results are obtained:

$$d_{mx} = d_{my} = d_{ex} = d_{ez} = 0,\ d_{ey} \approx -i\frac{4}{3}\pi\eta E_0 kr\, r^2,\ \text{and}\ d_{mz} \approx -i\frac{4}{3}\pi E_0 kr\, r^2. \tag{100}$$

Four of the dipole components are identically zero because of the polarization of the incident wave. Both $d_{ey}$ and $d_{mz}$ are proportional to $kr$ which is assumed to be small. It can be shown that for arbitrary incidence angle and polarization, all of the dipole components are proportional to $kr$.

## 4. CONCLUSIONS

Spherical scanning formulations have been presented for the general case where sources are present both inside and outside the measurement sphere. The spherical wave representation in the source-free measurement region includes both outgoing waves (due to interior sources) and standing waves (due to exterior sources). The coefficients of both sets of waves have been derived in terms of the transverse electric and magnetic fields. Related results have also been derived in terms of a uniqueness vector (a particular combination of perpendicular electric and magnetic fields) which can be sensed with an ideal Huygens' probe.

The special case of a low-frequency emitter (such as a timing device for explosives) has been studied in greater detail. The spherical wave expansion of this electrically small source can be truncated at $n = 1$, and the unknowns are three scalar components of the effective electric dipole moment and three scalar components of the effective magnetic dipole moment. Six additional coefficients of standing waves (for $n = 1$) represent the interfering fields of external sources. Formulations for the twelve unknowns are derived in terms of twelve values of either the radial or tangential fields. The example of an interfering plane wave (due to a distant source) shows that the formulations are successful in nulling out interference fields.

A number of extensions to this work are possible. The solutions in terms of transverse electric and magnetic fields or uniqueness vectors are representative of ideal electric and magnetic dipole probes or an ideal Huygens' probe. A more general and more practical measurement method would treat multiple scans with a real probe antenna and its associated probe correction [1,3]. The multiple scans could be carried out at different sphere radii or with probes pointing in and out. Sensitivity is a major issue for detection and characterization of weak, low-frequency emitters. Calculations and measurements would be useful for determining practical limitations for real sources in a noisy environment.

## ACKNOWLEDGMENTS

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Appendix: Dipole Components in Terms of the Transverse Electric Field

For the simpler case where no standing waves (due to exterior sources) exist, there are only six unknown dipole components to determine. Following the method of Section 3.2.2, the six transverse electric components provide sufficient information for a solution. Consider eqs (80) and (81) for the special case where the standing wave coefficients, $Q^{(1)}_{101}$, $Q^{(1)}_{2,-1,1}$, and $Q^{(1)}_{211}$, are zero. By taking the sum and difference of the simplified equations, the following solution for the coefficients of the outgoing waves is obtained:

$$Q^{(3)}_{101} = \frac{2\sqrt{\pi}}{\sqrt{6}\,k h^{(1)}_{1}(kr)}\left[E_{\phi}(r,\frac{\pi}{2},0) + E_{\phi}(r,\frac{\pi}{2},\pi)\right], \tag{A1}$$

$$Q^{(3)}_{2,-1,1} + Q^{(3)}_{211} = \frac{i2\sqrt{\pi}\sqrt{\eta}}{\sqrt{3}\,kR^{(3)}_{21}(kr)}\left[E_{\phi}(r,\frac{\pi}{2},0) - E_{\phi}(r,\frac{\pi}{2},\pi)\right]. \tag{A2}$$

The corresponding dipole components can be obtained by substituting eqs (A1) and (A2) into eq (55):

$$d_{mz} = \frac{i2\pi}{k^2 h_1^{(1)}(kr)} \left[ E_\phi(r, \frac{\pi}{2}, 0) + E_\phi(r, \frac{\pi}{2}, \pi) \right], \tag{A3}$$

$$d_{ey} = \frac{-2\pi\eta}{k^2 R_{21}^{(3)}(kr)} \left[ E_\phi(r, \frac{\pi}{2}, 0) - E_\phi(r, \frac{\pi}{2}, \pi) \right]. \tag{A4}$$

