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The Modulation Transfer Function for Two-Point and Periodic Objects Using Gaussian and Lorentzian Resolution Functions

U.S. DEPARTMENT OF COMMERCE
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THE MODULATION TRANSFER FUNCTION FOR TWO-POINT AND PERIODIC OBJECTS USING GAUSSIAN AND LORENTZIAN RESOLUTION FUNCTIONS

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U.S. DEPARTMENT OF COMMERCE, Malcolm Baldrige, *Secretary*
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The Modulation Transfer Function for Two-Point and Periodic Objects Using Gaussian and Lorentzian Resolution Functions

by

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This paper presents an analytical study of the effect of Gaussian- and Lorentzian-shaped line spread functions in non-coherent noise-free imaging systems. A mathematic development is given for the calculation of the Modulation Transfer Function (MTF). This technique is used to calculate the MTF for two-point and periodic objects using Gaussian and Lorentzian resolution functions. Figures and graphs are used to illustrate the comparison of the results. Relationships between the results obtained are developed that are useful in the interpretation of experiments used to determine the resolution of experimental systems. The development covers only noise-free, incoherent, one-dimensional systems.

Introduction

In the development of optical or electronic imaging systems some technique is usually employed to determine the resolution of the system--that is, its ability to present detail in the image. The resolving power of optical instruments is frequently given in terms of the ability to resolve two point sources. Rayleigh's criterion is probably the most widely known example of this approach. A technique that gives a much greater amount of information is the determination of the modulation transfer function of a system.[1-6] The modulation transfer function shows how well an imaging system will handle sinusoidal images as a function of the frequency of the sinusoidal image.

The work presented in this paper was done in connection with the development of an imaging system used in neutron radiography employing a microchannel plate electron multiplier. The resolution of the system was found to be dependent on a number of variables in the optical and electronic components of the system.[7] In this paper the mathematical development of the modulation transfer functions are given with comparisons to results for two-point image resolutions for mathematical models that closely approximate the experimental systems. The material developed in this paper was found to be useful in the development of the neutron radiography system and should be of value in other similar systems that are linear and stationary, having response functions that are real and symmetric.

The resolution of many systems can be given as a response function of the system. The image produced by the system is then obtained by a convolution of the object and response functions. The Lorentz distribution and the Gaussian distribution are analytic distributions that are similar in shape to response functions that are encountered in many systems. Many of the response functions one encounters have characteristics that lie between the Lorentzian and Gaussian distributions so that a study of the characteristics of these two functions is frequently helpful in predicting the characteristics of these systems.

To simplify the discussion and mathematics this paper will consider only one dimensional non-coherent noise-free imaging systems.

Tests of two-dimensional imaging systems are frequently done with patterns of lines that effectively reduce the system to one dimension. The resolution of a system is frequently described in terms of the system response to a single line. This single line--or impulse response is called a Line Spread Function. The two-dimensional analog is termed a Point Spread Function. If the impulse response is Gaussian in shape, the Line Spread Function and Point Spread Function are both Gaussian with the same width. This similarity is not true for resolution functions that are Lorentzian in shape.

The systems are assumed to be noncoherent in order to avoid the complexities of phase and interference effects. These effects can be ignored if the width of the line spread function is much greater than wavelength of the radiation that carries the image. Noise has also been ignored in these calculations but must, of course, be considered in cases where the signal to noise ratio is poor.

Two types of periodic objects and images are considered: square and sine waves. The square wave is used because it most nearly represents the sharp change in contrast of black and white objects frequently used as test patterns. Sine waves are considered because from them any other type of image contrast change can be generated by Fourier analysis.

In this paper the properties of systems having Gaussian- and Lorentzian-shaped Line Spread Functions when imaging objects with periodic sinusoidal and square wave intensity distributions are given. These results are compared with the imaging results obtained for objects having only two cycles. Experimentalists making resolution measurements should find many of the relationships developed useful in the interpretation of results.

Description

The Modulation Transfer Function (MTF) describes the response of an optical system in producing images of different spatial frequency. It is analogous to the frequency band pass curve used to describe the response of an electrical circuit to electrical signals. In analogy to the gain of an electrical circuit the MTF shows the "visibility" of an image as a function of spatial frequency. Visibility (V) is defined in terms of the relative intensity in the image:

$$V = (H_+ - H_-) / (H_+ + H_-)$$

where H_+ is the maximum intensity and H_- is the minimum intensity of the image. The MTF is the response of system to a periodic object of sinusoidal intensity variation. Because of the experimental interest the response to periodic objects having square wave intensity variation will also be developed. The spatial frequency is usually given in cycles per mm. The term "line pairs per mm" is equivalent to cycles per mm.

The image produced by a system for a delta function object is termed the Line Spread Function (LSF) of the system. The symbol G will be used to denote the LSF distribution. The LSF is defined to have unit area:

$$\int_{-\infty}^{+\infty} G(x)\, dx = 1 \;.$$

The image H, of any other object is the convolution of the line spread function, G and the object intensity distribution, I:

$$H(x) = G(x) \otimes I(x) = \int_{-\infty}^{+\infty} G(y)\, I\,(x-y)\, dy$$

Since it is usually difficult to get a delta function object of sufficient intensity to produce an experimentally useful image the LSF of an optical system is frequently determined by using a step function object ("knife edge"). The image of the "knife-edge" object will be the cumulative integral of the LSF, Q :

$$Q(x) = \int_{-\infty}^{x} G(y)\, dy$$

The derivative of Q is then the LSF, G.

$$\frac{\partial Q(x)}{\partial x} = G(x)$$

The MTF and the LSF are complementary ways of describing the response of a system. Both may be determined by experimental methods but it is easiest to measure the LSF. The MTF may then be obtained from the LSF by calculation.

Calculation of MTF for Sinusoidal Objects

For an object having a sinusoidal intensity variation of one frequency the intensity may be written: $I(x) = 0.5\,(1 + M_i \sin x)$

The image, H produced by an optical system will then have an intensity:

$$H(x) = 0.5\,(1 + M_h \sin x)$$

and the "visibility", V of H is given by:

$$V = \frac{H(\pi/2) - H(3\pi/2)}{H(\pi/2) + H(3\pi/2)} = \frac{M_h}{M_i}$$

note that:

$$H(x) = \int_{-\infty}^{+\infty} G(y)\{0.5 + 0.5 \sin(x-y)\}\, dy$$

where

$$H(\pi/2) - H(3\pi/2) = 0.5 \int_{-\infty}^{+\infty} G(y)\{\sin(\pi/2 - y) - \sin(3\pi/2 - y)\}\, dy$$

and

$$\sin(n\pi/2 - y) = \sin n\pi/2 \cos y - \cos n\pi/2 \sin y$$

Using the above relationships it can be shown that the visibility, V is the Fourier transform of the LSF:

$$V(p) = \int_{-\infty}^{+\infty} G(y) \cos py\, dy \equiv \mathcal{F}\{G(x)\} \equiv \mathcal{G}(p)$$

The most commonly encountered LSF forms are the Gaussian and the Lorentzian. The Fourier transform of the Gaussian is a Gaussian. The Fourier transform of a Lorentzian is an exponential. In Appendix A a number of properties of these functions and their transforms are given.

let the Fourier transform of I(x), the object distribution be $\mathcal{F}\{I(x)\} = \mathcal{I}(p)$

and the Fourier transform of H(x), the image distribution be $\mathcal{F}\{H(x)\} = \mathcal{H}(p)$

Then the MTF is defined to be:

$$MTF(p) = \mathcal{H}(p)/\mathcal{I}(p)$$

but $\mathcal{H}(p) = \mathcal{F}\{H(x)\} = \mathcal{F}\{I(x) \otimes G(x)\} = \mathcal{F}\{I(x)\}\mathcal{F}\{G(x)\} = \mathcal{I}(p)\mathcal{G}(p)$

thus

$$MTF(p) = \mathcal{G}(p)$$

and for objects with sinusoidal distribution the image visibility is the MTF.

$$V(p) = MTF(p)$$

Calculation of Visibility for Square Wave Objects

Strictly speaking the visibility is equivalent to the MTF only for sinusoidal objects. The frequency response of a system to a square wave input is frequently desired however, because most test object distributions are square wave in nature, i.e., alternate bands of black and white.

