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Piece-Wise Analytic Evaluation of the Radiative Tail From Elastic and Inelastic Electron Scattering

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ABSTRACT

We consider here the calculation of the radiative tail from the elastic peak in medium and high energy electron scattering as well as from a discrete inelastic level of the recoiling nucleus. We examine the method generally used for this calculation, viz., a numerical integration of the differential cross section over the angles of the unobserved photon, and discuss the difficulties inherent in this numerical integration due to the sharp peaking of the integrand. We present an alternative method for calculating the radiative tail, in which the region of integration is divided into an arbitrary number of subintervals, the structure functions are fitted by cubic spline functions in each subinterval, and the integrations are then performed analytically in closed form. This method has the advantages of greatly increased accuracy and a reduction of the computation time by a factor which can vary between 10 and 10^3, depending on the kinematics.

Key words: electron scattering cross sections; elastic electron scattering; elastic radiative tail; inelastic electron scattering; inelastic radiative tail; intermediate and high energy electron scattering.

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CONTENTS

I. Introduction

The analysis of medium and high energy electron scattering experiments involves, as is well known, the calculation of electromagnetic processes (e.g., radiative corrections, Landau straggling, radiative tails, etc.) which inevitably accompany and mask in some manner the electron-nucleus scattering of primary interest. The methods for evaluating the cross sections for these processes has been given in numerous articles and reviews familiar to experimenters, and are generally incorporated into the analysis of electron scattering experiments. We consider here the calculation of the radiative tail from the elastic peak, as well as from a discrete inelastic level of the recoiling nucleus.[1] We examine the method which is generally used for this calculation, namely a numerical integration of the differential cross section over the angles of the unobserved photon, and discuss the difficulties inherent in this numerical integration. Finally, we present an alternative method for calculating the radiative tail having the advantage of greatly increased accuracy and a reduction of the computation time by a factor which can vary between 10 and 10^3, depending on the kinematics (see Timing Table, p. 56).

The formalism used for the calculation of the radiative tail in almost all analyses of high-energy electron scattering experiments is that presented in a detailed series of papers by Tsai [1-3][2] and later refined by Miller [4]. In all of these papers the cross section for the scattering of an electron of initial momentum $|\underline{p}_1|$, final momentum $|\underline{p}_2|$ at an angle θ, with emission of a photon, is given by an integral of the form

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[1]The assumption that the final state of the recoiling nucleus is a discrete level does not constitute a significant restriction in practice, since in the analysis a final continuum state is generally treated as a sum of many discrete levels.

[2]Figures in brackets indicate literature references at the end of this paper.

$$\frac{d^2\sigma}{d\Omega d|\underline{p}_2|} = \int\{W_2 f_2 + W_1 f_1\}\, d\Omega_k \quad . \tag{1}$$

Here the functions f_2 and f_1 are known analytic functions of the initial and final electron momenta and scattering angle (which remain fixed) and the photon angles θ_k, φ_k, over which one integrates. The structure functions W_2 and W_1 are, for elastic scattering, a function of the four-momentum transfer $q^2 < 0$. They are, in general, given as the result of a numerical calculation in which a charge distribution has been entered into a phase shift calculation. (That is, although a Born approximation formalism is used to derive the cross section given in (1), one uses, in the integrand in that expression, "experimental" structure functions, defined either by the experimental cross section divided by the Mott cross section or by a phase shift calculation.) It is expected, therefore, that the calculation of the radiative tail from (1) requires a numerical integration. However, the numerical integration of the expression appearing in the integrand in (1) is not completely trivial. Depending on the kinematics, the functions f_2 and f_1 can be very sharply peaked, — a reflection of the fact that the photons are emitted preferentially in the direction of either the initial or the final electron. In addition, since the functions f_2 and f_1 have an overall factor $\frac{1}{q^4}$, the integrand may rise sharply for q^2 near its minimum value, achieved (in the coordinate system defined following eqs (9)) when $\cos\theta_k = 1$. These peaks are discussed in detail both in Maximon and Isabelle [5] and in Mo and Tsai [2]. In general, the height of the peaks is of order (E/mc^2) relative to the background and the width is of order (mc^2/E) relative to the width of the region over which one integrates. The contribution from the peaks is thus of the same order of magnitude as that from the background, and for high energies great care must be taken to ensure that the numerical integration procedure has indeed converged and gives the desired accuracy. The result of the peaking is thus that the numerical integration may be quite lengthy, or of dubious accuracy, or both. Further, given the large number of intervals that must be chosen for the integration in the region of the peaks, the numerical evaluation of the structure functions W_2 and W_1 in each integration interval would be prohibitive in computation time, and rather useless since the functions W_2 and W_1 vary little over even the entire region of a peak. The procedure generally followed[3] is therefore to divide the entire region of integration into a fixed number of subintervals ($1 \leq i \leq N$) (say with N of the order of 100) and to fit W_2 and W_1 by spline functions in each of these subintervals, e.g., by a cubic function of $-q^2$:

$$\begin{aligned} W_2 &= a_i + b_i(-q^2) + c_i(-q^2)^2 + d_i(-q^2)^3 \\ W_1 &= a_i' + b_i'(-q^2) + c_i'(-q^2)^2 + d_i'(-q^2)^3 \quad . \end{aligned} \tag{2}$$

With the overall region of integration divided into these subintervals (some of which contain the peaks) and W_2 and W_1 represented by these simple analytic expressions in each subinterval, the numerical integration is then carried out as just discussed, the peak rendering the integration costly in time and questionable in accuracy.

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[3]This is, essentially, the procedure followed by R. Altemus in the analysis of the data on ^{56}Fe taken at Bates (reported in Phys. Rev. Lett. 44, 965 (1980)), by M. Deady in the analysis of the data on ^{40,48}Ca taken at Bates (reported in Phys. Rev. C28, 631 (1983)), and by Z. Meziani in the analysis of the data on ^{40,48}Ca and ^{56}Fe taken at Saclay (to be published shortly).

We note that the integration procedure just described consists in fact of two parts which are quite independent. In the first, each of the structure functions is fitted by a spline (or other useful function) in each of the N subintervals into which the integration region is subdivided for this purpose. (For example, for a point charge with spin zero we have $W_2 = Z^2$ and $W_1 = 0$, so that we may choose N = 1.) In the second part, the numerical integration of (1) is performed in each of the N subintervals, with W_2 and W_1 now given by (2), using some suitable integration technique (e.g., Simpson's rule or Gaussian integration). In the event that a peak occurs in a given subinterval, then clearly this subinterval must be further subdivided, and the convergence of the integration procedure must be tested.

The essence of the present report is contained in the observation that once we have carried out the fitting of W_2 and W_1 by functions of the form given in (2) (the first part of the procedure just described), there is no longer any need to perform the integration numerically. With W_2 and W_1 given by finite polynomials in q^2 the functions f_2 and f_1 are sufficiently simple that the integration can be performed in each subinterval in closed form. (The resulting functions involve only algebraic functions and logarithms.) The integration over the entire region then consists merely of summing the contribution from each of the N subintervals. The problems related to these peaks are thereby avoided completely. The presence of sharp peaks does not necessitate a further division of the subintervals. It is thus by eliminating the numerical integration, the second part of the procedure usually followed, that we reduce the time for the entire calculation by a factor which varies between 10 and 10^3, depending on the specific kinematics. The accuracy of the entire calculation is also greatly increased, since it is now given by the precision of the computer rather than the accuracy of the particular numerical integration method.

As a last general comment before entering into the details, we observe that in this note the method, which we call "piece-wise analytic" evaluation, is applied to the integration of the doubly differential cross section over the angles of the photon. However, it is equally applicable (with, of course, different resulting expressions) to the situation of interest for the analysis of experiments using photons as the electromagnetic probe. Then one integrates this same doubly differential cross section over the angles of the electron: The expressions which result in this latter case will be presented in a future report, in which we consider in particular the cross section for polarized photons (integrated over the angles of the final electron).

We next present the details of the calculation of the radiative tail given in this report. We divide this presentation into three parts: In the first we give preliminaries to the method presented here. Notation is defined, the differential cross section which forms the basis of our work is presented, and a few comments are made. In the second we give the details of the calculation as reflected directly in the computer program, a copy of which is included with this report.[4] Finally, in the Appendix we give details of the algebra involved in obtaining our final expression for the cross section, given by eq (38) and the subsidiary equations defining the quantities in it.

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[4]The Fortran program is available from the authors. Requests should be accompanied by a magnetic tape and a specification of the desired format.

II. Preliminaries for the Piece-Wise Analytical Evaluation of the Radiative Tail

The cross section under consideration here is the radiative tail to both elastic and inelastic scattering of electrons from a nucleus. In this report (as in the work of Tsai) we consider emission of photons only by the electron. Furthermore, the formalism used is that of the Born approximation. Thus in terms of diagrams we have

We do not consider here the contribution due to radiation by the recoiling nucleus. That has been treated in an approximate manner by Miller [4] and will be examined by us in a report to appear later.

The notation used here is as follows: Energies, momenta, and masses are all understood to be in energy units. When four-vectors are written, the metric is as in Bjorken and Drell [6], viz.,

$$p_1 \cdot p_2 = \varepsilon_1 \varepsilon_2 - \underline{p}_1 \cdot \underline{p}_2 \tag{3}$$

$p_1 = (\varepsilon_1, \underline{p}_1)$:		four-momentum of the incident electron
$p_2 = (\varepsilon_2, \underline{p}_2)$:		four-momentum of the scattered electron
$k = (\omega, \underline{k})$:		four-momentum of the emitted (real) photon
$q = (q_0, \underline{q})$:		four-momentum of the virtual photon exchanged between the electron and the target particle
$P_1 = (E_1, \underline{P}_1)$:		four-momentum of the target particle
$= (M, 0)$		in the lab system
$P_2 = (E_2, \underline{P}_2)$:		four-momentum of the recoiling nucleus
$= (M + q_0, \underline{q})$		in the lab system
$m =$		rest mass of the electron
$M =$		rest mass of the target particle.
$M^* =$		rest mass of the final, recoiling nucleus

$$\mu \equiv \frac{M^{*2} - M^2}{2M} \quad \left(= (M^* - M)\left(1 + \frac{M^* - M}{2M}\right) \approx M^* - M \text{ for } \frac{M^* - M}{2M} << 1\right). \tag{4}$$

Then from energy-momentum conservation

$$p_1 + P_1 = p_2 + P_2 + k \quad . \tag{5}$$

The momentum transferred to the nucleus is then

$$q = P_2 - P_1 = p_1 - p_2 - k \quad . \tag{6}$$

Further, it is useful to define the four-vector u by

$$u = (u_0, \underline{u}) \equiv p_1 + P_1 - p_2 = P_2 + k \tag{7}$$

Then in the lab system

$$u_0 = M + \varepsilon_1 - \varepsilon_2 \tag{8a}$$

and

$$\underline{u} = \underline{p}_1 - \underline{p}_2 \tag{8b}$$

We note that u_0 and $\underline{u}$ are independent of both photon energy and angles.

