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A Near-Optimal Starting Solution for Polynomial Approximation of a Continuous Function in the L_1 Norm

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ABSTRACT

This paper presents a method of selecting a near-optimal starting solution for a large class of discrete polynomial approximation problems in the L_1 norm. While it is possible to prove the optimality of these advanced starting solutions for only a small class of continuous polynomial approximation problems, empirical evidence indicates the starting bases will be nearly optimal for a much larger class of discrete problems. This paper presents the method used to determine the starting basis and a heuristic justification backed by empirical results supporting its use.

I. INTRODUCTION

The investigation of a near-optimal starting solution for discrete polynomial approximation problems in the L_1 norm resulted from a study to develop a methodology for comparing and evaluating mathematical programming software.[1] In this study four L_1 approximation algorithms[2,3,4,5] were tested on a large set of problems with diverse problem structures and characteristics. Included were 80 problems used to test code performance in the polynomial approximation of a continuous function, a specific type of approximation problem. In these problems, the continuous function being approximated, the degree of the approximating polynomial, and the number of observation points were independently varied. In all problems, the observations were equally spaced over the same interval of approximation, [0,1].

When the interpolation points defining the best L_1 solution were tabulated, with the degree of the polynomial and the number of observations fixed, but varying the functional form, a recurrent pattern emerged which indicated that the points of interpolation remained relatively stationary within the normalized interval regardless of the functional form being approximated. This pattern suggested a method of selecting a starting basis which would be nearly optimal for a large number of approximation problems.

After further study, an analytic expression was identified that can be used to calculate a feasible starting solution which, when used in the continuous analog of the discrete approximation problem, is the optimal solution in the L_1 norm.[6] Calculating this starting solution and using it in the discrete approximation problem also produced impressive empirical results, both in terms of its relative distance to the optimal solution and also in the marked reduction in the number of iterations needed to reach the optimal solution. Furthermore, the integration of this method of selecting the starting basis into existing L_1 algorithms is easy. An explanation of the success of this method can be found in the underlying approximation theory and is presented in the next two sections. The final section reports the results of using this near-optimal starting basis on a large set of test problems.

II. THE DISCRETE POLYNOMIAL APPROXIMATION PROBLEM

The general form of the discrete L_1 approximation problem can be formulated as follows.

Given n sets of observations on a dependent variable y_i and a single independent variable x_i

$$\{x_i, y_i\} ,$$

determine m+1 parameters

$$\overline{\beta} = \beta_o, \beta_1, \beta_2, \ldots , \beta_m ,$$

which minimize

$$Z = \sum_{i=1}^{n} \left| y_i - \sum_{j=0}^{m} \beta_j x_i^j \right| .$$

This is the polynomial approximation of a continuous function when $y_i = f(x_i)$, where $f(x)$ is a continuous function defined over an approximation interval I. Introducing variables d and d , corresponding to the positive and negative deviation, the approximation problem can be posed as the following linear programming problem.

$$\text{Minimize} \quad \sum_{i=1}^{n} d_i^{+} + d_i^{-} \quad ,$$

$$\text{subject to} \qquad d_i^{+} - d_i^{-} + y_i - \sum_{j=0}^{m} \beta_j \, x_i^{j} = 0 \quad ,$$

$$d_i^{+} \geq 0 \; , \; d_i^{-} \geq 0 \quad \text{for } i = 1, 2, 3, \ldots, n \quad .$$

For this class of problems it is not unreasonable to assume the problems are of full rank (independent columns) and overdetermined (the number of observations strictly greater than m, the number of β parameters in the problem). A further condition necessary for one to "predict" the best solution to any problem of this type is that the solution be unique. When the uniqueness condition is satisfied and the number of observations sufficiently large, the starting solution we suggest is very close to the optimum. Unfortunately, uniqueness cannot be guaranteed for certain approximation problems.

III. FUNCTIONAL ANALYSIS AND PROPERTIES OF L_1 APPROXIMATION OF A CONTINUOUS FUNCTION

This section presents the theoretical basis behind a unique best L_1 approximation solution for a special class of continuous functions. The section begins by presenting several properties of the polynomial approximation of a continuous function in the L_1 norm. These properties establish the existence, uniqueness and optimality of a solution. A description of a class of functions which have these properties is developed and it will be shown that the starting basis we propose will be the optimal L_1 solution. An alternate representation of the optimal L_1 solution, which serves as a useful guide in determining the "goodness" of an approximation is also presented as is a method for calculating the starting basis easily and efficiently.

Since the best L_1 polynomial approximation of degree m to a continuous function, f(x), intersects the function in at least m+1 points[7], we will examine the additional characteristics a function must possess to insure the uniqueness of the optimal solution. The first assumption on the function requires the existence of a non-trivial approximation problem, i.e., f(x) is different from the L_1 approximation almost everywhere. Had this not been assumed, any m+1 observations would have defined the best L_1 approximation. By requiring that f(x) have a m+1st derivative that is continuous and non-constant, a nontrivial approximation problem is guaranteed. We characterize any such function as a "higher-order function" than the polynomial of degree m. This is closely analogous to the term "higher-order polynomials" though more general.

The uniqueness of the final solution and ultimately the optimality of the suggested starting basis is established by making one further requirement on the derivatives of f(x). If the $m+1^{st}$ derivative, $f^{m+1}(x)$, is nonvanishing over the approximation interval, then it will be shown that the best L_1 polynomial approximation to f(x) will intersect at exactly m+1 observations and, therefore, will be unique. With this condition satisfied, one can then find the m+1 observations which define the optimal L_1 approximation to the continuous problem, evaluate the function at these points and enter these observations to the discrete problem basis. Solving the corresponding $(m+1)\cdot(m+1)$ system of equations defined by this new basis will locate a near-optimal L_1 solution to the discrete problem.