The square wave distribution can be expanded in terms of a Fourier series of sinusoidal components. Consider the square wave given by the following conditions:

$$I(x) = 1.; \ -\pi/2 \leq x < \pi/2$$
$$I(x) = 0; \pi/2 < x \leq 3\pi/2$$
$$I(x+2\pi m) = I(x), \ m = 1, 2, 3.....$$

The cosine expansion for this square wave is given by:

$$I(x) = 0.5\left(1 + \sum_{m=1}^{\infty} b_m \cos mx\right) \quad m = 1, 3, 5, 7, \ldots$$

where

$$b_m = 4(-1)^{\frac{m-1}{2}} / (m\pi)$$

note that for $I(o) = 1. = 0.5\ [1 + \frac{4}{\pi}(1 - 1/3 + 1/5 - 1/7 + 1/9 + ...)]$ yields the known series: $\pi = 4(1 - 1/3 + 1/5 - 1/7 + 1/9 + ...)$.

Note that $$1 = \sum_{m=1}^{\infty} b_m \quad m = 1, 3, 5, 7, \ldots$$

The image H(x) is the convolution of the LSF and I(x)

$$H(x) = G(x) \otimes I(x).$$

Expand H(x) into its sinusoidal components:

$$H(x) = \sum_{m=0}^{\infty} H_m(x) \quad m = 0, 1, 3, 5, 7, \ldots$$

let $H_o = 0.5$, $H_m = 0.5\ b_m\ (G(x) \otimes \cos mx)$

but $G(x) \otimes \cos mx = G(mx) \otimes \cos x$

thus

$$H_m(x) = \int_{-\infty}^{+\infty} G(mZ) \cos (x-Z)\, dZ$$

and $\cos(x-Z) = \cos x \cos Z + \sin x \sin Z.$

max: $$H_m(0) = \int_{-\infty}^{+\infty} G(mz) \cos z\, dz = \mathcal{F}\{G(mz)\} = \mathcal{G}(mp)$$

min: $$H_m(\pi) = \int_{-\infty}^{+\infty} G(mz) \cos z\, dz = -\mathcal{F}\{G(mz)\} = -\mathcal{G}(mp)$$

thus $$V(p) = \frac{H(0)-H(\pi)}{H(0)+H(\pi)} = \sum_{\substack{m=1 \\ m \text{ odd}}}^{\infty} b_m \mathcal{G}(mp)$$

Square Wave Visibility for Lorentzian LSF

The Lorentzian LSF is given in Appendix A as $G_\ell(x) = a\pi/(a^2+x^2)$. The Fourier transform is: $\mathcal{F}\{G_\ell(x)\} = \mathcal{L}(y) = \exp(-2\pi a y)$

The visibility of a square wave object is then

$$V(y) = \sum_{m=1}^{\infty} b_m \mathcal{L}(my) = \sum_{m=1}^{\infty} b_m [\exp(-2\pi a m y)], \quad m = 1, 3, 5, \cdots$$

or $$V(y) = \sum_{m=1}^{\infty} b_m [\exp(-2\pi a y)]^m$$

The values of V(p) are calculated in Appendix B. The results are plotted in Fig. 1 and shown in Table I. The abscissa p is:

$$p = ya = y \cdot \frac{\Gamma}{2}$$

where y is the number of cycles per unit length (usually given as line pairs per mm) and Γ is the full-width half-maximum of the LSF: $\Gamma = 2a$.

Thus to find the visibility for the value of line pairs per mm that is the reciprocal of the full-width half maximum of the LSF we let $p = ya = \frac{1}{\Gamma} \cdot \frac{\Gamma}{2} = 0.5$ and get a value of $V(0.5) = 0.055$.

This corresponds to a density ratio of

$$D = -\log_{10}(I_f / I_i) = -\log_{10}(0.055) = 1.26.$$

For comparison purposes Fig. 1 also shows the visibility of a sinusoidal object.

Square Wave Visibility for Gaussian LSF

The Gaussian LSF is given in Appendix A as

$$G_g(x) = B\exp(-u^2).$$

The Fourier transform is

$$\mathcal{F}\{G_g(x)\} = \mathcal{G}(p) = \exp(-p^2)$$ and

the visibility of a square wave object is then:

$$V(p) = \sum_{m=1}^{\infty} b_m \mathcal{G}(mp) = \sum_{m=1}^{\infty} b_m \{\exp[-(mp)^2]\} \; ; \; m = 1, 3, 5, 7, \ldots$$

The values of V(p) are calculated in Appendix C, plotted in Fig. 2 and shown in Table I.

The abscissa p is:

$$p = \Gamma y/c$$

where y is the number of cycles per unit length, Γ is the full width half-maximum of the Gaussian LSF and the constant $c = 2\sqrt{|\ln(.5)|} = 1.67$.

To find the visibility for the value of y that is the reciprocal of Γ we let

$$p = \frac{\Gamma}{c} \cdot \frac{1}{\Gamma} = 0.6$$

and get a value of $V(0.6) = 0.036$.

This corresponds to a density ratio of 1.44.

Comparison of Sinusoidal and Square Wave Visibility

Because of the different width parameters used in the normal representation of the Lorentzian and Gaussian LSF it is not easy to compare their relative visibilities as shown in Figs. 1 and 2. A common basis of comparison is the full-width half-maximum of the LSF distributions, Γ. Figure 3 shows the visibility for both distributions plotted as a function of p where

$$p = \Gamma y$$

and y is the number of cycles per unit length. Note that the visibility for the Gaussian LSF is greater than that for the Lorentzian distribution for values of p less than 0.9. For values above that the visibility for the Gaussian distribution drops rapidly in comparison to the Lorentzian.

The ratio of visibility of the sine wave image to the visibility of the square wave image is shown for the two LSF in Fig. 4. Note that for $p = \Gamma y$ greater than about 0.6 the ratio for both approaches $\pi/4$ because by then only the first term in the square wave expansion is contributing significantly.

Visibility of Two-Cycle Objects

Up to this point the image visibility has been considered for periodic objects having a large number of cycles. In many cases it is interesting to know what the resolution of the system is for discerning the existence of two objects.

The formalism used up to this point does not easily allow for the calculation of the two cycle visibility. Numerical calculations of the convolution of different LSF and object distributions were performed. The cases of Gaussian and Lorentzian LSF and sine and square wave object distributions of several cycles were calculated and were found to agree with calculations based on the analytic expressions derived in the previous sections. The computer program was then modified to do the case of the two cycle object distributions. The image amplitude distributions were obtained for different values of the width of the LSF. Figure 5 shows the image amplitude distributions obtained for a convolution of the Lorentzian LSF and a two cycle square wave object. The visibility, V, is defined in the same manner as before:

$$V = (H_+ - H_-)/(H_+ + H_-)$$

with H_- being measured at the midpoint of the distribution. When the curvature at the midpoint becomes zero, V is taken to be zero. The visibility is not defined for cases of negative curvature at the midpoint. The visibility as determined from Fig. 5 is shown as a function of cycles per unit length y as a dashed line labeled L_2 in Fig. 3. Note that for values of $p = y\Gamma$ less than 1 there is little difference between the results for a multi-cycle object and a two cycle object. The difference becomes quite large however as p approaches 1.5, where $V = 0$ for the two cycle case.

Figure 6 shows the image amplitude distributions obtained for a convolution of the Gaussian LSF and a two cycle square wave. The visibility as determined from Fig. 6 is shown as a dashed line labeled G_2 in Fig. 3. In this case the visibility goes to zero for $p = 1.03$.

Most test objects have a pattern with a square wave intensity distribution. If the intensity distribution has a rectangular wave shape--i.e., other than a 50% duty cycle, the visibility of the image will be greater and the value of y for which the visibility goes to zero will be slightly altered.

Figure 7 shows the image amplitude distribution obtained for a convolution of the Gaussian LSF and a two cycle delta function. The visibility is about 40% greater than for the comparable square wave and the visibility goes to zero at $p = 1.1$.

