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AN APPLICATION OF MEASUREMENT ERROR PROPAGATION THEORY TO TWO MEASUREMENT SYSTEMS USED TO CALCULATE THE POSITION AND HEADING OF A VEHICLE ON A FLAT SURFACE

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An Application of Measurement Error Propagation Theory to Two Measurement Systems Used to Calculate the Position and Heading of a Vehicle on a Flat Surface

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Abstract

This work applies known results on the propagation of measurement error statistics to two types of measurement systems, an angle measuring system and a length measuring system. Such systems can be used to measure the position and heading of a vehicle (or any rigid object) on a flat surface. For example, one might need to sense position and heading of a vehicle in order to control its motion. The functional relationships to go from angle and length measurements to position and heading are discussed and derived. Measurement error propagation theory is given in a way that can apply to any measurement system. The theory thus described is applied to the two types of measurement systems. Specific examples of these systems are given. The stated examples are currently being used by the US Bureau of Mines (BOM) to measure the position and heading of certain underground coal mining machines in an effort to move equipment operators away from the more dangerous areas of the mine. Simulation results for these specific examples in the form of density maps are given which plot the error in position and heading for various positions (heading constant) of the vehicle.

1. Introduction

It is often important to determine accurately the position and heading of a vehicle (or any rigid object) on a flat surface. This paper examines two methods of gaining this information. The first measurement system is an angle measuring system, the second, a length measuring system. The vector relations, geometry, and any constraints on the configuration are discussed for each system. The theory of the propagation of error (how errors in the measured quantities 'propagate' to errors in position and heading) is briefly given and applied to these systems. The theory is not specific and gives measurement error propagation equations that apply to these and other systems that use "noisy" measurements to calculate desired quantities that are functionally related to these measurements. The results in this paper should have impact in the area of autonomous robotic vehicles, but is not at all limited to that field and should also apply to human operated vehicles or tools.

This work assumes that correctable errors (errors of repeatable bias) in the measurement systems have been removed by appropriate calibration. Therefore, the only remaining errors in the measurement devices themselves are, for all practical purposes, unknown to the user; unknown, that is, except in terms of mean values and variances.

2. Definitions and Derivation of Closed Form Expressions for Position and Heading for an Angle Measuring System

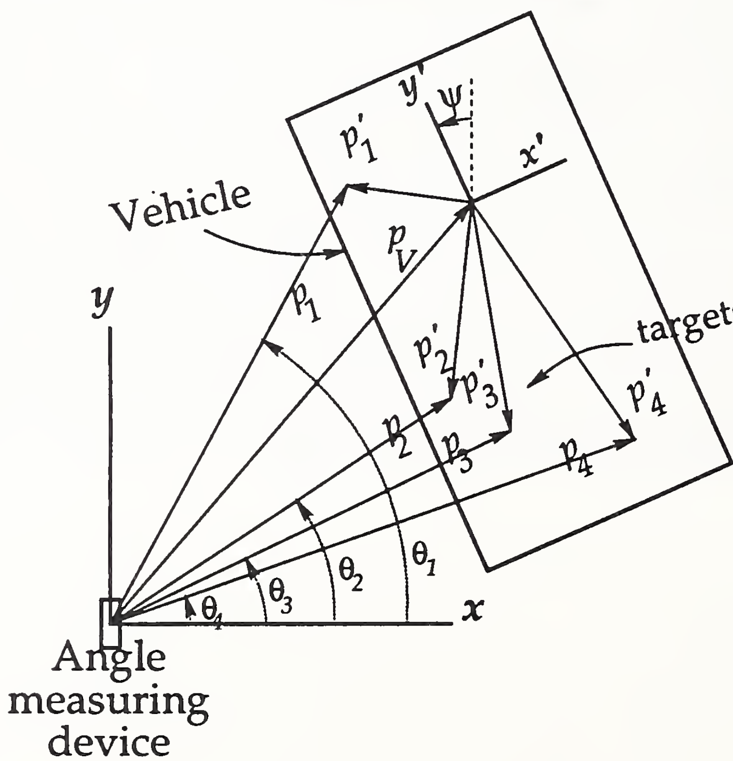

figure 1

Figure 1 represents what might be a typical scenario along with the appropriate definitions of the coordinate systems and vectors.

We seek the position, $p_V = (x,y)$, and the heading, ψ, of the vehicle. The unprimed coordinate frame is located at some known position and orientation. The primed coordinate frame is attached to the vehicle. Its position and orientation are known with respect to the vehicle but unknown with respect to the unprimed frame of reference. It is required that, when the vehicle pivots, it does so around the origin of the primed frame. θ_i for $i = 1,\ldots,4$ are the measured angles. $p_i' = (x_i', y_i')$ are the vectors corresponding to the known placements of the four targets on the vehicle and are defined in the primed reference frame. The unknown vectors, $p_i = (x_i, y_i)$, are defined in the unprimed reference frame.

Since there are only three unknowns, (x, y, ψ), that we seek to obtain, we will first attempt to obtain these unknowns using only three measured angles. $p_i' = (x_i', y_i')$ for $i = 1, 2, 3$. It will be shown that one can get closed form solutions for these unknowns, with the restriction that ψ must be within a predetermined 180° interval.

The effect of occluded targets is now introduced. If a target is completely blocked, the independent information from the occluded target is lost. This phenomenon must be examined for each configuration of the angle measurement system and will or will not be allowed depending on the nature of the angle measurement system. We will initially assume that the i^{th} target is in some way labeled so that the angle measuring device has some way of determining that it is, in fact, the i^{th} target, independent of the order in which the device encounters it. This may not be the case depending on the particulars of the measuring device. There might be several ways to get (or to eliminate the need for) target labeling: 1) Assure that each target has its own unique 'signature'. 2) Constrain position and heading of the vehicle so that there is never any occlusion of targets. 3) Write 'intelligent' software that will keep a real-time 'map' of the position of the targets so that, in effect, there is target labeling.

2.1 Closed Form Expressions for Position and Heading Using an Angle Measuring System: Three Targets and One Measurement Device

In this section, we seek to obtain the unknowns using only three measured angles.

With the definitions in figure 1 we have the following set of three vector equations for $i = 1,2,3$,

$$p_V = p_i - \begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix} p_i' \tag{1}$$

Noting that $p_i = \left(|p_i|\cos\theta_i,\ |p_i|\sin\theta_i\right)$, we have,

$$\begin{bmatrix} x \\ y \end{bmatrix} = |p_i| \begin{bmatrix} \cos\theta_i \\ \sin\theta_i \end{bmatrix} - \begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix} \begin{bmatrix} x_i' \\ y_i' \end{bmatrix} \tag{2}$$

which are six equations and six unknowns. It is helpful to eliminate $|p_i|$ to get the following three (non-linear, transcendental) equations with three unknowns,

$$\begin{bmatrix} -\tan\theta_1 & 1 & y_1'\tan\theta_1 + x_1' \\ -\tan\theta_2 & 1 & y_2'\tan\theta_2 + x_2' \\ -\tan\theta_3 & 1 & y_3'\tan\theta_3 + x_3' \end{bmatrix} \begin{bmatrix} x \\ y \\ \sin\psi \end{bmatrix} = \cos\psi \begin{bmatrix} x_1'\tan\theta_1 - y_1' \\ x_2'\tan\theta_2 - y_2' \\ x_3'\tan\theta_3 - y_3' \end{bmatrix} \tag{3}$$

If we restrict ψ to be from -90° to 90°, we can get unambiguous, closed form expressions for x, y, and ψ by first solving equation 3 and getting a single scalar expression for $\sin\psi$ in terms of $\cos\psi$ (depending on how we define ψ, we can make the restriction on ψ within *any* 180° range). From this we get an expression for $\tan\psi$, which is a one-to-one map on the range -90° to 90°. Having found $\tan\psi$, we have that $\cos\psi = \cos(\arctan(\tan\psi))$, so we can get x and y. x and y can be anywhere in the plane as long as the angle measurements, θ_i, can be made in the full range from 0° to 360°. If this full range of measurement is not available, x and y will also be restricted.

It is important to recognize that during the time any target is occluded, the equations fail. In this case, one might depend on information from some other sensor. Another option might be to use two targets and two devices which will be examined later.

One might reasonably guess that four targets and one device might yield equations which place no restriction on ψ. This is not the case since the following matrix equation is obtained

$$\begin{bmatrix} \tan\theta_1 & -1 & x_1'\tan\theta_1 - y_1' & -y_1'\tan\theta_1 - x_1' \\ \tan\theta_2 & -1 & x_2'\tan\theta_2 - y_2' & -y_2'\tan\theta_2 - x_2' \\ \tan\theta_3 & -1 & x_3'\tan\theta_3 - y_3' & -y_3'\tan\theta_3 - x_3' \\ \tan\theta_4 & -1 & x_4'\tan\theta_4 - y_4' & -y_4'\tan\theta_4 - x_4' \end{bmatrix} \begin{bmatrix} x \\ y \\ \cos\psi \\ \sin\psi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{4}$$

yielding only the trivial solution, $(x, y, \cos\psi, \sin\psi) = (0, 0, 0, 0)$. So, adding more targets than three simply allows three redundant computations of equation 3 with its 180° restriction on ψ. We will now see that with two devices (in separate locations) we only need two targets and need not require any restrictions on ψ (albeit, there are some constraints on x, y, and ψ to avoid occluding targets).

2.2 Closed Form Expressions for Position and Heading Using an Angle Measuring System: Two Targets and Two Measurement Devices

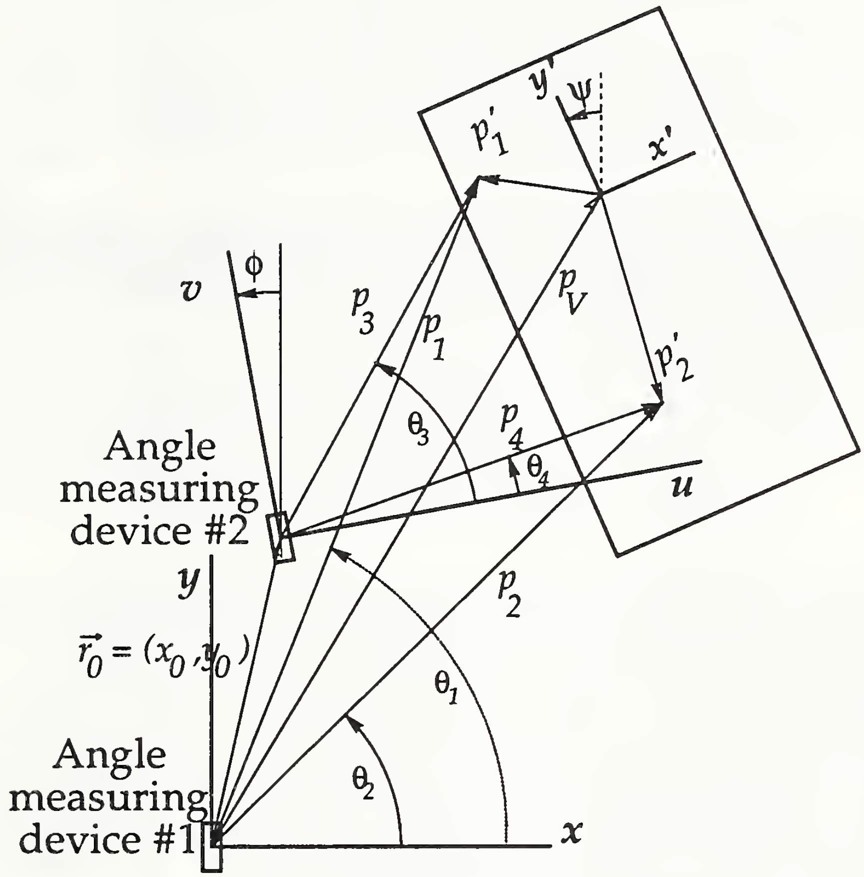

figure 2

Figure 2 represents what might be a typical scenario with only two targets and two measurement devices along with the appropriate definitions of the coordinate systems and vectors. It can be easily shown that we get and can solve the following matrix equation if $(x_0, y_0) \neq (0,0)$,

$$\begin{bmatrix} \tan\theta_1 & -1 & x_1'\tan\theta_1 - y_1' & -y_1'\tan\theta_1 - x_1' \\ \tan\theta_2 & -1 & x_2'\tan\theta_2 - y_2' & -y_2'\tan\theta_2 - x_2' \\ \tan(\phi+\theta_3) & -1 & x_1'\tan(\phi+\theta_3) - y_1' & -y_1'\tan(\phi+\theta_3) - x_1' \\ \tan(\phi+\theta_4) & -1 & x_2'\tan(\phi+\theta_4) - y_2' & -y_2'\tan(\phi+\theta_4) - x_2' \end{bmatrix} \begin{bmatrix} x \\ y \\ \cos\psi \\ \sin\psi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ x_0\tan(\phi+\theta_3) - y_0 \\ x_0\tan(\phi+\theta_4) - y_0 \end{bmatrix} \tag{5}$$

When a target is occluded from the perspective of one device, one might be able to assume that the measured angle is the same for the occluded target as for the occluding target (i.e. $\theta_1 = \theta_2$ or $\theta_3 = \theta_4$). Alternately, one might use a different set of equations. In this case, one would simply use something like the following three target matrix equation for two devices,

$$
\begin{bmatrix} -\tan\theta_1 & 1 & y_1'\tan\theta_1 + x_1' \\ -\tan(\phi+\theta_3) & 1 & y_1'\tan(\phi+\theta_3)+x_1' \\ -\tan(\phi+\theta_4) & 1 & y_2'\tan(\phi+\theta_4)+x_2' \end{bmatrix} \begin{bmatrix} x \\ y \\ \sin\psi \end{bmatrix}
= \begin{bmatrix} (x_1'\tan\theta_1 - y_1')\cos\psi \\ (x_1'\tan(\phi+\theta_3)-y_1')\cos\psi - x_0\tan(\phi+\theta_3)+y_0 \\ (x_2'\tan(\phi+\theta_4)-y_2')\cos\psi - x_0\tan(\phi+\theta_4)+y_0 \end{bmatrix} \tag{6}
$$

This equation assumes that target 2 was occluded from the vantage point of device 1. Slightly different equations are obtained for other target occlusion situations. We solve these equations similarly as we did in section 2.1 except that we must use the following formula to get $\sin\psi$,

$$
\begin{aligned}
a\sin\psi + b\cos\psi = c \Rightarrow \frac{a\sin\psi}{\sqrt{a^2+b^2}} + \frac{b\cos\psi}{\sqrt{a^2+b^2}} = \frac{c}{\sqrt{a^2+b^2}} \\
\cos\alpha\sin\psi + \sin\alpha\cos\psi = \sin(\alpha+\psi) = \frac{c}{\sqrt{a^2+b^2}}
\end{aligned} \tag{7}
$$

Still on the subject of occlusion, there exists two lines in (x, y, ψ) space, such that both devices will simultaneously experience occluded targets. In this case we would assume that $\theta_1 = \theta_2$ and $\theta_3 = \theta_4$ during the time of occlusion.

Depending on the specific angle measuring system used, it might be convenient not to allow any occlusion. If limited to two targets, one would simply be faced with tighter constraints when choosing the initial configuration of devices and targets. Another possibility is to add a third target with two devices. Still another would be two targets and three or more devices. This would assure that in all of (x, y, ψ) space there is always a solution.

2.3 The LASERNET™ Angle Measuring System

A specific example of an angle measuring system is one used by the Bureau of Mines (BOM) to locate the position and heading of a mining machine. Special cases of the equations derived above and a more detailed look at the application of an angle measurement device to coal mining can be found in [4]. BOM chose a system called LASERNET™ (manufactured by NAMCO Controls, 7567 Tyler Blvd., Mentor, OH 44060)[1] as its angle measuring system. The LASERNET™ device employs a beam of laser light that scans the scene when reflected off a rotating mirror. Cylindrical targets with a special retroreflective coating are mounted on the mining machine. At the start of each scan (based on some reference angle), a digital counter commences. The device continuously seeks to detect reflected laser light of sufficient amplitude signifying the presence of a target at that angular location. The values on the digital counter correspond to the angles from the reference angle to the 'edges' of each cylindrical target. The angle output of the device is the average between the angles to the edges. It can measure angles up to about 113° from the reference angle.

The device cannot distinguish between targets (i.e. targets are not 'labelled'). Occlusion causes two targets to be seen as one, at an angle that is between the two and, in general, is neither of the two. It seems that the only simple way of handling this problem is to restrict the movement of the mining machine so that there is never any occlusion of targets. A more challenging solution might be to employ 'intelligent' software that will keep a real-time 'map' of the position of the targets and monitor the commands to the mining machine so that, in effect, there is target labeling and one can correct for the erroneous angle.

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1 Reference to specific commercial vendors does not imply endorsement by either NIST or BOM.