θ = scattering angle of the electron $\left(\theta = \measuredangle\, (\underline{p}_1,\underline{p}_2)\right)$

θ_k = angle between $\underline{u}$ and $\underline{k}$

θ_s = angle between $\underline{u}$ and $\underline{p}_1$

θ_p = angle between $\underline{u}$ and $\underline{p}_2$. (9)

As in Mo and Tsai [2] the integration over photon angles is performed in a coordinate system such that the z axis is in the direction of $\underline{u} = \underline{p}_1 - \underline{p}_2$. As observed there, this coordinate system has the particular advantage that q^2 is then independent of the photon azimuthal angle φ_k. The structure functions $W_2(-q^2)$ and $W_1(-q^2)$ can then be taken outside of the integral over φ_k, which can then be performed analytically. We show this explicitely in the following paragraph.

From the notation just introduced, since the recoiling nucleus is assumed to be a free particle of mass M^*, we have

$$M^{*2} = P_2{}^2 = (u - k)^2$$

$$= u^2 - 2k \cdot u \tag{10}$$

from which

$$\omega = \frac{u^2 - M^{*2}}{2(u_0 - |\underline{u}|\cos\theta_k)} \quad . \tag{11}$$

Thus the photon energy, ω, depends on the photon direction, but in the particular coordinate system we have chosen it is a function only of θ_k, and not of φ_k. (Note, however, from the expression for ω given below, viz., $\omega = \varepsilon_1 - \varepsilon_2 - \mu + \frac{q^2}{2M}$, that in the limit $\frac{q^2}{2M} \to 0$, we have $\omega = \varepsilon_1 - \varepsilon_2 - \mu$; that is, in the absence of recoil the photon energy is independent of photon direction.)

Similarly, we can write

$$M^{*2} = P_2{}^2 = (P_1 + q)^2$$

$$= M^2 + q^2 + 2q \cdot P_1 \quad . \tag{12}$$

Thus in the lab system

$$q_0 = -\frac{q^2}{2M} + \mu \quad . \tag{13}$$

On the other hand, substituting $q = p_1 - p_2 - k$ in $2q \cdot P_1$ above gives

$$2P_1 \cdot k = 2P_1 \cdot (p_1 - p_2) - M^{*2} + M^2 + q^2 \quad , \tag{14}$$

so that in the lab system

$$\omega = \varepsilon_1 - \varepsilon_2 - \mu + \frac{q^2}{2M} \quad . \tag{15}$$

Thus in the coordinate system with z-axis in the direction of $\underline{u}$, both ω and q^2 are independent of φ_k. Only the products $p_1 \cdot k$ and $p_2 \cdot k$ depend on φ_k.

Our starting point is then the doubly differential cross section for scattering of an electron with the emission of a photon, essentially as given by Tsai [1] and Mo and Tsai [2]. We make two minor changes in the expressions presented in these references, discussed immediately following the cross section written below.

$$\frac{d^2\sigma}{d\Omega d|\underline{p}_2|} = \frac{\alpha^3}{(2\pi)^2} \frac{|\underline{p}_2|^2}{\varepsilon_2 |\underline{p}_1|} \int_{-1}^{+1} \frac{2M\omega \, d(\cos\theta_k)}{q^4(u_0 - |\underline{u}|\cos\theta_k)} \int_0^{2\pi} d\varphi_k \left\{ \quad \right\} \tag{16}$$

where

$$
\begin{aligned}
\{\ \ \} = W_2(-q^2)\Bigg[& -\frac{m^2}{(p_1\cdot k)^2}[2\varepsilon_2(\varepsilon_1-\omega)+\tfrac{1}{2}q^2] - \frac{m^2}{(p_2\cdot k)^2}[2\varepsilon_1(\varepsilon_2+\omega)+\tfrac{1}{2}q^2] \\
& + \frac{2}{(p_1\cdot k)(p_2\cdot k)}\left(m^2(p_1\cdot p_2-\omega^2)+(p_1\cdot p_2)[2\varepsilon_1\varepsilon_2-p_1\cdot p_2+\omega(\varepsilon_1-\varepsilon_2)]\right) \\
& - \frac{1}{(p_1\cdot k)}[2(\varepsilon_1\varepsilon_2-\varepsilon_2\omega+\varepsilon_1{}^2)+\tfrac{1}{2}q^2-p_1\cdot p_2-m^2] \\
& + \frac{1}{(p_2\cdot k)}[2(\varepsilon_1\varepsilon_2+\varepsilon_1\omega+\varepsilon_2{}^2)+\tfrac{1}{2}q^2-p_1\cdot p_2-m^2] \\
& - 2\Bigg] \\
+ W_1(-q^2)\Bigg[& m^2(2m^2+q^2)\left(\frac{1}{(p_1\cdot k)^2}+\frac{1}{(p_2\cdot k)^2}\right) \\
& + \frac{4p_1\cdot p_2(p_1\cdot p_2-2m^2)}{(p_1\cdot k)(p_2\cdot k)} \\
& - \left(\frac{1}{(p_1\cdot k)}-\frac{1}{(p_2\cdot k)}\right)(2p_1\cdot p_2+2m^2-q^2) \\
& + 4\Bigg] \qquad (17)
\end{aligned}
$$

The first modification we have made in the expression of Tsai concerns the structure functions. We use W_2 and W_1, defined by the elastic scattering cross section, which at high energies is given in terms of these functions by

$$
\frac{d\sigma_{el}}{d\Omega} = \frac{\alpha^2\cos^2\tfrac{1}{2}\theta[W_2+2\tan^2\tfrac{1}{2}\theta\, W_1]}{4\varepsilon_1{}^2\sin^4\tfrac{1}{2}\theta\ [1+\frac{2\varepsilon_1}{M}\sin^2\tfrac{1}{2}\theta]} \ . \qquad (18)
$$

(Note that for a point charge with spin zero, $W_2(0) = Z^2$, $W_1(0) = 0$.) This is the same as the change of notation introduced by Stein et al. [7] in Appendix A of their paper. That appendix is a useful reference in that it is essentially a summary of Tsai's SLAC-PUB 848 [3]. The structure functions F_j and G_j used by Tsai are related to W_2 and W_1 by

$$F_j = 4W_2$$

$$G_j = 4M^2W_1 \quad . \tag{19}$$

The second modification made in the expression of Tsai concerns the term $\frac{|\underline{p}_2|^2}{\varepsilon_2|\underline{p}_1|}$ which appears as a factor to the integral just written. In the articles of Tsai this factor has been replaced by its high energy limit, viz., $\frac{\varepsilon_2}{\varepsilon_1}$. While this replacement is clearly of little significance numerically for energies of experimental interest, we retain the factor $\frac{|\underline{p}_2|^2}{\varepsilon_2|\underline{p}_1|}$ out of a sense of consistency, since no high energy approximations are made anywhere else in the expression for the cross section. Our expression is then identical to that given by Nguyen-Ngoc and Perez-y-Jorba [8], apart from a trivial difference in the normalization of the structure functions: they use F and G, which are related to our W_2 and W_1 by

$$F = \frac{W_2}{M^2}$$

$$G = - W_1 \ . \tag{20}$$

In any event, this term is closely related to the physics: Writing $\frac{|\underline{p}_2|^2}{\varepsilon_2|\underline{p}_1|} = \left(\frac{|\underline{p}_2|}{\varepsilon_2}\right)\frac{|\underline{p}_2|}{|\underline{p}_1|}$, we note that the factor $\frac{|\underline{p}_2|}{\varepsilon_2}$ comes from the fact that we have written the cross section as $\frac{d^2\sigma}{d\Omega d|\underline{p}_2|}$ rather than $\frac{d^2\sigma}{d\Omega d\varepsilon_2}$. The term $|\underline{p}_1|$ comes from the incident flux, and the final factor $|\underline{p}_2|$ comes from the density of final states for the scattered electron: (Note that in Stein et al. [7] they write $\frac{d^2\sigma}{d\Omega d\varepsilon_2}$, so that only $\frac{|\underline{p}_2|}{|\underline{p}_1|}$ should appear, but they replace this with $\frac{\varepsilon_2}{\varepsilon_1}$.) This factor without high energy approximation has also been retained to facilitate comparison with the Bethe-Heitler cross section in the limit $M \to \infty$.

Finally, we note that in Stein et al. [7] the structure functions W_2 and W_1 have been replaced by $\tilde{W}_2$ and $\tilde{W}_1$, defined by (A42) and (A43) of Stein et al.:

$$\tilde{W}_2(q^2) = \tilde{F}(q^2)W_2(q^2)$$

$$\tilde{W}_1(q^2) = \tilde{F}(q^2)W_1(q^2) \tag{21}$$

to account for both thick target bremsstrahlung (equivalent radiator) and that part of the radiative correction which does not depend on the resolution ΔE. In addition, in the work of Miller [4] these factors are further modified to account (in an approximate manner) for the multiple emission of soft photons from both the electron and the target particle. The important point to retain here is that all these factors depend on the photon direction only through the four-momentum transfer q^2 (or via the photon energy, which is again a linear function of q^2, as we have just seen). All these effects can therefore be accounted for simply by an appropriate modification of the structure functions which leaves them independent of φ_k.