The square wave and the delta function are the most extreme cases of the rectangular wave shape so that the results for other values of the duty cycle will lie between these cases.

FIGURES

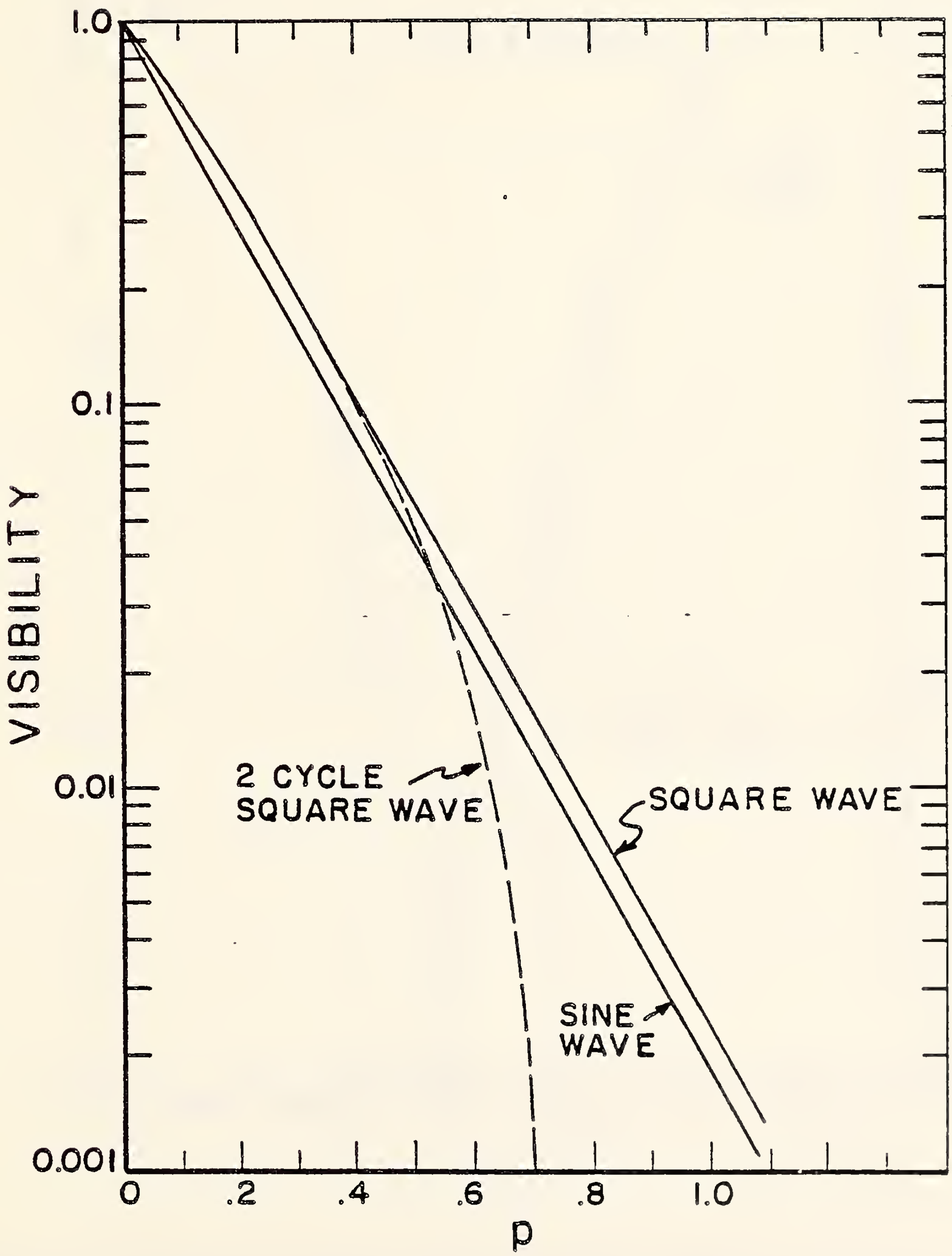

Figure 1. Visibility of image formed by convolution with Lorentzian line spread fuction. Results are shown for sinusoidal and square wave objects. The abscissa is $p = \Gamma y/2$.

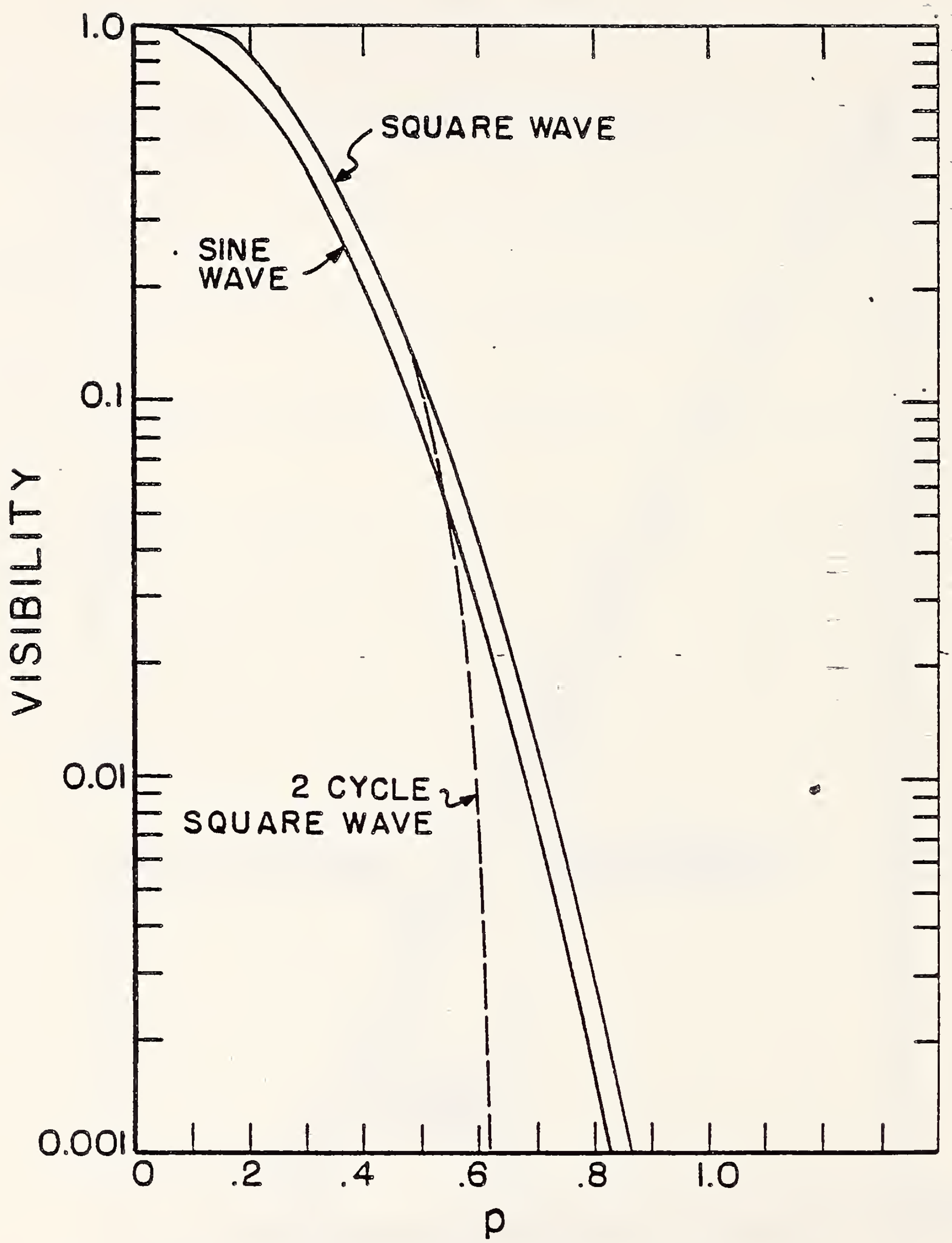

Figure 2. Visibility of image formed by convolution with Gaussian line spread function. Results are shown for sinusoidal and square wave objects. The abscissa is $p = \Gamma y / 2\sqrt{|\ln(0.5)|}$.

