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ON THE EVALUATION OF THE INTEGRAL

$$I_{\ell,\ell'}(k,k') = \int_0^\infty j_\ell(kr) j_{\ell'}(k'r)\, r^2\, dr$$

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$$I_{\ell,\ell'}(k,k') = \int_0^\infty j_\ell(kr) j_{\ell'}(k'r)\, r^2 dr$$

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Abstract

The integral $I_{\ell,\ell'}(k,k') = \int_0^\infty j_\ell(kr) j_{\ell'}(k'r)\, r^2 dr$, in which the spherical Bessel functions $j_\ell(kr)$ are the radial eigenfunctions of the three-dimensional wave equation in spherical coordinates, is evaluated in terms of distributions, in particular step functions and delta functions. We show that the behavior of $I_{\ell,\ell'}$ is very different in the cases $\ell - \ell'$ even $(0, \pm 2, \pm 4, \ldots)$ and $\ell - \ell'$ odd $(\pm 1, \pm 3, \ldots)$. For $\ell - \ell'$ even it is expressed in terms of the delta function, step functions, and Legendre polynomials. For $\ell - \ell'$ odd it is expressed in terms of Legendre functions of the second kind and step functions; no delta functions appear.

Key words: delta functions; distributions; integrals of Bessel functions; non-convergent integrals; spherical Bessel functions; step functions.

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* Guest Scientist

ON THE EVALUATION OF THE INTEGRAL

$$I_{\ell,\ell'}(k,k') = \int_0^\infty j_\ell(kr) j_{\ell'}(k'r)\, r^2 dr \quad . \tag{1}$$

I. Introduction

The integral $I_{\ell,\ell'}$ as defined in (1) arises in connection with the solutions of the three-dimensional wave equation in spherical coordinates, for which the radial eigenfunctions are the spherical Bessel functions $j_\ell(kr)$. Although the integral does not converge, it can be expressed in terms of distributions, in particular step functions and delta functions. For $\ell = \ell'$ the result is well-known; in this case it can be expressed in terms of the delta function:

$$I_{\ell,\ell}(k,k') = \frac{\pi}{2kk'} \delta(k-k') \quad . \tag{2}$$

In this note we extend this result by evaluating (1) for arbitrary (integer) ℓ and ℓ'. We show that the behavior of $I_{\ell,\ell'}$ is very different in the cases $\ell - \ell'$ even (0, ± 2, ± 4, ...) and $\ell - \ell'$ odd (± 1, ± 3, ...). For $\ell - \ell'$ even, the integral $I_{\ell,\ell'}$ may be written in terms of step functions and the delta function. For $\ell - \ell'$ odd, it may be expressed in terms of step functions alone – no delta function appears. In both cases, however, there is a singularity when $k = k'$, as has been noted for the related integral

$$I^\lambda_{\mu,\nu}(a,b) \equiv \int_0^\infty \frac{J_\mu(at) J_\nu(bt)}{t^\lambda}\, dt \tag{3}$$

which is discussed in some detail in [1], pp. 398-410. Although we will make use of the results given there, it should be noted that the integral, (3), is considered in [1] only under the conditions which are sufficient for convergence [$Re(\mu+\nu+1) > Re(\lambda) > -1$ for $a \neq b$ and $Re(\mu+\nu+1) > Re(\lambda) > 0$ for $a = b$]. Thus our integral $I_{\ell,\ell'}(k,k')$ does not follow directly from the results in [1]. More importantly, the presence (or lack) of a delta function in $I_{\ell,\ell'}(k,k')$ when $k = k'$ does not appear at all in the analysis in [1].

The form of our result for $\ell - \ell'$ even is

$$
\begin{aligned}
I_{\ell,\ell'}(k,k') &= \frac{\pi}{2kk'}\,\{g_{\ell,\ell'}(k,k')\theta(k-k') + (-1)^{(\ell-\ell')/2}\delta(k'-k)\} \quad , \quad \ell > \ell' \\
&= \frac{\pi}{2kk'}\,\{g_{\ell',\ell}(k',k)\theta(k'-k) + (-1)^{(\ell'-\ell)/2}\delta(k'-k)\} \quad , \quad \ell' > \ell \qquad (4) \\
&= \frac{\pi}{2kk'}\,\delta(k'-k) \qquad\qquad \ell = \ell'
\end{aligned}
$$

in which the step function $\theta(k'-k)$ is defined by

$$
\theta(k'-k) = \begin{cases} 1 & k' > k \\ 0 & k' < k \end{cases} \tag{5}
$$

and the delta function $\delta(k'-k)$ by

$$
\delta(k'-k) = 0 \qquad \text{for} \qquad k' \neq k \tag{6}
$$

and

$$
\int_{-\infty}^{\infty} \delta(k'-k)dk' = 1 \quad . \tag{7}
$$

For $\ell - \ell'$ even, the functions $g_{\ell,\ell'}(k,k')$ are finite polynomials in k/k'; we show that they are related to Legendre polynomials,* P_ℓ.

On the other hand, for $\ell - \ell'$ odd the integral has the form

$$I_{\ell,\ell'}(k,k') = g_{\ell,\ell'}(k,k')\theta(k-k') + g_{\ell',\ell}(k',k)\theta(k'-k) \quad . \tag{8}$$

For $\ell - \ell'$ odd, however, the functions $g_{\ell,\ell'}(k,k')$ are related to Legendre functions* of the second kind, Q_ℓ. Specific expressions for the functions $g_{\ell,\ell'}(k,k')$ for $\ell - \ell'$ even and for $\ell - \ell'$ odd are given in (53) and (56) and in (69) and (70), respectively.