The following examples and subsequent theorems will solidify the theoretical underpinnings bounding the number of intersections between the continuous function and the best approximating polynomial. First consider the problem of locating all the possible intersections between a second-order polynomial $p_2(x)$ and a first-order polynomial $p_1(x)$, (shown graphically in figure 1).

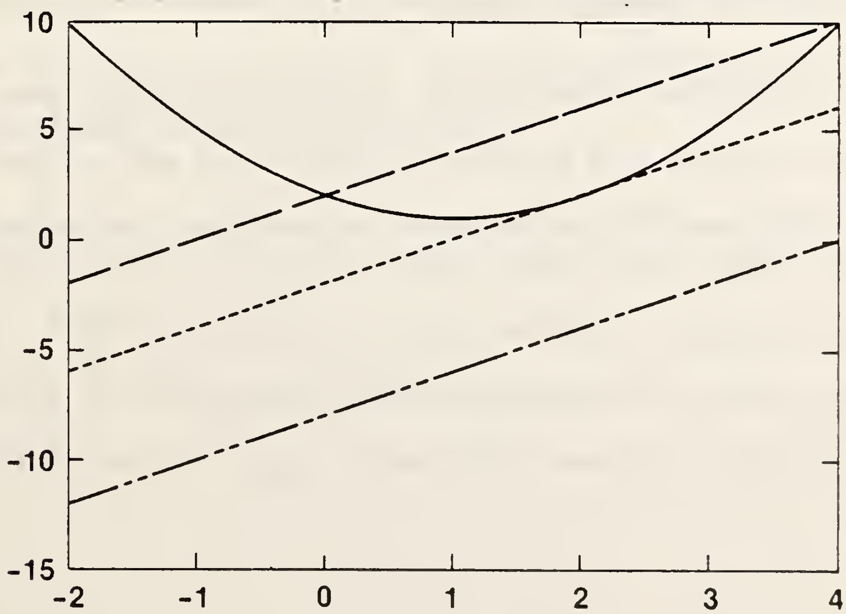

FIGURE 1

Three possible situations exist: first, $p_1(x)$ and $p_2(x)$ fail to intersect in $[a, b]$; second, $p_1(x)$ and $p_2(x)$ are tangent which will produce only one point of intersection; or finally, $p_1(x)$ and $p_2(x)$ intersect at two points. The problem of locating the points of intersection of $p_1(x)$ and $p_2(x)$ is equivalent to that of finding all possible roots of a residual function defined as their difference in $[a, b]$,

$$r(x) = p_2(x) - p_1(x) = 0, \ x \in [a, b].$$

Since $r(x)$ is binomial, it can have at most two real-valued roots. This argument can easily be extended for any two polynomials, $p_m(x)$ and $p_{m+1}(x)$, of degree m and $m+1$ respectively, with no loss of generality.

The task of establishing the number of intersections of $p_m(x)$ and a polynomial of degree $m+k$, $p_{m+k}(x)$, for $k > 1$, or any function of higher order is slightly more difficult. Again, the behavior of the derivatives will determine the number of roots the residual function

$$r(x) = p_{m+k}(x) - p_m(x),$$

will have over $[a, b]$. By assuming a restriction on p_{m+k}, it is again possible to limit the number of roots $r(x)$ may have to at most $m+1$ and therefore define a larger class of problems which will have a unique best L_1 solution.

Lemma 1. Suppose a function $f(x)$ is continuously differentiable at least k times in an interval [a, b]. Suppose further that $f^k(x) > 0$ (<0) in [a, b]. Then $f^{k-1}(x)$ is strictly monotone increasing (decreasing) in [a, b], and there is only one possible zero of $f^{k-1}(x)$ in [a, b].

Together with the following theorem it will be possible to establish upper bounds on the number of real-valued roots for each successively lower-ordered derivative.

Theorem 1 [McCormick[8]]. Suppose a function $f(x)$ is continuously differentiable at least k times in an interval [a, b]. Suppose that $f^k(x)$ has q zeros in that interval and that they are known and ordered as $a \leq z_1 < z_2 < z_3 \ldots < z_q \leq b$. Then $f^{k-1}(x)$ is strictly monotone in $[a, z_1]$, $[z_i, z_{i+1}]$, (for $i=1,\ldots,q-1$), and $[z_q, b]$. There are at most q+1 zeros of $f^{k-1}(x)$ in [a, b]. Specifically, there may be one in $[a, z_1]$, one in each of $[z_i, z_{i+1}]$, (for $i=1,\ldots,q-1$) and one in $[z_q, b]$. If $f^{k-1}(a)\cdot f^{k-1}(z_1) > 0$ there is no zero in that interval, otherwise there is exactly one there. If (for $i=1, 2,\ldots,q-1$) $f^{k-1}(z_i)\cdot f^{k-1}(z_{i+1}) > 0$, there is no zero in that interval. If $f^{k-1}(z_q)\cdot f^{k-1}(b) > 0$, there is no zero in $[z_q, b]$. Otherwise there is exactly one.

Proof. Since $f^k(z_i) = f^k(z_{i+1}) = 0$ and there are no zeros between z_i and z_{i+1} then $f^k(x) > 0$ for all x in (z_i, z_{i+1}), or $f^k(x) < 0$ in that interval. Thus the hypotheses of the previous lemma are satisfied and the appropriate conclusion follows for $[z_i, z_{i+1}]$. The other cases are identical.