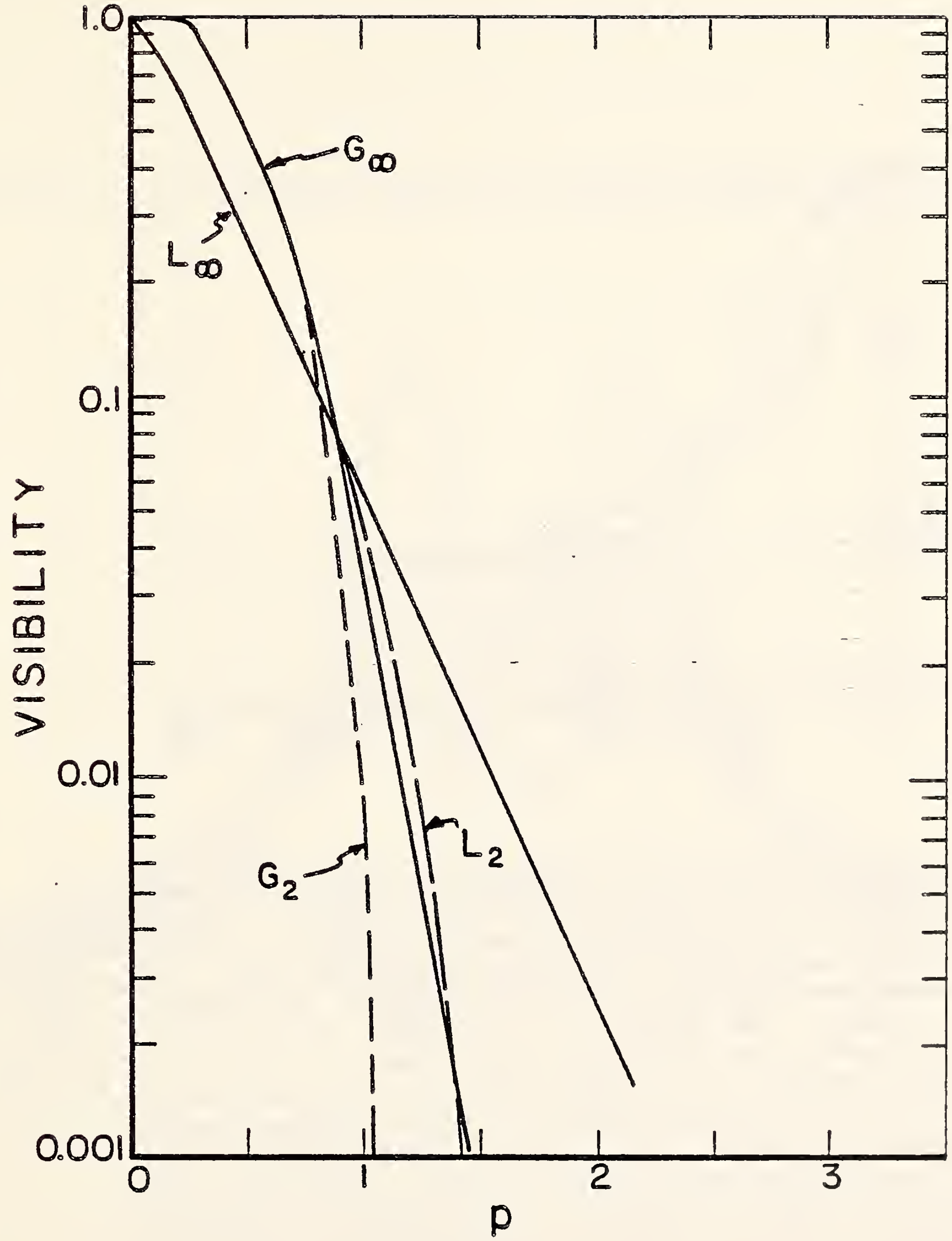

Figure 3. Comparison of visibility of image of square wave object for Gaussian and Lorentzian line spread functions. The curves labeled G_∞ and L_∞ show the image visibility for a square wave object with an infinite number of cycles for Gaussian and Lorentzian line spread functions. The dashed curves labeled G_2 and L_2 show the image visibility for objects having only two cycles. The abscissa $p = \Gamma y$.

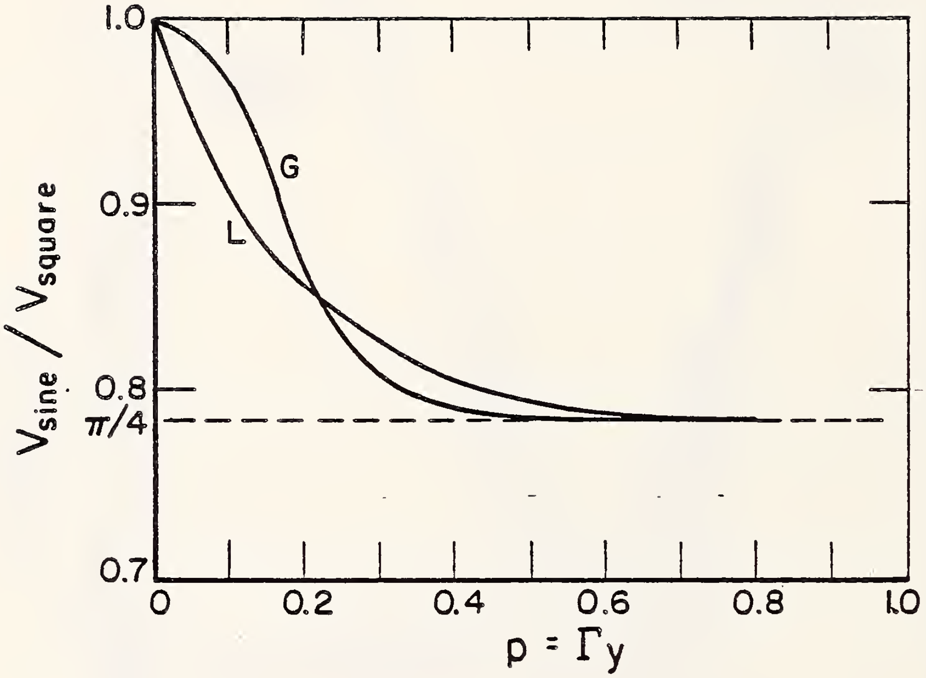

Figure 4. Ratio of visibility of sinusoidal and square wave objects for Gaussian and Lorentzian Line Spread Functions. The ordinate is the ratio of visibility of the sine wave object to the visibility of the square wave object. The curves labeled G and L refer respectively to the ratio obtained for Gaussian and Lorentzian line spread functions. The abscissa $p = y\Gamma$ where Γ is the full-width half-maximum of the line spread function and y is the line pairs per unit length. It is interesting to note that the shape of the ratio curves replicates the shape of the line spread functions.