We next perform the integration over φ_k, as in Mo and Tsai [2], Tsai [3], and Stein et al. [7], and obtain

$$\frac{d^2\sigma}{d\Omega d|\underline{p}_2|} = \frac{\alpha^3}{2\pi}\frac{|\underline{p}_2|^2}{\varepsilon_2|\underline{p}_1|}\int_{-1}^{+1}\frac{2M\omega\, d(\cos\theta_k)}{q^4(u_0 - |\underline{u}|\cos\theta_k)}\left\{\quad\right\} \tag{22}$$

where

$$\begin{aligned}\left\{\quad\right\} = W_2(-q^2)\Bigg[&-\frac{am^2}{x^3}[2\varepsilon_1(\varepsilon_2+\omega) + \tfrac{1}{2}q^2] - \frac{a'm^2}{y^3}[2\varepsilon_2(\varepsilon_1-\omega) + \tfrac{1}{2}q^2] \\ &+ 2\nu\left(\frac{1}{x} - \frac{1}{y}\right)\{m^2(p_1\cdot p_2-\omega^2) + (p_1\cdot p_2)[2\varepsilon_1\varepsilon_2 - p_1\cdot p_2+\omega(\varepsilon_1-\varepsilon_2)]\} \\ &+ \frac{1}{x}[2(\varepsilon_1\varepsilon_2+\varepsilon_1\omega+\varepsilon_2^2) + \tfrac{1}{2}q^2 - p_1\cdot p_2 - m^2] \\ &- \frac{1}{y}[2(\varepsilon_1\varepsilon_2-\varepsilon_2\omega+\varepsilon_1^2) + \tfrac{1}{2}q^2 - p_1\cdot p_2 - m^2] \\ &- 2\Bigg] \qquad\qquad (23) \\ + W_1(-q^2)\Bigg[&\left(\frac{a}{x^3} + \frac{a'}{y^3}\right)m^2(2m^2+q^2) + 4\nu\left(\frac{1}{x} - \frac{1}{y}\right)(p_1\cdot p_2)(p_1\cdot p_2-2m^2) \\ &+ \left(\frac{1}{x} - \frac{1}{y}\right)(2p_1\cdot p_2+2m^2-q^2) + 4\Bigg]\end{aligned}$$

in which[5,6]

$$a = \omega(\varepsilon_2 - |\underline{p}_2|\cos\theta_p\cos\theta_k)$$

$$a' = \omega(\varepsilon_1 - |\underline{p}_1|\cos\theta_s\cos\theta_k)$$

$$b = -\omega|\underline{p}_2|\sin\theta_p\sin\theta_k$$

$$\nu = (a' - a)^{-1}$$

$$x = (a^2 - b^2)^{1/2}$$

$$y = (a'^2 - b^2)^{1/2}$$

$$\cos\theta_p = \frac{|\underline{p}_1|\cos\theta - |\underline{p}_2|}{|\underline{u}|}$$

$$\cos\theta_s = \frac{|\underline{p}_1| - |\underline{p}_2|\cos\theta}{|\underline{u}|} \quad . \tag{24}$$

[5]The symbol x which is defined in (24) and used in (23) is employed at this point out of consistency with the notation of Tsai [3] and Stein et al. [7]. It is used nowhere else in this report as defined in (24). Elsewhere throughout this report we define $x = -q^2$.

[6]The expressions given here in (24) are those given in Tsai's SLAC-PUB 848 [3] and Stein et al. [7]. They represent a considerable simplification of the expressions given originally in Mo and Tsai [2].

III. Details of the Piece-Wise Analytical Evaluation of the Radiative Tail

At this point we depart from the procedure followed in these references. Since we perform the remaining integration analytically, the most appropriate variable of integration is q^2 rather than $\cos\theta_k$, in view of the functions $W_2(-q^2)$ and $W_1(-q^2)$. Indeed, the form of the integrand itself is then somewhat simpler. To give just one example, referring to the overall factor in the integrand in above expressions for the radiative tail, we now have, with q^2 as integration variable,

$$\frac{2M\,\omega\, d(\cos\theta_k)}{q^4(u_0 - |\underline{u}|\cos\theta_k)} = \frac{1}{|\underline{u}|}\,\frac{d(q^2)}{q^4} \quad . \tag{25}$$

We will not, however, encumber this section of the report with these algebraic manipulations. They are presented at the end in the Appendix. Here we give only the final form of our expression for the radiative tail, which lends itself directly to the computer implementation of the calculation.

As variable of integration we choose finally a positive quantity, viz.,

$$x = -q^2 > 0 \quad . \tag{26}$$

The cross section for the radiative tail is then given by

$$\frac{d^2\sigma}{d\Omega d|\underline{p}_2|} = \frac{\alpha^3}{\pi}\,\frac{|\underline{p}_2|^2}{\varepsilon_2|\underline{p}_1|}\int_{x_m}^{x_M} dx\,[W_2(x)\{\ \} + W_1(x)\{\ \}']\tag{27}$$

where

$$
\begin{aligned}
\{\ \ \} = \; & - \frac{1}{x^2 D_1^{3/2}(x)} [\beta_0 + \beta_1 x + \beta_2 x^2] \\
& - \frac{1}{x^2 D_2^{3/2}(x)} [\delta_0 + \delta_1 x + \delta_2 x^2] \\
& - \frac{\eta}{|\underline{u}| x^2} - \frac{1}{(2\lambda - x)} \left(\frac{1}{D_1^{1/2}(x)} - \frac{1}{D_2^{1/2}(x)} \right) \gamma \\
& + \frac{\tau_1}{x^2 D_1^{1/2}(x)} - \frac{\tau_2}{x^2 D_2^{1/2}(x)} \\
& - \frac{\rho_1}{x D_1^{1/2}(x)} + \frac{\rho_2}{x D_2^{1/2}(x)}
\end{aligned}
\tag{28}
$$

and where $\{\ \ \}'$ is obtained from $\{\ \ \}$ merely by replacing the parameters $\beta_0, \beta_1, \ldots$ by the corresponding parameters with primes:

$$\{\beta_0, \beta_1, \beta_2, \delta_0, \delta_1, \delta_2, \eta, \gamma, \tau_1, \tau_2, \rho_1, \rho_2\}$$

$$\rightarrow \{\beta_0', \beta_1', \beta_2', \delta_0', \delta_1', \delta_2', \eta', \gamma', \tau_1', \tau_2', \rho_1', \rho_2'\}$$

In (28) the quadratic functions $D_1(x)$ and $D_2(x)$ are related to the quantities a, a', and b of (24) by eqs (A19) and (A25). Referring to the terms in the integrand with factor $\frac{1}{(2\lambda - x)}$, we note in the appendix that the point $x = 2\lambda$ indeed lies in the range of integration:

$$x_m < 2\lambda < x_M \ , \tag{29}$$

but that $D_1(2\lambda) = D_2(2\lambda)$, so that there is in fact no singularity in the integrand at the point $x = 2\lambda$. (This is shown explicitely later in this report, in the Appendix.)

The parameters appearing in the integral in (27) and (28) are defined shortly below. However, we first outline our piece-wise analytic evaluation of the cross section. We divide the range of integration (x_m, x_M) into N subintervals (which can be of different and arbitrary length) writing

$$\int_{x_m}^{x_M} dx \ldots = \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} dx \ldots \tag{30}$$

with

$$x_0 = x_m \quad , \quad x_N = x_M \ . \tag{31}$$

In each subinterval (x_{i-1}, x_i) we represent $W_2(x)$ and $W_1(x)$ by a third order polynomial in x: The coefficients of the polynomial will, in general, be different in each subinterval.

In (x_{i-1}, x_i)

$$W_2(x) = a_i + b_i x + c_i x^2 + d_i x^3$$

$$W_1(x) = a_i' + b_i' x + c_i' x^2 + d_i' x^3 \tag{32}$$

We may then write

$$\int_{x_m}^{x_M} dx\,[W_2(x)\{\ \} + W_1(x)\{\ \}'] = \sum_{i=1}^{N}\int_{x_{i-1}}^{x_i} dx\,[W_2(x)\{\ \} + W_1(x)\{\ \}']$$

$$= \sum_{i=1}^{N}\int_{x_{i-1}}^{x_i} dx\Big[(a_i + b_i x + c_i x^2 + d_i x^3)\ \{\ \}$$

$$+ (a_i' + b_i' x + c_i' x^2 + d_i' x^3)\ \{\ \}'\Big]\,. \tag{33}$$

Considering the expressions just given for $\{\ \}$ and $\{\ \}'$ (the functions $D_1(x)$ and $D_2(x)$ are quadratic functions of x), the integrals can now all be performed analytically. The integrations are all straightforward — none result in functions any more complicated than the logarithm. However, care must be taken in arranging the terms, and in particular the arguments of the logarithms, so that at high energies there are no cancellations between almost equal terms, leaving a remainder of relative order $(m/\varepsilon_1)^2$, with a consequent loss in precision.