3. Definitions and Derivation of Closed Form Expressions for Position and Heading Using a Length Measuring System

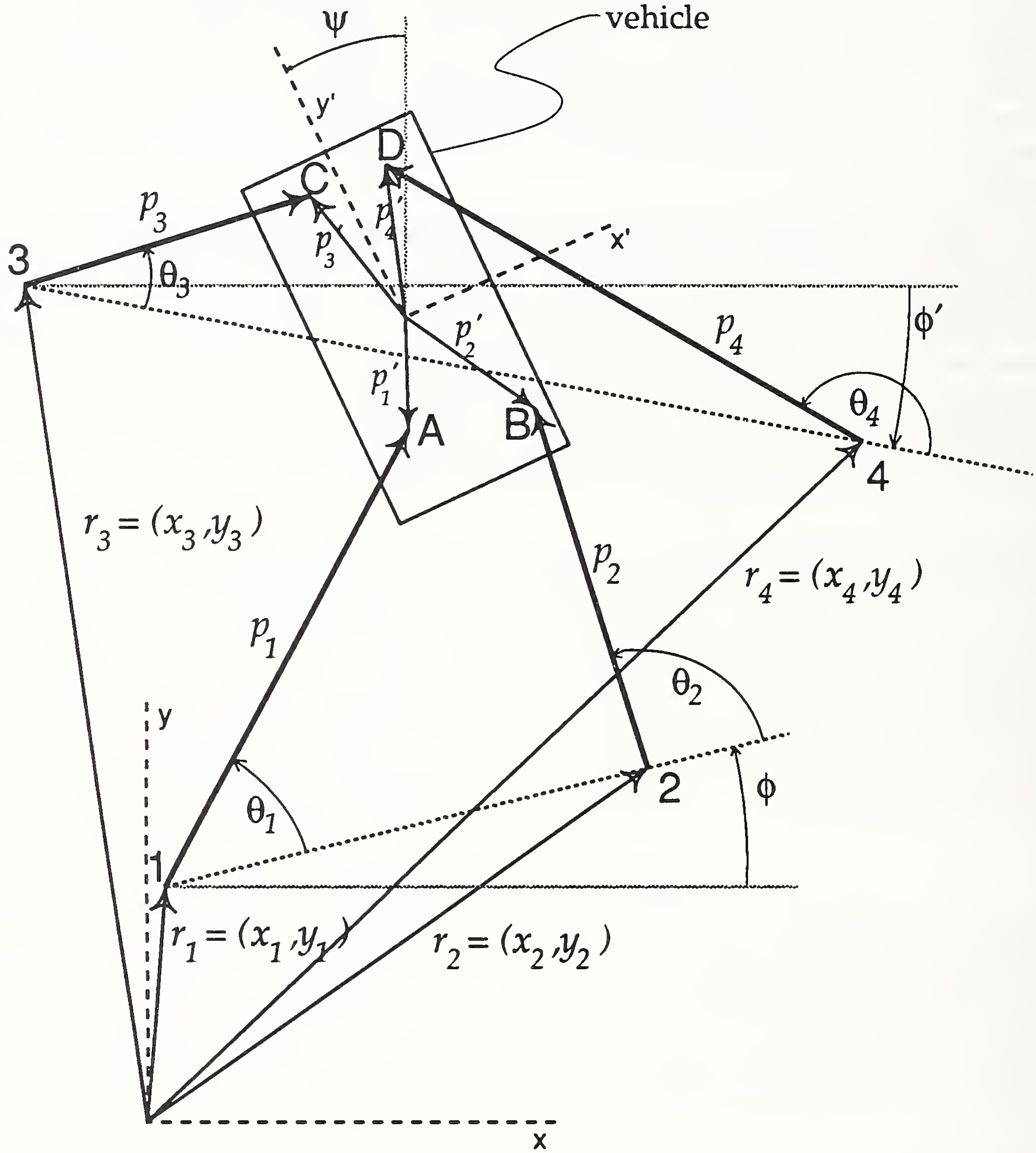


figure 3

3.1 Seeking a General Solution

Figure 3 represents what might be a typical scenario for a length measuring system in its most arbitrary configuration. Included are all the appropriate definitions of the coordinate systems and vectors. Positions **1**, **2**, **3**, and **4** are the fixed and known locations of the length measuring devices (therefore, vectors r_1, r_2,

r_3, and r_4 and angles ϕ and ϕ' are known). Positions **A**, **B**, **C**, and **D** are the fixed and known locations on the vehicle of the 'target' points of each length measuring device (therefore, vectors $p_i' = (x_i', y_i')$ are known). Lengths of the vectors p_1, p_2, p_3, and p_4 are the outputs of the length measuring devices. The vehicle is at an unknown position $p = (x, y)$ and heading ψ.

We should be able to get position and heading of the vehicle with only three devices. With three devices and the definitions given above we get the following six non-linear, transcendental equations with six unknowns (x, y, ψ, θ_1, θ_2, and θ_3)

$$\begin{aligned}
x &= x_1 + |p_1|\cos\phi\cos\theta_1 - |p_1|\sin\phi\sin\theta_1 - x_1'\cos\psi + y_1'\sin\psi \\
x &= x_2 + |p_2|\cos\phi\cos\theta_2 - |p_2|\sin\phi\sin\theta_2 - x_2'\cos\psi + y_2'\sin\psi \\
y &= y_1 + |p_1|\sin\phi\cos\theta_1 + |p_1|\cos\phi\sin\theta_1 - y_1'\cos\psi - x_1'\sin\psi \\
y &= y_2 + |p_2|\sin\phi\cos\theta_2 + |p_2|\cos\phi\sin\theta_2 - y_2'\cos\psi - x_2'\sin\psi \\
x &= x_3 + |p_3|\cos\phi'\cos\theta_3 - |p_3|\sin\phi'\sin\theta_3 - x_3'\cos\psi + y_3'\sin\psi \\
y &= y_3 + |p_3|\sin\phi'\cos\theta_3 + |p_3|\cos\phi'\sin\theta_3 - y_3'\cos\psi - x_3'\sin\psi
\end{aligned} \qquad (8)$$

These equations can be solved iteratively, but for real-time systems a closed form solution might be preferred. However, if the vehicle is moving slow enough (or not moving at all), the iterative solution may be the best since one has greater freedom in choosing geometries than if constrained to a closed form solution, as we will see. We should note that adding a fourth device still will not easily and simply allow a closed form solution.

We might first attempt to continue to use only three devices while co-locating the 'target' points of positions **1** and **2**. This way we form a single triangle with the lengths of sides known and the laws of cosines and sines can be used. Unfortunately, this yields an ambiguity in the position and heading (gives exactly two solutions). The only way to get a non-ambiguous, closed form solution, seems to be to use all four devices connected to the vehicle in such a way that two triangles are formed with known lengths of sides. This means that we might attach positions **1** and **2** to point **A** and positions **3** and **4** to point **C**.

3.2 A Special Case Allowing a Closed Form Solution

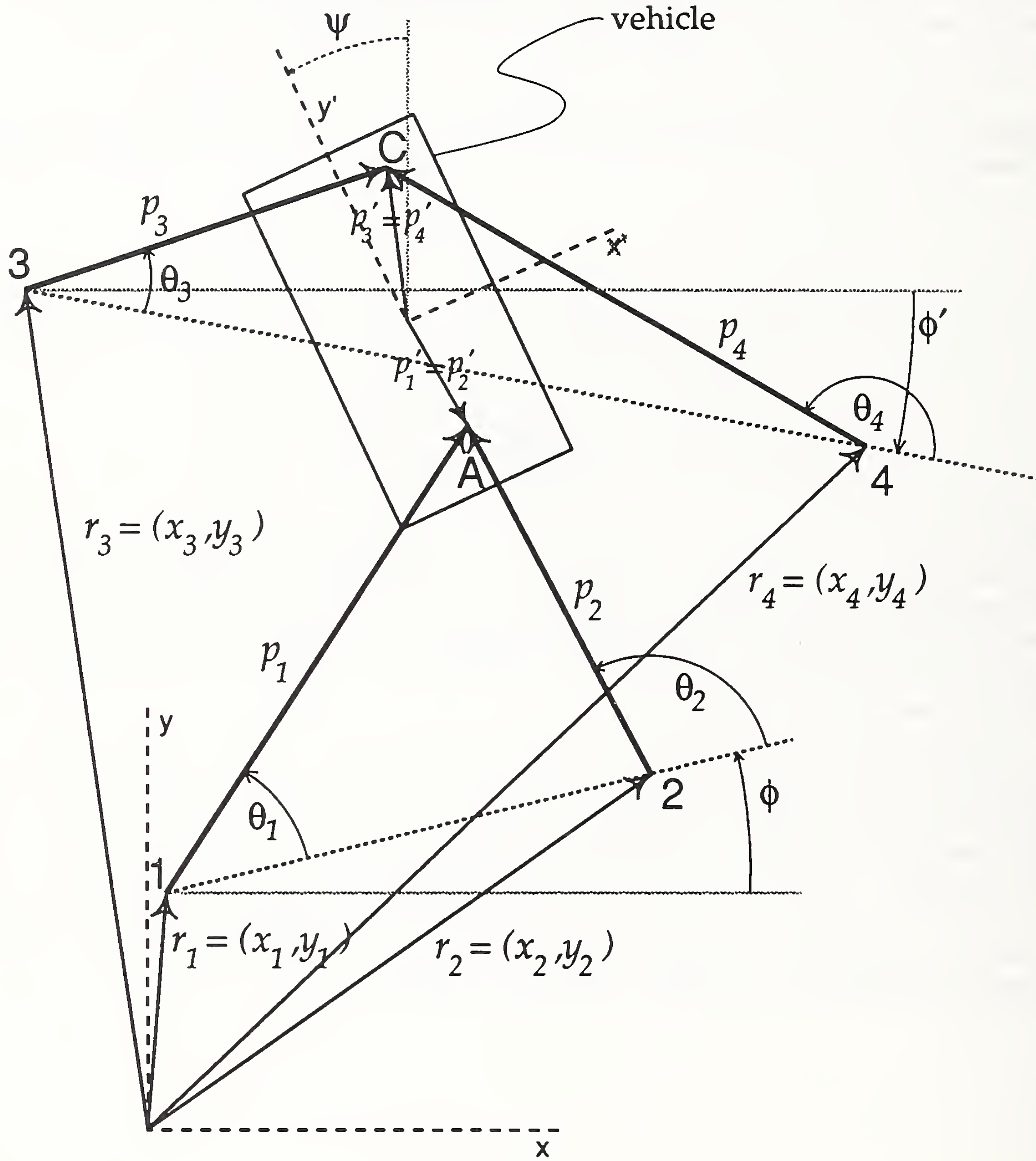


figure 4

Figure 4 gives a four device scenario in which two triangles are formed fully specifying the value of $\cos\theta_i$. Up until now we have stated that this configuration leads to an unambiguous solution. This is only true if we know the location of target point **C** with respect to the line formed by points **3** and **4** (line **34**) and the location of target point **A** with respect to the line formed by points **1** and **2** (line **12**). This is because the laws of sines and cosines do not give us the sign of $\sin\theta_i$ and we do not have enough independent equations to determine the sign of $\sin\theta_i$

(as we will demonstrate). Nonetheless, with certain minimal constraints, we can easily get around this problem and specify the position and heading of the vehicle anywhere in the plane.

Using the laws of sines and cosines and assuming we know initially the location of target point **C** with respect to the line formed by points **3** and **4** and the location of target point **A** with respect to the line formed by points **1** and **2** (which gives us the sign of $\sin\theta_i$ for all i), we get the following four independent equations expressed here in matrix form (there are actually eight equations, with only four being independent),

$$\begin{bmatrix} 1 & 0 & x_1' & -y_1' \\ 1 & 0 & x_3' & -y_3' \\ 0 & 1 & y_1' & x_1' \\ 0 & 1 & y_3' & x_3' \end{bmatrix} \begin{bmatrix} x \\ y \\ \cos\psi \\ \sin\psi \end{bmatrix} = \begin{bmatrix} x_1 + |p_1|\cos\phi\cos\theta_1 - |p_1|\sin\phi\sin\theta_1 \\ x_3 + |p_3|\cos\phi'\cos\theta_3 - |p_3|\sin\phi'\sin\theta_3 \\ y_1 + |p_1|\sin\phi\cos\theta_1 + |p_1|\cos\phi\sin\theta_1 \\ y_3 + |p_3|\sin\phi'\cos\theta_3 + |p_3|\cos\phi'\sin\theta_3 \end{bmatrix} = \begin{bmatrix} x_1 + |p_1|\cos(\phi+\theta_1) \\ x_3 + |p_3|\cos(\phi'+\theta_3) \\ y_1 + |p_1|\sin(\phi+\theta_1) \\ y_3 + |p_3|\sin(\phi'+\theta_3) \end{bmatrix} \tag{9}$$

This can be solved easily for x, y, and ψ. ψ is unconstrained here because we can get both sine and cosine of ψ. Again, as long as we know which regions points **A** and **C** are in, we can solve for x, y, and ψ with ψ unconstrained. The knowledge of where points **A** and **C** are in the plane may be hard to obtain internal to this system, particularly if there is dithering of movement when points **A** and **C** are near lines **12** and **34**. We promised a solution to this problem (with some minimal constraints) which is now given.

3.3 Getting the Sign of the Sine at Region Boundaries

It will not be hard to detect when the vehicle is in a position where the sign of $\sin\theta_i$ is uncertain, since the value for $\cos\theta_i$ will be close to 1. Therefore, when $\cos\theta_i$ is close to 1, we can simply constrain ψ to be within an appropriate 180° range (one could redefine ψ based on its current value and then recompute x_i' and y_i') and solve equation 9 for the sine of the appropriate angle (θ_1 or θ_3) to resolve the ambiguity in position. For now assume we know that point **C** is known to be near the line defined by positions **3** and **4** (line **34**). Then we would use equation 9, however, this time we solve for x, y, $\cos\psi$, and $\sin\theta_3$ in terms of $\sin\psi$,

$$\begin{bmatrix} 1 & 0 & x_1' & 0 \\ 1 & 0 & x_3' & |p_3|\sin\phi' \\ 0 & 1 & y_1' & 0 \\ 0 & 1 & y_3' & -|p_3|\cos\phi' \end{bmatrix} \begin{bmatrix} x \\ y \\ \cos\psi \\ \sin\theta_3 \end{bmatrix} = \begin{bmatrix} x_1 + y_1'\sin\psi + |p_1|\cos(\phi+\theta_1) \\ x_3 + y_3'\sin\psi + |p_3|\cos\phi'\cos\theta_3 \\ y_1 - x_1'\sin\psi + |p_1|\sin(\phi+\theta_1) \\ y_3 - x_3'\sin\psi + |p_3|\sin\phi'\cos\theta_3 \end{bmatrix} \tag{10}$$

The solution to this matrix equation gives an expression for $\cos\psi$ that is a linear function of $\sin\psi$ and known quantities (including $\cos\theta_3$ but excluding $\sin\theta_3$). Therefore, if we let a and b represent the known quantities, we have, $a\sin\psi + \cos\psi = b$. Letting $\cos\alpha = a/\sqrt{a^2+1}$ and $\sin\alpha = 1/\sqrt{a^2+1}$,

$$\psi = \arcsin\left[\frac{b}{\sqrt{a^2+1}}\right] - \alpha \tag{11}$$

Now one can proceed to get x, y, and $\sin\theta_3$.

Once the vehicle is safely outside the boundary region, one can return to equation 9 which is 'ψ-unconstrained'. A similar scheme to that outlined above is gotten when point **A** on the vehicle enters a region such that the sign of $\sin\theta_1$ is uncertain.

Finally, consider the less likely event that points **A** and **C** enter a region such that the signs of both $\sin\theta_1$ and $\sin\theta_3$ are uncertain. One can handle this problem easily as long as there is access to the type of move that the vehicle was commanded to perform. For example, if the command was to move straight forward (no turning), one can assume that ψ is known for the short period while the vehicle is within the ambiguous region. In this case, we use the following matrix equation,

$$\begin{bmatrix} 1 & 0 & |p_1|\sin\phi & 0 \\ 1 & 0 & 0 & |p_3|\sin\phi' \\ 0 & 1 & -|p_1|\cos\phi & 0 \\ 0 & 1 & 0 & -|p_3|\cos\phi' \end{bmatrix} \begin{bmatrix} x \\ y \\ \sin\theta_1 \\ \sin\theta_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_1'\cos\psi + y_1'\sin\psi + |p_1|\cos\phi\cos\theta_1 \\ x_3 - x_3'\cos\psi + y_3'\sin\psi + |p_3|\cos\phi'\cos\theta_3 \\ y_1 - y_1'\cos\psi - x_1'\sin\psi + |p_1|\sin\phi\cos\theta_1 \\ y_3 - y_3'\cos\psi - x_3'\sin\psi + |p_3|\sin\phi'\cos\theta_3 \end{bmatrix} \tag{12}$$

If the command was to pivot, one can assume that x and y are known and solve the following matrix equation.

$$\begin{bmatrix} x_1' & -y_1' & |p_1|\sin\phi & 0 \\ x_3' & -y_3' & 0 & |p_3|\sin\phi' \\ y_1' & x_1' & -|p_1|\cos\phi & 0 \\ y_3' & x_3' & 0 & -|p_3|\cos\phi' \end{bmatrix} \begin{bmatrix} \cos\psi \\ \sin\psi \\ \sin\theta_1 \\ \sin\theta_3 \end{bmatrix} = \begin{bmatrix} -x + x_1 + |p_1|\cos\phi\cos\theta_1 \\ -x + x_3 + |p_3|\cos\phi'\cos\theta_3 \\ -y + y_1 + |p_1|\sin\phi\cos\theta_1 \\ -y + y_3 + |p_3|\sin\phi'\cos\theta_3 \end{bmatrix} \tag{13}$$

If the command was to turn (i.e. x, y and ψ change at each time step), one must do some prediction and monitor $\sin\theta_1$ and $\sin\theta_3$ based on one or both of equations 12 and 13.

3.4 Handling Gross or Catastrophic Errors

It is shown in [3] that ambiguous solutions using all combinations of only three devices are useful in a special configuration of the four device scenario (this configuration is examined in section 7.1.1) to detect when one or more of the length measurements is grossly in error. To handle this problem nicely, one simply computes pairs of solutions (since ψ is ambiguous and has exactly two solutions) for the four three-device equations and makes sure that they are within maximum bounds of the (random) error values developed in this paper.

This method works also for more general configurations as in figure 4. If we eliminate device 4, we need to solve the following for x, y, ψ, and θ_3

$$\begin{bmatrix} 1 & 0 & -y_1' & 0 \\ 1 & 0 & x_1' & 0 \\ 0 & 1 & -y_3' & 0 \\ 0 & 1 & x_3' & -|p_3| \end{bmatrix} \begin{bmatrix} x \\ y \\ \sin\psi \\ \sin\theta_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_1'\cos\psi + |p_1|\cos\theta_1 \\ x_3 - y_1'\cos\psi + |p_1|\sin\theta_1 \\ x_3 - x_3'\cos\psi + |p_3|\cos\theta_3 \\ y_3 - y_3'\cos\psi \end{bmatrix} \tag{14}$$

This gives two simultaneous equations of the form

$$\begin{aligned} \sin(\alpha + \theta_3) &= \frac{a_2}{\sqrt{1+a_1^2}}\cos\psi + \frac{a_3}{\sqrt{1+a_1^2}} \\ \sin(\beta + \psi) &= \frac{a_5}{\sqrt{1+a_4^2}}\cos\theta_3 + \frac{a_6}{\sqrt{1+a_4^2}} \end{aligned} \tag{15}$$

(where α, β, and a_i are known constants) which can be solved for ψ and θ_3 (x and y are then easily gotten).