II. Derivation of integrals for $k \neq k'$

We carry out the analysis for $k \neq k'$ in terms of the ordinary Bessel functions $J_\mu(x)$, which are related to the spherical Bessel functions $j_\ell(kr)$ by

$$j_\ell(kr) = \sqrt{\frac{\pi}{2kr}}\, J_{\ell+\frac{1}{2}}(kr) \quad . \tag{9}$$

In order to secure convergence we start with the more general integral

$$I(\epsilon) = \int_0^\infty J_{\ell+\frac{1}{2}}(kr)\, J_{\ell'+\frac{1}{2}}(k'r) r^{1-\epsilon}\, dr \quad , \quad \epsilon > 0 \quad . \tag{10}$$

Here in the integrand, $r^{-\epsilon}$ is a convergence factor. Further, we assume $0 < k' < k$. We note that the integral is convergent provided $k \neq k'$ and $\epsilon > 0$. After writing the expression for this integral for $\epsilon > 0$, we let $\epsilon \to 0^+$. From [1], p. 401 (2), we have

* Throughout this paper, unless specifically noted, the Legendre functions P_ν^μ and Q_ν^μ as used here are those defined in [2], p. 122 (3) and (5).

$$\int_0^\infty J_{\ell+\frac{1}{2}}(kr)\, J_{\ell'+\frac{1}{2}}(k'r)\, r^{1-\epsilon}\, dr \;=\; 2^{1-\epsilon}\, \frac{k'^{\ell'+(1/2)}}{k^{\ell'+(5/2)-\epsilon}}\, \frac{\Gamma(\frac{\ell+\ell'+3-\epsilon}{2})}{\Gamma(\ell'+\frac{3}{2})\,\Gamma(\frac{\ell-\ell'+\epsilon}{2})}$$

$$\times\, {}_2F_1\left(\tfrac{\ell+\ell'+3-\epsilon}{2}, \tfrac{\ell'-\ell-\epsilon}{2}+1; \ell'+\tfrac{3}{2}; \frac{k'^2}{k^2}\right) \tag{11}$$

where ${}_2F_1(a,b;c;z)$ is the hypergeometric function ([2], Chapter II). We then let $\epsilon \to 0^+$, giving, for the integral in (1),

$$I_{\ell,\ell'}(k,k') = g_{\ell,\ell'}(k,k') \equiv \frac{\pi k'^{\ell'}}{k^{\ell'+3}}\, \frac{\Gamma(\frac{\ell+\ell'+3}{2})}{\Gamma(\ell'+\frac{3}{2})\,\Gamma(\frac{\ell-\ell'}{2})}$$

$$\times\, {}_2F_1\left(\tfrac{\ell+\ell'+3}{2}, \tfrac{\ell'-\ell}{2}+1; \ell'+\tfrac{3}{2}; \frac{k'^2}{k^2}\right) \qquad \text{for} \quad k' < k \quad . \tag{12}$$

We note from (1) that $I_{\ell,\ell'}(k,k')$ is invariant under the interchange $\ell \rightleftharpoons \ell'$, $k \rightleftharpoons k'$. Thus in the case $k < k'$ we have

$$I_{\ell,\ell'}(k,k') = g_{\ell',\ell}(k',k) = \frac{\pi k^{\ell}}{k'^{\ell+3}}\, \frac{\Gamma(\frac{\ell+\ell'+3}{2})}{\Gamma(\ell+\frac{3}{2})\,\Gamma(\frac{\ell'-\ell}{2})}$$

$$\times\, {}_2F_1\left(\tfrac{\ell+\ell'+3}{2}, \tfrac{\ell-\ell'}{2}+1; \ell+\tfrac{3}{2}; \frac{k^2}{k'^2}\right) \qquad \text{for} \quad k < k' \quad . \tag{13}$$

Now, as noted in [1], "it so happens that the expressions on the right in [(12) and (13)] are not the analytic continuations of the same functions." This may in fact be seen directly. In the present case, if we consider the function $I_{\ell,\ell'}$ as defined in (12), then the desired analytic continuation is given in [2], p. 107 (34) together with the expressions given in [2], p. 105 (1), (9), and (13), namely,

$$
{}_2F_1(a,b;c;z) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(c-a)\Gamma(b)} (-z)^{-a} \; {}_2F_1(a, a+1-c; a+1-b; z^{-1})
$$

$$
+ \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(c-b)\Gamma(a)} (-z)^{-b} \; {}_2F_1(b, b+1-c; b+1-a; z^{-1}) \quad . \tag{14}
$$

Applying (14) to the hypergeometric function in (12), we find that the analytic continuation of $I_{\ell,\ell'}(k,k')$ as defined by (12) is, in the region $\frac{k'}{k} > 1$, given by

$$
g_{\ell,\ell'}(k,k') = g_{\ell',\ell}(k',k) - \frac{\pi k^{\ell+2}}{k'^{\ell+5}} \frac{\Gamma(\ell+\frac{1}{2})\, e^{i\varepsilon\pi(\ell-\ell')/2}}{\Gamma(\frac{\ell-\ell'}{2})\,\Gamma(\frac{\ell+\ell'+1}{2})}
$$

$$
\times \; {}_2F_1\left(\tfrac{\ell'-\ell}{2}+1, \tfrac{-\ell-\ell'+1}{2}; -\ell+\tfrac{1}{2}; \frac{k^2}{k'^2}\right) \tag{15}
$$

where

$$
\varepsilon = \begin{cases} +1 & \text{for} \quad Im(\frac{k'^2}{k^2}) > 0 \\ -1 & \text{for} \quad Im(\frac{k'^2}{k^2}) < 0 \end{cases} .
$$

The difference between the case $\ell-\ell'$ even and $\ell-\ell'$ odd is illustrated most strikingly in the expressions (12), (13), and (15). Thus, for $\ell-\ell'$ even, if $\ell' < \ell$ then from (12) $I_{\ell,\ell'}(k,k')$ is a finite polynomial for $k' < k$, whereas from (13) it is identically zero for $k < k'$ in view of the factor $\Gamma(\frac{\ell'-\ell}{2})$ in the denominator. The analytic continuation of $I_{\ell,\ell'}(k,k')$ in the region $\frac{k'}{k} > 1$ is then the second term on the right hand side of (15), which is also a finite polynomial, as it must be since the polynomial given on the right hand side of (12) has no singularity at $\frac{k'}{k} = 1$. On the other hand, if $\ell - \ell'$ is odd, then neither $g_{\ell,\ell'}(k,k')$ nor $g_{\ell',\ell}(k',k)$ is zero, but again $g_{\ell',\ell}(k',k)$ is not the analytic continuation of $g_{\ell,\ell'}(k,k')$; there is the added term in (15). Again the ${}_2F_1$ function there is a finite polynomial. However, the entire second term is then purely imaginary in view of the factor $e^{i\varepsilon\pi(\ell-\ell')/2} = \pm i$. We shall see this most clearly in some of the examples given later.

At this point in our analysis we consider separately the cases $\ell - \ell'$ even and $\ell - \ell'$ odd.