Referring to the integrand, the integrals to be performed analytically are thus of the the form

$$I_{1,i}^{(1)} = \int_{x_{i-1}}^{x_i} \frac{dx}{x^2 D_1{}^{3/2}(x)}\,[a_i + b_i x + c_i x^2 + d_i x^3]\ [\beta_0 + \beta_1 x + \beta_2 x^2] \tag{34a}$$

$$I_i^{(2)} = \int_{x_{i-1}}^{x_i} \frac{dx}{x^2}\,[a_i + b_i x + c_i x^2 + d_i x^3] \tag{34b}$$

$$I_{1,i}^{(3)} = \int_{x_{i-1}}^{x_i} \frac{dx}{x^2 D_1{}^{1/2}(x)}\,[a_i + b_i x + c_i x^2 + d_i x^3] \tag{34c}$$

$$I_{1,i}^{(4)} = \int_{x_{i-1}}^{x_i} \frac{dx}{xD_1^{1/2}(x)} \left[a_i + b_i x + c_i x^2 + d_i x^3\right] \tag{34d}$$

and

$$I_i^{(5)} = \int_{x_{i-1}}^{x_i} \frac{dx}{(2\lambda - x)} \left(\frac{1}{D_1^{1/2}(x)} - \frac{1}{D_2^{1/2}(x)}\right) \left[a_i + b_i x + c_i x^2 + d_i x^3\right]. \tag{34e}$$

The integrals $I_{2,i}^{(1)}$, $I_{2,i}^{(3)}$, and $I_{2,i}^{(4)}$ follow from $I_{1,i}^{(1)}$, $I_{1,i}^{(3)}$, and $I_{1,i}^{(4)}$ merely by replacing $D_1(x)$ by $D_2(x)$ in the integrands (and replacing $\beta_0, \beta_1, \beta_2$ in $I_{1,i}^{(1)}$ by $\delta_0, \delta_1, \delta_2$). We then write

$$I_i = - I_{1,i}^{(1)} - I_{2,i}^{(1)} - \frac{\eta}{|\underline{u}|} I_i^{(2)} + \tau_1 I_{1,i}^{(3)} - \tau_2 I_{2,i}^{(3)}$$

$$- \rho_1 I_{1,i}^{(4)} + \rho_2 I_{2,i}^{(4)} - \gamma I_i^{(5)} \tag{35}$$

and define I_i' by substituting throughout the corresponding parameters with primes:

$$\{\beta_0, \beta_1, \ldots\} \rightarrow \{\beta_0', \beta_1', \ldots\}$$

and

$$\{a_i, b_i, c_i, d_i\} \rightarrow \{a_i', b_i', c_i', d_i'\} \quad . \tag{36}$$

The integral given for the cross section is then

$$\int_{x_m}^{x_M} dx \ \ldots \ = \sum_{i=1}^{N} \left[I_i + I_i'\right] \tag{37}$$

and our piece-wise analytic expression for the cross section is

$$\frac{d^2\sigma}{d\Omega d|\underline{p}_2|} = \frac{\alpha^3}{\pi} \frac{|\underline{p}_2|^2}{\varepsilon_2|\underline{p}_1|} \sum_{i=1}^{N} [I_i + I_i'] \ . \tag{38}$$

Again we stress that the integrations are all performed analytically. The sum above reflects only the subdivision such that $W_2(x)$ and $W_1(x)$ can be approximated with sufficient accuracy by third order polynomials in each subinterval.

Referring to the integrals $I_{1,i}^{(1)}$, $I_{2,i}^{(1)}$, $I_i^{(2)}$, $I_{1,i}^{(3)}$, $I_{2,i}^{(3)}$, $I_{1,i}^{(4)}$, $I_{2,i}^{(4)}$, and $I_i^{(5)}$, we see that the important integrals to be evaluated are

$$J_{1,m} = \int_{x_{i-1}}^{x_i} \frac{x^m}{D_1^{1/2}(x)} dx$$

$$m = -2,-1,0,1,2 \tag{39}$$

$$J_{2,m} = \int_{x_{i-1}}^{x_i} \frac{x^m}{D_2^{1/2}(x)} dx$$

$$K_{1,m} = \int_{x_{i-1}}^{x_i} \frac{x^m}{D_1^{3/2}(x)} dx$$

$$m = -2,-1,0,1,2 \tag{40}$$

$$K_{2,m} = \int_{x_{i-1}}^{x_i} \frac{x^m}{D_2^{3/2}(x)} dx$$

and

$$L = \int_{x_{i-1}}^{x_i} \frac{1}{(2\lambda - x)} \left(\frac{1}{D_1^{1/2}(x)} - \frac{1}{D_2^{1/2}(x)} \right) dx \tag{41}$$

In terms of these integrals we have

$$\begin{aligned} I_{1,i}^{(1)} = {} & a_i \beta_0 K_{1,-2} + (b_i \beta_0 + a_i \beta_1) K_{1,-1} \\ & + (c_i \beta_0 + b_i \beta_1 + a_i \beta_2) K_{1,0} \\ & + (d_i \beta_0 + c_i \beta_1 + b_i \beta_2) K_{1,1} \\ & + (d_i \beta_1 + c_i \beta_2) K_{1,2} + d_i \beta_2 \, K_{1,3} \end{aligned} \tag{42a}$$

$$\begin{aligned} I_{2,i}^{(1)} = {} & a_i \delta_0 K_{2,-2} + (b_i \delta_0 + a_i \delta_1) K_{2,-1} \\ & + (c_i \delta_0 + b_i \delta_1 + a_i \delta_2) K_{2,0} \\ & + (d_i \delta_0 + c_i \delta_1 + b_i \delta_2) K_{2,1} \\ & + (d_i \delta_1 + c_i \delta_2) K_{2,2} + d_i \delta_2 \, K_{2,3} \end{aligned} \tag{42b}$$

$$I_i^{(2)} = a_i \left(\frac{1}{x_{i-1}} - \frac{1}{x_i} \right) + b_i \ln\left(\frac{x_i}{x_{i-1}} \right) + c_i (x_i - x_{i-1}) + \tfrac{1}{2} d_i (x_i^2 - x_{i-1}^2) \tag{42c}$$

$$I_{1,i}^{(3)} = a_i J_{1,-2} + b_i J_{1,-1} + c_i J_{1.0} + d_i J_{1,1} \tag{42d}$$

$$I_{2,i}^{(3)} = a_i J_{2,-2} + b_i J_{2,-1} + c_i J_{2.0} + d_i J_{2,1} \tag{42e}$$

$$I_{1,i}^{(4)} = a_i J_{1,-1} + b_i J_{1,0} + c_i J_{1,1} + d_i J_{1,2} \tag{42f}$$

$$I_{2,i}^{(4)} = a_i J_{2,-1} + b_i J_{2,0} + c_i J_{2,1} + d_i J_{2,2} \tag{42g}$$

$$\begin{aligned} I_1^{(5)} &= [a_i + 2\lambda b_i + (2\lambda)^2 c_i + (2\lambda)^3 d_i]\, L \\ &- [b_i + 2\lambda c_i + (2\lambda)^2 d_i](J_{1,0} - J_{2,0}) \\ &- [c_i + 2\lambda d_i](J_{1,1} - J_{2,1}) \\ &- d_i(J_{1,2} - J_{2,2}) \,. \end{aligned} \tag{42h}$$

As just mentioned, the integrals $I_{1,i}^{(1)'}$, $I_{2,i}^{(1)'}$, $I_{1,i}^{(2)'}$, $I_{1,i}^{(3)'}$, $I_{2,i}^{(3)'}$, $I_{1,i}^{(4)'}$, $I_{2,i}^{(4)'}$, and $I_i^{(5)'}$ follow from the above expressions by substituting throughout the corresponding parameters with primes:

$$\{\beta_0, \beta_1, \ldots\} \to \{\beta_0', \beta_1', \ldots\}$$

and

$$\{a_i, b_i, c_i, d_i\} \to \{a_i', b_i', c_i', d_i'\} \quad . \tag{36}$$

Finally, we give the explicit expressions for the parameters which appear in the integral in terms of the input variables for the analysis, viz., $|\underline{p}_1|$, $|\underline{p}_2|$, θ, m, M, and M^*, as well as the expressions for the integrals $J_{1,m}$,

$J_{2,m}$, $K_{1,m}$, $K_{2,m}$, and L in terms of these variables. Again we leave the details of the evaluation to the Appendix, giving here only the final results. It is to be noted that in each case, the argument of the logarithm has been written so that there are no cancellations — all terms in the argument are positive.

For ease of writing we now write p_1 and p_2 for $|\underline{p}_1|$ and $|\underline{p}_2|$, respectively. Then with

$$\begin{aligned} \varepsilon_1 &= \sqrt{p_1^2 + m^2} \\ \varepsilon_2 &= \sqrt{p_2^2 + m^2} \end{aligned} \tag{43}$$

$$\mu = \frac{M^{*2} - M^2}{2M} \tag{44}$$

we define

$$\lambda = \varepsilon_1 \varepsilon_2 - p_1 p_2 \cos\theta - m^2$$

$$|\underline{u}| = (p_1^2 - 2p_1 p_2 \cos\theta + p_2^2)^{1/2}$$

$$\kappa = \varepsilon_1 - \varepsilon_2 - \mu - \frac{\lambda}{M} \tag{45}$$

$$x_m = \frac{2[\lambda + (\varepsilon_1 - \varepsilon_2 - \mu)(\varepsilon_1 - \varepsilon_2 - |\underline{u}|)]}{1 + \dfrac{\varepsilon_1 - \varepsilon_2 - |\underline{u}|}{M}}$$

$$x_M = \frac{2[\lambda + (\varepsilon_1 - \varepsilon_2 - \mu)(\varepsilon_1 - \varepsilon_2 + |\underline{u}|)]}{1 + \dfrac{\varepsilon_1 - \varepsilon_2 + |\underline{u}|}{M}} \tag{46}$$

$$A_1 = \left(p_2 + \frac{\varepsilon_1 p_2 - \varepsilon_2 p_1 \cos\theta}{M}\right)^2 + \left(\frac{mp_1 \sin\theta}{M}\right)^2$$

$$B_1 = 2p_1[p_1 p_2 - (\varepsilon_1\varepsilon_2 - m^2)\cos\theta]\left[p_2 + \frac{\varepsilon_1 p_2 - \varepsilon_2 p_1 \cos\theta}{M}\right] + \frac{2m^2 p_1{}^2(\varepsilon_1 - \varepsilon_2)\sin^2\theta}{M}$$

$$-2\mu\left[p_2(\varepsilon_1 p_2 - \varepsilon_2 p_1 \cos\theta) + \frac{\lambda(\lambda + 2m^2)}{M}\right]$$

$$C_1 = 4[p_1\lambda - \mu(p_1\varepsilon_2 - \varepsilon_1 p_2 \cos\theta)]^2 + 4(\mu m p_2 \sin\theta)^2$$