Thus, for any function f(x) with a nonvanishing and monotonic $(m+1)^{st}$ derivative Theorem 1 can be applied successively to construct upper bounds to each successively lower-order derivative until a upper bound of m+1 can be established for the number of real valued roots to f(x) over the interval [a,b]. The upper bound on the number of roots of the residual function r(x) is, therefore, dependent only on the behavior of $f^{m+1}(x)$ since the $(m+1)^{st}$ derivative of the approximating polynomial $p_m(x)$ is zero valued over the real line, thus

$$\begin{aligned} r^{m+1}(x) &= f^{m+1}(x) - p_m^{m+1}(x) \\ &= f^{m+1}(x). \end{aligned}$$

Under the assumptions above and by requiring that the associated approximation problem is of full rank, the existence of a unique optimal solution to the discrete polynomial approximation problem defined by exactly m+1 observations is guaranteed. The final question remains. How to select the m+1 observations close to the optimal solution?

As shown in Rivilin[10] and Rice [11], there exists a large class of continuous functions for which the optimal L_1 solution in the continuous approximation problem can be determined a priori whenever it is known a priori that the optimal solution is unique and that the interpolation of these two functions occur at m+1 points. The necessary and sufficient requirements on f(x) for this to occur are 1) f(x) is continuous, 2) the difference of the function f(x) and the L_1 approximating polynomial $P_m(x)$is different from zero almost everywhere on [a,b], 3) the residual function, $r(x)=f(x)-P_m(x)$, changes in sign at a unique set of m+1 points in the interval [a,b]. These points of interpolation are determined in the following theorem.

Theorem 2: If f(x) is continuous and differentiable, and f(x) and $p_m(x)$ intersect m+1 times in the interval [-1, 1], then the least L_1 approximation is the unique polynomial $p_m{}^*(x)$ which satisfies

$$p_m{}^*\left(\cos\frac{j\Pi}{m+2}\right) = f\left(\cos\frac{j\Pi}{m+2}\right), \text{ for } j=1, 2, \ldots, m+1.$$ [11]

This optimal set of observations are the roots of the m^{th} degree Chebyshev polynomial of the second kind, Um(x). The explicit expression for this family of polynomials is listed below:

$$U_m(x) = \sum_{i=1}^{m/2} (-1)^i \frac{(m-i)!}{i!(m-2i)!} (2x)^{m-2i};$$

or equivalently

$$U_m(\cos\Theta) = \frac{\sin(m+1)\Theta}{\sin\Theta} \text{ for } x=\cos\Theta\ ;$$

which have the following recurrence relation

$$U_{m+1}(x) = 2xU_m(x) - U_{m-1}(x).$$ [12]

The first four of these polynomials are:

$$U_0(x) = 1,$$

$$U_1(x) = 2x,$$

$$U_2(x) = 4x2-1;$$

$$U_3(x) = 8x^3-4x.$$

The series in which these polynomials are used in approximation to a continuous function f(x),

$$f(x) = \sum_{j=0}^{k} b_j U_j(x),$$

are called Chebyshev Series and if

$$\lim_{i\to\infty} b_i = 0,$$

the series is termed a Chebyshev series expansion of the function f(x).

Chebyshev series expansions have long been known to converge very quickly to the target function compared to other series approximations and can be a useful tool in determining the significance of the neglected term in the L_1 polynomial approximation. Generating the equivalent Chebyshev representation to the best L_1 approximating polynomial provides the user with information on the relative significance of the last term of the Chebyshev representation. When the coefficient of the higher order terms are very close to zero, the user can assume little accuracy will be gained by increasing the degree of the approximating polynomial. Conversely, if the last term of the Chebyshev approximation are of the same order as the previous coefficients, attempts to include more terms may provide a better approximation, or give the user an indication that the target function f(x) may not have a converging Chebyshev series expansion and an alternate approximation method should be used.

IV. GENERATING THE STARTING BASIS

The method used in calculating the starting basis involves determining the zero-valued points of the Chebyshev polynomial. Using the trigonometric representation of the Chebyshev polynomials,

$$U_m(x) = U_m(\cos\theta) = \frac{\sin(m+1)\theta}{\sin\theta}$$

it is clear that the zero values of $U_m(x)$ occur at

$$x_i^* = \cos\frac{k\pi}{m+1} \text{ for } k = 1, 2, \ldots, m \text{ and } -1 \leq x_i \leq 1 .$$

In our work, the zero-valued points of $U_m(x)$ were translated into the interval of approximation, evaluated by the continuous function, and entered as constraints to the problem. Results of using these points as a starting basis are presented in the next section.

V. COMPUTATIONAL RESULTS

In our experiment, 256 approximation problems were used in testing the performance of the advanced starting method described. All problems were produced by an L_1 polynomial approximation test-problem generator[13] using eight different continuous functions, varying both in form and difficulty (see Table 1a of the appendix). The problems generated ranged in degree from 3 to 10, with 100, 200, 400, and 800 observations over the closed interval [0,1]. Each problem was solved twice. One run of all 256 problems did not use the advanced solution method but included together with the sorted equidistant observations, the m+1 generated observations $\{x_i^*\}$ used in the new starting solution method, thereby guaranteeing identical problems in both runs. The 256 problems were then rerun. The observations $\{x_i^*\}$ now formed the first m+1 constraints to the problem and were selected to enter the starting basis by the L_1 code on the first m+1 pivots. The method used to solve all problems was double-precision L_1-approximation algorithm developed by N. N. Abdelmalek.[14]

In this section, the results from solving the problem with the new starting solution method are compared to results from the original method. Two performance measures were used in the comparison. One measured the distance between the objective function value Z at the near-optimal starting point, and the optimum objective function value Z^*, normalized by the distance from the starting objective function value, Z^s (where $\bar{\beta}=0$), to Z^*. Or, mathematically, the normalized distance is

$$D_i = \frac{Z_i - Z_i^*}{Z^s - Z_i^*} \qquad i = 1, 2, \ldots, 256 \quad .$$

A second measure was difference in iteration count between using the near-optimal "starting basis" and the starting basis defined by $\bar{\beta}=0$.