A. Integral for $\ell - \ell'$ even, $k \neq k'$

From (12) and (13) we then have, for $k \neq k'$,

$$I_{\ell,\ell'}(k,k') = \frac{\pi k'^{\ell'}}{k^{\ell'+3}} \frac{(\ell' + \frac{3}{2})_{n+1}}{n!}$$
$$\times {}_2F_1\left(-n, \ell' + \tfrac{5}{2} + n; \ell' + \tfrac{3}{2}; \frac{k'^2}{k^2}\right) \theta(k - k') \quad \text{for} \quad \ell > \ell' \tag{16}$$

and

$$I_{\ell,\ell'}(k,k') = \frac{\pi k^{\ell}}{k'^{\ell+3}} \frac{(\ell + \frac{3}{2})_{n+1}}{n!}$$
$$\times {}_2F_1\left(-n, \ell + \tfrac{5}{2} + n; \ell + \tfrac{3}{2}; \frac{k^2}{k'^2}\right) \theta(k' - k) \quad \text{for} \quad \ell < \ell' \tag{17}$$

where

$$n = \tfrac{1}{2} \mid \ell - \ell' \mid -1$$

and

$$(\ell + \tfrac{3}{2})_{n+1} = (\ell + \tfrac{3}{2})\,(\ell + \tfrac{5}{2}) \cdots (\ell + \tfrac{3}{2} + n) \quad . \tag{18}$$

The expressions in (16) and (17) can be combined in the single result

$$I_{\ell,\ell'}(k,k') = \frac{\pi k_<^{\ell_<}}{k_>^{\ell_<+3}} \frac{(\ell_< + \frac{3}{2})_{n+1}}{n!}$$

$$\times {}_2F_1\left(-n, \ell_< + \tfrac{5}{2} + n; \ell_< + \tfrac{3}{2}; \left(\frac{k_<}{k_>}\right)^2\right) \tag{19}$$

where

$$\begin{aligned} \ell_< &= \min(\ell, \ell') \\ k_< &= \min(k, k') \\ k_> &= \max(k, k') \quad . \end{aligned} \tag{20}$$

For $\ell = \ell'$, we note from both (12) and (13) that

$$I_{\ell,\ell}(k,k') = 0 \qquad \text{for} \quad k \neq k' \quad . \tag{21}$$

B. Integral for $\ell - \ell'$ odd, $k \neq k'$

In the case $\ell - \ell'$ odd, the hypergeometric functions in (12) and (13) do not reduce to finite polynomials. The integral in (1) is now given by (12) for $k' < k$ and by (13) for $k' > k$. These two expressions can again be written more compactly using the step function, *viz.*, we can write, for $k \neq k'$,

$$I_{\ell,\ell'}(k,k') = g_{\ell,\ell'}(k,k')\theta(k-k') + g_{\ell',\ell}(k',k)\theta(k'-k) \quad . \tag{22}$$

Although the hypergeometric functions in (12) and (13) appear to be rather intractible for $\ell - \ell'$ odd, we show in Section IV that they can be expressed in terms of Legendre functions of the second kind, Q_ℓ, i.e., in terms of polynomials and the logarithm, $\log \mid \frac{k+k'}{k-k'} \mid$. Explicit expressions for specific values of ℓ and ℓ' are given at the end of this note.

III. The region $k' \approx k$

We now consider the region around $k = k'$ and a possible term $\delta(k'-k)$ in $I_{\ell,\ell'}(k,k')$. Note that thus far we have assumed $k' \neq k$, for which the delta function would not appear. We therefore examine, for arbitrarily small positive ϵ_1 and ϵ_2, the integral *

$$I_\delta = \int_{k-\epsilon_1}^{k+\epsilon_2} g(k')dk' \int_0^\infty j_\ell(kr) j_{\ell'}(k'r) r^2 dr \quad . \tag{23}$$

where $g(k')$ is assumed to be a continuous function having at least one derivative in the region $k - \epsilon_1 \leq k' \leq k + \epsilon_2$, but otherwise arbitrary. Inverting the order of integration we have

$$I_\delta = \int_0^\infty r^2\, dr\, j_\ell(kr) \int_{k-\epsilon_1}^{k+\epsilon_2} j_{\ell'}(k'r) g(k') dk' \quad . \tag{24}$$

Now it is clear that the contribution to the outer integral from any finite region of r goes to zero when $\epsilon_1 \longrightarrow 0^+, \;\; \epsilon_2 \longrightarrow 0^+$.

Thus we can write

$$I_\delta = \int_N^\infty r^2\, dr\, j_\ell(kr) \int_{k-\epsilon_1}^{k+\epsilon_2} j_{\ell'}(k'r) g(k') dk' \quad . \tag{25}$$

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* Although we do not write it explicitly, one may consider the inner integral over r in (23) to have a convergence factor $e^{-\varepsilon r}$. The remaining steps are then justified, and the limit $\varepsilon \longrightarrow 0$ can be taken at the end.

with N arbitrarily large but finite. Now the arguments of both of the spherical Bessel functions in (25) are arbitrarily large, and hence we can substitute their asymptotic expansions and obtain

$$I_\delta = \frac{1}{k}\int_N^\infty dr\ \sin(kr - \tfrac{1}{2}\ell\pi)\int_{k-\epsilon_1}^{k+\epsilon_2} \frac{\sin(k'r - \frac{1}{2}\ell'\pi)}{k'}\, g(k')dk' \quad . \tag{26}$$

(Note that we need to retain only the first term in each of the asymptotic expansions; the terms of higher order in $1/k'r$ or $1/kr$ give contributions which go to zero when $\epsilon_1 \to 0^+, \epsilon_2 \to 0^+$.) Writing $h(k') = g(k')/k'$ in the inner integral in (26), an integration by parts gives

$$\int_{k-\epsilon_1}^{k+\epsilon_2} \sin(k'r - \tfrac{1}{2}\ell'\pi)h(k')dk' = \frac{-\cos(k'r - \frac{1}{2}\ell'\pi)}{r}\, h(k')\Big|_{k-\epsilon_1}^{k+\epsilon_2}$$

$$+ \frac{1}{r}\int_{k-\epsilon_1}^{k+\epsilon_2} \cos(k'r - \tfrac{1}{2}\ell'\pi)h'(k')dk' \quad . \tag{27}$$

We may now neglect the integral on the right-hand side of (27), since its contribution to the integral over r goes to zero when $\epsilon_1 \to 0^+$, $\epsilon_2 \to 0^+$. We then have

$$I_\delta = \frac{1}{k}\int_N^\infty \frac{dr}{r}\ \sin(kr - \tfrac{1}{2}\ell\pi)$$

$$\times \left[h(k-\epsilon_1)\cos\big((k-\epsilon_1)r - \tfrac{1}{2}\ell'\pi\big) - h(k+\epsilon_2)\cos\big((k+\epsilon_2)r - \tfrac{1}{2}\ell'\pi\big)\right] \quad . \tag{28}$$