3.5 A Specific Linear Transducer Length Measuring System

A specific example of a length measuring system is one used by the Bureau of Mines (BOM) to locate the position and heading of a mining machine. A somewhat less general case of the equations derived above and a more detailed look at the application of a length measurement system to coal mining can be found in [3]. BOM is using a system of linear position transducers which utilize physical wires. Each transducer device measures the length of the extent of its wire. A difficulty with this arrangement is that not all positions and headings of the vehicle can be allowed since certain lines, of physical necessity, will not be allowed to cross others.

4. The Variance of Propagated Error

In this section we simply state the definitions and results of error propagation theory.

4.1 Definitions

$u = (u_1, \ldots, u_n) =$ the n-dimensional random vector denoting the measured quantities.

$\mu = E[u] = (E[u_1], \ldots, E[u_n]) = (\mu_1, \ldots, \mu_n)$ the limiting mean value of the n-dimensional random vector of measured quantities. This assumes (ideally) that one can take an infinite number of measurements in the determination of this value (μ will, of course, be approximated by some sample mean formed from a finite number of measurements).

$v = (v_1, \ldots, v_m) = g(u_1, \ldots, u_n) = (g_1(u), \ldots, g_m(u)) =$ the m-dimensional (known) function (g: $\Re^n \to \Re^m$) that computes the desired result from the measured values. In both measurement systems under consideration, the function g can be known from elementary trigonometry. g varies depending on what type of measurement system is employed.

$$\frac{\partial g}{\partial u}(\mu) = \begin{bmatrix} \frac{\partial g_1}{\partial u_1}(\mu) & \cdots & \frac{\partial g_1}{\partial u_n}(\mu) \\ \vdots & \ddots & \vdots \\ \frac{\partial g_m}{\partial u_1}(\mu) & \cdots & \frac{\partial g_m}{\partial u_n}(\mu) \end{bmatrix}$$ = the matrix of partial derivatives evaluated at the limiting mean.

$\sigma_v^2 = \left(\sigma_{v_1}^2, \ldots, \sigma_{v_m}^2\right)$ = the vector of variances of each calculated result.

$u - \mu = \Delta u = \left(\Delta u_1, \ldots, \Delta u_n\right)$ = the vector of measurement errors (relatively small compared to μ) in the measurement vector, u, expressed as a random vector of arbitrary but known distribution. For the purpose of this paper, we assume that the random variables are, in general, correlated.

$\sigma_{ij} = E\left[\Delta u_i \Delta u_j\right]$ = the covariance of the random variables, Δu_i and Δu_j.

$\sigma_i^2 = E\left[\Delta u_i^2\right]$ = the variance of the random variable, Δu_i.

$\rho_{ij} = \sigma_{ij} / \sigma_i \sigma_j$ = the correlation coefficient (equals ± 1 if the random variables of measurement error, Δu_i and Δu_j, are completely correlated and equals 0 if completely independent). [1]

Before leaving the subject of definitions, it must be stated that μ, σ_i and σ_{ij} are ideal values requiring an infinite number of measurements and, for real measurement systems, need to be approximated. Below are given standard formulas for determining the approximations to the ideal. Of course, very large values for N will give good approximations.

$\mu_i \approx \overline{u}_i = \frac{1}{N} \sum_{k=1}^{N} u_{i_k}$ = the estimated mean of the i^{th} component of u based on N sample values of the random measurement, u_i.

$\sigma_i \approx s_i = \sqrt{\frac{1}{N-1} \sum_{k=1}^{N} \left(u_{i_k} - \mu_i\right)^2}$ = the estimated standard deviation of the i^{th} component of u based on N sample values of the measurement random variable, u_i.

$\sigma_{ij} \approx s_{ij} = \frac{1}{N-1}\sum_{k=1}^{N}\left(u_{i_k} - \mu_i\right)\left(u_{j_k} - \mu_j\right) =$ the estimated covariance of the i^{th} component of u with the j^{th} component of u using N sample values of the measurement random variables, u_i and u_j.

4.2 Error Propagation Theory

The multi-dimensional Taylor series theorem says

$$v = g(\mu + \Delta u) \approx g(\mu) + \frac{\partial g}{\partial u}(\mu) \cdot \Delta u \tag{16}$$

This is usually a very good approximation, if Δu is small (if not, the measurement system is poor). In this derivation, we are assuming that Δu is a random vector of known but arbitrary distribution. Our goal is to determine the variance $\sigma_v^2 = \left(\sigma_{v_1}^2, \ldots, \sigma_{v_m}^2\right)$ of the random vector, $v = g(\mu + \Delta u)$. From [1] and [2]

$$\sigma_{v_k}^2 \approx \sum_{i=1}^{n}\left(\frac{\partial g_k}{\partial u_i}(\mu)\right)^2 \sigma_i^2 + 2\sum_{i=1}^{n}\sum_{j=i+1}^{n}\frac{\partial g_k}{\partial u_i}(\mu)\frac{\partial g_k}{\partial u_j}(\mu)\rho_{ij}\sigma_i\sigma_j \quad \text{for } k = 1,\ldots,m \tag{17}$$

If the vector of random variables of measurement error is an independent sequence of random variables (equivalently, $\rho_{ij} = 0$)

$$\sigma_{v_k}^2 \approx \sum_{i=1}^{n}\left(\frac{\partial g_k}{\partial u_i}(\mu)\right)^2 \sigma_i^2 \quad \text{for } k = 1,\ldots,m \tag{18}$$

On the other hand, if the vector of random variables of measurement error is a completely dependent (in the same direction) sequence of random variables ($\rho_{ij} = 1$)

$$\sigma_{v_k} \approx \left|\sum_{i=1}^{n}\frac{\partial g_k}{\partial u_i}(\mu)\sigma_i\right| \quad \text{for } k = 1,\ldots,m \tag{19}$$

Note that $\sum\left(\partial g_k / \partial u_i\right)^2 \sigma_i^2$ is unaffected by the sign of the partial derivatives, but $\left|\sum \partial g_k / \partial u_i\, \sigma_i\right|^2$ is. For example, say $v = u_1 + u_2$ and $\sigma_1 = \sigma_2 = \sigma$. If $\rho_{12} = 1$, $\sigma_v^2 \approx \sigma_1^2 + \sigma_2^2 + 2\sigma_1\sigma_2 = 4\sigma^2$. If $\rho_{12} = 0$ (independence), $\sigma_v^2 \approx \sigma_1^2 + \sigma_2^2 = 2\sigma^2$. On the other hand, say $v = u_1 - u_2$. In this case $\rho_{12} = 1$ implies that $\sigma_v^2 \approx \sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2 = 0$.

5. The Variance of the Random Variables of Position and Heading: Length Measuring Systems

Because each length measuring device of the total of each length measuring system operates independently of all the others, it is reasonable to expect that the sequence of measurement errors is independent. So we take equation 18 and apply it to the length measuring system. With the following definitions (for simplicity of notation, let $p_i \triangleq |p_i|$), $\mu = p = (p_1, p_2, p_3, p_4)$ and $(g_1(p), g_2(p), g_3(p)) = (x, y, \psi)$, we have that

$$\begin{aligned}
\sigma_x^2 &= \left(\frac{\partial x}{\partial p_1}(p)\right)^2 \sigma_1^2 + \left(\frac{\partial x}{\partial p_2}(p)\right)^2 \sigma_2^2 + \left(\frac{\partial x}{\partial p_3}(p)\right)^2 \sigma_3^2 + \left(\frac{\partial x}{\partial p_4}(p)\right)^2 \sigma_4^2 \\
\sigma_y^2 &= \left(\frac{\partial y}{\partial p_1}(p)\right)^2 \sigma_1^2 + \left(\frac{\partial y}{\partial p_2}(p)\right)^2 \sigma_2^2 + \left(\frac{\partial y}{\partial p_3}(p)\right)^2 \sigma_3^2 + \left(\frac{\partial y}{\partial p_4}(p)\right)^2 \sigma_4^2 \qquad (20) \\
\sigma_\psi^2 &= \left(\frac{\partial \psi}{\partial p_1}(p)\right)^2 \sigma_1^2 + \left(\frac{\partial \psi}{\partial p_2}(p)\right)^2 \sigma_2^2 + \left(\frac{\partial \psi}{\partial p_3}(p)\right)^2 \sigma_3^2 + \left(\frac{\partial \psi}{\partial p_4}(p)\right)^2 \sigma_4^2
\end{aligned}$$

6. The Variance of the Random Variables of Position and Heading: Angle Measuring Systems

We expect that the measurement errors of an angle measuring system, single device configuration, will be correlated, but not fully correlated. Let's assume that we have three targets and one angle measurement device and are using the solutions to equation 3. With the following definitions, $\mu = \theta = (\theta_1, \theta_2, \theta_3)$ and $(g_1(\theta), g_2(\theta), g_3(\theta)) = (x, y, \psi)$, we have that

$$\begin{aligned}
\sigma_x^2 &= \sum_{i=1}^{3}\left(\frac{\partial x}{\partial \theta_i}\right)^2 \sigma_i^2 + 2\left(\frac{\partial x}{\partial \theta_1}\frac{\partial x}{\partial \theta_2}\rho_{12}\sigma_1\sigma_2 + \frac{\partial x}{\partial \theta_1}\frac{\partial x}{\partial \theta_3}\rho_{13}\sigma_1\sigma_3 + \frac{\partial x}{\partial \theta_2}\frac{\partial x}{\partial \theta_3}\rho_{23}\sigma_2\sigma_3\right) \\
\sigma_y^2 &= \sum_{i=1}^{3}\left(\frac{\partial y}{\partial \theta_i}\right)^2 \sigma_i^2 + 2\left(\frac{\partial y}{\partial \theta_1}\frac{\partial y}{\partial \theta_2}\rho_{12}\sigma_1\sigma_2 + \frac{\partial y}{\partial \theta_1}\frac{\partial y}{\partial \theta_3}\rho_{13}\sigma_1\sigma_3 + \frac{\partial y}{\partial \theta_2}\frac{\partial y}{\partial \theta_3}\rho_{23}\sigma_2\sigma_3\right) \qquad (21) \\
\sigma_\psi^2 &= \sum_{i=1}^{3}\left(\frac{\partial \psi}{\partial \theta_i}\right)^2 \sigma_i^2 + 2\left(\frac{\partial \psi}{\partial \theta_1}\frac{\partial \psi}{\partial \theta_2}\rho_{12}\sigma_1\sigma_2 + \frac{\partial \psi}{\partial \theta_1}\frac{\partial \psi}{\partial \theta_3}\rho_{13}\sigma_1\sigma_3 + \frac{\partial \psi}{\partial \theta_2}\frac{\partial \psi}{\partial \theta_3}\rho_{23}\sigma_2\sigma_3\right)
\end{aligned}$$

For the two device scenario, we expect that that angle measurements within each device will be correlated, but that the angle measurements across devices will not be correlated. In this case, with the following definitions, $u = \theta = (\theta_1, \theta_2, \theta_3, \theta_4)$ and $(g_1(\theta), g_2(\theta), g_3(\theta)) = (x, y, \psi)$, we get the following equations

$$\sigma_x^2 = \sum_{i=1}^{4}\left(\frac{\partial x}{\partial \theta_i}\right)^2 \sigma_i^2 + 2\left(\frac{\partial x}{\partial \theta_1}\frac{\partial x}{\partial \theta_2}\rho_{12}\sigma_1\sigma_2 + \frac{\partial x}{\partial \theta_3}\frac{\partial x}{\partial \theta_4}\rho_{34}\sigma_3\sigma_4\right)$$
$$\sigma_y^2 = \sum_{i=1}^{4}\left(\frac{\partial y}{\partial \theta_i}\right)^2 \sigma_i^2 + 2\left(\frac{\partial y}{\partial \theta_1}\frac{\partial y}{\partial \theta_2}\rho_{12}\sigma_1\sigma_2 + \frac{\partial y}{\partial \theta_3}\frac{\partial y}{\partial \theta_4}\rho_{34}\sigma_3\sigma_4\right) \qquad (22)$$
$$\sigma_\psi^2 = \sum_{i=1}^{4}\left(\frac{\partial \psi}{\partial \theta_i}\right)^2 \sigma_i^2 + 2\left(\frac{\partial \psi}{\partial \theta_1}\frac{\partial \psi}{\partial \theta_2}\rho_{12}\sigma_1\sigma_2 + \frac{\partial \psi}{\partial \theta_3}\frac{\partial \psi}{\partial \theta_4}\rho_{34}\sigma_3\sigma_4\right)$$

7. Simulation

The solutions to matrix equations 3, 5, 9, and 11 have been verified comparing numerical results with graphical measurements. Relevant symbolic partial derivatives have also been generated and error propagation values computed for 1) two configurations of a length measuring system and 2) three configurations of an angle measuring system, each with constraints and distances appropriate to an underground coal mine. The general correctness of the error propagation values has also been verified using standard Monte Carlo techniques. Density maps of the errors in the position and heading of the vehicle are now given. These results might be used by a sensor fusion module that takes position and/or heading values from various sensors and determines an optimum position and heading value.

Each configuration is accompanied by three error maps that give the 'maximum' errors ($3\sigma_x$, $3\sigma_y$, and $3\sigma_\psi$) in x, y, and ψ, respectively, for various positions x and y of the mining machine in a 20' by 60' area of an underground coal mine (a typical 'entry'). The heading, ψ, is at 0° for all positions. The device error is assumed to be 0.5 inches = $3\sigma_i$ per device for all length measuring devices and is 0.05° = $3\sigma_i$ per device for all angle measuring devices (these values were gotten using emperical data from the systems described in sections 2.3 and 3.5). The error maps are given as density plots. The dark areas correspond to the highest errors in the stated range and the light areas to the lowest errors. The error maps give values for mining machine positions such that the front end of the mining machine extends 60 feet past the origin of the x-y axes.

7.1 Error Maps for Length Measuring Systems Assuming a Coal Mining Scenario

7.1.1 Error Maps for a Certain Configuration of a Length Measuring System: Case 1

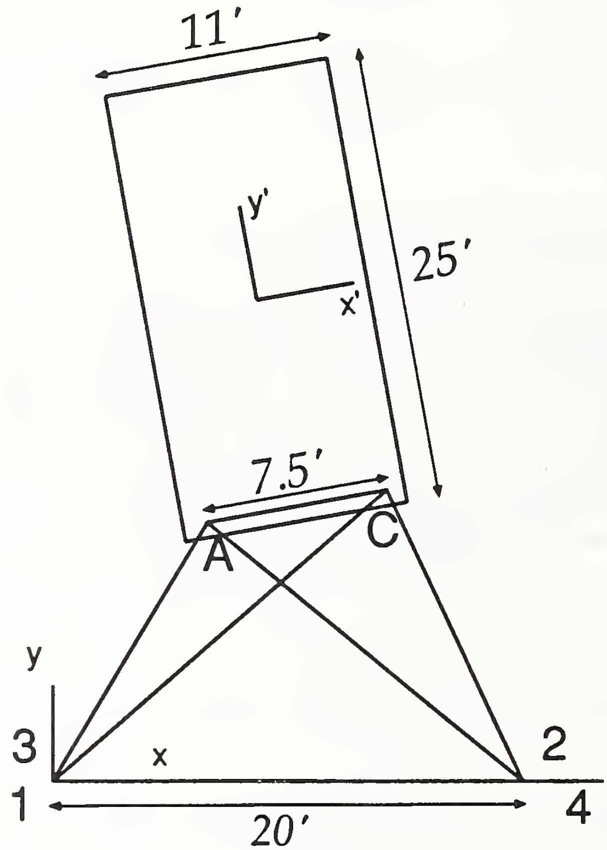


figure 5

Figure 5 gives a configuration for a length measuring system currently being used at the US Bureau of Mines to determine the position and heading of a mining machine. Lengths one would expect in a coal mine are given (in English units). The axes of the error plots correspond to the position of the origin of the x'-y' axes (the center of the mining machine) in the x-y coordinate frame.