The remaining four dipole components can be obtained from eqs (A3) and (A4) by rotation:

$$d_{my} = \frac{i2\pi}{k^2 h_1^{(1)}(kr)} \left[ -E_\phi(r, 0, \frac{\pi}{2}) + E_\phi(r, \pi, \frac{\pi}{2}) \right], \tag{A5}$$

$$d_{ex} = \frac{-2\pi\eta}{k^2 R_{21}^{(3)}(kr)} \left[ -E_\phi(r, 0, \frac{\pi}{2}) - E_\phi(r, \pi, \frac{\pi}{2}) \right], \tag{A6}$$

$$d_{mx} = \frac{i2\pi}{k^2 h_1^{(1)}(kr)} \left[ -E_\theta(r, \frac{\pi}{2}, \frac{\pi}{2}) + E_\theta(r, \frac{\pi}{2}, \frac{3\pi}{2}) \right], \tag{A7}$$

$$d_{ez} = \frac{-2\pi\eta}{k^2 R_{21}^{(3)}(kr)} \left[ -E_\theta(r, \frac{\pi}{2}, \frac{\pi}{2}) - E_\theta(r, \frac{\pi}{2}, \frac{3\pi}{2}) \right]. \tag{A8}$$

Similar equations could be obtained for the dipole components in terms of magnetic field values by starting with eqs (83) and (84).

Equations (A3) through (A8) are simpler than eqs (88) through (93), and they offer the measurement advantage of requiring only electric field samples. They do not reject the $n = 1$ standing waves, but by going through a plane-wave example as in Section 3.2.2, they can be shown to offer plane-wave rejection comparable to eqs (88) through (93). The tradeoffs between the two formulations merit further study.

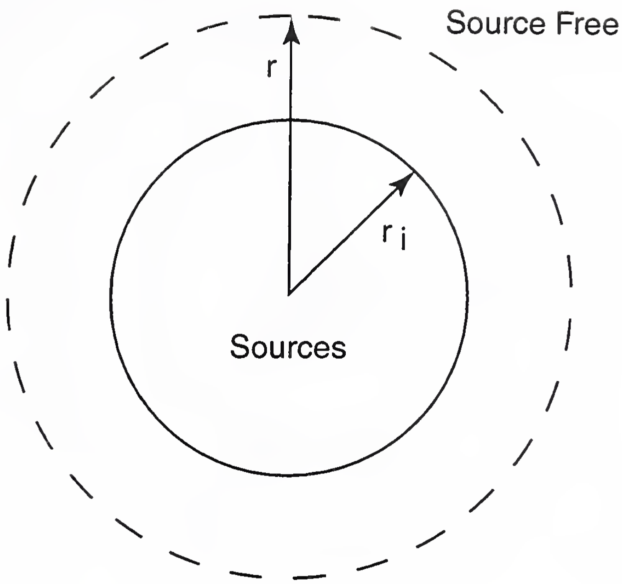


Figure 1. Geometry for a measurement sphere of radius $r$ with sources located inside a sphere of radius $r_i$.

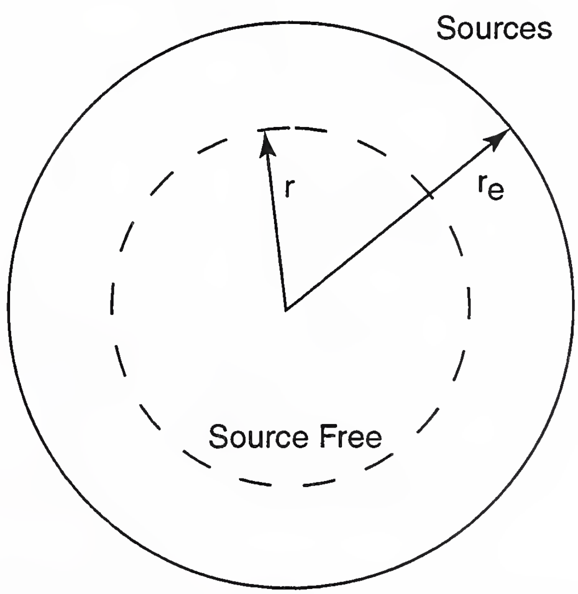


Figure 2. Geometry for a measurement sphere of radius $r$ with sources located outside a sphere of radius $r_e$.

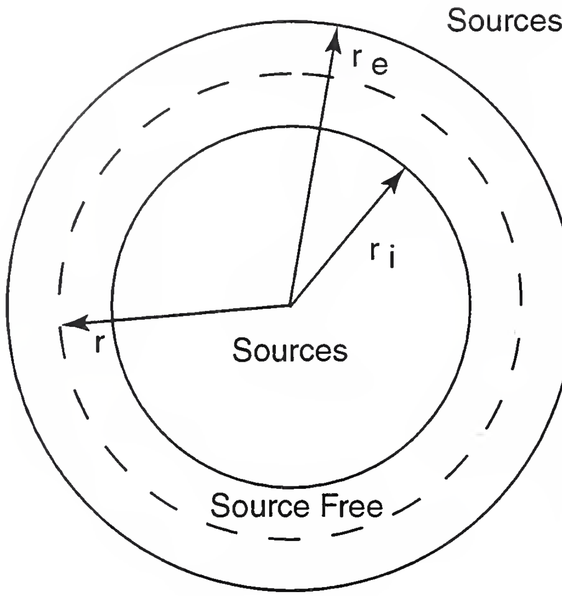


Figure 3. Geometry for a measurement sphere of radius $r$ with sources located inside a sphere of radius $r_i$ and outside a sphere of radius $r_e$.