Here, although $\epsilon_1 << k$ and $\epsilon_2 << k$, we cannot take the terms $\epsilon_1 r$ and $\epsilon_2 r$ to be small since $r \to \infty$ in the integrand. We then obtain, for the terms in [ ] in (28),

$$\frac{1}{k}\cos(kr - \tfrac{1}{2}\ell'\pi)\,\big(h(k-\epsilon_1)\cos\epsilon_1 r \;-\; h(k+\epsilon_2)\cos\epsilon_2 r\big)$$

$$+\,\frac{1}{k}\sin(kr - \tfrac{1}{2}\ell'\pi)\,\big(h(k-\epsilon_1)\sin\epsilon_1 r \;+\; h(k+\epsilon_2)\sin\epsilon_2 r\big) \quad . \tag{29}$$

Next we combine these terms with the remaining factor in the integrand, writing

$$\sin(kr - \tfrac{1}{2}\ell\pi)\cos(kr - \tfrac{1}{2}\ell'\pi) \;=\; \tfrac{1}{2}\left[\sin(\ell'-\ell)\tfrac{\pi}{2} + \sin 2kr\,\cos(\ell'+\ell)\tfrac{\pi}{2} - \cos 2kr\,\sin(\ell'+\ell)\tfrac{\pi}{2}\right]$$

and

$$\sin(kr - \tfrac{1}{2}\ell\pi)\sin(kr - \tfrac{1}{2}\ell'\pi) \;=\; \tfrac{1}{2}\left[\cos(\ell'-\ell)\tfrac{\pi}{2} - \sin 2kr\,\sin(\ell'+\ell)\tfrac{\pi}{2} - \cos 2kr\,\cos(\ell'+\ell)\tfrac{\pi}{2}\right] \quad . \tag{30}$$

The integrals to be evaluated are then, apart from factors independent of ϵ_1 and ϵ_2, of the form

$$J_1 \;\equiv\; \int_N^\infty dr\,\frac{\cos\epsilon_1 r - \cos\epsilon_2 r}{r}\,(1,\ \sin 2kr,\ \cos 2kr)$$

$$J_2 \;\equiv \epsilon \int_N^\infty dr\,\frac{\cos\epsilon_1 r}{r}\,(1,\ \sin 2kr,\ \cos 2kr) \tag{31}$$

$$J_3 \;\equiv (1,\epsilon) \int_N^\infty dr\,\frac{\sin\epsilon_1 r}{r}\,(1,\ \sin 2kr,\ \cos 2kr)$$

and integrals obtained from J_2 and J_3 with ϵ_1 replaced by ϵ_2. Here, in J_2 and J_3, the factor ϵ represents a term of order ϵ_1 or ϵ_2, e.g., $h(k-\epsilon_1) - h(k)$ or $h(k+\epsilon_2) - h(k)$.

In the integrands in (31), terms with factors $\sin 2kr$ or $\cos 2kr$ can be combined, writing

$$\begin{aligned}
\cos \epsilon_1 r \sin 2kr &= \tfrac{1}{2}[\sin(2k+\epsilon_1)r + \sin(2k-\epsilon_1)r] \\
\sin \epsilon_1 r \cos 2kr &= \tfrac{1}{2}[\sin(2k+\epsilon_1)r - \sin(2k-\epsilon_1)r] \\
\cos \epsilon_1 r \cos 2kr &= \tfrac{1}{2}[\cos(2k+\epsilon_1)r + \cos(2k-\epsilon_1)r] \\
\sin \epsilon_1 r \sin 2kr &= \tfrac{1}{2}[-\cos(2k+\epsilon_1)r + \cos(2k-\epsilon_1)r] \quad .
\end{aligned} \tag{32}$$

Thus the terms in J_2 are all of the form

$$\epsilon \int_N^\infty \frac{dr}{r} \cos \epsilon_1 r \quad ,$$

$$\epsilon \int_N^\infty \frac{dr}{r} \sin(2k \pm \epsilon_1) r \quad ,$$

or

$$\epsilon \int_N^\infty \frac{dr}{r} \cos(2k \pm \epsilon_1) r \quad . \tag{33}$$

In the limit of small ϵ_1, these integrals are, respectively, of order $\epsilon \log(N\epsilon_1)$, ϵ, and ϵ, and hence go to zero when $\epsilon_1 \to 0^+$, $\epsilon_2 \to 0^+$. Next, the terms in J_3 are of the form

$$(1, \epsilon) \int_N^\infty \frac{\sin \epsilon_1 r}{r} dr \quad ,$$

$$(1, \epsilon) \int_N^\infty \frac{[\cos(2k-\epsilon_1)r - \cos(2k+\epsilon_1)r]}{r} dr$$

or

$$(1, \epsilon) \int_N^\infty \frac{[\sin(2k+\epsilon_1)r - \sin(2k-\epsilon_1)r]}{r} dr \quad . \tag{34}$$

All three integrals here have finite integrands as $r \to 0$, and the contribution to the integrals from finite r goes to zero when we take the limit $\epsilon_1 \to 0^+$. We may thus extend the lower limit of these integrals to $r = 0$ and evaluate them exactly. The integrals in (34) are then

$$\int_0^\infty \frac{\sin \epsilon_1 r}{r}\, dr \;=\; \frac{\pi}{2} \tag{35}$$

and

$$\int_0^\infty \frac{\cos(2k-\epsilon_1)r - \cos(2k+\epsilon_1)r}{r} \;=\; \log\left|\frac{2k+\epsilon_1}{2k-\epsilon_1}\right| \quad . \tag{36}$$

Thus in the limit $\epsilon_1 \to 0^+$, the last two integrals in (34) are zero and the first is $\frac{\pi}{2}$. Moreover we may neglect the terms with a factor ϵ multiplying the first integral in (34).