$$x_1 = \frac{B_1}{A_1} \tag{47}$$

$$A_2 = \left(p_1 - \frac{(\varepsilon_2 p_1 - \varepsilon_1 p_2 \cos\theta)}{M}\right)^2 + \left(\frac{mp_2 \sin\theta}{M}\right)^2$$

$$B_2 = 2p_2[p_1 p_2 - (\varepsilon_1\varepsilon_2 - m^2)\cos\theta]\left[p_1 - \frac{(\varepsilon_2 p_1 - \varepsilon_1 p_2 \cos\theta)}{M}\right] + \frac{2m^2 p_2{}^2(\varepsilon_1 - \varepsilon_2)\sin^2\theta}{M}$$

$$+2\mu\left[p_1(\varepsilon_2 p_1 - \varepsilon_1 p_2 \cos\theta) + \frac{\lambda(\lambda + 2m^2)}{M}\right]$$

$$C_2 = 4[p_2\lambda + \mu(p_2\varepsilon_1 - \varepsilon_2 p_1 \cos\theta)]^2 + 4(\mu m p_1 \sin\theta)^2$$

$$x_2 = \frac{B_2}{A_2} \tag{48}$$

$$\zeta = 2mp_1 p_2 \kappa \sin\theta \tag{49a}$$

$$\eta_1 = \frac{\zeta}{A_1} \tag{49b}$$

$$\eta_2 = \frac{\zeta}{A_2} \tag{49c}$$

$$D_1(x) = A_1[(x-x_1)^2 + \eta_1{}^2] = A_1 x^2 - 2B_1 x + C_1 \tag{50a}$$

$$D_2(x) = A_2[(x-x_2)^2 + \eta_2{}^2] = A_2 x^2 - 2B_2 x + C_2 \tag{50b}$$

$$\beta_0 = 8m^2\varepsilon_1(\varepsilon_1-\mu)\lambda[p_1(p_1-p_2\cos\theta) - (\varepsilon_1+\varepsilon_2)\mu]$$

$$\beta_1 = -2m^2\Big[2\varepsilon_1(\varepsilon_1-\mu)\Big(p_1(p_1-p_2\cos\theta) - (\varepsilon_1+\varepsilon_2)(\kappa+\mu)\Big)$$

$$+ \lambda\Big(p_1(p_1-p_2\cos\theta) - (\varepsilon_1+\varepsilon_2)\mu\Big)\Big(1 + \frac{2\varepsilon_1}{M}\Big)\Big]$$

$$\beta_2 = m^2[p_1(p_1-p_2\cos\theta) - (\varepsilon_1+\varepsilon_2)(\kappa+\mu)](1 + \frac{2\varepsilon_1}{M})$$

$$\delta_0 = -8m^2\varepsilon_2(\varepsilon_2+\mu)\lambda[p_2(p_2-p_1\cos\theta) + (\varepsilon_1+\varepsilon_2)\mu]$$

$$\delta_1 = 2m^2\Big[2\varepsilon_2(\varepsilon_2+\mu)\Big(p_2(p_2-p_1\cos\theta) + (\varepsilon_1+\varepsilon_2)(\kappa+\mu)\Big)$$

$$+ \lambda\Big(p_2(p_2-p_1\cos\theta) + (\varepsilon_1+\varepsilon_2)\mu\Big)\Big(1 - \frac{2\varepsilon_2}{M}\Big)\Big]$$

$$\delta_2 = -m^2[p_2(p_2-p_1\cos\theta) + (\varepsilon_1+\varepsilon_2)(\kappa+\mu)](1 - \frac{2\varepsilon_2}{M})$$

$$\tau_1 = \frac{1}{\lambda}\,[\lambda^2 - 4m^2\varepsilon_1\varepsilon_2 - 2\mu m^2(\varepsilon_1-\varepsilon_2-\mu) - 2\mu\varepsilon_2\lambda]$$

$$\tau_2 = \frac{1}{\lambda}\,[\lambda^2 - 4m^2\varepsilon_1\varepsilon_2 - 2\mu m^2(\varepsilon_1-\varepsilon_2-\mu) + 2\mu\varepsilon_1\lambda]$$

$$\rho_1 = \frac{1}{\lambda^2}\begin{bmatrix}(\lambda+m^2)(2\varepsilon_1\varepsilon_2-\lambda) + \lambda\kappa^2 + (\lambda+m^2)\big((\lambda/M) + \mu\big)\kappa \\ + (m^2/M^2)\lambda^2 + \tfrac{1}{2}\lambda^2 + (\lambda^2\varepsilon_1/M)\end{bmatrix}$$

$$\rho_2 = \frac{1}{\lambda^2}\begin{bmatrix}(\lambda+m^2)(2\varepsilon_1\varepsilon_2-\lambda) + \lambda\kappa^2 + (\lambda+m^2)\big((\lambda/M) + \mu\big)\kappa \\ + (m^2/M^2)\lambda^2 + \tfrac{1}{2}\lambda^2 - (\lambda^2\varepsilon_2/M)\end{bmatrix}$$

$$\gamma = \frac{1}{\lambda^2}\,[(\lambda+m^2)(2\varepsilon_1\varepsilon_2-\lambda) + \lambda\kappa^2 + (\lambda+m^2)\big((\lambda/M) + \mu\big)\kappa]$$

$$\eta = 1 \qquad (51)$$

$$\beta_0' = -8m^4\lambda[p_1(p_1-p_2\cos\theta) - (\varepsilon_1+\varepsilon_2)\mu]$$

$$\beta_1' = 4m^2[m^2(p_1(p_1-p_2\cos\theta) - (\varepsilon_1+\varepsilon_2)(\kappa+\mu)) + \lambda(p_1(p_1-p_2\cos\theta) - (\varepsilon_1+\varepsilon_2)\mu)]$$

$$\beta_2' = -2m^2[p_1(p_1-p_2\cos\theta) - (\varepsilon_1+\varepsilon_2)(\kappa+\mu)]$$

$$\delta_0' = 8m^4\lambda[p_2(p_2-p_1\cos\theta) + (\varepsilon_1+\varepsilon_2)\mu]$$

$$\delta_1' = -4m^2[m^2(p_2(p_2-p_1\cos\theta) + (\varepsilon_1+\varepsilon_2)(\kappa+\mu)) + \lambda(p_2(p_2-p_1\cos\theta) + (\varepsilon_1+\varepsilon_2)\mu)]$$

$$\delta_2' = 2m^2[p_2(p_2-p_1\cos\theta) + (\varepsilon_1+\varepsilon_2)(\kappa+\mu)]$$

$$\tau_1' = \tau_2' = -2\,\frac{[\lambda^2 - 2m^2(\lambda + m^2)]}{\lambda}$$

$$\rho_1' = \rho_2' = \frac{(\lambda^2 - 2m^4)}{\lambda^2}$$

$$\gamma' = \frac{2(\lambda^2 - m^4)}{\lambda^2}$$

$$\eta' = -2 \quad . \tag{52}$$

The important integrals are then

$$J_{1,0} = \frac{1}{\sqrt{A_1}}\,\mathrm{sgn}(A_1x_i - B_1)\,\ln\left(\frac{|A_1x_i - B_1| + D_1^{\frac{1}{2}}(x_i)\sqrt{A_1}}{\zeta}\right)$$

$$- \frac{1}{\sqrt{A_1}}\,\mathrm{sgn}(A_1x_{i-1} - B_1)\,\ln\left(\frac{|A_1x_{i-1} - B_1| + D_1^{\frac{1}{2}}(x_{i-1})\sqrt{A_1}}{\zeta}\right) \tag{53}$$

and

$$J_{1,-1} = \frac{1}{\sqrt{C_1}} \operatorname{sgn}(B_1 x_i - C_1) \ \ln\left(\frac{|B_1 x_i - C_1| + D_1^{\frac{1}{2}}(x_i)\sqrt{C_1}}{x_i \zeta}\right)$$

$$- \frac{1}{\sqrt{C_1}} \operatorname{sgn}(B_1 x_{i-1} - C_1) \ \ln\left(\frac{|B_1 x_{i-1} - C_1| + D_1^{\frac{1}{2}}(x_{i-1})\sqrt{C_1}}{x_{i-1} \zeta}\right) \tag{54}$$

where

$$\operatorname{sgn}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases} \quad . \tag{55}$$

Then from $J_{1,0}$ and $J_{1,-1}$ we have

$$J_{1,1} = \frac{1}{A_1} \left[D_1^{\frac{1}{2}}(x_i) - D_1^{\frac{1}{2}}(x_{i-1})\right] + \frac{B_1}{A_1} J_{1,0} \tag{56}$$

$$J_{1,2} = \frac{1}{2A_1} \left[x_i D_1^{\frac{1}{2}}(x_i) - x_{i-1} D_1^{\frac{1}{2}}(x_{i-1})\right] + \frac{3B_1}{2A_1} J_{1,1} - \frac{C_1}{2A_1} J_{1,0} \tag{57}$$

$$J_{1,-2} = - \frac{1}{C_1} \left[\frac{D_1^{\frac{1}{2}}(x_i)}{x_i} - \frac{D_1^{\frac{1}{2}}(x_{i-1})}{x_{i-1}}\right] + \frac{B_1}{C_1} J_{1,-1} \tag{58}$$

Then the functions $K_{1,m}$ are given by

$$K_{1,0} = \frac{1}{\zeta^2} \left[\frac{(A_1 x_i - B_1)}{D_1^{\frac{1}{2}}(x_i)} - \frac{(A_1 x_{i-1} - B_1)}{D_1^{\frac{1}{2}}(x_{i-1})}\right] \tag{59}$$

$$K_{1,1} = - \frac{1}{\zeta^2} \left[\frac{(-B_1 x_i + C_1)}{D_1^{\frac{1}{2}}(x_i)} - \frac{(-B_1 x_{i-1} + C_1)}{D_1^{\frac{1}{2}}(x_{i-1})}\right] \tag{60}$$

$$K_{1,2} = \frac{1}{A_1} [J_{1,0} + 2B_1K_{1,1} - C_1K_{1,0}] \tag{61}$$

$$K_{1,3} = \frac{1}{A_1} \left[\frac{x_i^{\,2}}{D_1^{\frac{1}{2}}(x_i)} - \frac{x_{i-1}^{\,2}}{D_1^{\frac{1}{2}}(x_{i-1})} \right] + \frac{3B_1}{A_1} K_{1,2} - \frac{2C_1}{A_1} K_{1,1} \tag{62}$$

$$K_{1,-1} = \frac{1}{C_1} \left[\frac{1}{D_1^{\frac{1}{2}}(x_i)} - \frac{1}{D_1^{\frac{1}{2}}(x_{i-1})} \right] + \frac{1}{C_1} J_{1,-1} + \frac{B_1}{C_1} K_{1,0} \tag{63}$$

$$K_{1,-2} = -\frac{1}{C_1} \left[\frac{1}{x_i D_1^{\frac{1}{2}}(x_i)} - \frac{1}{x_{i-1} D_1^{\frac{1}{2}}(x_{i-1})} \right] + \frac{3B_1}{C_1} K_{1,-1} - \frac{2A_1}{C_1} K_{1,0} \,. \tag{64}$$

The integrals $J_{2,m}$ and $K_{2,m}$ can all be obtained from $J_{1,m}$ and $K_{2,m}$ merely by replacing all subscripts 1 by the subscript 2 in the expressions above.