The results in measuring the performance of the advanced solution method by the normalized objective function distance are given in Table 2 of the appendix and summarized in Figure 2.*

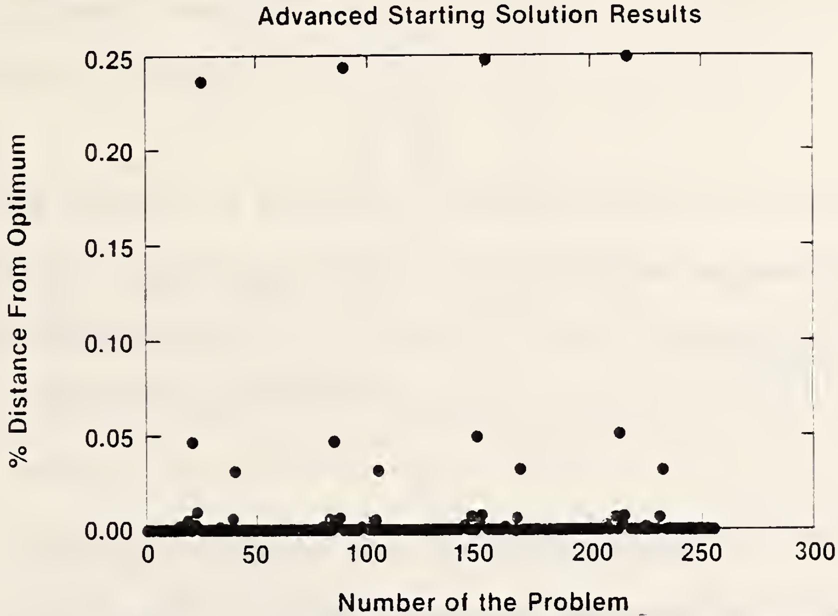

FIGURE 2. Relative Distance from the Optimum

Of the total 256 problems, only 12 were more than 1 percent from the optimum. Closer examination of the outliers reveals that all outliers of magnitude greater than 1 percent are related to problems produced by two of the eight continuous functions being approximated. Closer investigation reveals that the distance D is worst in the approximations made by polynomials of lower degree. The functions (illustrated in Figures 3 and 4) connected with these outlier values are highly nonlinear and approximations by lesser degree polynomials are very poor.

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*The regular pattern of the outliers in Figure 2 illustrate that problems of a specific function and degree are the same relative distance from the optimum for 100, 200, 400, and 800 observations.

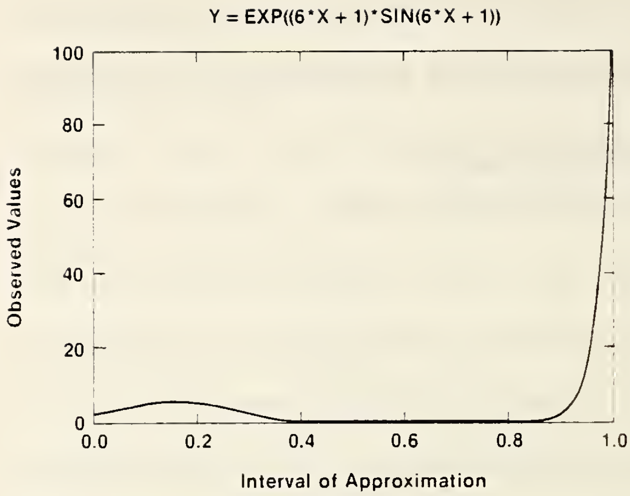

FIGURE 3. FUNCTION 3:

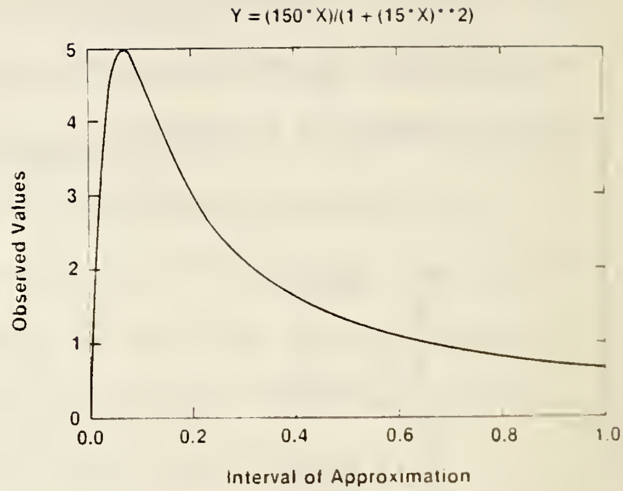

FIGURE 4. FUNCTION 5:

Though selection of a low degree polynomial was unwise by the user, the starting basis did reduce the number of iterations required to solve these problems in 13 of the 16 cases, as shown in Table 1. Similar data for problems of 200, 400, and 800 observations are given in Table 2a of the appendix.

DEGREE OF POLYNOMIAL	10		9		8		7		6		5		4		3	
STARTING METHOD	NO	YES	NO	YES	NO	YES	NO	YES	NO	YES	NO	YES	NO	YES	NO	YES
FUNCTION #3	28	14	42	27	19	14	24	8	20	23	24	19	13	11	14	15
FUNCTION #5	26	15	19	14	24	8	18	7	14	6	19	17	13	10	23	31

TABLE 1. ITERATIONS NEEDED TO OBTAIN AN OPTIMAL SOLUTION (NUMBER OF OBSERVATIONS = 100).