We have, finally, the integrals in J_1. Here, as with the integrals in J_3, the integrands are finite as $r \to 0$ and the contribution from finite r goes to zero as $\epsilon_1 \to 0^+$, $\epsilon_2 \to 0^+$. We may thus again extend the lower limit of these integrals and evaluate them exactly. We then find

$$\int_0^\infty \frac{\cos \epsilon_1 r - \cos \epsilon_2 r}{r}\,(1, \sin 2kr, \cos 2kr)\, dr \;=\; \left(\log\left|\frac{\epsilon_2}{\epsilon_1}\right|, 0, \tfrac{1}{2}\log\left|\frac{(2k+\epsilon_1)(2k-\epsilon_1)}{(2k+\epsilon_2)(2k-\epsilon_2)}\right|\right) \quad . \tag{37}$$

The last term here gives zero in the limit $\epsilon_1 \to 0^+$, $\epsilon_2 \to 0^+$, independently of the order in which ϵ_1 and ϵ_2 go to the limit. We thus have, now including the relevant factors from (28) and (30), and substituting $h(k) = g(k)/k$,

$$I_6 \;=\; \frac{1}{2k^2}\, g(k) \log\left|\frac{\epsilon_2}{\epsilon_1}\right| \sin(\ell' - \ell)\tfrac{\pi}{2} \;+\; \frac{\pi}{2k^2}\, g(k) \cos(\ell' - \ell)\tfrac{\pi}{2} \quad . \tag{38}$$

Thus, for $\ell - \ell'$ even we have

$$I_\delta = \frac{\pi}{2k^2} (-1)^{(\ell'-\ell)/2} g(k) \tag{39}$$

i.e., for $\ell - \ell'$ even, $I_{\ell,\ell'}(k,k')$ has a term $\frac{\pi}{2k^2}$ $(-1)^{(\ell'-\ell)/2}$ $\delta(k'-k)$, which we may write more symmetrically as

$$\frac{\pi}{2kk'} (-1)^{(\ell'-\ell)/2} \delta(k'-k) \quad . \tag{40}$$

On the other hand, for $\ell - \ell'$ odd we have

$$I_\delta = \frac{(-1)^{(\ell'-\ell-1)/2}}{2k^2} \log\left|\frac{\epsilon_2}{\epsilon_1}\right| g(k) \quad . \tag{41}$$

This contribution is infinite if ϵ_1 and ϵ_2 approach their limits independently and can have any finite value if ϵ_1 and ϵ_2 are related linearly. However, it is zero if $\epsilon_1 = \epsilon_2$, that is, referring to the definition of I_δ given in (23), we get no contribution from an integration over the region $k' = k$ if that integral is considered as a principle part. This conclusion is clear if we consider the expansion of $I_{\ell,\ell'}(k,k')$ and $I_{\ell',\ell}(k',k)$ in the neighborhood of $k' = k$ for $\ell - \ell'$ odd. The expansion of the hypergeometric functions in (12) and (13) about the point $\frac{k'}{k} = 1$ for the case $\ell - \ell'$ odd is given in [2], p. 75 (4):

$$I_{\ell,\ell'}(k,k') = \frac{\sin(\ell-\ell')\frac{\pi}{2}}{k(k^2-k'^2)} + R \qquad k' < k$$

and

$$I_{\ell',\ell}(k',k) = \frac{\sin(\ell'-\ell)\frac{\pi}{2}}{k'(k'^2-k^2)} + R' \qquad k' > k \tag{42}$$

where R and R' are terms which are finite as $\frac{k'}{k} \to 1^-$ and $\frac{k}{k'} \to 1^-$, respectively. Thus, in the neighborhood of $k' = k$, for both $k' < k$ and $k' > k$, we can write, for $\ell - \ell'$ odd,

$$I_{\ell,\ell'}(k,k') = I_{\ell',\ell}(k',k) = \frac{(-1)^{(\ell'-\ell-1)/2}}{k} \cdot \frac{1}{k'^2 - k^2} + \text{finite terms} \quad . \tag{43}$$

The behavior of $I_{\ell,\ell'}(k,k')$ for $\ell - \ell'$ even is very different. Although there is a delta function $\delta(k'-k)$, the limits $k' \to k-0$ and $k' \to k+0$ are finite: From (12) and (13), for $\ell > \ell'$

$$\begin{aligned} \lim_{k' \to k-0} I_{\ell,\ell'}(k,k') &= \frac{\pi(-1)^{\frac{1}{2}(\ell-\ell')-1}}{4k^3} (\ell-\ell')(\ell+\ell'+1) \\ \lim_{k' \to k+0} I_{\ell,\ell'}(k,k') &= 0 \end{aligned} \tag{44}$$

while for $\ell' > \ell$

$$\begin{aligned} \lim_{k' \to k-0} I_{\ell,\ell'}(k,k') &= 0 \\ \lim_{k' \to k+0} I_{\ell,\ell'}(k,k') &= \frac{\pi(-1)^{\frac{1}{2}(\ell'-\ell)-1}}{4k^3} (\ell'-\ell)(\ell+\ell'+1) \quad . \end{aligned} \tag{45}$$

IV. Expression of $I_{\ell,\ell'}(k,k')$ in terms of Legendre functions

In this section we show that the integral, (1), can be expressed in terms of Legendre functions. Specifically, for $\ell - \ell'$ even it can be expressed in terms of Legendre polynomials, P_ℓ, and for $\ell - \ell'$ odd it can be expressed in terms of Legendre functions of the second kind, Q_ℓ, which can in turn be expressed in terms of elementary functions and the logarithm,

$\log\left|\frac{k+k'}{k-k'}\right|$. The expressions for $I_{\ell,\ell'}(k,k')$ may be written in several forms, both for $\ell-\ell'$ even and for $\ell-\ell'$ odd. Since all of these expressions have the form of derivatives of the Legendre function, we choose that form which has the least number of derivatives for a given set of indices,* ℓ, ℓ'. The results then fall into four categories:

(1) $\ell-\ell'$ even , (a) $\ell<\ell'$, (b) $\ell>\ell'$

(2) $\ell-\ell'$ odd , (a) $\ell<\ell'$, (b) $\ell>\ell'$.

(1) $\ell-\ell'$ even

(a) $\ell<\ell'$

In this case the integral, (1), is given by $I_{\ell,\ell'}(k,k')$ in (13), for $k<k'$, and is zero for $k'<k$, from (12). We first transform the hypergeometric function in (13) using [2], p. 102 (5):

$$\frac{(-1)^n(a)_n(c-b)_n}{(c)_n}(1-z)^{a-1}\,{}_2F_1(a+n,b;c+n;z) = \frac{d^n}{dz^n}\left[(1-z)^{a+n-1}\,{}_2F_1(a,b;c;z)\right] \quad . \tag{46}$$

We then have, for the hypergeometric function in (13), with $n=\ell$,

$$\begin{aligned}{}_2F_1\left(\tfrac{\ell+\ell'+3}{2}, \tfrac{\ell-\ell'}{2}+1; \ell+\tfrac{3}{2}; x^2\right) = &\frac{(-1)^\ell(\frac{3}{2})_\ell(1-x^2)^{(\ell-\ell'-1)/2}}{\left(\frac{\ell'-\ell+3}{2}\right)_\ell\left(\frac{\ell'-\ell+1}{2}\right)_\ell} \\ &\times \frac{d^\ell}{d(x^2)^\ell}\left[(1-x^2)^{(\ell+\ell'+1)/2}\,{}_2F_1\left(\tfrac{\ell'-\ell+3}{2}, \tfrac{\ell-\ell'}{2}+1; \tfrac{3}{2}; x^2\right)\right]\end{aligned} \tag{47}$$