Finally, we have the integral L. In order to write this without cancellations, we define[7]

$$\beta \equiv D_1^{\frac{1}{2}}(2\lambda) = D_2^{\frac{1}{2}}(2\lambda) = 2\kappa\sqrt{\lambda(\lambda + 2m^2)} \tag{65}$$

and

$$\xi_{1,i} = (2\lambda A_1 - B_1)x_i - 2\lambda B_1 + C_1$$

$$\xi_{2,i} = (2\lambda A_2 - B_2)x_i - 2\lambda B_2 + C_2$$

$$\xi_{1,i-1} = (2\lambda A_1 - B_1)x_{i-1} - 2\lambda B_1 + C_1$$

$$\xi_{2,i-1} = (2\lambda A_2 - B_2)x_{i-1} - 2\lambda B_2 + C_2 \tag{66}$$

[7]See pages 51 and 52 of the appendix, leading to eq. (A.59).

Then

$$
\begin{aligned}
L = {} & \frac{1}{\beta}\,\mathrm{sgn}(\xi_{1,i})\ \ln\left(\frac{|\xi_{1,i}| + \beta D_1^{1/2}(x_i)}{\zeta}\right) \\
& - \frac{1}{\beta}\,\mathrm{sgn}(\xi_{1,i-1})\ \ln\left(\frac{|\xi_{1,i-1}| + \beta D_1^{1/2}(x_{i-1})}{\zeta}\right) \\
& - \frac{1}{\beta}\,\mathrm{sgn}(\xi_{2,i})\ \ln\left(\frac{|\xi_{2,i}| + \beta D_2^{1/2}(x_i)}{\zeta}\right) \\
& + \frac{1}{\beta}\,\mathrm{sgn}(\xi_{2,i-1})\ \ln\left(\frac{|\xi_{2,i-1}| + \beta D_2^{1/2}(x_{i-1})}{\zeta}\right) \\
& - \frac{1}{\beta}\left(\mathrm{sgn}(\xi_{1,i}) - \mathrm{sgn}(\xi_{2,i})\right)\ln|x_i - 2\lambda| \\
& + \frac{1}{\beta}\left(\mathrm{sgn}(\xi_{1,i-1}) - \mathrm{sgn}(\xi_{2,i-1})\right)\ln|x_{i-1} - 2\lambda| \quad . \qquad (67)
\end{aligned}
$$

APPENDIX

Algebraic Details

In this appendix we present some of the detailed algebra leading to our final expression for the cross section, in particular to eqs (27), (28), (38), and (44)-(67).

Our starting point is the cross section integrated over the photon azimuthal angle φ_k, given here by eqs (22) and (23). As we have noted, this is the expression given by Mo and Tsai [2], and Tsai [3], and Stein et al. [7], apart from the minor modifications and typographical errors which we have already discussed. Since we perform the remaining integration analytically, the most appropriate variable of integration is q^2 rather than $\cos\theta_k$ in view of the functions $W_2(-q^2)$ and $W_1(-q^2)$. Further, as we have already noted, the form of the integrand itself is then also somewhat simpler. Referring to eqs (22), (23), and (24), we see that our first task is then to express the quantities listed in (24), as well as the photon energy, ω, in terms of the variable of integration, q^2, and to derive the differential change of variable given by (25). We see that in (24) the variable $\cos\theta_k$ enters only either as ω or as $\omega\cos\theta_k$. (The quantity b enters only as

$$b^2 = p_2{}^2\sin^2\theta_p\omega^2\sin^2\theta_k$$

$$= p_2{}^2\sin^2\theta_p[\omega^2 - (\omega\cos\theta_k)^2]$$

Thus again we have either ω or $\omega\cos\theta_k$.)

Further, we note that the expressions for $\cos\theta_p$ and $\cos\theta_s$ listed in (24) follow from

$$\underline{p}_1 \cdot \underline{u} = \underline{p}_1 \cdot (\underline{p}_1 - \underline{p}_2)$$

$$= p_1^2 - p_1 p_2 \cos\theta$$

$$= p_1 |\underline{u}| \cos\theta_s \qquad (p_1 = |\underline{p}_1|) \qquad \text{(A.1)}$$

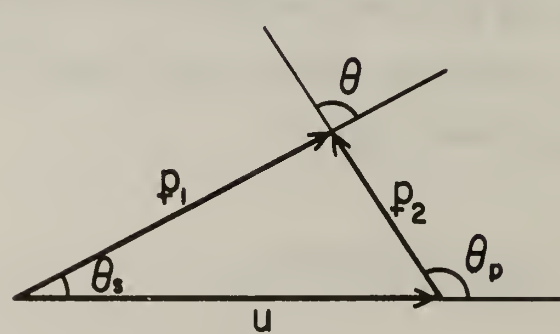
and

$$\underline{p}_2 \cdot \underline{u} = \underline{p}_2 \cdot (\underline{p}_1 - \underline{p}_2)$$

$$= p_1 p_2 \cos\theta - p_2^2$$

$$= p_2 |\underline{u}| \cos\theta_p \qquad (p_2 = |\underline{p}_2|) \ . \qquad \text{(A.2)}$$

In addition, it is useful to note from $\underline{u} = \underline{p}_1 - \underline{p}_2$ and the figure below,



that

$$\theta_p = \theta_s + \theta \qquad \text{(A.3)}$$

and

$$\frac{p_2}{\sin\theta_s} = \frac{p_1}{\sin\theta_p} = \frac{|\underline{u}|}{\sin\theta} \quad . \tag{A.4}$$

Thus

$$p_2 \sin\theta_p = p_1 \sin\theta_s \tag{A.5}$$

and

$$|\underline{u}|\sin\theta_p = p_1 \sin\theta \tag{A.6}$$

$$|\underline{u}|\sin\theta_s = p_2 \sin\theta \quad . \tag{A.7}$$

The Jacobian expressed by eq (25) follows directly from eqs (11) and (15). From eq (11), viz.,

$$\omega = \frac{u^2 - M^{*2}}{2(u_0 - |\underline{u}|\cos\theta_k)} \quad ,$$

we have

$$d\omega = \frac{(u^2 - M^{*2})|\underline{u}|d(\cos\theta_k)}{2(u_0 - |\underline{u}|\cos\theta_k)^2}$$

$$= \frac{\omega|\underline{u}|\ d(\cos\theta_k)}{(u_0 - |\underline{u}|\ \cos\theta_k)} \quad ,$$

and from eq (15), viz.,

$$\omega = \varepsilon_1 - \varepsilon_2 - \mu + \frac{q^2}{2M} \quad ,$$

we have

$$d\omega = \frac{1}{2M} d(q^2) \quad ,$$

so that

$$\frac{\omega|\underline{u}|\ d(\cos\theta_k)}{(u_0 - |\underline{u}|\ \cos\theta_k)} = \frac{1}{2M} d(q^2)$$

or

$$\frac{2M\omega\ d(\cos\theta_k)}{q^4(u_0 - |\underline{u}|\cos\theta_k)} = \frac{1}{|\underline{u}|} \frac{d(q^2)}{q^4} \quad ,$$

which is eq (25).

Next we express ω and $\omega\cos\theta_k$ in terms of q^2. In eqs (12)-(15) we have shown that

$$\omega = \varepsilon_1 - \varepsilon_2 - \mu + \frac{q^2}{2M} \quad , \tag{A.8}$$

in which the quantities $\varepsilon_1, \varepsilon_2, \mu$, and M are defined in (4). Next, from eq (6) we have (four-vectors are implied in the next two lines)

$$q^2 = (p_1 - p_2 - k)^2$$

$$= (p_1 - p_2)^2 - 2k\cdot(p_1 - p_2)$$

$$= -2\lambda - 2\omega(\varepsilon_1 - \varepsilon_2) + 2\omega|\underline{u}|\cos\theta_k \qquad (A.9)$$

where, as in (45),

$$\lambda = \varepsilon_1\varepsilon_2 - \underline{p}_1\cdot\underline{p}_2 - m^2 \quad .$$

Thus, using (A.8),

$$\omega\cos\theta_k = \frac{1}{|\underline{u}|}\left[\lambda + (\varepsilon_1-\varepsilon_2)(\varepsilon_1-\varepsilon_2-\mu) + \tfrac{1}{2}q^2\left(1 + \frac{\varepsilon_1-\varepsilon_2}{M}\right)\right] \quad . \qquad (A.10)$$

From (A.8) and (A.10) we note that both ω and $\omega\cos\theta_k$ are linear functions of q^2. Thus, referring to (24), we see that a and a' are also linear functions of q^2, and that $a^2 - b^2$ and $a'^2 - b^2$ are quadratic functions of q^2.