Increasing the degree of the approximating polynomial, in these outlier cases, did reduce the D_i values substantially, with 4 of the 16 problems listed above having an advanced starting objective function value identical to the optimal, to 8 significant digits.

Over all 256 problems, a total of 62 problems using the advanced method were at the optimum, requiring no additional pivots for an accuracy of 10^{-8}. A total of 5558 iterations were necessary to solve all 256 problems using the standard method. Solving the 256 problems using the starting method reduced the total number of iterations needed to solve all problems by 2028, a reduction of 36 percent. In all, 241 problems had a reduced iteration count, 2 marked no change, and 13 increased.

In problems where the degree of the polynomial was overspecified, a degenerate problem much more difficult to solve, is produced. We noted several problems of this type which had an increased iteration count but also had a very small normalized distance measure D_i. This suggests there are many other solutions with objective function values very close to the optimal solution. This could force the L_1 problem solver to test a large number of solutions of nearly identical objective function value before obtaining the optimal objective function value. Solving these problems with a polynomial of lesser degree improves iteration count, and in several cases required no additional pivots. From this we can infer that in the problems where a lesser degree polynomial fit the function quite well, the Chebyshev expansion converged very quickly, thereby producing a good approximation to the problem. Inferences drawn from the results such as those described above may provide the user insight into the problem concerning "goodness of fit" and "sufficiency of degree" of the approximating polynomial.

The results of this computational study indicate ways in which L_1-approximation codes used for polynomial approximation problems can be improved. They are collected below.

1. Use the zeroes of the Chebyshev polynomials as the observations in the starting basis.

2. Convert the L_1 polynomial coefficients into the coefficients of the equivalent Chebyshev polynomial representation to provide the user with information on the rate of convergence of the Chebyshev terms and the correctness of the user's degree specification.

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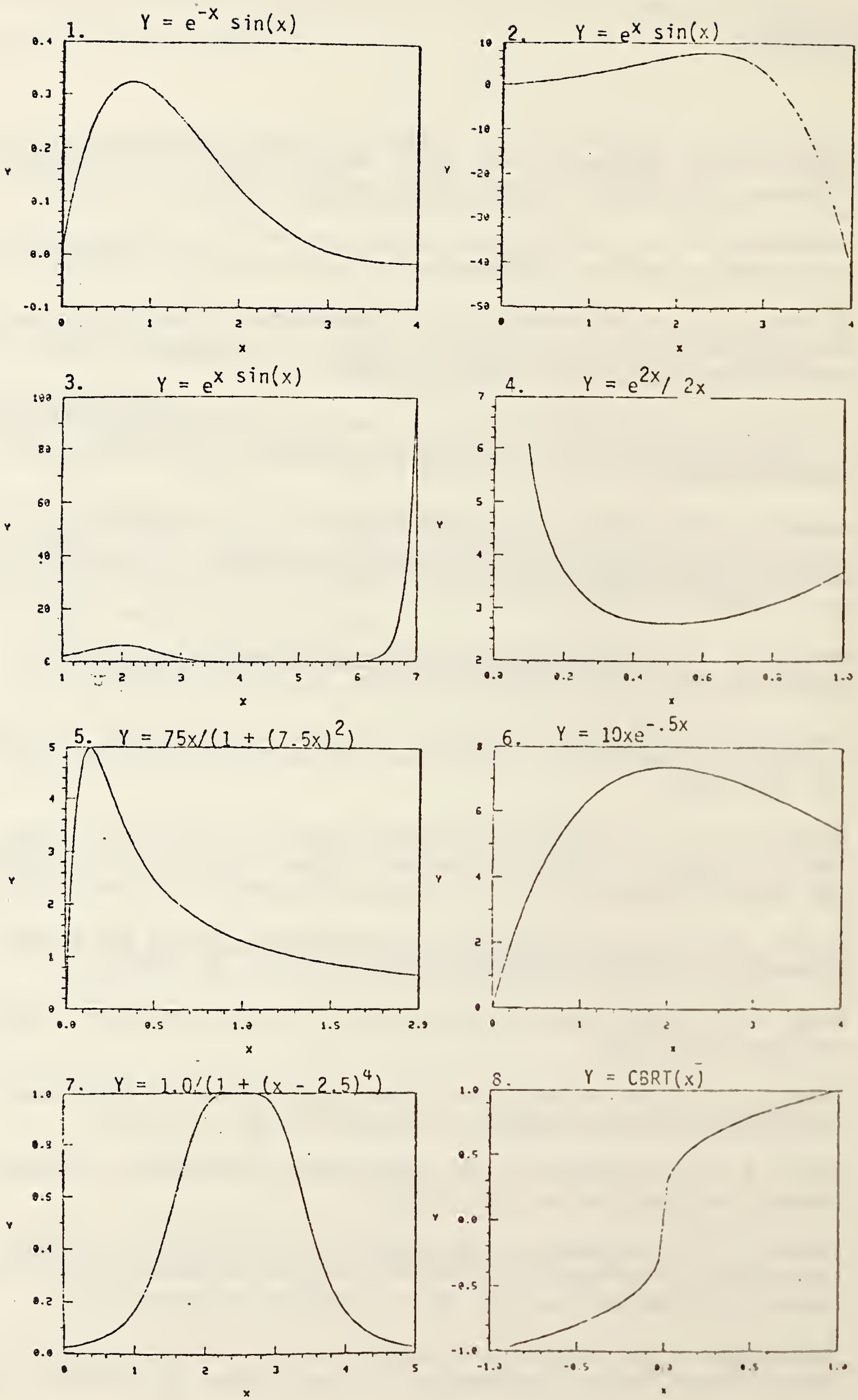