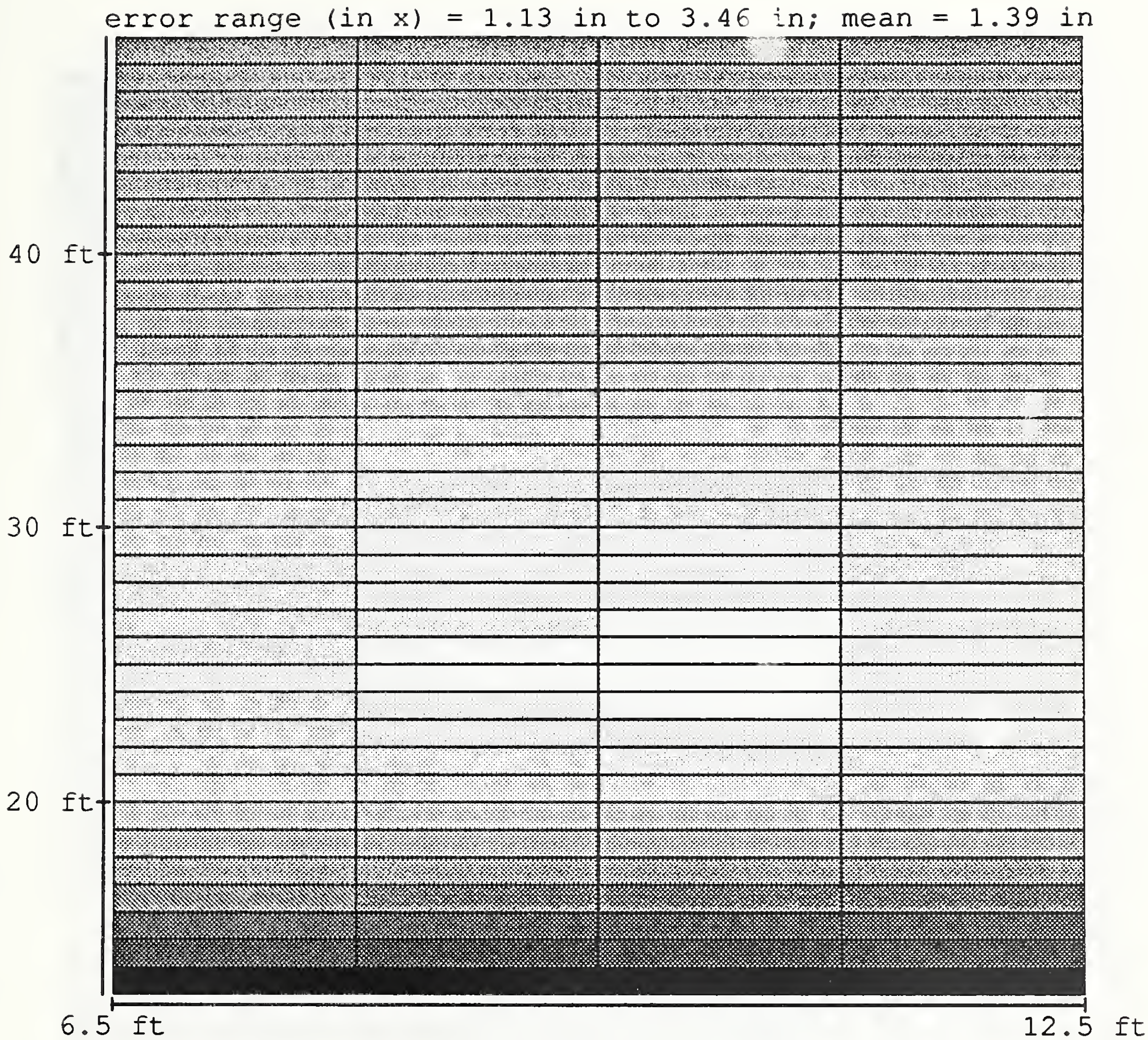

error range (in x) = 1.13 in to 3.46 in; mean = 1.39 in
40 ft
30 ft
20 ft
6.5 ft
12.5 ft

figure 6

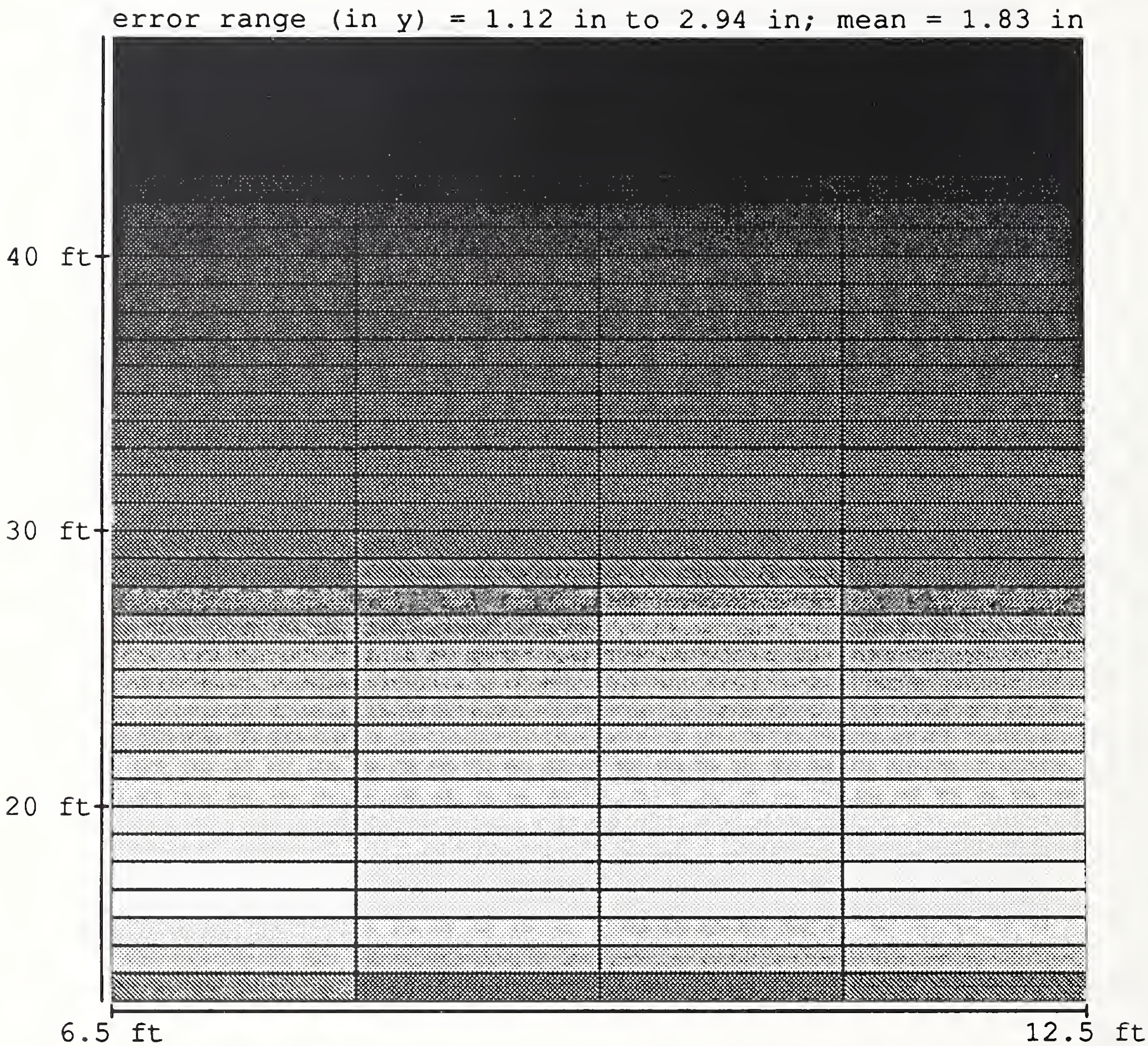

error range (in y) = 1.12 in to 2.94 in; mean = 1.83 in
40 ft
30 ft
20 ft
6.5 ft
12.5 ft

figure 7

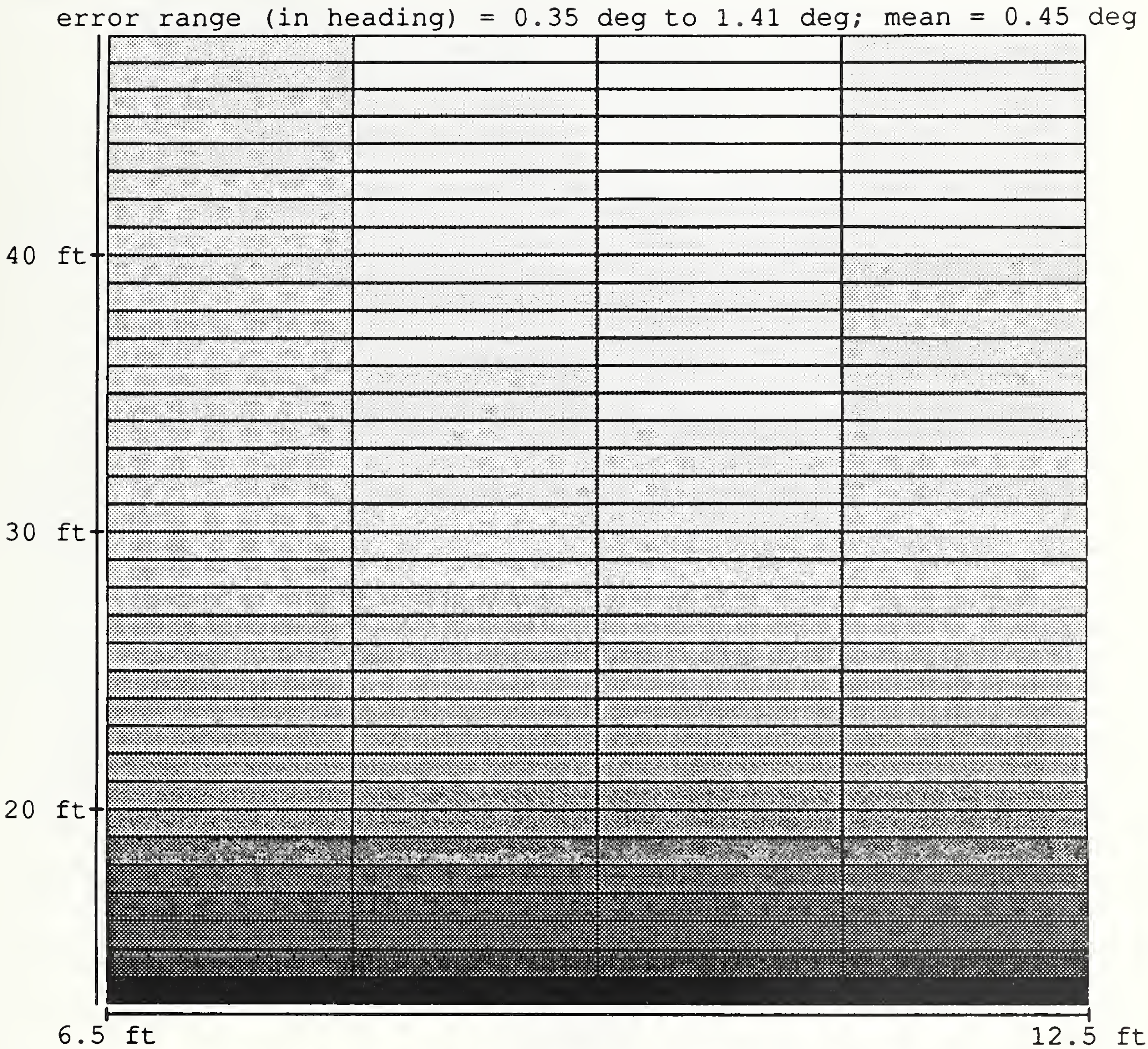

error range (in heading) = 0.35 deg to 1.41 deg; mean = 0.45 deg
40 ft
30 ft
20 ft
6.5 ft
12.5 ft

figure 8

7.1.2 Error Maps for a Certain Configuration of a Length Measuring System: Case 2

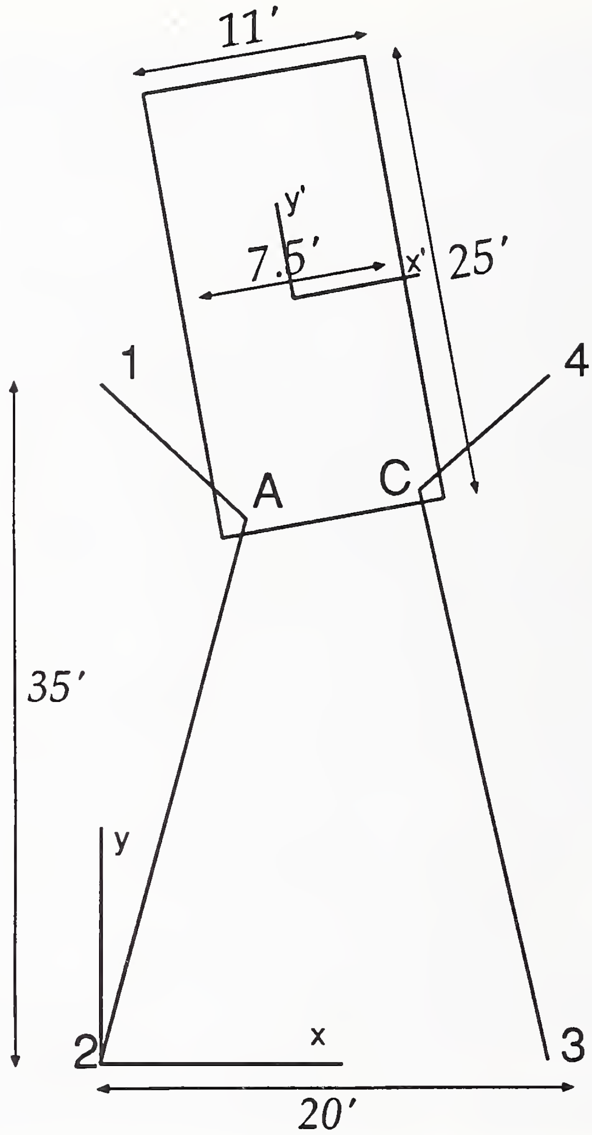


figure 9

Figure 9 gives another configuration for a length measuring system that can be used to determine the position and heading of the mining machine. Lengths one would expect in a coal mine are given (in English units). The axes of the error plots correspond to the position of the origin of the x'-y' axes (the center of the mining machine) in the x-y coordinate frame.

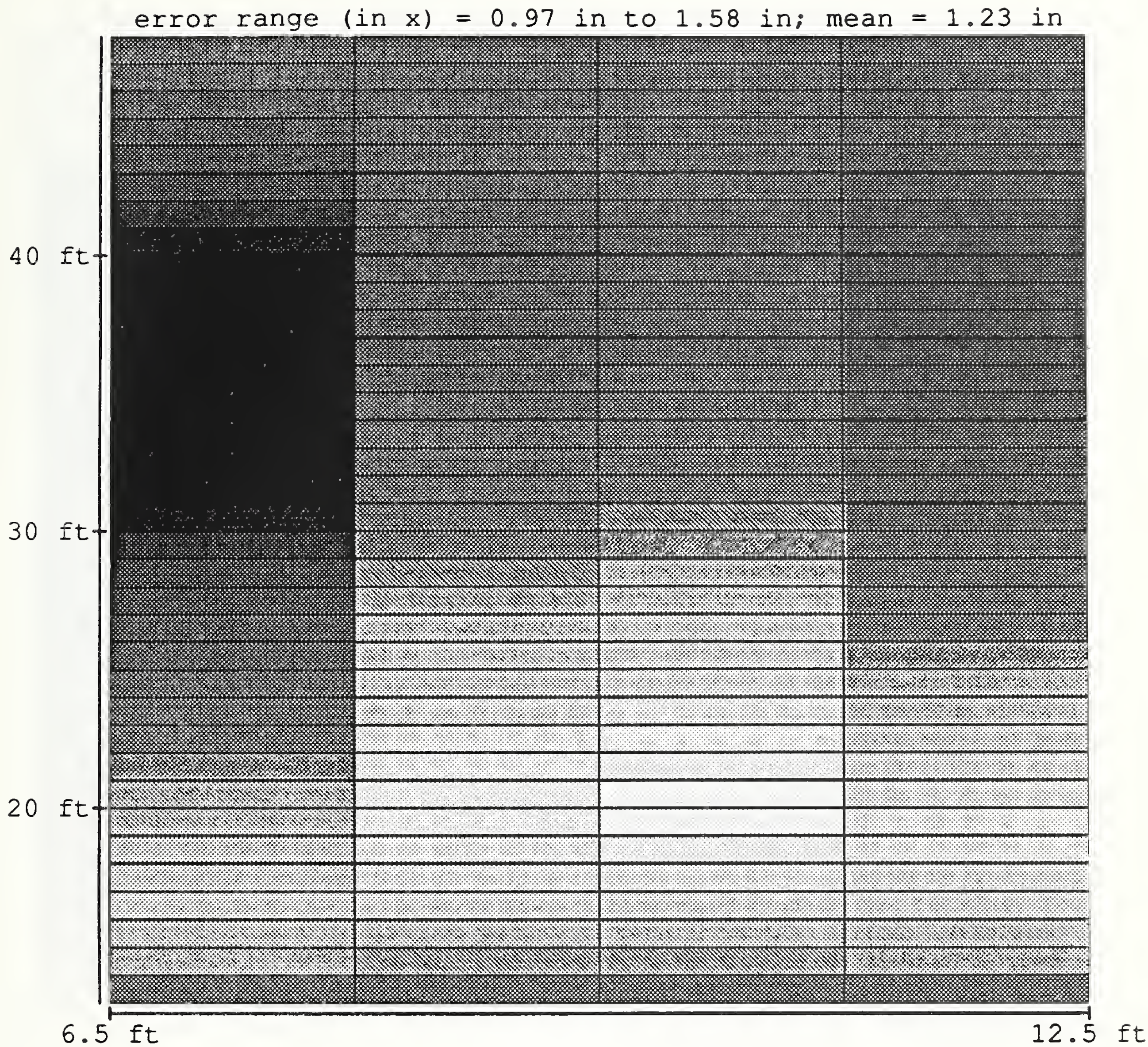

error range (in x) = 0.97 in to 1.58 in; mean = 1.23 in
40 ft
30 ft
20 ft
6.5 ft
12.5 ft