* Specifically, the minimum number of derivatives is $\ell_< + 1$.

where

$$0 < x = \frac{k}{k'} < 1 \quad . \tag{48}$$

Now for $\ell - \ell'$ even and $\ell < \ell'$ the hypergeometric function on the right-hand side of (47) can be expressed directly in terms of Legendre polynomials. Using [2], pp. 126, 127 (22),

$$P^{-1}_{\ell'-\ell}(x) = \frac{-\pi^{1/2} x (x^2-1)^{1/2}}{\Gamma\left(1+\frac{\ell'-\ell}{2}\right)\Gamma\left(\frac{\ell-\ell'+1}{2}\right)} \, {}_2F_1\left(\tfrac{\ell'-\ell+3}{2},\, \tfrac{\ell-\ell'}{2}+1;\, \tfrac{3}{2};\, x^2\right) \quad . \tag{49}$$

Further, from [2], p. 140 (7),

$$P^{-1}_{\ell'-\ell}(x) = \frac{\Gamma(\ell'-\ell)}{\Gamma(\ell'-\ell+2)} \, P^{1}_{\ell'-\ell}(x) \tag{50}$$

and from [2], p. 148 (4),

$$P^{1}_{\ell'-\ell}(x) = (x^2-1)^{1/2} \, \frac{d}{dx} \, P_{\ell'-\ell}(x) \quad . \tag{51}$$

Thus, from (49)-(51),

$${}_2F_1\left(\tfrac{\ell'-\ell+3}{2},\, \tfrac{\ell-\ell'}{2}+1;\, \tfrac{3}{2};\, x^2\right) = -\frac{\Gamma\left(\frac{\ell'-\ell}{2}+1\right)\Gamma\left(\frac{\ell-\ell'+1}{2}\right)\Gamma(\ell'-\ell)}{\pi^{1/2}\, x\, \Gamma(\ell'-\ell+2)} \, \frac{d}{dx} P_{\ell'-\ell}(x) \quad . \tag{52}$$

The question of the phase of $(x^2 - 1)$ in (49) and (51) does not, therefore, enter our final expression. Substituting (52) in (47) and then substituting this in (13) we then have, after considerable simplification of the gamma functions,*

$$I_{\ell,\ell'}(k,k') = \frac{\pi(-1)^{\frac{1}{2}(\ell+\ell')-1}}{2kk'^2\left(\frac{\ell'-\ell}{2}+\frac{1}{2}\right)_\ell}\, x^{\ell+1}\,(1-x^2)^{(\ell-\ell'-1)/2}$$

$$\times\ \frac{d^\ell}{d(x^2)^\ell}\left[x^{-1}(1-x^2)^{(\ell+\ell'+1)/2}\,\frac{d}{dx}\,P_{\ell'-\ell}(x)\right]\cdot\theta(k'-k) \qquad (53)$$

for $\ell-\ell'$ even, $\ell<\ell'$, with $x=k/k'$.

In particular, for $\ell=0$, (53) reduces to the very simple result

$$I_{0,\ell'}(k,k') = \frac{\pi(-1)^{\frac{1}{2}\ell'-1}}{2kk'^2}\,\frac{d}{dx}\,P_{\ell'}(x)\cdot\theta(k'-k)\quad . \qquad (54)$$

Next we consider

(1) $\ell-\ell'$ even

(b) $\ell>\ell'$

In this case the integral, (1), is given by $I_{\ell,\ell'}(k,k')$ in (12), for $k'<k$, and is.zero for $k'>k$, from (13). We now transform the hypergeometric function in (12), again using (46), but now set $n=\ell'$. Observing that (12) may be obtained from (13) by the interchange $\ell\rightleftharpoons\ell'$, $k\rightleftharpoons k'$, we follow the identical steps leading from (13) to (53) and obtain, for $\ell-\ell'$ even, $\ell>\ell'$, with

$$0<y=\frac{k'}{k}<1 \qquad (55)$$

* We use, extensively, $\Gamma(z)\,\Gamma(1-z)=\pi/\sin\pi z$ and $\Gamma(2z)=2^{2z-1}\,\pi^{-1/2}\,\Gamma(z)\,\Gamma(z+\frac{1}{2})$.

$$I_{\ell,\ell'}(k,k') = \frac{\pi(-1)^{\frac{1}{2}(\ell+\ell')-1}}{2k'k^2\left(\frac{\ell-\ell'}{2}+\frac{1}{2}\right)}\; y^{\ell'+1}\,(1-y^2)^{(\ell'-\ell-1)/2}$$

$$\times\; \frac{d^{\ell'}}{d(y^2)^{\ell'}}\left[y^{-1}(1-y^2)^{(\ell+\ell'+1)/2}\,\frac{d}{dy}\,P_{\ell-\ell'}(y)\right]\cdot\theta(k-k') \quad . \tag{56}$$

Now, for $\ell' = 0$, (56) reduces to

$$I_{\ell,0}(k,k') = \frac{\pi(-1)^{\frac{1}{2}\ell-1}}{2k'k^2}\,\frac{d}{dy}\,P_\ell(y)\cdot\theta(k-k') \quad . \tag{57}$$

Next we consider

(2) $\ell - \ell'$ odd

(a) $\ell < \ell'$

For $\ell - \ell'$ odd, the integral (1) is given by $I_{\ell,\ell'}(k,k')$ in (22). We again make use of (46), and now apply it to both of the hypergeometric functions in (22), in both cases with $n = \ell$. For the second hypergeometric function in (22) we then have the result given in (47). (Note, however, that now none of the parameters in the hypergeometric functions in (47) is a negative integer or zero, since now $\ell - \ell'$ is odd.) For the first hypergeometric function in (22) we have

$${}_2F_1\left(\tfrac{\ell+\ell'+3}{2},\, \tfrac{\ell'-\ell}{2}+1;\, \ell'+\tfrac{3}{2};\, y^2\right)$$

$$= \frac{(-1)^\ell(\ell'-\ell+\frac{3}{2})_\ell\,(1-y^2)^{(\ell-\ell'-1)/2}}{\left(\frac{\ell'-\ell+3}{2}\right)_\ell\left(\frac{\ell'-\ell+1}{2}\right)_\ell}$$

$$\times\; \frac{d^\ell}{d(y^2)^\ell}\left[(1-y^2)^{(\ell+\ell'+1)/2}\,{}_2F_1\left(\tfrac{\ell'-\ell+3}{2},\, \tfrac{\ell'-\ell}{2}+1;\, \ell'-\ell+\tfrac{3}{2};\, y^2\right)\right] \quad . \tag{58}$$