Appendix Table 1a

D_i	Z_i^s	Z_i^*	w/Starting Basis	w/o Starting Basis
.38473593-07	.77892444-05	.82891135-05	21	28
.24290186-07	.30697751-04	.31013893-04	16	25
.30442865-06	.72850767-03	.73247530-03	14	19
.00000000	.50022952-02	.50022952-02	8	24
.00000000	.16919402-01	.16919402-01	7	18
.19496985-03	.29877542-01	.32425934-01	16	20
.34377840-04	.51255188+00	.51298533+00	7	12
.61263081-04	.22661790+01	.22668457+01	6	10
.46469465-08	.83252204-03	.83575393-03	15	28
.35962814-08	.37304331-02	.37329244-02	16	26
.26644684-06	.80368631-02	.82202058-02	23	20
.00000000	.23934237+00	.23934237+00	8	24
.00000000	.16633715+01	.16633715+01	7	18
.00000000	.52384065+01	.52384065+01	6	14
.21836045-02	.62510170+01	.77182238+01	14	15
.43829752-04	.89270914+02	.89296236+02	6	10
.27054200-03	.32746095+02	.32872835+02	14	28
.56691379-02	.41832502+02	.44398650+02	27	42
.16575215-02	.77596401+02	.78276267+02	14	19
.00000000	.13421441+03	.13421441+03	8	24
.46974042-01	.15037904+03	.16559020+03	23	20
.29302081-02	.21768489+03	.21841852+03	19	24
.10249503-01	.30288454+03	.30452431+03	11	13
.23603782+00	.31174017+03	.34658324+03	15	14
.16690546-04	.12055663+01	.12115803+01	15	24
.47125368-05	.18255402+01	.18272093+01	16	26
.63001109-04	.27663362+01	.27885449+01	13	20
.00000000	.42724219+01	.42724219+01	8	24
.00000000	.65400433+01	.65400433+01	8	17
.00000000	.10006316+02	.10006316+02	6	14
.40739098-04	.15307436+02	.15321172+02	8	11
.17944713-04	.23685636+02	.23691525+02	6	10
.15599856-02	.16947877+01	.19729329+01	20	25
.16769122-03	.28028235+01	.28325441+01	15	26
.26872698-03	.58560272+01	.59028487+01	14	19
.00000000	.10005554+02	.10005554+02	8	24
.00000000	.14561430+02	.14561430+02	7	18
.00000000	.18221696+02	.18221696+02	6	14
.66197410-02	.19271834+02	.20338684+02	17	19
.31524878-01	.20571445+02	.25601976+02	10	13
.37064981-08	.52429049-05	.74095537-05	31	23
.38463739-08	.59751246-05	.82265572-05	23	25
.63290565-09	.33727479-04	.34098439-04	15	21
.00000000	.52173636-03	.52173636-03	8	24
.00000000	.71661680-02	.71661680-02	7	18
.00000000	.83928718-01	.83928718-01	6	14
.12098219-05	.81092013+00	.81163161+00	7	12
.30978829-05	.61682657+01	.61706745+01	6	10
.16690499-04	.10908414+01	.10962107+01	16	23
.47126418-05	.16518173+01	.16533276+01	16	26
.63001231-04	.25030841+01	.25231795+01	14	19
.00000000	.38658469+01	.38658469+01	8	24
.00000000	.59176758+01	.59176758+01	7	18
.00000000	.90540892+01	.90540892+01	6	14
.40739214-04	.13850741+02	.13863170+02	8	11
.17944476-04	.21431650+02	.21436979+02	6	10
.11657442-03	.26188930-05	.36036563-05	28	28
.20359740-09	.52117261-05	.58908322-05	23	23
.49581164-09	.49071724-04	.49490803-04	15	17
.00000000	.22282864-03	.22688364-03	8	24
.00000000	.33706960-02	.35706960-02	7	18
.34145425-06	.67682363-02	.70570059-02	15	19
.23825944-06	.23514312+00	.23534460+00	8	11

Appendix Table 2a

D_i	Z_i^s	Z_i^*	Iteration Count w/Starting Basis	Iteration Count w/o Starting Basis
.63731453-04	.27161526+00	.32551653+00	11	15