figure 10

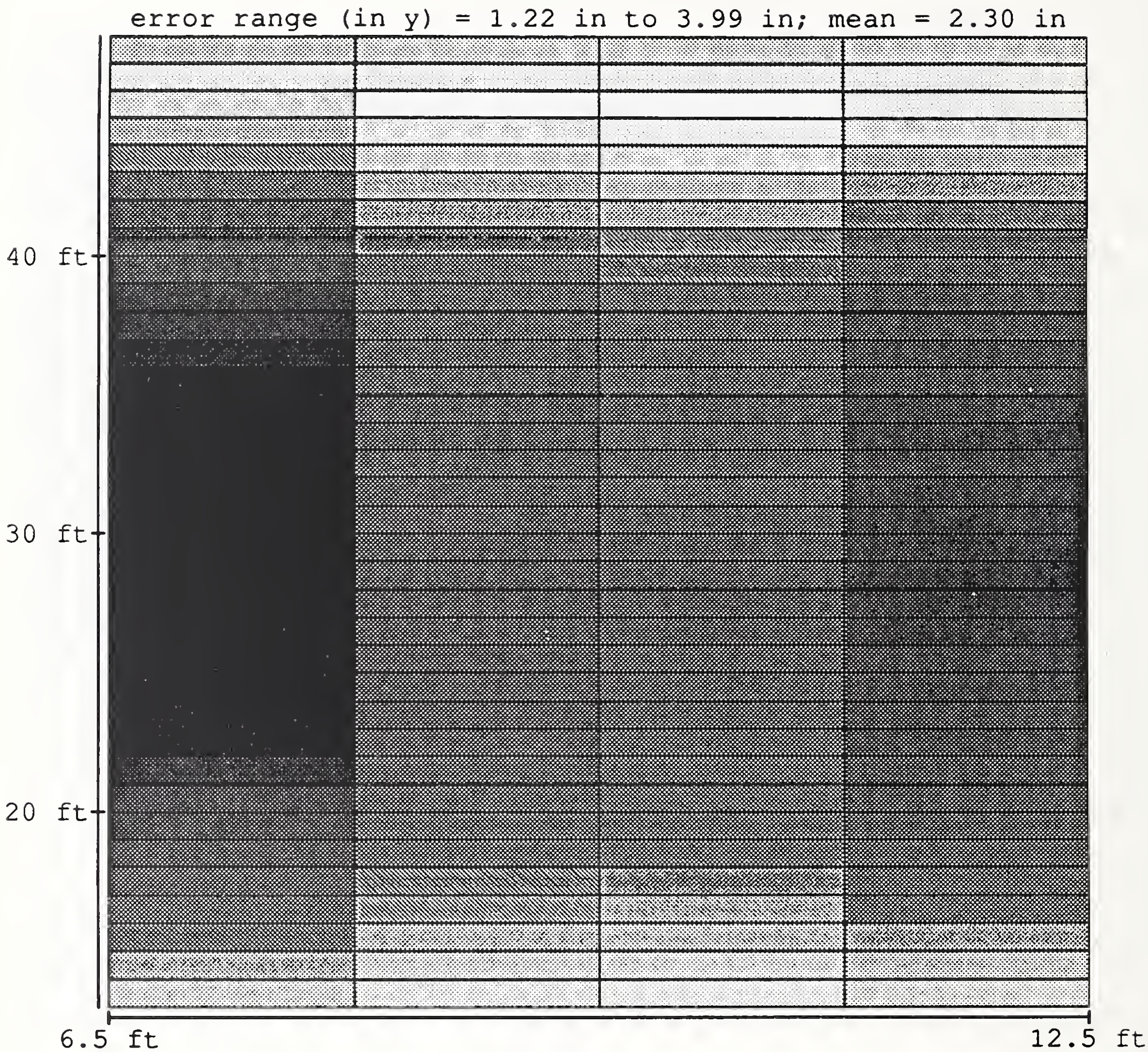

error range (in y) = 1.22 in to 3.99 in; mean = 2.30 in
40 ft
30 ft
20 ft
6.5 ft
12.5 ft

figure 11

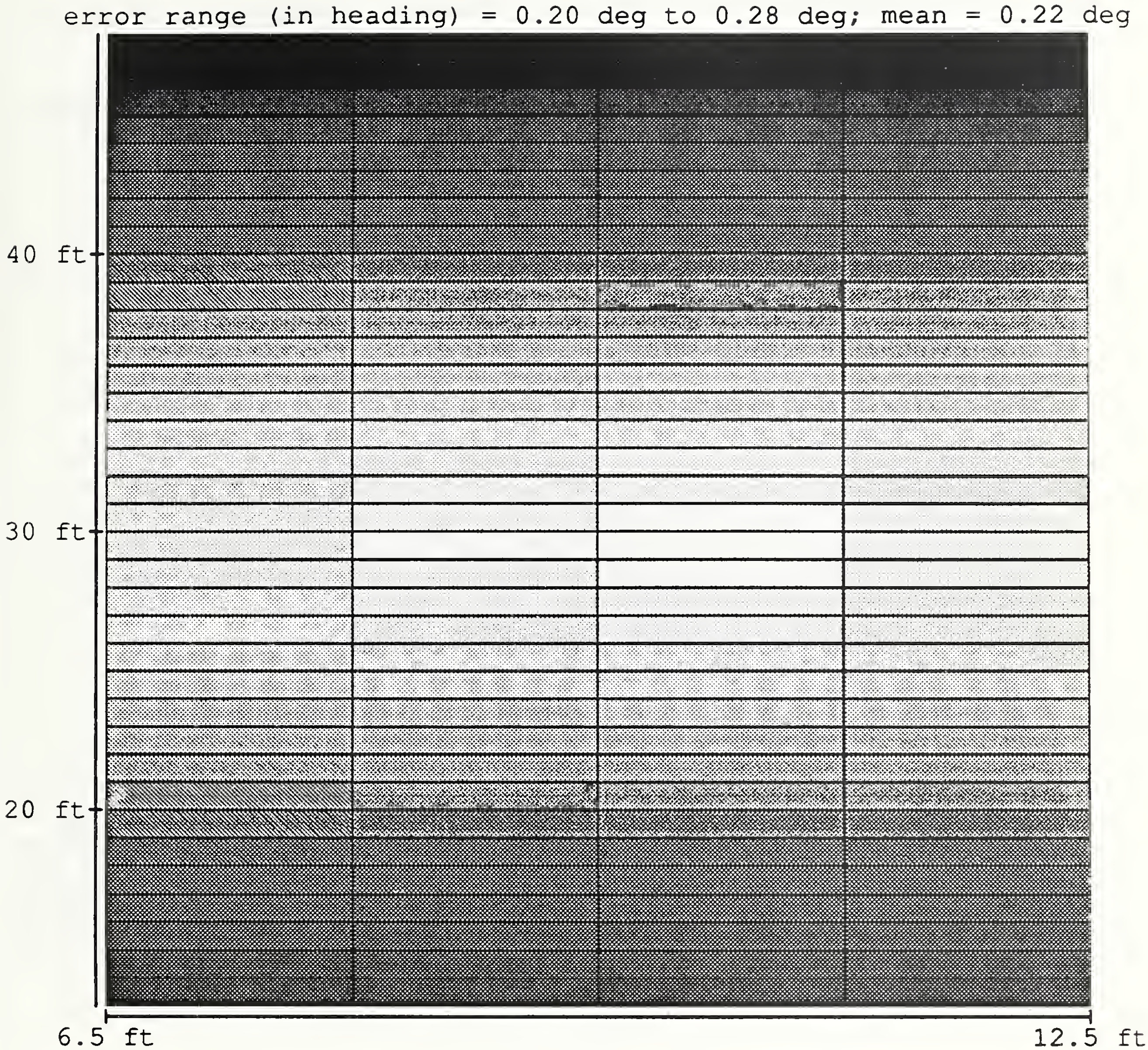

error range (in heading) = 0.20 deg to 0.28 deg; mean = 0.22 deg
40 ft
30 ft
20 ft
6.5 ft
12.5 ft

figure 12

7.2 Error Maps for Angle Measuring Systems Assuming a Coal Mining Scenario

7.2.1 Error Maps for a Certain Configuration of an Angle Measuring System: One Device, Three Targets

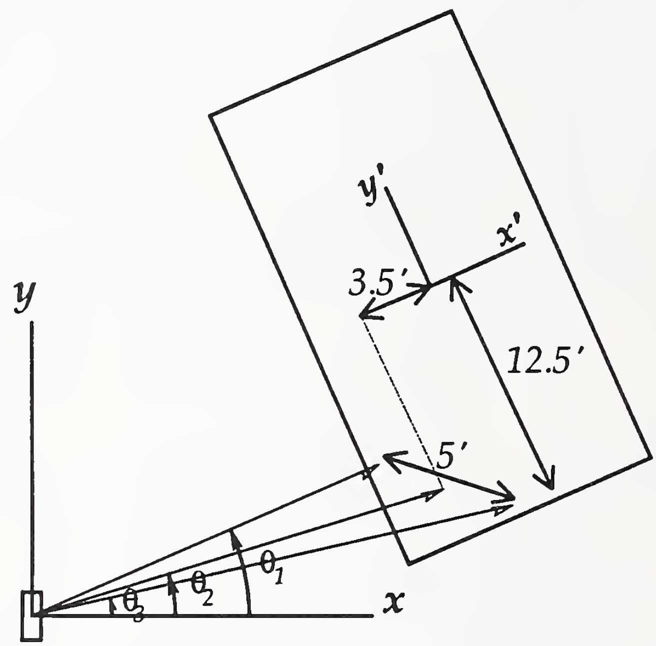


figure 13

Figure 13 gives a configuration for an angle measuring system that can be used to determine the position and heading of a mining machine. Lengths one would expect in a coal mine are given (in English units). The correlation coefficients, ρ_{12}, ρ_{13}, and ρ_{23} in equation 21 are all given the value 0.75. The axes of the error plots correspond to the position of the origin of the x'-y' axes (the center of the mining machine) in the x-y coordinate frame.

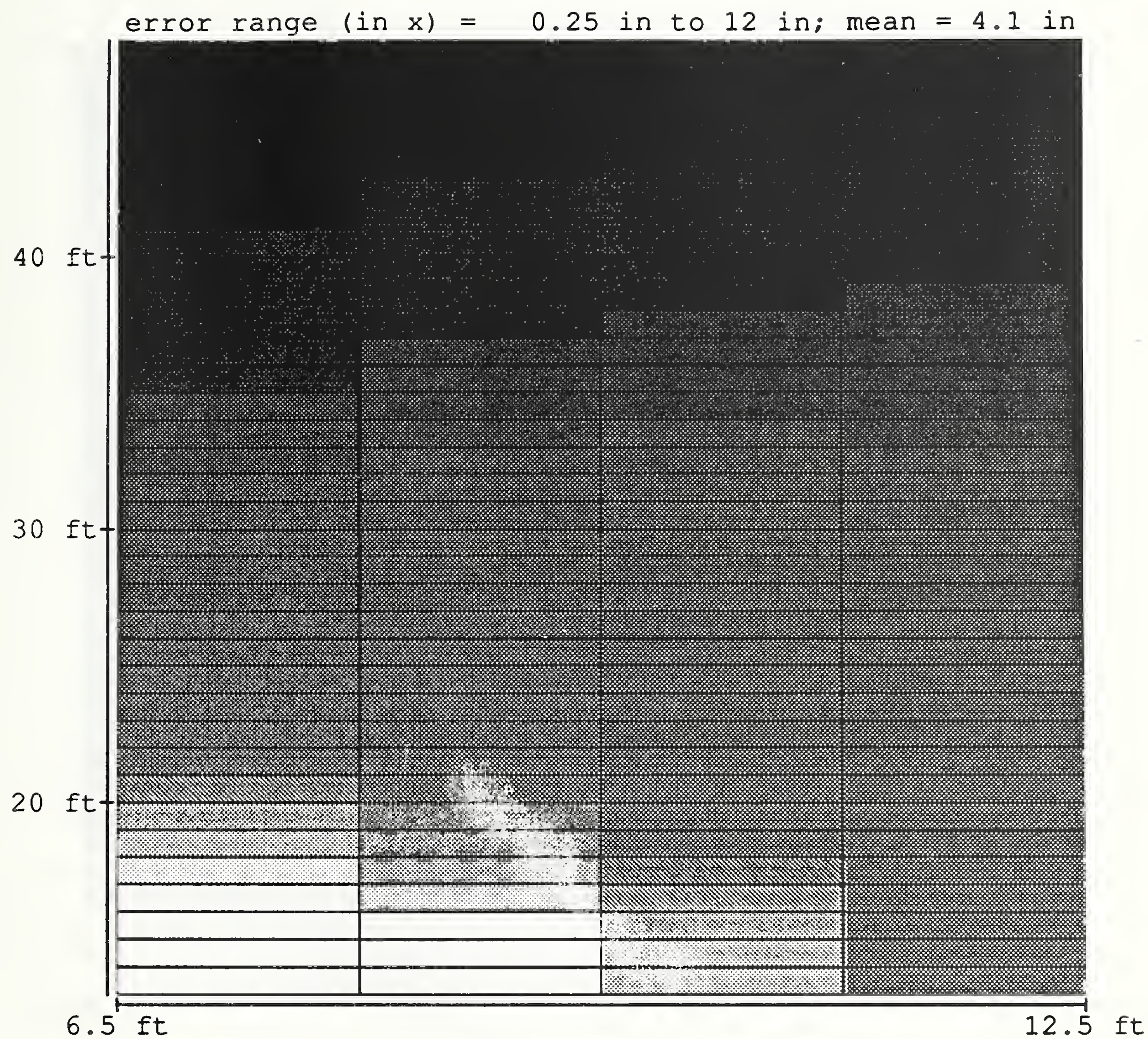

error range (in x) = 0.25 in to 12 in; mean = 4.1 in
40 ft
30 ft
20 ft
6.5 ft
12.5 ft

figure 14

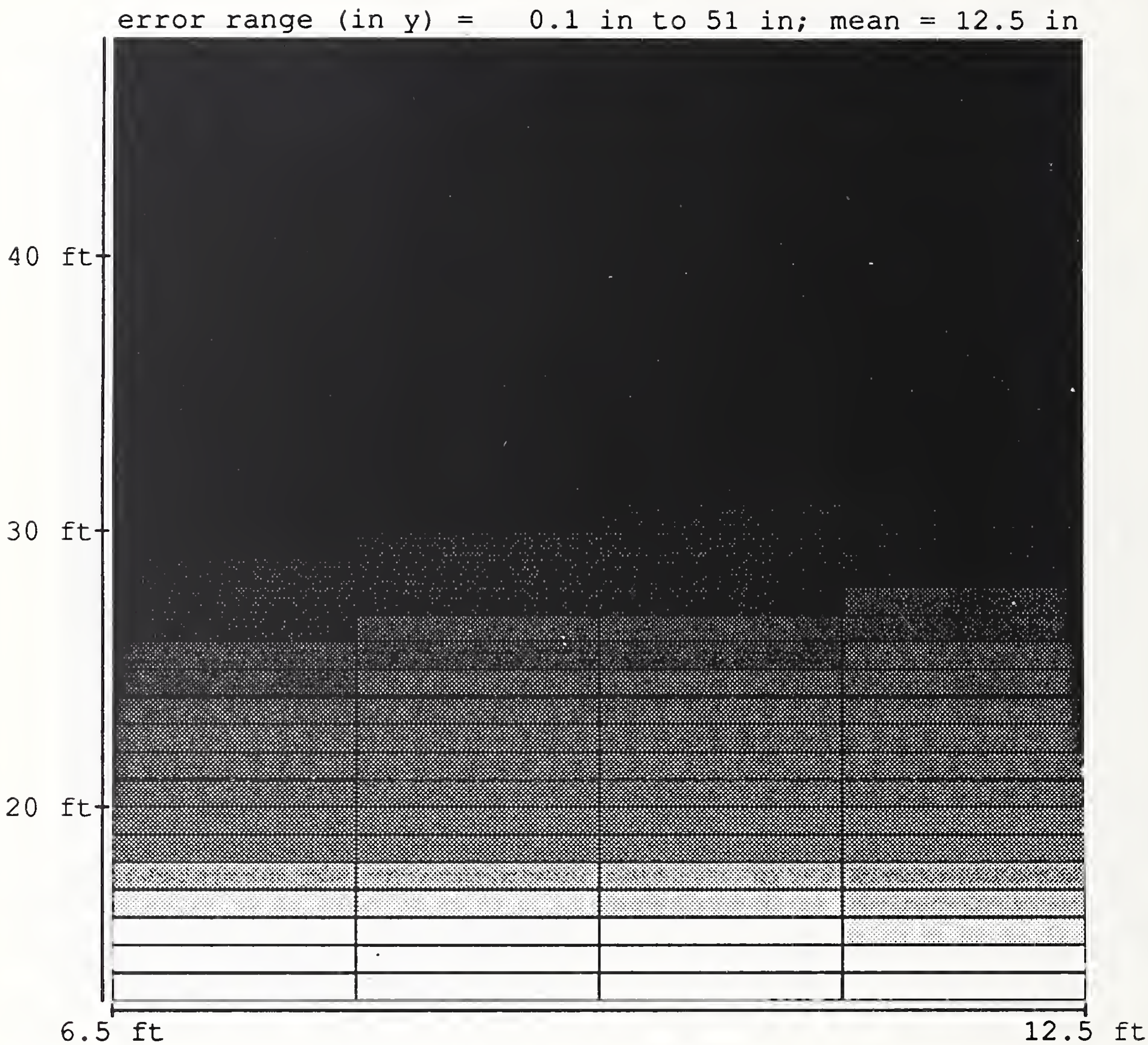

error range (in y) = 0.1 in to 51 in; mean = 12.5 in
40 ft
30 ft
20 ft
6.5 ft
12.5 ft