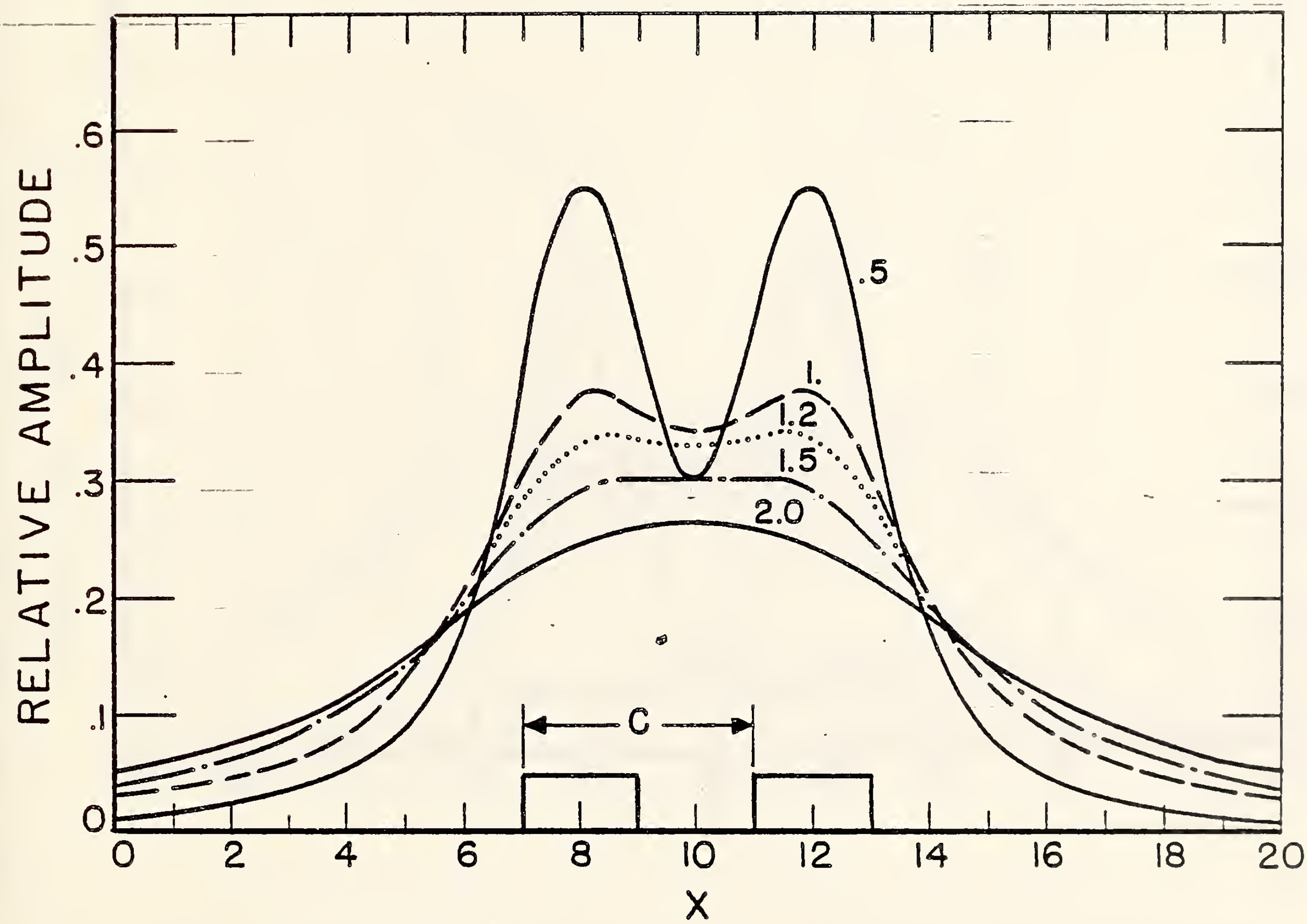

Figure 5. Convolution of Lorentzian line spread function and two cycles of a square wave object. The wavelength of the square wave is $c = 1/y$. Amplitude of the convolution is shown as a function of position for five different values of $p = \Gamma y$. Note that the curvature of the convolution at the midpoint of the two square waves goes to zero for $p = 1.5$. The visibility as a function of p obtained from these curves is shown as the dashed line in Fig. 1.

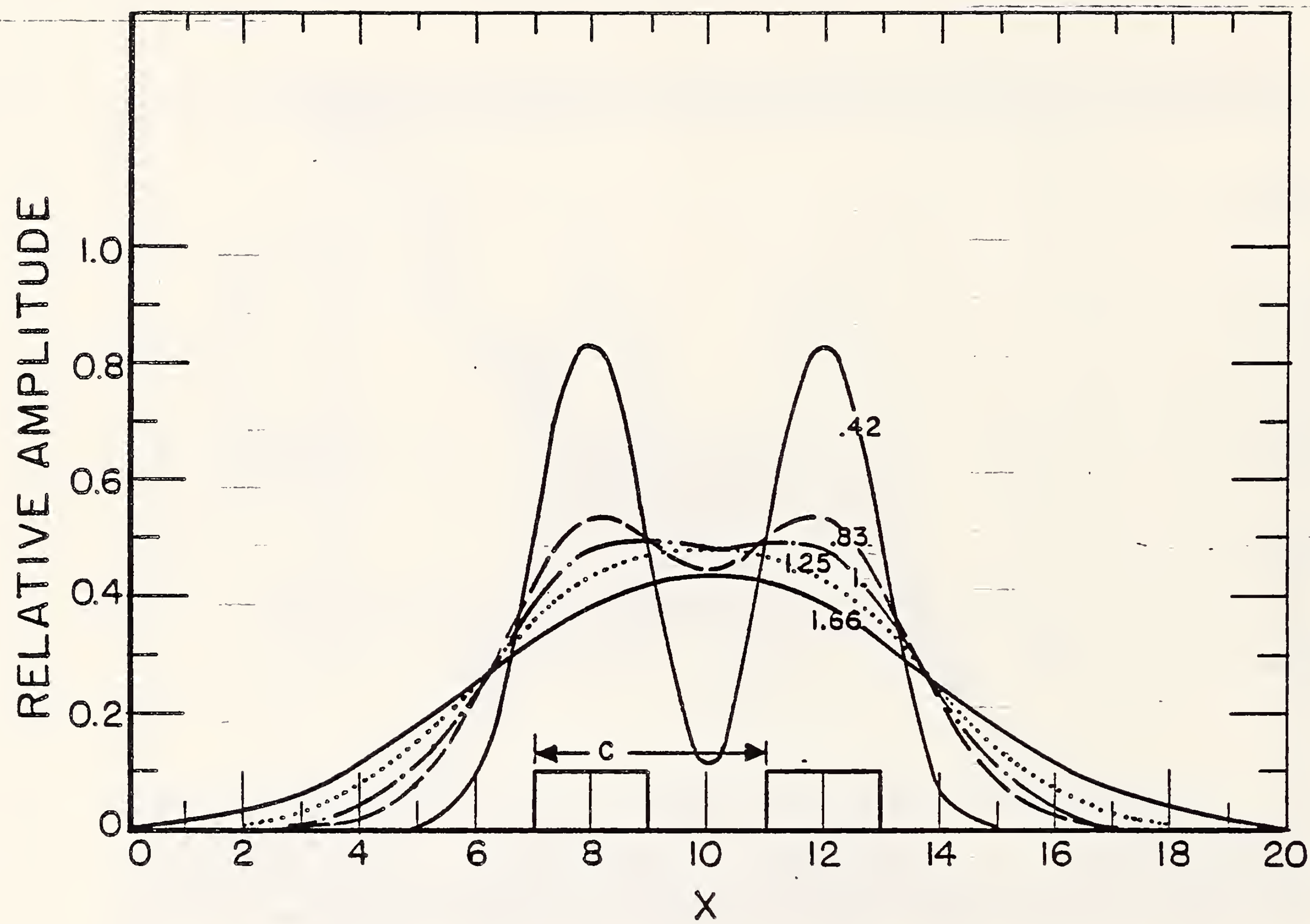

Figure 6. Convolution of Gaussian line spread function and two cycles of a square wave object. The wavelength of the square wave is $c = 1/y$. Amplitude of the convolution is shown as a function of position for five different values of $p = \Gamma y$. Note that the curvature of the convolution at the midpoint of the two square waves is closest to zero for $p = 1$. Analytic extrapolation indicates the curvature is zero at $p = 1.03$. The visibility as a function of p obtained from these curves is shown as the dashed line in Fig. 2.