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D_i	Z_i^s	Z_i^*	Iteration Count w/Starting Basis	Iteration Count w/o Starting Basis
.21004880-07	.16247862-04	.16799285-04	24	35
.42279109-08	.61684384-04	.61795471-04	14	24
.17598517-07	.14533588-02	.14538215-02	14	18
.35726395-07	.10507884-01	.10508823-01	10	22
.89239756-06	.33444651-01	.33468128-01	11	20
.19671199-03	.59182688-01	.64356405-01	18	17
.00000000	.10254544+01	.10254544+01	5	12
.00000000	.45302921+01	.45302921+01	4	11
.30399493-08	.16522184-02	.16563432-02	15	24
.35505721-09	.73658535-02	.73663344-02	14	25
.20685818-06	.16098261-01	.16377379-01	20	25
.31631776-07	.47535440+00	.47539703+00	10	22
.16496500-05	.32998579+01	.33020734+01	11	20
.26332218-07	.10395911+02	.10395946+02	8	15
.22628885-02	.12173192+02	.15176597+02	14	16
.00000000	.17912035+03	.17912035+03	4	11
.20555875-03	.63012531+02	.63188113+02	15	24
.65983953-02	.78178966+02	.83670844+02	33	40
.64168452-04	.14684077+03	.14688935+03	14	18
.29622031-04	.25518024+03	.25519925+03	10	22
.47283814-01	.28130349+03	.31009830+03	25	33
.35493094-02	.40575267+03	.40745069+03	22	29
.76980047-02	.57451566+03	.57685961+03	12	16
.24363090+00	.58351554+03	.65465576+03	16	13
.12050066-04	.21347636+01	.21432264+01	17	24
.26423726-07	.33070339+01	.33070524+01	13	25
.18237533-05	.51165475+01	.51178202+01	13	19
.78270031-06	.79030838+01	.79036273+01	10	22
.14893419-04	.12206046+02	.12216316+02	12	19
.42302934-07	.18898755+02	.18898784+02	8	15
.00000000	.29206237+02	.29206237+02	6	11
.00000000	.45617926+02	.45617926+02	4	11
.16184729-02	.31287643+01	.37075731+01	25	33
.79636709-04	.59232876+01	.59515485+01	13	27
.13677088-04	.11825580+02	.11830353+02	14	18
.54511249-05	.19515598+02	.19517459+02	10	22
.90881341-04	.27905852+02	.27936128+02	11	20
.67157074-07	.34563443+02	.34563465+02	8	15
.67798080-02	.35959146+02	.38164130+02	26	16
.32060690-01	.38351684+02	.48692715+02	13	16
.33097175-08	.13128588-04	.17736472-04	31	34
.47033935-08	.12294069-04	.17841040-04	32	33
.12878231-08	.66770196-04	.68290005-04	18	20
.26604323-09	.10340947-02	.10344089-02	12	22
.78893543-08	.14219366-01	.14228689-01	11	20
.53879191-09	.16706186+00	.16706249+00	8	15
.00000000	.16179027+01	.16179027+01	5	12
.00000000	.12329318+02	.12329318+02	4	11
.12058265-04	.19316136+01	.19332712+01	17	24
.26475262-07	.29923276+01	.29923444+01	13	25
.18236596-05	.45296439+01	.46387954+01	14	18
.78267105-06	.71510062+01	.71514590+01	10	22
.14398313-04	.11044487+02	.11053780+02	11	20
.42521793-07	.17100301+02	.17100327+02	8	15
.00000000	.26426836+02	.26426896+02	5	11
.00000000	.41276287+02	.41276807+02	4	11
.12125279-08	.57796269-05	.76410703-05	35	33

Appendix Table 2a (Cont'd.)

D_i	Z_i^s	Z_i^*	w/Starting Basis	w/o Starting Basis
.95660747-09	.10472558-04	.12091049-04	27	36
.11298588-09	.97007961-04	.97199140-04	15	20
.23385776-10	.45287743-03	.45091700-03	10	22
.30245494-08	.78882029-02	.78933216-02	11	20
.33519471-06	.13421031-01	.13988357-01	17	15
.00000000	.46991806+00	.46991806+00	6	11
.65629549-04	.53222314+00	.64328949+00	13	16

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D_i	Z_i^s	Z_i^*	w/Starting Basis	w/o Starting Basis
.16612774-07	.31313732-04	.32190352-04	26	39
.15912153-08	.12428100-03	.12436501-03	13	27
.36629285-08	.29042042-02	.29043976-02	11	20
.00000000	.20974218-01	.20974218-01	8	23
.00000000	.66723283-01	.66723283-01	8	18
.19708016-03	.11799288+00	.12839040+00	17	22
.78956468-06	.20529683+01	.20530084+01	8	12
.00000000	.90549833+01	.90549833+01	4	11
.32262322-09	.32973465-02	.32982110-02	16	28
.11511588-09	.14675742-01	.14676051-01	12	28
.20932624-06	.32179432-01	.32738769-01	27	28
.00000000	.94985769+00	.94985769+00	8	23
.00000000	.65940033+01	.65940033+01	7	19
.00000000	.20737085+02	.20737085+02	6	15
.23123426-02	.24047035+02	.30147300+02	17	22
.00000000	.35866493+03	.35866493+03	4	11
.14060882-05	.12473923+03	.12474152+03	14	28
.72737923-02	.15166338+03	.16325416+03	34	58
.14187570-04	.28520285+03	.28522346+03	11	20
.00000000	.49936010+03	.49936010+03	8	22
.49577814-01	.54330817+03	.60189273+03	33	31
.36843154-02	.78472106+03	.78816273+03	29	34
.81957905-02	.11199060+04	.11247727+04	13	14
.24741680+00	.11290011+04	.12728065+04	16	20
.42699385-08	.40806182+01	.40806242+01	14	28
.59051080-07	.63306454+01	.63307276+01	11	28
.39143837-06	.98341317+01	.98346753+01	12	19
.00000000	.15260824+02	.15260824+02	8	23
.00000000	.23689052+02	.23689052+02	8	18
.00000000	.36804467+02	.36804467+02	6	15
.93365117-06	.57144825+02	.57146074+02	7	13
.00000000	.89556905+02	.89556905+02	4	11
.15890469-02	.61239235+01	.72617351+01	29	44
.32432366-04	.12151671+02	.12174700+02	15	29
.29174664-05	.23698554+02	.23700592+02	11	22
.00000000	.38637351+02	.38637351+02	8	23
.00000000	.54851924+02	.54851924+02	8	18
.00000000	.67392214+02	.67392214+02	6	15
.68206483-02	.69554892+02	.74009003+02	32	18
.32224167-01	.74032214+02	.94986865+02	12	13
.41029544-08	.24256655-04	.33966580-04	42	36
.51438585-08	.26802009-04	.38979333-04	38	40
.11251222-08	.13454819-03	.13721263-03	17	20
.33688665-09	.20662418-02	.20670399-02	10	23
.00000000	.28411315-01	.28411315-01	7	19
.00000000	.33372850+00	.33372850+00	6	15
.27788277-07	.32347908+01	.32348566+01	8	12
.00000000	.24640777+02	.24640777+02	4	11
.42707056-08	.36922970+01	.36923024+01	14	28
.59019133-07	.57282041+01	.57282785+01	11	28
.39143342-06	.88982896+01	.88987814+01	11	20

Appendix Table 2a (Cont'd.)