figure 15

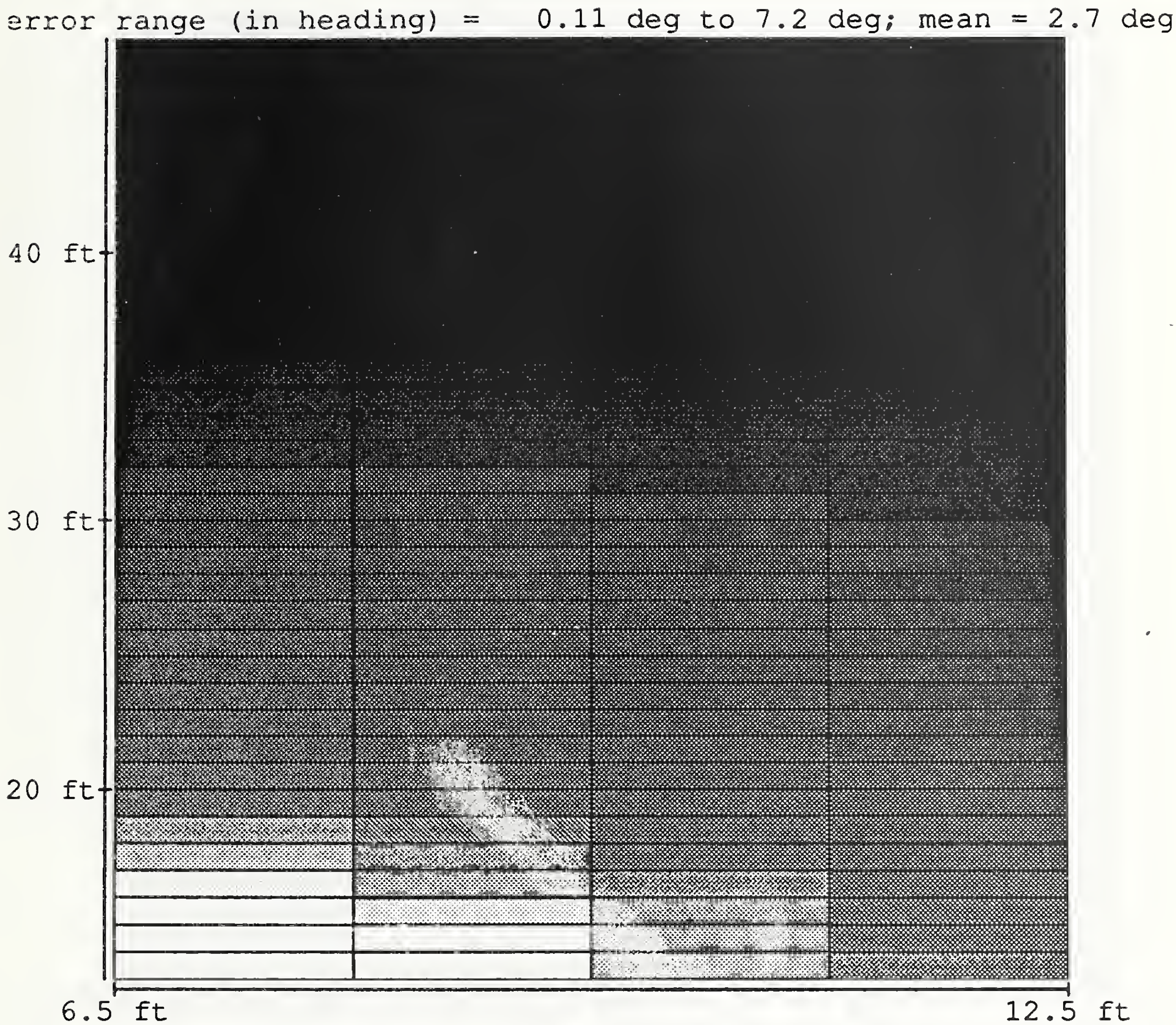
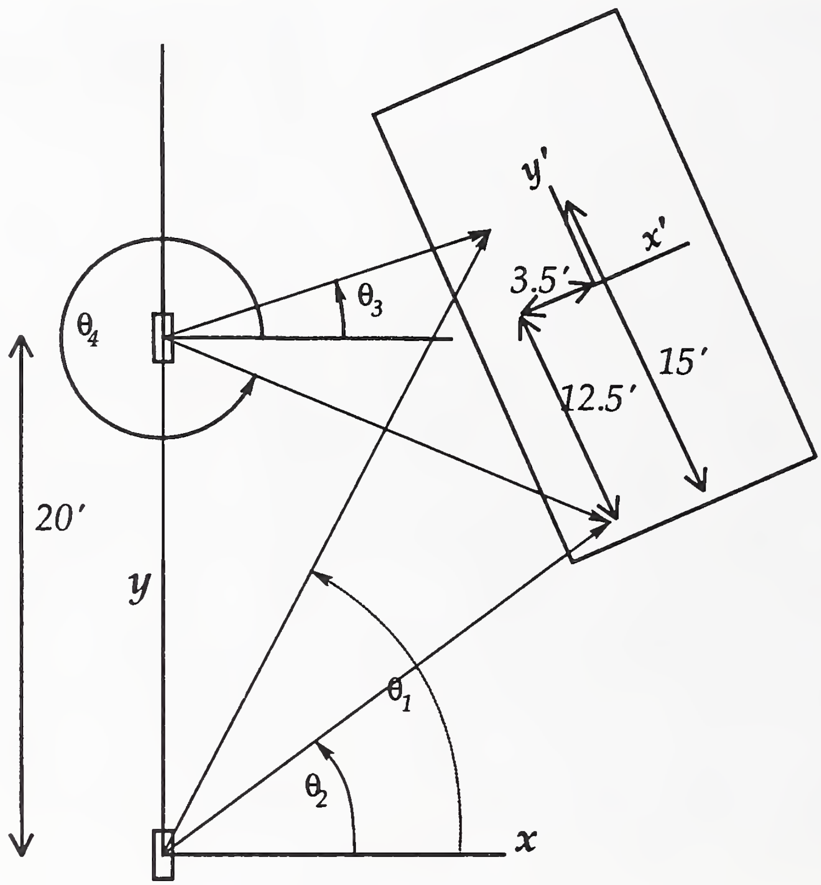

error range (in heading) = 0.11 deg to 7.2 deg; mean = 2.7 deg
40 ft
30 ft
20 ft
6.5 ft
12.5 ft

figure 16

7.2.2 Error Maps for a Certain Configuration of an Angle Measuring System: Two Devices, Two Targets; Configuration One



Two Device Configuration

figure 17

Figure 17 gives a two device configuration for an angle measuring system to determine the position and heading of a mining machine. Lengths are those that one would expect in a coal mine (in English units). The correlation coefficients, ρ_{12} and ρ_{34} in equation 22 are both given the value 0.75. The axes of the error plots correspond to the position of the origin of the x'-y' axes (the center of the mining machine) in the x-y coordinate frame.

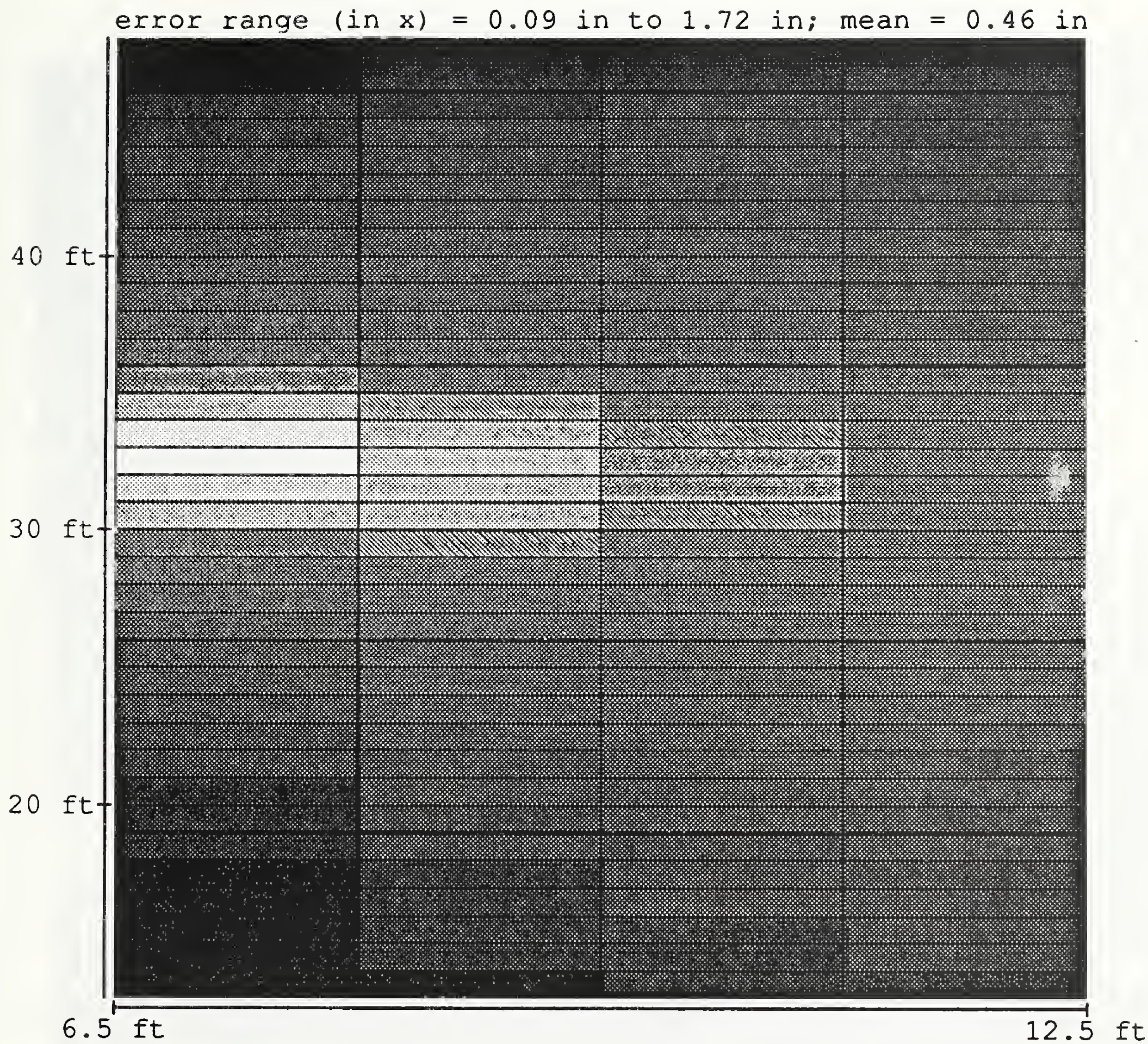

error range (in x) = 0.09 in to 1.72 in; mean = 0.46 in
40 ft
30 ft
20 ft
6.5 ft
12.5 ft

figure 18

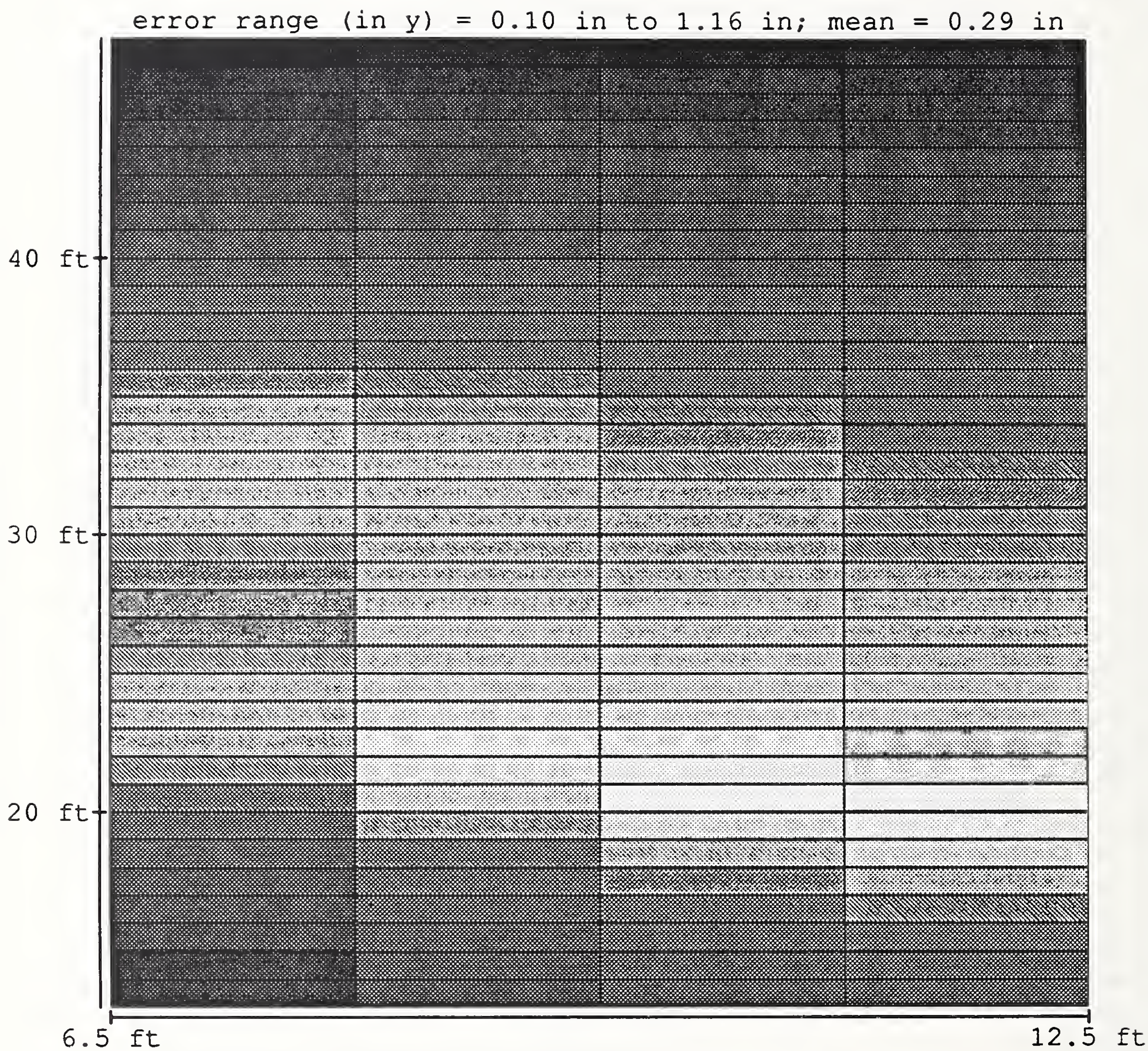

error range (in y) = 0.10 in to 1.16 in; mean = 0.29 in
40 ft
30 ft
20 ft
6.5 ft
12.5 ft

figure 19

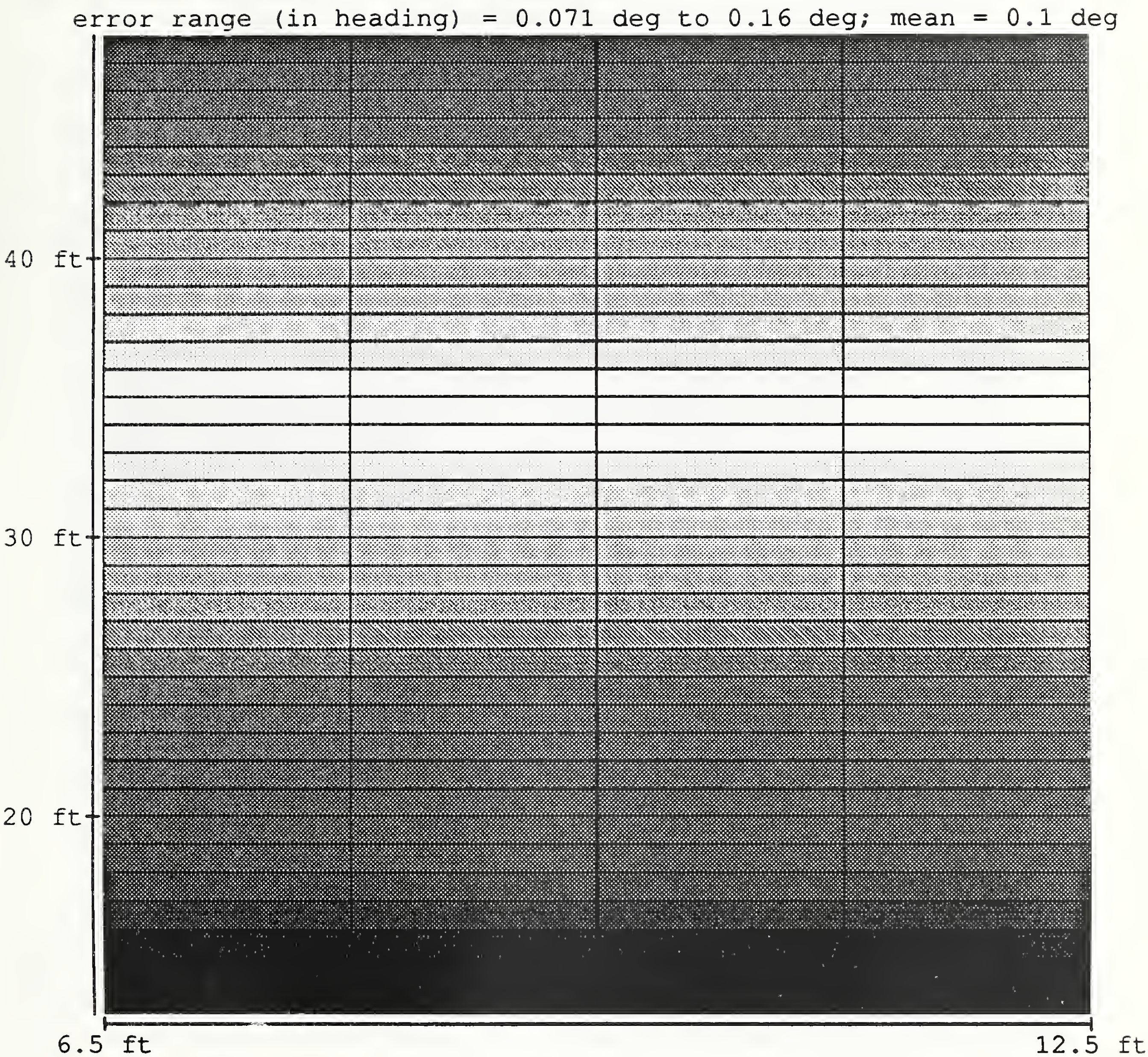

error range (in heading) = 0.071 deg to 0.16 deg; mean = 0.1 deg
40 ft
30 ft
20 ft
6.5 ft
12.5 ft