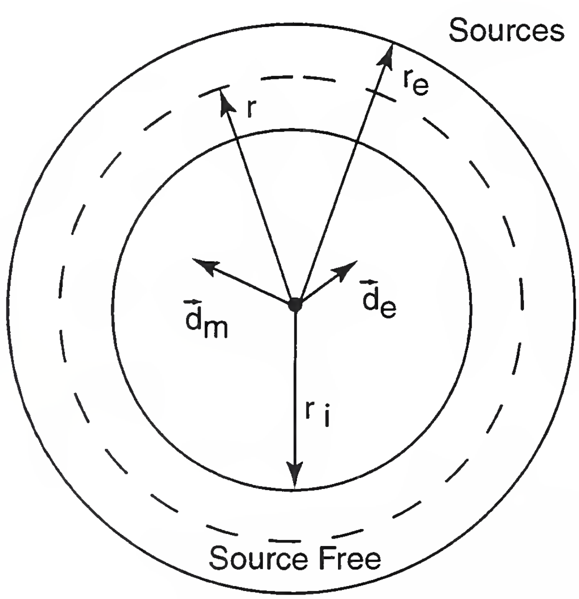


Figure 4. Geometry for a measurement sphere of radius $r$ with equivalent electric and magnetic dipoles located inside a sphere of radius $r_i$ and exterior sources located outside a sphere of radius $r_e$.

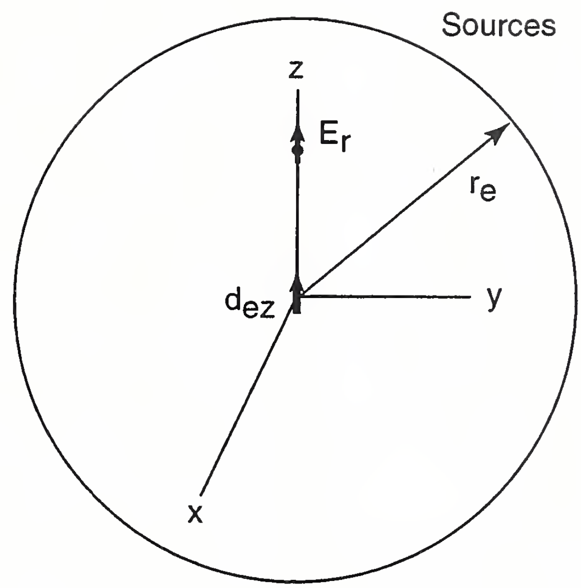


Figure 5. Geometry for a $z$-directed electric dipole located at the origin and exterior sources located outside a sphere of radius $r_e$.

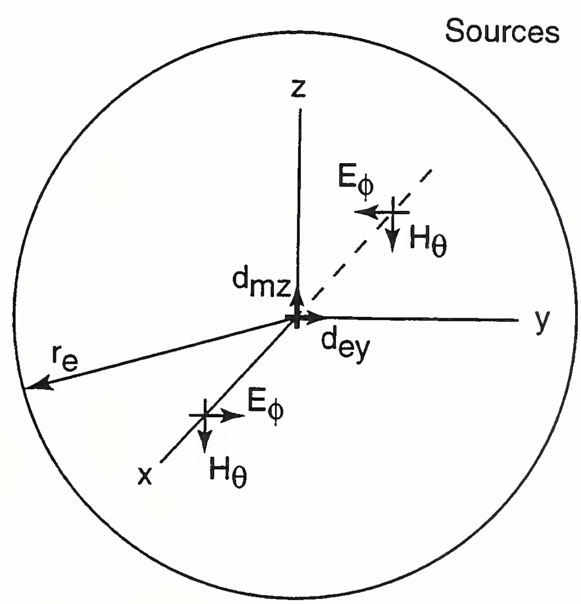


Figure 6. Electric and magnetic field samples for determining two dipole components in the presence of exterior sources.