D_i	Z_i^s	Z_i^*	Iteration w/Starting Basis	Count w/o Starting Basis
.00000000	.13808564+02	.13808564+02	8	23
.00000000	.21434741+02	.21434741+02	8	18
.00000000	.33302061+02	.33302061+02	6	15
.93342327-06	.51706775+02	.51707905+02	7	13
.00000000	.81034440+02	.81034440+02	4	11
.13504562-08	.11300624-04	.15872453-04	37	33
.87631514-09	.20014545-04	.22981357-04	25	43
.13304253-09	.19375719-03	.19420763-03	13	19
.26528405-10	.90194237-03	.90203219-03	10	23
.24066876-11	.15773307-01	.15773315-01	9	18
.33499489-06	.26752013-01	.27886359-01	19	21
.55085913-08	.94027323+00	.94029188+00	7	13
.66374334-04	.10539569+01	.12786638+01	14	17

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D_i	Z_i^s	Z_i^*	Iteration w/Starting Basis	Count w/o Starting Basis
.80046733-08	.64293980-04	.65140856-04	22	40
.18345249-08	.24815461-03	.24834874-03	16	29
.36629733-08	.58039728-02	.58043604-02	12	22
.00000000	.41925787-01	.41925787-01	8	23
.42976012-07	.13330225+00	.13330679+00	9	22
.19840216-03	.23556860+00	.25653379+00	20	25
.54559471-06	.41074362+01	.41074918+01	8	12
.00000000	.18106070+02	.18106070+02	4	11
.21255559-09	.65718409-02	.65729728-02	17	33
.55336955-10	.29315385-01	.29315679-01	12	27
.20795124-06	.64390530-01	.65496352-01	26	32
.00000000	.18995941+01	.18995941+01	8	23
.79240228-07	.13184441+02	.13184861+02	9	22
.22102331-06	.41430966+02	.41432025+02	10	19
.23349790-02	.47798595+02	.60080621+02	21	22
.00000000	.71781141+03	.71781141+03	4	11
.59461223-05	.24805682+03	.24887570+03	16	33
.76641692-02	.29893534+03	.32282294+03	52	60
.22071987-04	.56253665+03	.56259948+03	11	22
.00000000	.98860999+03	.98860999+03	8	26
.51131724-01	.10679217+04	.11869372+04	41	35
.35918012-02	.15442489+04	.15508765+04	23	31
.81619533-02	.22116443+04	.22212159+04	12	17
.24939864+00	.22210719+04	.25103070+04	24	26
.49781212-06	.79781977+01	.79795832+01	18	29
.50334836-07	.12403474+02	.12403614+02	12	27
.73321779-06	.19297719+02	.19299751+02	12	21
.00000000	.30017495+02	.30017495+02	8	23
.75345121-06	.46694571+02	.46696637+02	10	21
.79633189-06	.72671841+02	.72674003+02	10	19
.64270557-06	.11304775+03	.11304947+03	7	13
.00000000	.17749944+03	.17749944+03	4	11
.15972459-02	.12130429+02	.14418929+02	30	52
.11299182-04	.24589022+02	.24605071+02	18	30
.31935591-05	.47439826+02	.47444269+02	11	23
.00000000	.76916878+02	.76916878+02	8	23
.46469532-05	.10877446+03	.10878067+03	9	22
.46161086-05	.13311416+03	.13311943+03	10	19
.62623490-02	.13673110+03	.14571201+03	27	29
.32464467-01	.14544337+03	.18763437+03	15	15
.49282159-08	.48391491-04	.71763759-04	48	46
.52833163-08	.50820778-04	.77802320-04	33	39
.51699431-09	.27059446-03	.27434478-03	20	26
.12656333-09	.41254558-02	.41263403-02	12	23
.32665635-09	.56810172-01	.56811722-01	9	22

Appendix Table 2a (Cont'd.)

D_i	Z_i^s	Z_i^*	Iteration Count w/Starting Basis	Iteration Count w/o Starting Basis
.32665635-09	.56810172-01	.56811722-01	9	22
.34004779-08	.66722302+00	.66723916+00	10	19
.19198617-07	.64677122+01	.64678032+01	8	12
.00000000	.49268394+02	.49268394+02	4	11
.49774244-06	.72189723+01	.72202258+01	18	29
.50269660-07	.11223128+02	.11223255+02	12	27
.73329243-06	.17461298+02	.17463136+02	11	22
.00000000	.27160951+02	.27160951+02	8	23
.75330588-06	.42250996+02	.42252865+02	9	22

Appendix Table 2a (Cont'd.)

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16. ABSTRACT *(A 200-word or less factual summary of most significant information. If document includes a significant bibliography or literature survey, mention it here.)*

This paper presents a method of selecting a near-optimal starting basis for a large class of polynomial approximation problems in the L_1 norm. While it is possible to prove the optimality of these advanced starting solutions for only a small class of problems, empirical evidence indicates the starting bases are nearly optimal for a much larger class of problems. This paper presents the method used to determine the starting basis and a heuristic justification backed by empirical results supporting its use.

17. KEY WORDS *(six to twelve entries; alphabetical order; capitalize only the first letter of the first key word unless a proper name; separated by semicolons)*

Approximation; Chebyshev polynomials; computational experiment; least absolute deviation; near-optimal starting basis; polynomial approximation

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