figure 20

7.2.3 Error Maps for a Certain Configuration of an Angle Measuring System: Two Devices, Two Targets; Configuration Two

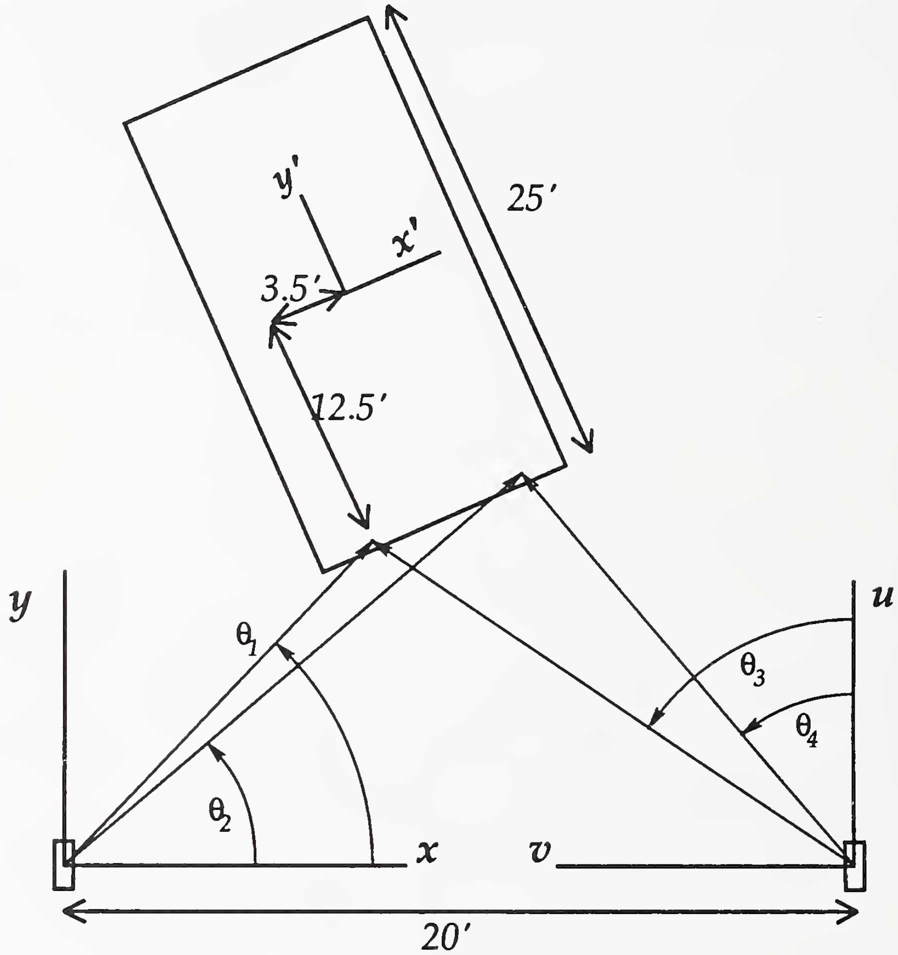
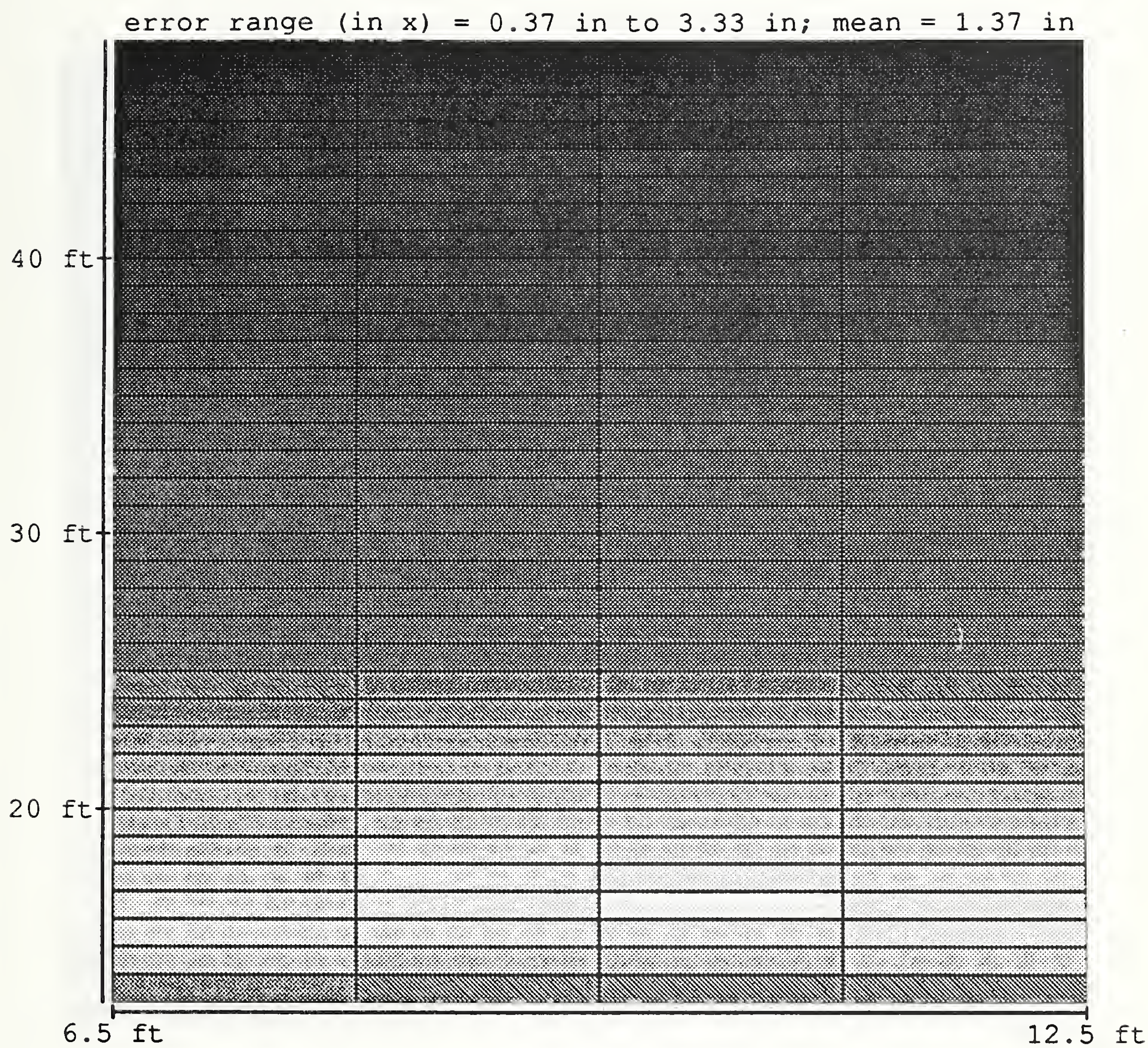


figure 21

Figure 21 gives another two device configuration for an angle measuring system to determine the position and heading of a mining machine. This configuration allows a more competitive comparison with the length measuring system configurations examined in sections 7.1.1 and 7.1.2 as can easily be seen from figure 21. Lengths one would expect in a coal mine are given (in English units). The correlation coefficients, ρ_{12} and ρ_{34} in equation 17 are both given the value 0.75.



error range (in x) = 0.37 in to 3.33 in; mean = 1.37 in
40 ft
30 ft
20 ft
6.5 ft
12.5 ft

figure 22

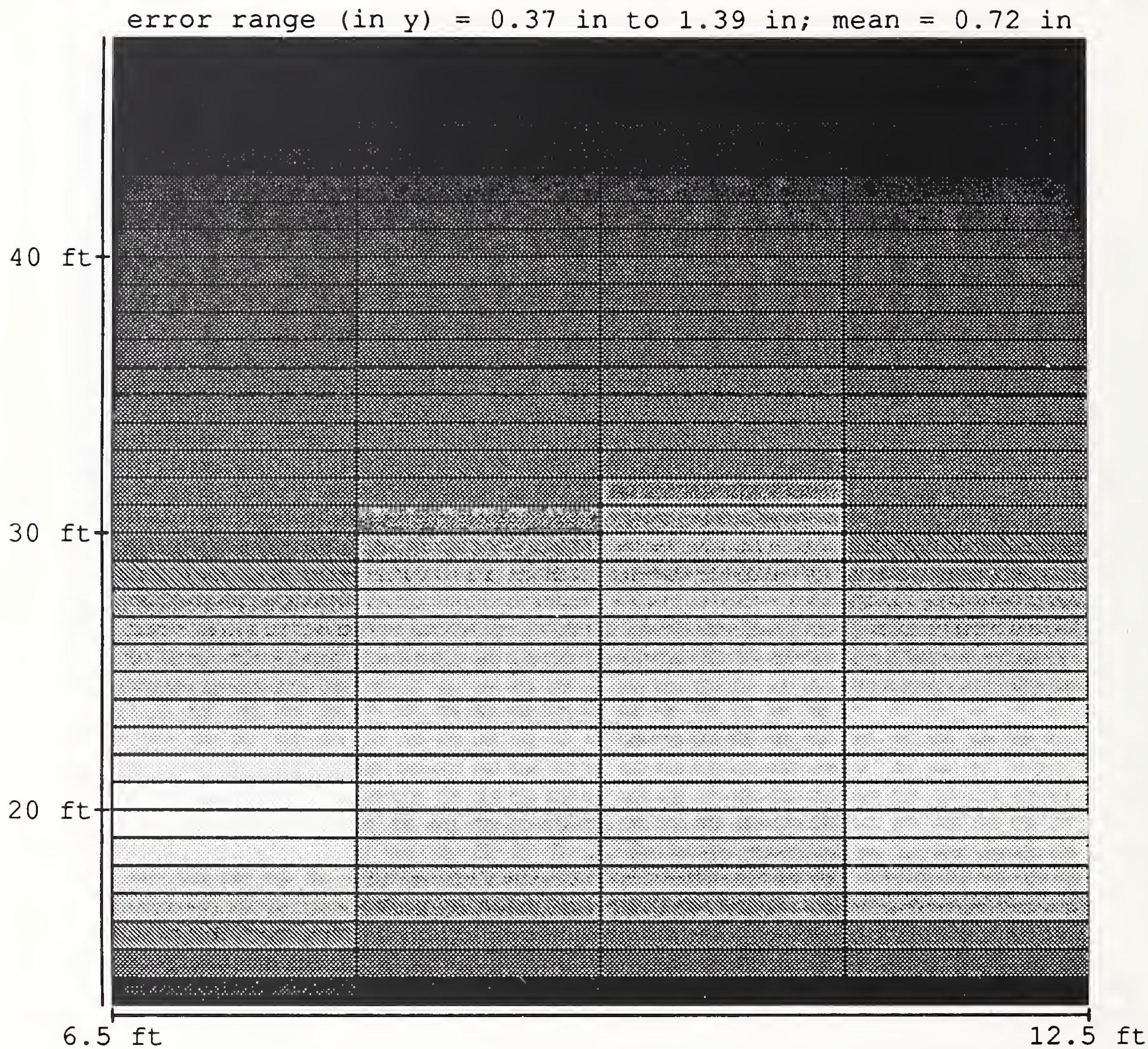

error range (in y) = 0.37 in to 1.39 in; mean = 0.72 in
40 ft
30 ft
20 ft
6.5 ft
12.5 ft

figure 23

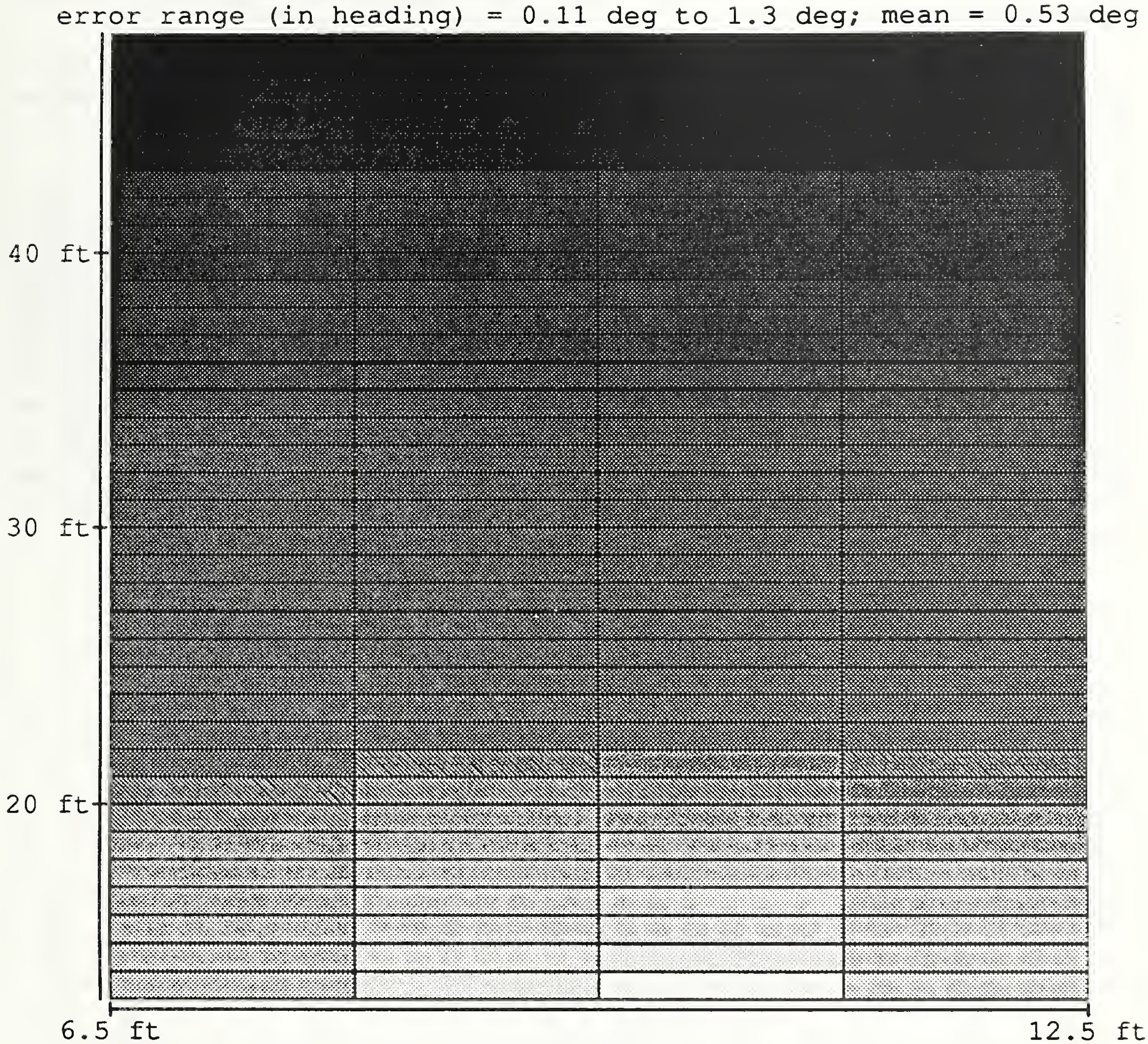


figure 24

7.3 Discussion of the Error Maps for Length Measuring Systems

The two configurations are seen to have essentially comparable error performance in the regions where their error is best. However, case 1 seems to be somewhat more stable as the y component of position gets large but gets unstable as y gets close to 12.5'. The choice between the two would be a decision as to which configuration is easiest to maneuver and cut coal with. For example, with the case 2 configuration the machine is free to move (while using the same equation) ahead of or behind the lines between points **2** and **3** and between points **1** and **4**. The case 1 configuration will not easily allow this. However, if we are using physical wires, the case 2 situation could more easily have a wire bumping into uncut coal, causing a catastrophic failure.

Another possible configuration with similar error performance to the case 2 configuration is one in which points **1**, **2**, **3**, and **4** are on the machine and points **A** and **C** are off.

7.4 Discussion of the Error Maps for Angle Measuring Systems

The fact that the single device configuration degrades with increasing distance from the device should not be surprising. However, the extent and nature of the degradation is remarkable: propagated errors are quite bad if the angle measurement errors are completely uncorrelated and quite good if the errors are completely correlated. It is therefore natural to ask what the dependance of error on correlation is for the single device configuration of an angle measurement system (as in figure 13). Figures 25-27 denote three times the standard deviation in x, y, and ψ, respectively, as a function of the correlation coefficient value for the single device scenario when the center of the mining machine is 48 feet from the device. The 3σ value of angle measurement error is 0.05° as before.