The algebra involved in our calculation is reduced considerably by noting that upon making the substitutions

$$\text{subscripts } (1 \to 2,\ 2 \to 1), \quad \mu \to -\mu, \quad M \to -M \qquad (A.11)$$

we have, from (A.8) and (A.10),

$$\omega \to -\omega$$

and

$$\omega\cos\theta_k \to \omega\cos\theta_k \qquad \text{(i.e., } \omega\cos\theta_k \text{ does not change),}$$

and from (A.1) and (A.2)

$$\cos\theta_p \to -\cos\theta_s \quad , \quad \cos\theta_s \to -\cos\theta_p .$$

Thus when a and a' in (24) are expressed in terms of q^2 as the sole variable, we may obtain a' from a simply by making the substitutions indicated in (A.11) and introducing an overall minus sign. That is, under (A.11)

$$a \to -a' \quad , \quad a' \to -a \quad .$$

In similar fashion we see from (A.5) that b^2 as defined in (24) remains unchanged under the substitutions of (A.11). Thus under (A.11) we also have

$$a^2 - b^2 \to a'^2 - b^2 \quad .$$

We thus need only write a and $a^2 - b^2$ in terms of q^2, and then use (A.11)-(A.13) to obtain a' and $a'^2 - b^2$.

We start then with a as given in (24), viz.,

$$a = \varepsilon_2\omega - p_2\cos\theta_p\,\omega\cos\theta_k$$

In this expression we then substitute ω as given by (A.8), $\cos\theta_p$ as given by (24) and $\omega\cos\theta_k$ as given by (A.10). After some tedious but straightforward algebra we find

$$a = \frac{1}{\underline{u}^2}\left\{\begin{array}{l}[p_1(p_1-p_2\cos\theta) - (\varepsilon_1+\varepsilon_2)\mu]\lambda \\ + \frac{1}{2}q^2[p_1(p_1-p_2\cos\theta) - (\varepsilon_1+\varepsilon_2)(\kappa+\mu)]\end{array}\right\} \quad . \qquad (A.14)$$

Then, applying (A.11) and (A.12) we have

$$a' = -\frac{1}{\underline{u}^2}\left\{\begin{array}{l}[p_2(p_2-p_1\cos\theta) + (\varepsilon_1+\varepsilon_2)\mu]\lambda \\ \\ + \frac{1}{2}q^2[p_2(p_2-p_1\cos\theta) + (\varepsilon_1+\varepsilon_2)(\kappa+\mu)]\end{array}\right\} \tag{A.15}$$

where λ and κ are given by eq (43) in the body of this report, viz.,

$$\lambda = \varepsilon_1\varepsilon_2 - p_1p_2\cos\theta - m^2$$

$$\kappa = \varepsilon_1 - \varepsilon_2 - \mu - \frac{\lambda}{M} \tag{A.16}$$

We next consider $a^2 - b^2$, which, as we have noted, is a quadratic function of q^2. The quantity b is given in (24), viz.,

$$b = -\omega p_2\sin\theta_p\sin\theta_k$$

and from (A.6) this may be written as

$$b = -\frac{\omega p_1p_2}{|\underline{u}|}\sin\theta\sin\theta_k \quad ,$$

from which

$$b^2 = \frac{(p_1p_2\sin\theta)^2}{\underline{u}^2}\left[\omega^2 - (\omega\cos\theta_k)^2\right] \quad .$$

We then substitute in this expression ω and $\omega\cos\theta_k$ as given by (A.8) and (A.10), and obtain b^2 as a quadratic function of q^2. With a as given by (A.14) we then have then have, after much tedious but straightforward algebra, the following expression for $a^2 - b^2$:

$$a^2 - b^2 = \frac{1}{4\underline{u}^2} \{A_1(q^2)^2 + 2B_1q^2 + C_1\} \tag{A.17}$$

where A_1, B_1, and C_1 are given by eqs (47). It should be noted from eqs (47) (in particular for the integrations that follow) that

$$A_1 > 0 \quad , \quad C_1 > 0 \quad . \tag{A.18}$$

With

$$x = -q^2 > 0$$

as defined earlier in eq (26), we then write

$$a^2 - b^2 = \frac{1}{4\underline{u}^2} D_1(x) \tag{A.19}$$

where

$$D_1(x) = A_1x^2 - 2B_1x + C_1$$

$$= A_1[(x - x_1)^2 + \eta_1{}^2] \tag{A.20}$$

with

$$x_1 = \frac{B_1}{A_1} \tag{A.21}$$

and

$$\eta_1{}^2 = \frac{A_1C_1 - B_1{}^2}{A_1{}^2} \quad . \tag{A.22}$$

With A_1, B_1, and C_1 as given by (47) we find, after considerable algebra,

$$A_1C_1 - B_1{}^2 = (2mp_1p_2\kappa \sin\theta)^2 \equiv \zeta^2 > 0 \quad . \tag{A.23}$$

Again making the substitutions (A.11), we obtain

$$a'^2 - b^2 = \frac{1}{4\underline{u}^2} \{A_2(q^2)^2 + 2B_2q^2 + C_2\} \tag{A.24}$$

with A_2, B_2, and C_2 as given by eqs (48). We now write

$$a'^2 - b^2 = \frac{1}{4\underline{u}^2} D_2(x) \tag{A.25}$$

where

$$D_2(x) = A_2x^2 - 2B_2x + C_2$$

$$= A_2[(x - x_2)^2 + \eta_2{}^2] \tag{A.26}$$

with

$$x_2 = \frac{B_2}{A_2} \tag{A.27}$$

and

$$\eta_2{}^2 = \frac{A_2C_2 - B_2{}^2}{A_2{}^2} \quad . \tag{A.28}$$

From (A.23) we see that $A_1C_1 - B_1{}^2$ is unchanged under the substitutions (A.11). Thus

$$A_2C_2 - B_2{}^2 = A_1C_1 - B_1{}^2 = \zeta^2 \quad . \tag{A.29}$$

Next we consider the terms in (23) with factor ν. From eqs (24), (A.14), and (A.15) we have

$$\begin{aligned} \nu &= \frac{1}{a'-a} \\ &= \frac{-\underline{u}^2}{(\tfrac{1}{2}q^2 + \lambda)\left[p_2(p_2-p_1\cos\theta) + p_1(p_1-p_2\cos\theta)\right]} \\ &= \frac{-2}{q^2 + 2\lambda} \end{aligned} \tag{A.30}$$

since, from eq (8b),

$$\underline{u}^2 = p_1{}^2 - 2p_1p_2\cos\theta + p_2{}^2 \quad .$$

Thus the terms with factor ν in the integral (22) have an overall factor

$$\frac{1}{q^4(q^2 + 2\lambda)} \quad .$$

As may be seen by referring to these terms in (23), they also have a coefficient of the form $c_0 + c_1 q^2 + c_2 (q^2)^2$. (For the terms multiplying $W_1(-q^2)$ we have $c_1 = c_2 = 0$.) Then, to obtain our expression (28), we write

$$\frac{c_0 + c_1 q^2 + c_2 (q^2)^2}{q^4 (q^2 + 2\lambda)} = \left(\frac{c_0 - 2\lambda c_1 + (2\lambda)^2 c_2}{(2\lambda)^2} \right) \frac{1}{(q^2 + 2\lambda)}$$

$$- \left(\frac{c_0 - 2\lambda c_1}{(2\lambda)^2} \right) \frac{1}{q^2} + \left(\frac{c_0}{2\lambda} \right) \frac{1}{(q^2)^2} \quad . \tag{A.31}$$

Finally we substitute (A.14), (A.15), (A.19), (A.25), and (A.30) with $x = - q^2$, in the cross section given by (22) and (23), using (A.31) for the terms with factor ν. Collecting the terms with similar denominators then gives the expression (28).

We derive next the expression given in (46) for the limits of integration, x_m and x_M. From (11), viz.,

$$\omega = \frac{u^2 - M^{*2}}{2\left(u_0 - |\underline{u}| \cos\theta_k\right)} \quad , \tag{11}$$

we see that ω is an increasing function of $\cos\theta_k$, and from (15), viz.,

$$\omega = \varepsilon_1 - \varepsilon_2 - \mu + \frac{q^2}{2M} \quad . \tag{15}$$

it follows that q^2 is also an increasing function of $\cos\theta_k$, and hence that $x = -q^2$ is a decreasing function of $\cos\theta_k$. Thus the minimum value of x, viz.,

$$x = x_m = -q_m{}^2 \qquad \text{corresponds to } \cos\theta_k = + 1$$

and the maximum value of x, viz.,

$$x = x_M = -q_M{}^2 \qquad \text{corresponds to } \cos\theta_k = -1 \ .$$

We now compare eq (15) given just above with eq (A.10), viz.,

$$\omega\cos\theta_k = \frac{1}{|\underline{u}|}[\lambda + (\varepsilon_1-\varepsilon_2)(\varepsilon_1-\varepsilon_2-\mu) + \tfrac{1}{2}q^2(1 + \frac{\varepsilon_1-\varepsilon_2}{M})] \ . \qquad \text{(A.10)}$$

When $\cos\theta_k = +1$ these two equations give

$$\varepsilon_1-\varepsilon_2-\mu + \frac{q_m{}^2}{2M} = \frac{1}{|\underline{u}|}[\lambda + (\varepsilon_1-\varepsilon_2)(\varepsilon_1-\varepsilon_2-\mu) + \tfrac{1}{2}q_m{}^2(1 + \frac{\varepsilon_1-\varepsilon_2}{M})]$$

or

$$x_m = \frac{2[\lambda + (\varepsilon_1-\varepsilon_2-|\underline{u}|)(\varepsilon_1-\varepsilon_2-\mu)]}{1 + \frac{\varepsilon_1-\varepsilon_2-|\underline{u}|}{M}}$$

as in (46). When $\cos\theta_k = -1$ these two equations give

$$\varepsilon_1-\varepsilon_2-\mu + \frac{q_M{}^2}{2M} = -\frac{1}{|\underline{u}|}[\lambda + (\varepsilon_1-\varepsilon_2)(\varepsilon_1-\varepsilon_2-\mu) + \tfrac{1}{2}q_M{}^2(1 + \frac{\varepsilon_1-\varepsilon_2}{M})]$$

or

$$x_M = \frac{2[\lambda + (\varepsilon_1-\varepsilon_2+|\underline{u}|)(\varepsilon_1-\varepsilon_2-\mu)]}{1 + \frac{\varepsilon_1-\varepsilon_2+|\underline{u}|}{M}}$$

as in (46).