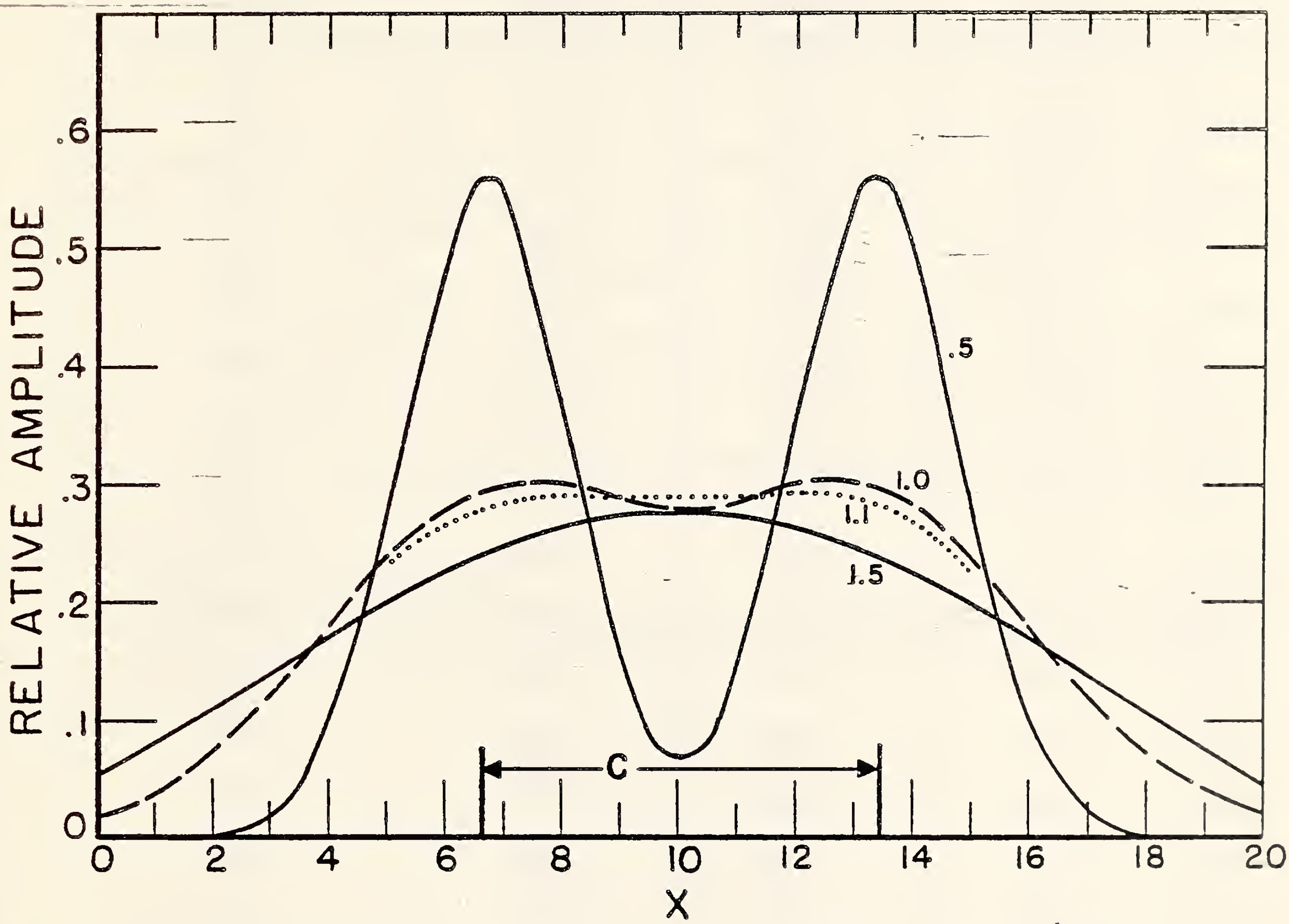

Figure 7. Convolution of Gaussian line spread function and two cycle delta function object. The amplitude of the convolution is shown as a function of position for four different values of $p = \Gamma y = \Gamma/c$ where c is the spacing between the delta functions. Note that the visibility goes to zero at $p = 1.1$.

TABLE I. Visibility of Images Derived From Convolution of LSF With Sinusoidal and Square Wave Objects*

LSF	Gaussian		Lorentzian	
P	V(sine)	V(square)	V(sine)	V(square)
0	1.00	1.00	1.00	1.00
0.05	0.98	0.99	0.73	0.81
0.10	0.91	0.99	0.53	0.62
0.15	0.80	0.96	0.39	0.47
0.20	0.67	0.85	0.28	0.35
0.25	0.54	0.69	0.21	0.26
0.30	0.41	0.52	0.15	0.19
0.35	0.30	0.38	0.11	0.14
0.40	0.21	0.26	0.081	0.10
0.45	0.14	0.17	0.059	0.075
0.50	0.085	0.11	0.043	0.054
0.55	0.051	0.064	0.032	0.040
0.60	0.029	0.036	0.023	0.029
0.65	0.015	0.020	0.017	0.021
0.70	0.008	0.010	0.012	0.016
0.75	0.004	0.005	0.009	0.011

* p = Γy/2 for the Lorentzian LSF and
p = Γy/1.67 for the Gaussian LSF.

APPENDIX A

INTRODUCTION

The line shape functions most commonly encountered are the Gaussian and Lorentzian. The Gaussian shape is the most frequently encountered and is closely associated with counting type experimental data and other situations where Poisson statistics obtain. It is also the result when many different distributions are convoluted together as might occur in an experiment where the total response function of the system is a result of a sequence of response functions affecting the observed result. The second function considered is the Lorentzian distribution, which is frequently the response function associated with experiments in which scattering plays a role in producing the observed result.

The Gaussian Distribution

Figure A shows the shape of the Gaussian distribution $G(x)$ and its integral $Q(x)$, where

$$Q(x) = \int_{-\infty}^{x} G(y)\,dy$$

$$\frac{\partial Q(x)}{\partial x} = G(x)$$

The form of the response function of a system may be determined by obtaining the image of a "knife edge" object. The "knife edge" is mathematically represented by 0 for $x < 0$ and 1. for $x \geq 0$. The image is the distribution $Q(x)$. The derivative of this distribution is then the line shape function $G(x)$. If the functional form of the line spread function is known to be Gaussian, the full-width half-maximum Γ characterizing the Gaussian may be obtained directly from the observed image Q without differentiation. Figure A shows that Γ is the distance between the 0.12 and 0.88 amplitude points on the image Q. It should be noted that the distance between the 0.25 and 0.75 amplitude points is $\Gamma/1.74$ for the Gaussian distribution.

The following table shows the relationship between a number of different mathematical forms of G(x) that are commonly encountered and their fourier transforms $\mathcal{G}(p)$. G(x) is normalized to unit area. The full-width half-maximum of G(x) is Γ.

$$G(x) = B\exp(-u^2), \mathcal{G}(p) = \exp(-p^2).$$

the Fourier transform of G(x) is $\mathcal{F}\{G(x)\} = \mathcal{G}(p)$

where $\mathcal{F}\{G(x)\} = \int e^{ixp}\, G(x)\, dx.$

$$\text{let } C = 2\sqrt{|\ln(.5)|} = 1.67$$

width parameter	full-width at half-maximum	standard deviation	B	u	p
σ	$\sqrt{2}\, C\sigma$	σ	$1/(\sqrt{2\pi}\,\sigma)$	$x/(\sqrt{2}\,\sigma)$	$\sqrt{2}\,\pi\sigma y$
Γ	Γ	$\Gamma/(\sqrt{2}\,C)$	$C/(\sqrt{\pi}\,\Gamma)$	Cx/Γ	$\pi\Gamma y/C$
λ	$C/\sqrt{\lambda}$	$1/\sqrt{2\lambda}$	$\sqrt{\lambda/\pi}$	$\sqrt{\lambda}\, x$	$\pi y/\sqrt{\lambda}$

The Fourier transform of the convolution of two functions is the product of the transforms:

$$G_1(x) \otimes G_2(x) = \mathcal{F}\{G_1(x)\}\, \mathcal{F}\{G_2(x)\} = \mathcal{G}_1(p)\, \mathcal{G}_2(p)$$

using this relationship it is easy to show that the convolution of two Gaussians characterized by Γ_1 and Γ_2 produces a Gaussian distribution characterized by width Γ_3 where

$$\Gamma_3^2 = \Gamma_1^2 + \Gamma_2^2 \, .$$

<u>The Lorentzian Distribution</u>

Figure B shows the shape of the Lorentzian distribution $L(x)$ and its integral $Q(x)$. Note that for the Lorentzian line shape function Γ is the distance between the 0.25 and 0.75 amplitude points on the "knife edge" image distribution Q.