The hypergeometric function on the right-hand side of (58) can now be expressed in terms of the Legendre functions of the second kind using [2], pp. 134, 135 (41):

$$-Q^{1}_{\ell'-\ell}(x) = 2^{-\ell'+\ell-1}\pi^{1/2}\frac{\Gamma(\ell'-\ell+2)}{\Gamma(\ell'-\ell+\frac{3}{2})}\, x^{-\ell'+\ell-2}(x^2-1)^{1/2}$$

$$_2F_1\left(\tfrac{\ell'-\ell+3}{2}, \tfrac{\ell'-\ell}{2}+1; \ell'-\ell+\tfrac{3}{2}; y^2\right) \quad . \tag{59}$$

We then use [2], p. 148 (5):

$$Q^{1}_{\ell'-\ell}(x) = (x^2-1)^{1/2}\,\frac{d}{dx}\,Q_{\ell'-\ell}(x) \tag{60}$$

from which

$$_2F_1\left(\tfrac{\ell'-\ell+3}{2}, \tfrac{\ell'-\ell}{2}+1; \ell'-\ell+\tfrac{3}{2}; y^2\right) = -\,\frac{2^{\ell'-\ell+1}}{\pi^{1/2}}\frac{\Gamma(\ell'-\ell+\frac{3}{2})}{\Gamma(\ell'-\ell+2)}\, x^{\ell'-\ell+2}\frac{d}{dx}Q_{\ell'-\ell}(x) \quad . \tag{61}$$

For $\ell-\ell'$ odd, the hypergeometric function in (47) can be expressed in terms of Legendre functions of the second kind and Legendre polynomials (rather than in terms of Legendre polynomials alone, as in (49)-(52)). From [2], pp. 134, 135 (40) and pp. 126, 127 (22), we have, for $\ell-\ell'$ odd and $\ell<\ell'$,

$$\begin{aligned}-Q^{-1}_{\ell'-\ell}(x) \;=\; & \frac{\pi^{1/2}\Gamma\left(\frac{\ell'-\ell}{2}\right)\, e^{\pm\frac{1}{2}i\pi(-\ell'+\ell-2)}\,(x^2-1)^{1/2}}{4\Gamma\left(\frac{\ell'-\ell+3}{2}\right)}\; {}_2F_1\left(\tfrac{\ell'-\ell}{2}+1, \tfrac{\ell-\ell'+1}{2}; \tfrac{1}{2}; x^2\right) \\ & + \frac{\pi^{1/2}\Gamma\left(\frac{\ell'-\ell+1}{2}\right)\, e^{\pm\frac{1}{2}i\pi(-\ell'+\ell-1)}\,x(x^2-1)^{1/2}}{2\Gamma\left(\frac{\ell'-\ell}{2}+1\right)}\; {}_2F_1\left(\tfrac{\ell'-\ell+3}{2}, \tfrac{\ell-\ell'}{2}+1; \tfrac{3}{2}; x^2\right) \quad .\end{aligned} \tag{62}$$

and

$$P^{-1}_{\ell'-\ell}(x) = \frac{\pi^{1/2}(x^2-1)^{1/2}}{2\Gamma\left(\frac{\ell-\ell'}{2}+1\right)\Gamma\left(\frac{\ell'-\ell+3}{2}\right)}\ {}_2F_1\left(\tfrac{\ell'-\ell}{2}+1, \tfrac{\ell-\ell'+1}{2}; \tfrac{1}{2}; x^2\right) \quad . \tag{63}$$

In (62), in the exponential, the upper sign is taken for $Im\, x > 0$, the lower sign for $Im\, x < 0$. This sign will, however, drop out of our final result. From (62) and (63) we now have, using (50), (51), (60), and from [2], p. 140 (2),

$$Q^{-1}_{\ell'-\ell}(z) = \frac{\Gamma(\ell'-\ell)}{\Gamma(\ell'-\ell+2)}\, Q^{1}_{\ell'-\ell}(z) \quad , \tag{64}$$

$${}_2F_1\left(\tfrac{\ell'-\ell+3}{2}, \tfrac{\ell-\ell'}{2}+1; \tfrac{3}{2}; x^2\right) = \frac{2}{\pi^{1/2}}(-1)^{(\ell'-\ell-1)/2}\, \frac{\Gamma(\frac{\ell'-\ell}{2}+1)\Gamma(\ell'-\ell)}{\Gamma(\frac{\ell'-\ell+1}{2})\Gamma(\ell'-\ell+2)x}$$

$$\times \frac{d}{dx}\left(Q_{\ell'-\ell}(x) \pm \frac{i\pi}{2} P_{\ell'-\ell}(x)\right) \quad . \tag{65}$$

Substituting (61) in (58) and (65) in (47), then substituting both of these in (22), and again simplifying the gamma functions, we have, for $I_{\ell,\ell'}(k,k')$ as given in (22), with $\ell-\ell'$ odd and $\ell < \ell'$,

$$
\begin{aligned}
I_{\ell,\ell'}(k,k') \;=\;& \frac{(-1)^{(\ell+\ell'-1)/2}\; y^{\ell'+1}\;(1-y^2)^{(\ell-\ell'-1)/2}}{k' k^2 (\frac{\ell'-\ell+1}{2})_\ell} \\
&\times\; \frac{d^\ell}{d(y^2)^\ell}\left[y^{\ell-\ell'-2}(1-y^2)^{(\ell+\ell'+1)/2}\,\frac{d}{dx}\,Q_{\ell'-\ell}(x)\right]\,\theta(k-k') \\
&+\; \frac{(-1)^{(\ell+\ell'-1)/2}\; x^{\ell+1}\;(1-x^2)^{(\ell-\ell'-1)/2}}{k k'^2 (\frac{\ell'-\ell+1}{2})_\ell} \\
&\times\; \frac{d^\ell}{d(x^2)^\ell}\left[x^{-1}(1-x^2)^{(\ell+\ell'+1)/2}\,\frac{d}{dx}\left(Q_{\ell'-\ell}(x)\pm\frac{i\pi}{2}P_{\ell'-\ell}(x)\right)\right]\cdot\theta(k'-k)
\end{aligned}
\tag{66}
$$