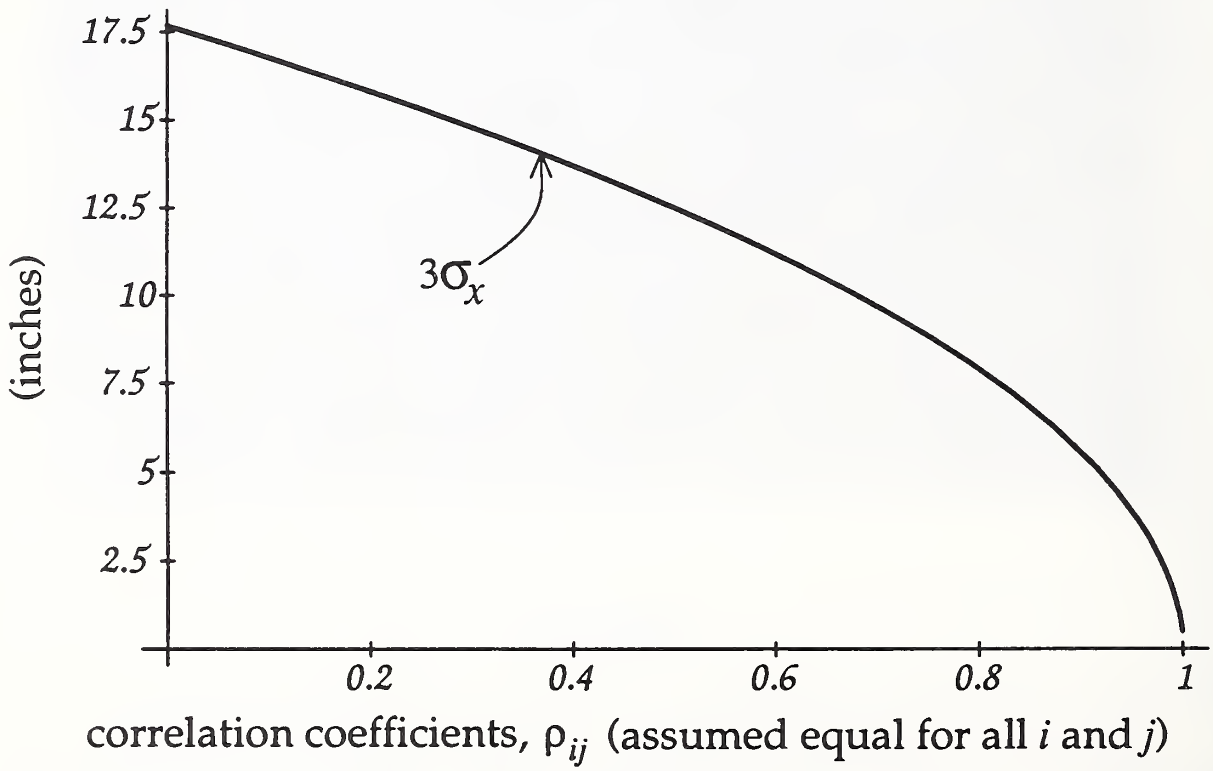
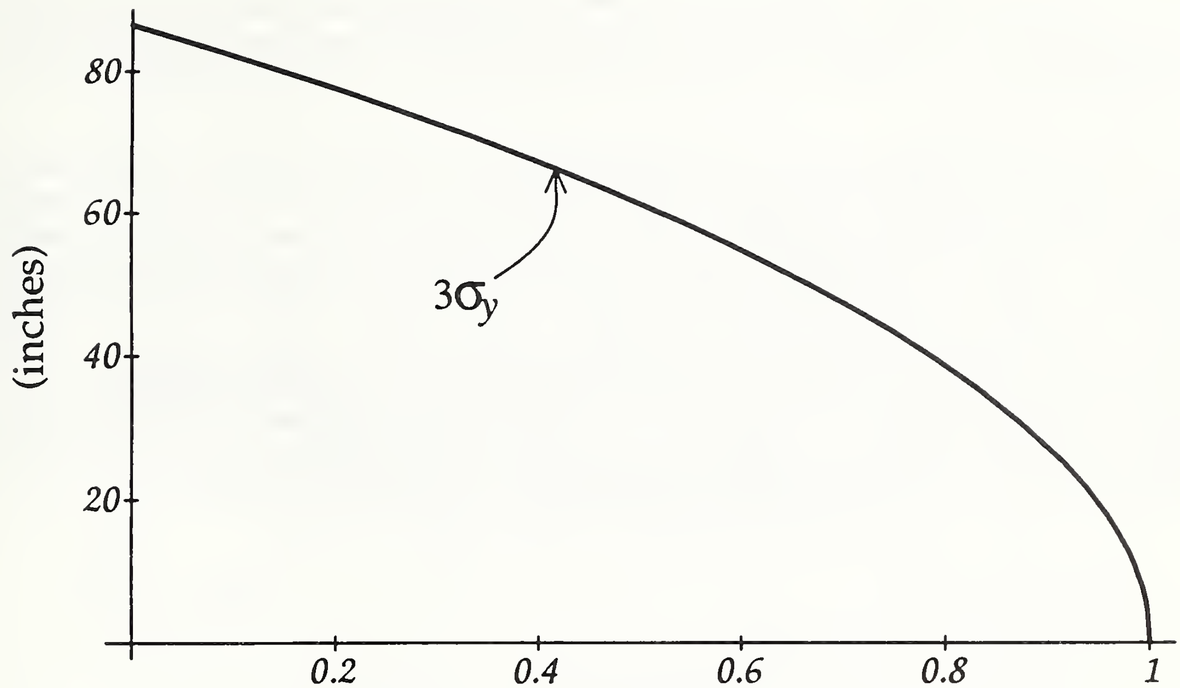
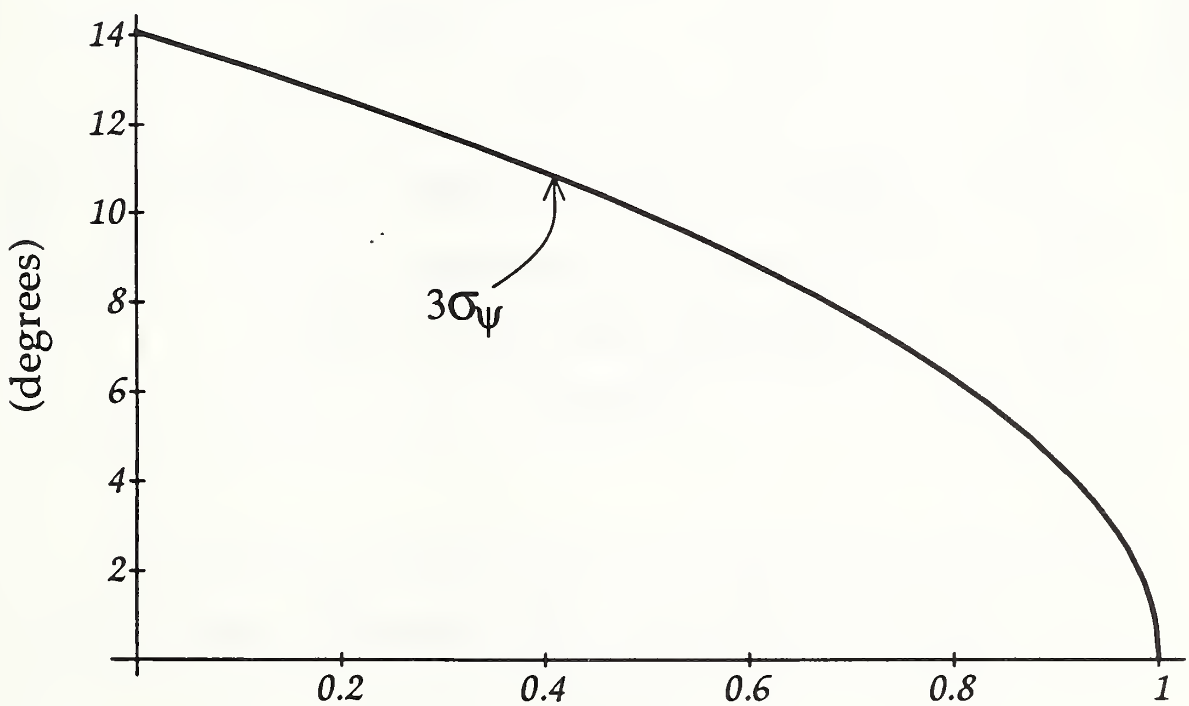


figure 25



correlation coefficients, ρ_{ij} (assumed equal for all i and j)

figure 26



correlation coefficients, ρ_{ij} (assumed equal for all i and j)

figure 27

It can be seen in figures 25-27 that, even though the position and heading error approaches zero when the value of the correlation coefficient approaches one, the magnitude of the error gets large quickly as the correlation coefficient gets less than one.

Also unexpected is the surprisingly good performance of both configurations using two devices and two targets (and that the error performance is comparable and often better than that of either of the configurations of the length measuring systems). These two configurations using two devices are seen to have somewhat comparable error performance. It is again natural to ask what the dependance of error on correlation is for the two device configuration (figure 21) of an angle measurement system. Figures 28-30 denote three times the standard deviation in x, y, and ψ, respectively, as a function of the correlation coefficient value for the two device scenario of figure 21 when the center of the mining machine is 48 feet from the device. The 3σ value of angle measurement error is 0.05° as before.

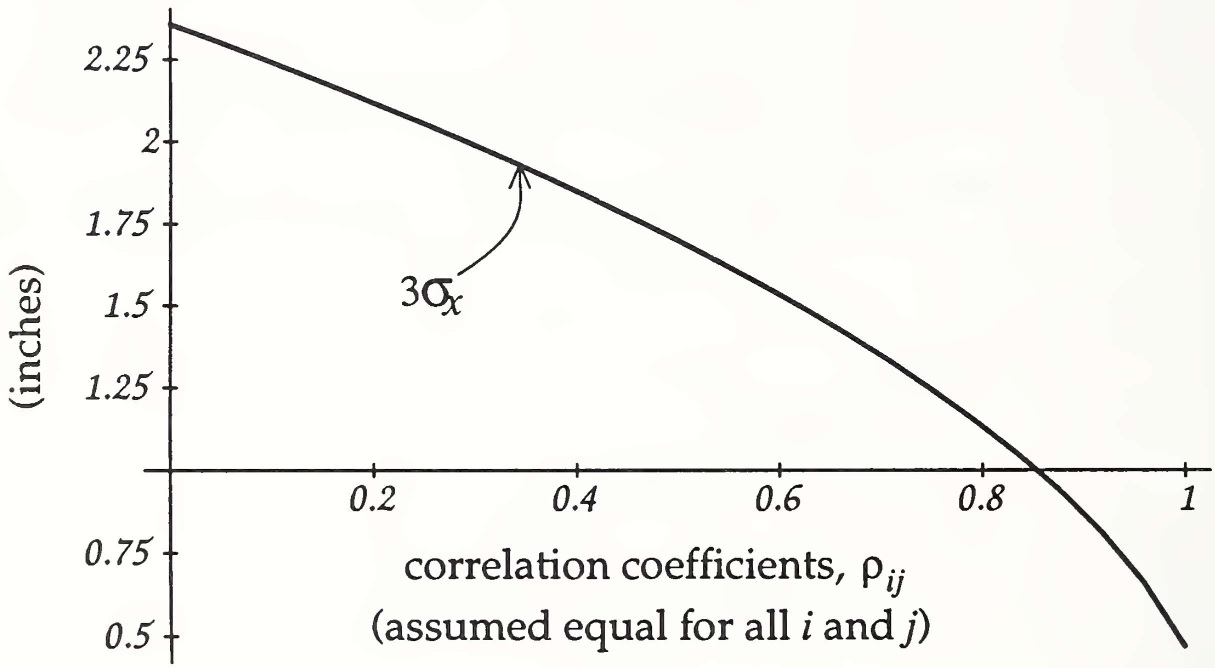


figure 28

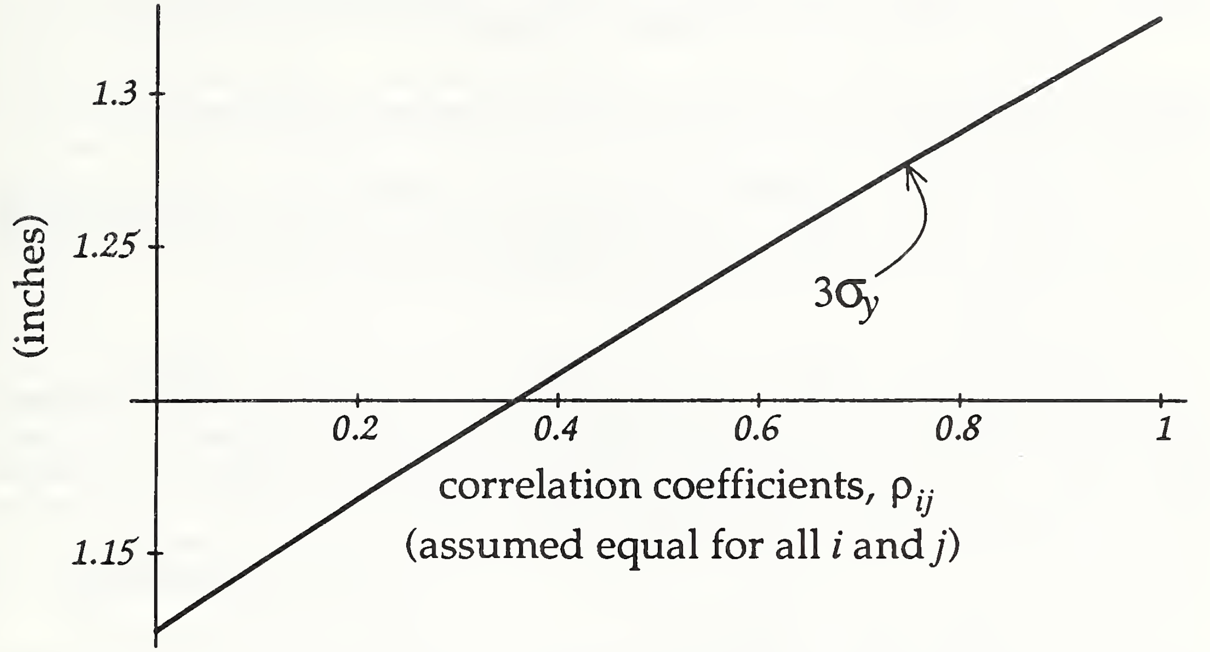


figure 29

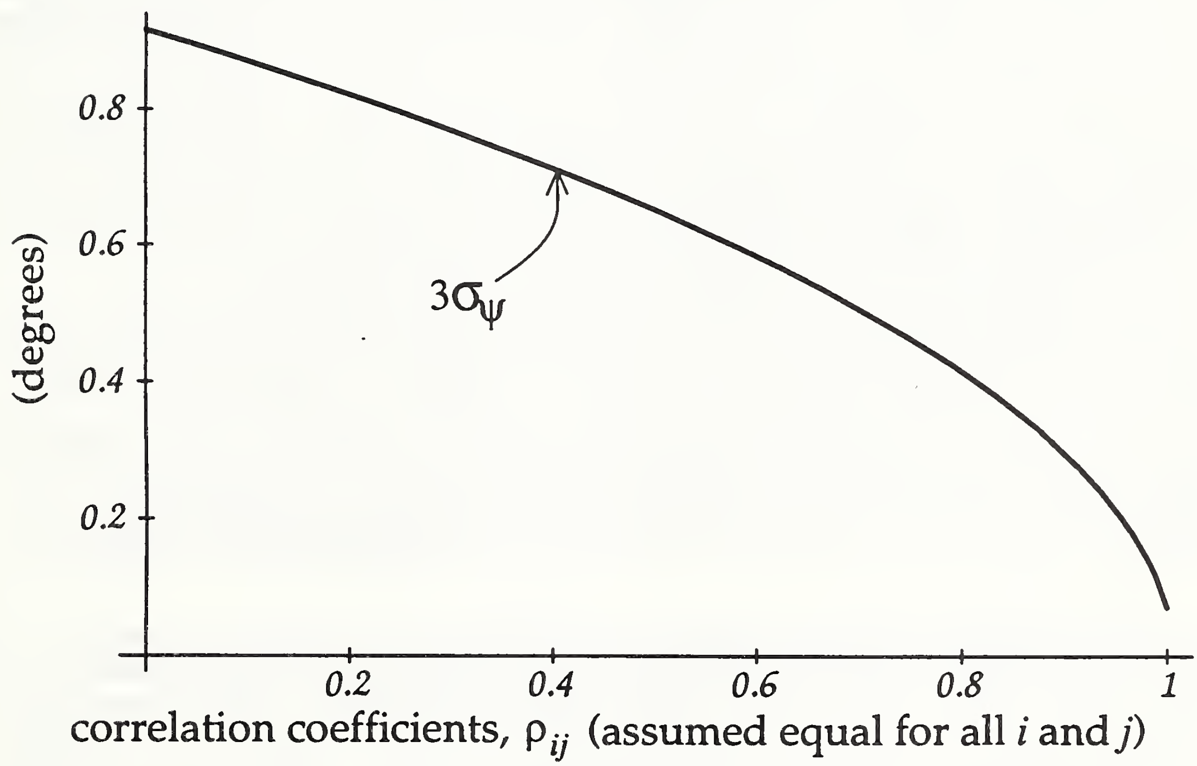


figure 30

It can be seen in figures 28-30 that for the two device scenario of figure 21, the error values are more stable with respect to the value of the correlation coefficient than for the one device scenario (figures 25-27). The magnitude of the error is also lower for most values of the correlation coefficient.

As with the length measuring configurations, the choice between the two configurations of the two device scenario would be a decision as to which configuration is easiest to operate with.

Another possible configuration with almost identical error performance to the two target, two device configurations is one in which the two angle measurement devices are on the machine and the two targets are off.

8. Conclusions

Standard tools necessary to analyze the error performance of angle and length measuring systems for position and heading measurements have been given along with realistic examples from underground coal mining. In pursuit of these tools, usable closed form solutions have been developed for all the matrix equations in their most general form. In addition, the presentation of the theory of measurement error propagation is completely generic and can be applied to virtually any measurement system.

For the single device scenario in an angle measuring system it was shown that the equations constrain the heading to a 180° range and adding a fourth target simply adds redundancy. The two device scenario has no such restraint if occlusion can be easily handled (not true with LASERNET™ system). However, if this analysis is correct, the debate between the one and the two device scenarios is overshadowed by the following: the error performance of the two device scenario is so markedly superior to the single device scenario that the use of two devices is to be preferred unless one can prove that the angle measurement errors are highly correlated (>98%). If target occlusion in the two device scenario is a problem, adding a third target might help (depending on the type of system used).

For length measuring systems it was shown that a closed form (non-iterative) solution to equation 8 requires the formation of triangles in the geometry (compare figures 3 and 4). Pairs of solutions can be obtained using only three lengths. Adding the fourth length eliminates one of the pair, but we are still left with the problem of boundary crossings. It was shown that, if one can constrain the heading to be within a 180° range, the detection of a boundary crossing can be easily detected. The occurrence of gross errors can also be dealt with by comparing the four three-device solutions.

In the specific configurations from underground coal mining specified in section 7 (Simulation), the two configurations of the length measuring devices have nearly similar error performance with certain differences. The two device configurations of angle measuring systems show an error performance roughly equivalent to that of the length measuring configurations. However, the single device configuration shows an unacceptably poor error performance for most positions of the vehicle.

The reader is reminded that every error result in this paper assumes that the vector-valued function, $g = (g_1(u), g_2(u), \ldots, g_n(u))$, can be considered linear in the region of error, $u + \Delta u$.

Discussion on the types of probability distributions (Gaussian, uniform, etc.) has been avoided until now, since knowing the error at the extreme limits ($\sim 3\sigma$) is often sufficient for engineering analysis even without knowledge of the type of distribution at the output. Knowing the mean and variance of the random vector, (x, y, ψ), and that the probability densities of measurement error are Gaussian, is sufficient to specify that the probability density function of that random vector is Gaussian with the specified variance. Nonetheless, even if the distribution of error is not Gaussian, since we assumed that v is a very nearly linear function of the measurement vector plus error, $u + \Delta u$, (the degree of linearity can be proved easily enough) we should be able to determine what the density function of v is if necessary. Knowing the density function of v, we can easily find the point where the amount of error in the random vector v is less than some acceptable minimum.

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11. ABSTRACT (A 200-WORD OR LESS FACTUAL SUMMARY OF MOST SIGNIFICANT INFORMATION. IF DOCUMENT INCLUDES A SIGNIFICANT BIBLIOGRAPHY OR LITERATURE SURVEY, MENTION IT HERE.)

This work applies known results on the propagation of measurement error statistics to two types of measurement systems, an angle measuring system and a length measuring system. Such systems can be used to measure the position and heading of a vehicle (or any rigid object) on a flat surface. For example, one might need to sense position and heading of a vehicle in order to control its motion. The functional relationships to go from angle and length measurements to position and heading are discussed and derived. Measurement error propagation theory is given in a way that can apply to any measurement system. The theory thus described is applied to the two types of measurement systems. Specific examples of these systems are given. The stated examples are currently being used by the US Bureau of Mines (BOM) to measure the position and heading of certain underground coal mining machines in an effort to move equipment operators away from the more dangerous areas of the mine. Simulation results for these specific examples in the form of density maps are given which plot the error in position and heading for various positions (heading constant) of the vehicle.

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