The most frequently encountered mathematical form for the Lorentzian distribution normalized to unit area is

$$L(x) = \frac{(a/\pi)}{(a^2 + x^2)}$$

where the full-width half-maximum $\Gamma = 2a$.

The Fourier transform of $L(x)$ is $\mathcal{L}(y)$:

$$\mathcal{L}(y) = \mathcal{F}\{L(x)\} = \exp(-2\pi a \; |y|)$$

The convolution of two Lorentzians of width Γ_1 and Γ_2 produces a Lorentzian of width Γ_3 where

$$\Gamma_3 = \Gamma_1 + \Gamma_2$$

APPENDIX B. Visibility of a Square Wave Folded with a Lorentzian

The visibility V of the image of a square wave object formed by a system with a Lorentzian line spread function is given by

$$V(p) = 4/\pi \sum_{m=1}^{odd} (-1)^{\frac{m-1}{2}} \left[\exp(-2\pi p)\right]^m / m$$

where $p = \Gamma y/2$.

Γ is the full width at half maximum of the Lorentzian line shape function

and y is the inverse wavelength of the object usually given as line pairs per mm.

The program listed for the Texas Instruments calculator model TI-59 will calculate five terms of the expansion for V(p). To operate the code load the starting value for p into storage register 01, press RST and R/S. When done the calculator will print four numbers:

1. P
2. V(p)
3. $\exp(-2\pi p)$
4. $R(L) = \exp(-2\pi p)/V(p)$

The value of $\exp(-2\pi p)$ is the visibility for a sine wave object. The ratio R(L) is then the ratio of the visibilities of sine and square wave objects.

After printing the four numbers the program will cause p to be incremented by 0.05 and restarted.

The program will continue until stopped externally.

Table I gives the numerical output produced by the code.

APPENDIX C. Visibility of a Square Wave Folded with a Gaussian.

The visibility V of the image of a square wave object formed by a system with a Gaussian line spread function is given by

$$V(p) = 4/\pi \sum_{m=1}^{odd} (-1)^{\frac{m-1}{2}} \exp[-(\pi m p)^2]/m$$

where $p = \Gamma y/(2\sqrt{|\ln(.5)|}) = \Gamma y/1.67 = \sqrt{2}\,\sigma y$

Γ is the full width at half maximum of the Gaussian line shape function

and y is the inverse wavelength of the object usually given as line pairs per mm.

The program listed for the Texas Instruments calculator model TI-59 will calculate seven terms of the expansion for V(p). To operate the code load the starting value of p into storage register 01, press RST and R/S. As with the program described in Appendix B; when done the calculator will print the four numbers:

1. P
2. V(p)
3. $\exp(-(\pi p)^2)$
4. $R(G) = \exp(-(\pi p)^2)/V(p)$

Table I gives the numerical output produced by the code.

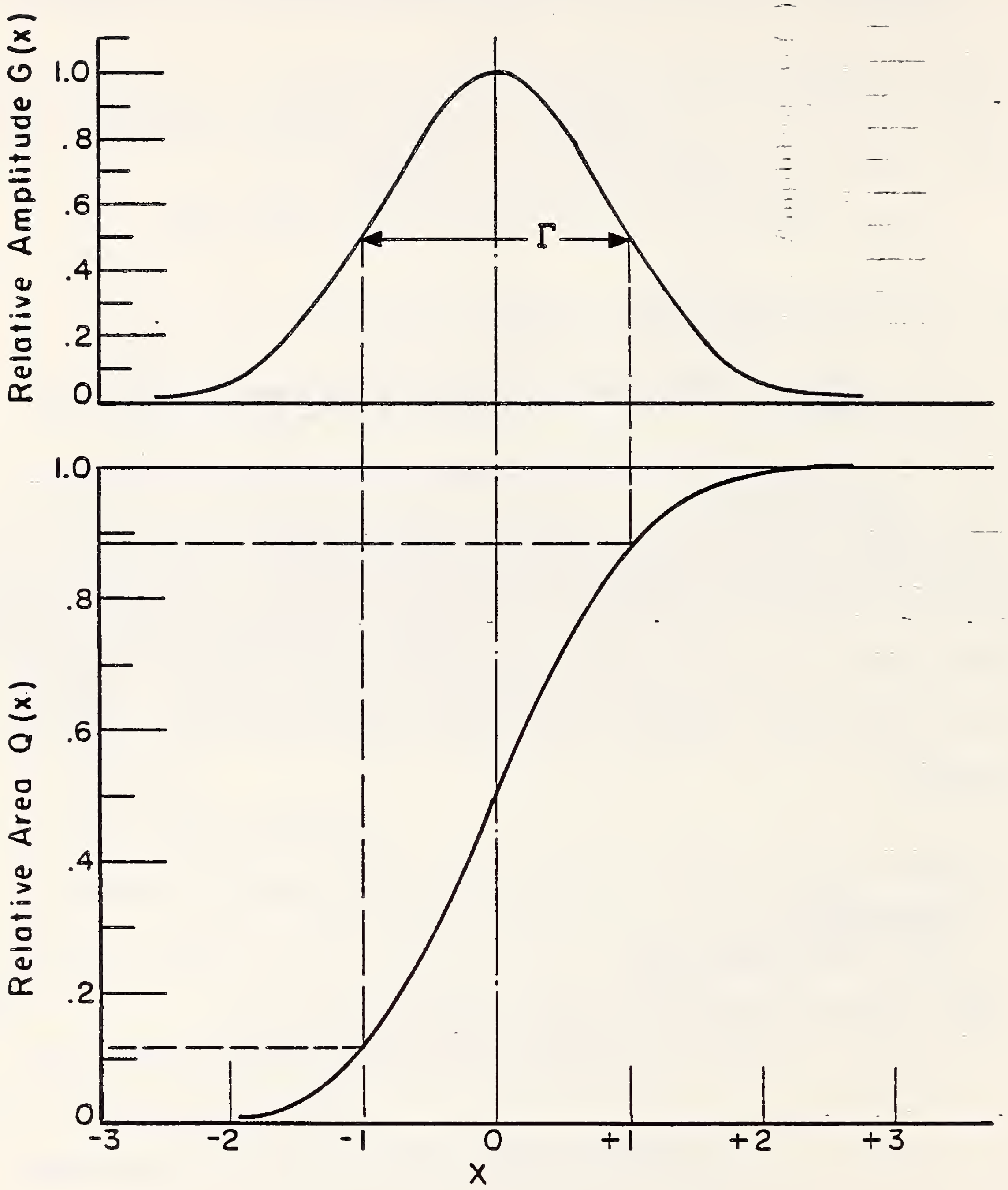

Fig. A. The Gaussian Line Shape Function G(x) and its integral Q(x). The abscissa scale is in units of half the full-width half-maximum, Γ. Dashed lines indicate that Γ is the distance between the 0.12 and 0.88 relative amplitude points on the Q distribution.