(Note the last term in (66), involving the term $P_{\ell'-\ell}(x)$, is identical to the result given in (53).) In (66), if $k > k'$ then the second term on the right-hand side is zero in view of the factor $\theta(k'-k)$, and, in the first term, $x = 1/y = k/k' > 1$. On the other hand, if $k < k'$ then the first term is zero, and, in the second term, $x = k/k' < 1$. Now from [2], p. 152 (24) we have

$$
Q_n(z) \;=\; \tfrac{1}{2}P_n(z)\log(\tfrac{z+1}{z-1}) \;-\; W_{n-1}(z) \tag{67}
$$

where $W_{n-1}(z)$ is a polynomial of degree $n-1$. For $z = x \pm i0$ where $0 < x < 1$ we can then write $\log(z-1) \;=\; \log(1-x) \;\pm i\pi$ and

$$
\begin{aligned}
Q_n(x\pm i0) \;\pm\; \frac{i\pi}{2}P_n(x) \;&=\; \tfrac{1}{2}P_n(x)\log(\tfrac{1+x}{1-x}) \;-\; W_{n-1}(x) \\
&=\; Q_n(x)
\end{aligned}
\tag{68}
$$

from [2], p. 153 (26).* We thus have, for $\ell - \ell'$ odd and $\ell < \ell'$,

$$I_{\ell,\ell'}(k,k') = \frac{(-1)^{(\ell+\ell'-1)/2}\, y^{\ell'+1}\,(1-y^2)^{(\ell-\ell'-1)/2}}{k'k^2(\frac{\ell'-\ell+1}{2})_\ell}$$

$$\times \frac{d^\ell}{d(y^2)^\ell}\left[y^{\ell-\ell'-2}\,(1-y^2)^{(\ell+\ell'+1)/2}\,\frac{d}{dx}\,Q_{\ell'-\ell}(x)\right]\theta(k-k')$$

$$+ \frac{(-1)^{(\ell+\ell'-1)/2}\, x^{\ell+1}\,(1-x^2)^{(\ell-\ell'-1)/2}}{kk'^2(\frac{\ell'-\ell+1}{2})_\ell}$$

$$\times \frac{d^\ell}{d(x^2)^\ell}\left[x^{-1}\,(1-x^2)^{(\ell+\ell'+1)/2}\,\frac{d}{dx}\,\mathrm{Q}_{\ell'-\ell}(x)\right]\theta(k'-k) \quad . \qquad (69)$$

Finally, for $\ell - \ell'$ odd and $\ell > \ell'$, since $I_{\ell,\ell'}(k,k')$ as given in (22) is invariant under the interchange $\ell \rightleftharpoons \ell'$, $k \rightleftharpoons k'$, we have

---

* We use Q_n to describe the function defined in [2], p. 143 (2):

$$\mathrm{Q}_n(x) = \tfrac{1}{2}[Q_n(x+i0) + Q_n(x-i0)] \quad , \quad -1 < x < 1.$$

$$I_{\ell,\ell'}(k,k') = \frac{(-1)^{(\ell+\ell'-1)/2}\, x^{\ell'+1}\, (1-x^2)^{(\ell'-\ell-1)/2}}{k k'^2 (\frac{\ell-\ell'+1}{2})_{\ell'}}$$

$$\times \frac{d^{\ell'}}{d(x^2)^{\ell'}} \left[x^{\ell'-\ell-2}\, (1-x^2)^{(\ell+\ell'+1)/2}\, \frac{d}{dy}\, Q_{\ell-\ell'}(y) \right] \theta(k'-k)$$

$$+ \frac{(-1)^{(\ell+\ell'-1)/2}\, y^{\ell'+1}\, (1-y^2)^{(\ell'-\ell-1)/2}}{k' k^2 (\frac{\ell-\ell'+1}{2})_{\ell'}}$$

$$\times \frac{d^{\ell'}}{d(y^2)^{\ell'}} \left[y^{-1}\, (1-y^2)^{(\ell+\ell'+1)/2}\, \frac{d}{dy}\, Q_{\ell-\ell'}(y) \right] \theta(k-k') \quad . \qquad (70)$$

In particular, for $\ell = 0$, (69) reduces to

$$I_{0,\ell'}(k,k') = \frac{(-1)^{(\ell'-1)/2}}{k k'^2}\, \frac{d}{dx}\, Q_{\ell'}(x) \cdot \theta(k-k')$$

$$+ \frac{(-1)^{(\ell'-1)/2}}{k k'^2}\, \frac{d}{dx}\, Q_{\ell'}(x) \cdot \theta(k'-k) \quad . \qquad (71)$$

Similarly, from (70), for $\ell' = 0$ we have

$$I_{\ell,0}(k,k') = \frac{(-1)^{(\ell-1)/2}}{k' k^2}\, \frac{d}{dy}\, Q_{\ell}(y) \cdot \theta(k'-k)$$

$$+ \frac{(-1)^{(\ell-1)/2}}{k' k^2}\, \frac{d}{dy}\, Q_{\ell}(y) \cdot \theta(k-k') \quad . \qquad (72)$$

If we define, for both $0 < x < 1$ and $x > 1$,

$$\widetilde{Q}_n(x) \equiv Re\, Q_n(x) = \tfrac{1}{2} P_n(x) \log \left| \frac{1+x}{1-x} \right| - W_{n-1}(x)$$

$$= \tfrac{1}{2} P_n(x) \log \left| \frac{k+k'}{k-k'} \right| - W_{n-1}(x) \qquad (73)$$

then we can write the expressions in (71) and (72) in the simpler form

$$I_{0,\ell'}(k,k') = \frac{(-1)^{(\ell'-1)/2}}{kk'^2} \frac{d}{dx} \widetilde{Q}_{\ell'}(x) \qquad (74)$$

$$I_{\ell,0}(k,k') = \frac{(-1)^{(\ell-1)/2}}{k'k^2} \frac{d}{dy} \widetilde{Q}_{\ell}(y) \quad . \qquad (75)$$

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The integral $I_{\ell,\ell'}(k,k') = \int_0^\infty j_\ell(kr) j_{\ell'}(k'r)\, r^2 dr$, in which the spherical Bessel functions $j_\ell(kr)$ are the radial eigenfunctions of the three-dimensional wave equation in spherical coordinates, is evaluated in terms of distributions, in particular step functions and delta functions. We show that the behavior of $I_{\ell,\ell'}$ is very different in the cases $\ell - \ell'$ even $(0, \pm 2, \pm 4, ...)$ and $\ell - \ell'$ odd $(\pm 1, \pm 3, ...)$. For $\ell - \ell'$ even it is expressed in terms of the delta function, step functions, and Legendre polynomials. For $\ell - \ell'$ odd it is expressed in terms of Legendre functions of the second kind and step functions; no delta functions appear.

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