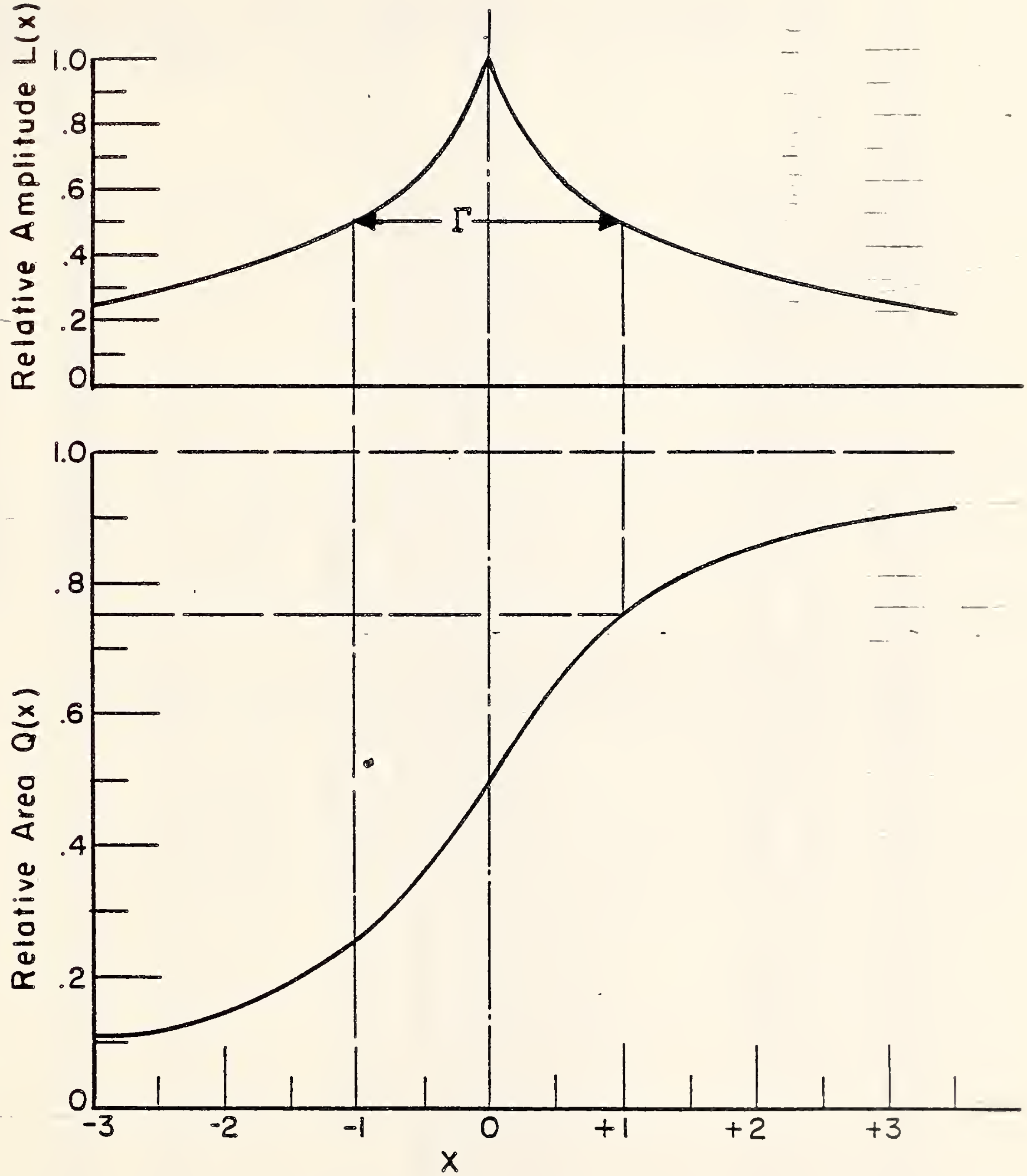

Fig. B. The Lorentzian Line Shape Function L(x) and its integral Q(x). The abscissa scale is in units of half the full-width half-maximum, Γ. Dashed lines indicate that Γ is the distance between the 0.25 and 0.75 relative amplitude points on the Q distribution.

Program listing for TI-59 to calculate the visibility of images produced by square and sine wave objects and a Gaussian line spread function.

```
000  43  RCL
001  01   01
002  65   ×
003  02   2
004  65   ×
005  89   π
006  95   =
007  42  STO
008  02   02
009  01   1
010  42  STO
011  04   04
012  71  SBR
013  15   E
014  42  STO
015  06   06
016  42  STO
017  03   03
018  03   3
019  42  STO
020  04   04
021  71  SBR
022  15   E
023  94  +/-
024  44  SUM
025  03   03
026  05   5
027  42  STO
028  04   04
029  71  SBR
030  15   E
031  44  SUM
032  03   03
033  07   7
034  42  STO
035  04   04
036  71  SBR
037  15   E
038  94  +/-
039  44  SUM
040  03   03
041  09   9
042  42  STO
043  04   04
044  71  SBR
045  15   E
046  44  SUM
047  03   03
048  01   1
049  01   1
050  42  STO
051  04   04
052  71  SBR
053  15   E
054  94  +/-
055  44  SUM
056  03   03
057  01   1
058  03   3
059  42  STO
060  04   04
061  71  SBR
062  15   E
063  55   ÷
064  02   2
065  95   =
066  44  SUM
067  03   03
068  43  RCL
069  03   03
070  65   ×
071  04   4
072  55   ÷
073  89   π
074  95   =
075  42  STO
076  05   05
077  98  ADV
078  43  RCL
079  01   01
080  65   ×
081  02   2
082  95   =
083  99  PRT
084  43  RCL
085  05   05
086  99  PRT
087  43  RCL
088  06   06
089  99  PRT
090  55   ÷
091  43  RCL
092  05   05
093  95   =
094  99  PRT
095  43  RCL
096  01   01
097  85   +
098  93   .
099  00   0
100  02   2
101  05   5
102  95   =
103  42  STO
104  01   01
105  61  GTO
106  00   00
107  00   00
108  76  LBL
109  15   E
110  53   (
111  53   (
112  43  RCL
113  02   02
114  65   ×
115  43  RCL
116  04   04
117  54   )
118  33  X²
119  94  +/-
120  22  INV
121  23  LNX
122  54   )
123  55   ÷
124  43  RCL
125  04   04
126  95   =
127  92  RTN
128  91  R/S
129  00   0
130  00   0
131  00   0
132  00   0
133  00   0
```

Program listing for TI-59 to calculate the visibility of images produced by square and sine wave objects and a Lorentzian line spread function.

000	43	RCL
001	01	01
002	65	×
003	02	2
004	65	×
005	89	π
006	95	=
007	94	+/-
008	22	INV
009	23	LNX
010	95	=
011	42	STO
012	02	02
013	43	RCL
014	02	02
015	45	YX
016	03	3
017	95	=
018	55	÷
019	03	3
020	95	=
021	94	+/-
022	42	STO
023	04	04
024	43	RCL
025	04	04
026	85	+
027	43	RCL
028	02	02
029	95	=
030	42	STO
031	03	03
032	43	RCL
033	02	02
034	45	YX
035	05	5
036	95	=
037	55	÷
038	05	5
039	95	=
040	44	SUM
041	03	03
042	43	RCL
043	02	02
044	45	YX
045	07	7
046	95	=
047	55	÷
048	07	7
049	95	=
050	94	+/-
051	44	SUM
052	03	03
053	43	RCL
054	02	02
055	45	YX
056	09	9
057	95	=
058	55	÷
059	09	9
060	95	=
061	44	SUM
062	03	03
063	43	RCL
064	03	03
065	65	×
066	04	4
067	55	÷
068	89	π
069	95	=
070	42	STO
071	05	05
072	98	ADV
073	43	RCL
074	01	01
075	99	PRT
076	43	RCL
077	05	05
078	99	PRT
079	43	RCL
080	02	02
081	99	PRT
082	55	÷
083	43	RCL
084	05	05
085	95	=
086	99	PRT
087	43	RCL
088	01	01
089	85	+
090	93	.
091	00	0
092	05	5
093	95	=
094	42	STO
095	01	01
096	61	GTO
097	00	00
098	00	00
099	91	R/S
100	00	0
101	00	0

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10. SUPPLEMENTARY NOTES

This work was done in connection with the development of the Microchannel Plate Neutron Detector.

☐ Document describes a computer program; SF-185, FIPS Software Summary, is attached.

11. ABSTRACT (A 200-word or less factual summary of most significant information. If document includes a significant bibliography or literature survey, mention it here)

This paper presents an analytical study of the effect of Gaussian- and Lorentzian-shaped line spread functions in non-coherent noise-free imaging systems. A mathematic development is given for the calculation of the Modulation Transfer Function (MTF). This technique is used to calculate the MTF for two-point and periodic objects using Gaussian and Lorentzian resolution functions. Figures and graphs are used to illustrate the comparison of the results. Relationships between the results obtained are developed that are useful in the interpretation of experiments used to determine the resolution of experimental systems. The development covers only noise-free, incoherent, one-dimensional systems.

12. KEY WORDS (Six to twelve entries; alphabetical order; capitalize only proper names; and separate key words by semicolons)

Gaussian; imaging systems; Lorentzian; modulation transfer function; resolution; visibility

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