We next show that the point $x = -q^2 = 2\lambda$, where, as in (45),

$$\lambda = \varepsilon_1\varepsilon_2 - p_1p_2\cos\theta - m^2 \quad ,$$

lies within the range of integration:

$$x_m < 2\lambda < x_M \quad ,$$

as noted in (29). Now from (A.9) we see that $x = -q^2 = 2\lambda$ implies

$$\cos\theta_k = \frac{\varepsilon_1 - \varepsilon_2}{|\underline{u}|} \quad . \tag{A.32}$$

Now as we show immediately below,

$$0 < \frac{\varepsilon_1 - \varepsilon_2}{|\underline{u}|} < 1 \quad . \tag{A.33}$$

Thus since the integration region, when we choose $\cos\theta_k$ as integration variable, is the entire range of possible values, $-1 < \cos\theta_k < 1$, we see from (A.32) and (A.33) that the point $x = 2\lambda$ corresponds to a point in the range of integration with x as integration variable (recall that x is a uniformly decreasing function of $\cos\theta_k$), i.e., that

$$x_m < 2\lambda < x_M \quad .$$

Further, since the photon energy, given by (15), must be positive for $\cos\theta_k$ within the range of integration, i.e.,

$$\omega = \varepsilon_1 - \varepsilon_2 - \mu + \frac{q^2}{2M} > 0 \quad ,$$

it follows that this is also true for the point $-q^2 = 2\lambda$ just considered. Thus

$$\kappa = \varepsilon_1 - \varepsilon_2 - \mu - \frac{\lambda}{M} > 0 \quad . \tag{A.34}$$

Returning now to the proof of (A.33), we note from $\varepsilon_2 < \varepsilon_1$ (and $p_2 < p_1$) that

$$\varepsilon_1 - p_1 = \frac{m^2}{\varepsilon_1 + p_1} < \frac{m^2}{\varepsilon_2 + p_2} = \varepsilon_2 - p_2 \quad .$$

Thus

$$0 < \varepsilon_1 - \varepsilon_2 < p_1 - p_2 \leq |\underline{p}_1 - \underline{p}_2| = |\underline{u}| \quad ,$$

from which (A.33) follows.

Finally, we conclude our presentation of algebraic details with a consideration of the expressions given in eqs (53)-(67) for the integrals defined in eqs (39)-(41). We begin with $J_{1,0}$, defined by eq (39), viz.,

$$J_{1,0} = \int_{x_{i-1}}^{x_i} \frac{dx}{D_1^{1/2}(x)} \tag{A.35}$$

where from eq (50a)

$$D_1(x) = A_1 x^2 - 2B_1 x + C_1 \quad .$$

A straighforward integration gives [9]

$$J_{1,0} = \frac{1}{\sqrt{A_1}} \ln\left(\frac{A_1 x_i - B_1 + D_1^{\frac{1}{2}}(x_i)\sqrt{A_1}}{c_0}\right)$$

$$- \frac{1}{\sqrt{A_1}} \ln\left(\frac{A_1 x_{i-1} - B_1 + D_1^{\frac{1}{2}}(x_{i-1})\sqrt{A_1}}{c_0}\right) \qquad (A.36)$$

where c_0 is an arbitrary constant. Now for reasons that will be apparent momentarily, we choose

$$c_0 = \sqrt{A_1 C_1 - B_1{}^2}$$

$$\equiv \zeta = 2mp_1 p_2 \kappa \sin\theta > 0 \qquad (A.37)$$

from (A.23) and (A.34). The arguments of the logarithms in (A.36) are then dimensionless and, from eqs. (47) and (A.23), are of order ε/m if $A_1 x - B_1 > 0$ (where $x = x_i$ or x_{i-1}). On the other hand, since

$$D_1(x)A_1 - (A_1 x - B_1)^2 = (A_1 x^2 - 2B_1 x + C_1)A_1 - (A_1 x - B_1)^2$$

$$= A_1 C_1 - B_1{}^2 \qquad (A.38)$$

we have

$$\frac{A_1 x - B_1 + D_1^{\frac{1}{2}}(x)\sqrt{A_1}}{\zeta} = \frac{\zeta}{-(A_1 x - B_1) + D_1^{\frac{1}{2}}(x)\sqrt{A_1}} \ . \qquad (A.39)$$

Thus if $A_1x - B_1 < 0$ we can write the arguments of the logarithms in (A.36) in the form given by the right hand side of (A.39), from which we see that the argument is then of order m/ε. Thus if $A_1x - B_1 < 0$ there is a cancellation, leaving terms of relative order $(m/\varepsilon)^2$. We will therefore write the argument so that there is no cancellation, i.e., as given by the left hand side of (A.39) if $A_1x - B_1 > 0$ and by the right hand side of (A.39) if $A_1x - B_1 < 0$. These can be combined into a single expression by writing

$$J_{1,0} = \frac{1}{\sqrt{A_1}} \operatorname{sgn}(A_1x_i - B_1)\ \ell n \left(\frac{|A_1x_i - B_1| + D_1^{\frac{1}{2}}(x_i)\sqrt{A_1}}{\zeta}\right)$$

$$- \frac{1}{\sqrt{A_1}} \operatorname{sgn}(A_1x_{i-1} - B_1)\ \ell n \left(\frac{|A_1x_{i-1} - B_1| + D_1^{\frac{1}{2}}(x_{i-1})\sqrt{A_1}}{\zeta}\right)$$

where

$$\operatorname{sgn}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases} .$$

This is the result given in (53).

The integral $J_{1,-1}$ is also defined by (39), viz.,

$$J_{1,-1} = \int_{x_{i-1}}^{x_i} \frac{dx}{xD_1^{\frac{1}{2}}(x)} \tag{A.40}$$

Again, a straightforward integration gives [9]

$$J_{1,-1} = -\frac{1}{\sqrt{C_1}}\,\ell n\left(\frac{-B_1x_i + C_1 + D_1^{\frac{1}{2}}(x_i)\sqrt{C_1}}{x_i c_0'}\right)$$

$$+\frac{1}{\sqrt{C_1}}\,\ell n\left(\frac{-B_1x_{i-1} + C_1 + D_1^{\frac{1}{2}}(x_{i-1})\sqrt{C_1}}{x_{i-1} c_0'}\right) \qquad \text{(A.41)}$$

where c_0' is an arbitrary constant. In analogy with (A.38) we note that

$$D_1(x)C_1 - (-B_1x + C_1)^2 = (A_1x^2 - 2B_1x + C_1)C_1 - (-B_1x + C_1)^2$$

$$= x^2(A_1C_1 - B_1^2)$$

$$= x^2\zeta^2 \quad . \qquad \text{(A.42)}$$

Thus we again choose $c_0' = \zeta$, and note that

$$\frac{-B_1x + C_1 + D_1^{\frac{1}{2}}(x)\sqrt{C_1}}{x\zeta} = \frac{x\zeta}{-(-B_1x - C_1) + D_1^{\frac{1}{2}}(x)\sqrt{C_1}} \quad . \qquad \text{(A.43)}$$

Thus if $-B_1x + C_1 < 0$ we can write the arguments of the logarithms in (A.41) in the form given by the right hand side of (A.43), thus avoiding the cancellation. As before, we now write $J_{1,-1}$ in a form which avoids the possible cancellation in the arguments of the logarithms, viz.,

$$J_{1,-1} = -\frac{1}{\sqrt{C_1}}\,\mathrm{sgn}(-B_1x_i + C_1)\ \ln\left(\frac{|-B_1x_i + C_1| + D_1^{\frac{1}{2}}(x_i)\sqrt{C_1}}{x_i\zeta}\right)$$

$$+\frac{1}{\sqrt{C_1}}\,\mathrm{sgn}(-B_1x_{i-1} + C_1)\ \ln\left(\frac{|-B_1x_{i-1} + C_1| + D_1^{\frac{1}{2}}(x_{i-1})\sqrt{C_1}}{x_{i-1}\zeta}\right)$$

which is equivalent to the result given in (54).

The integrals $K_{1,0}$ and $K_{1,1}$, defined by (40) and given explicitely by (59) and (60), are elementary and present no particular difficulty. The expression of $J_{1,m}$ and $K_{1,m}$ in terms of $J_{1,0}$, $J_{1,-1}$, $K_{1,0}$, and $K_{1,1}$, given in (56)-(64), is also elementary and may be found, for example, in the integral tables of Gröbner and Hofreiter [9].

Lastly, we have the integral L, defined in (41), viz.,

$$L = \int_{x_{i-1}}^{x_i} \frac{1}{(2\lambda - x)}\left(\frac{1}{D_1^{1/2}(x)} - \frac{1}{D_2^{1/2}(x)}\right)dx \quad . \qquad (A.44)$$

We first note that

$$D_1(2\lambda) = D_2(2\lambda) \quad , \qquad (A.45)$$

and thus the integrand has in fact no singularity at $x = 2\lambda$. This may be seen most easily as follows: From (A.30)

$$a' - a = -\tfrac{1}{2}(q^2 + 2\lambda)$$

$$= \tfrac{1}{2}(x - 2\lambda) \quad , \qquad (A.46)$$

and from (A.19) and (A.25)

$$D_1(x) - D_2(x) = 4\underline{u}^2(a^2 - a'^2) \quad .$$

Thus

$$D_1(x) - D_2(x) = -2\underline{u}^2(a + a')(x - 2\lambda) \tag{A.47}$$

and

$$D_1(2\lambda) = D_2(2\lambda) \quad . \tag{A.48}$$

We define[8]

$$\beta \equiv D_1{}^{1/2}(2\lambda) = D_2{}^{1/2}(2\lambda) \quad . \tag{A.49}$$

In analogy with the procedure followed for $J_{1,0}$ and $J_{1,-1}$ we first perform the integration straightforwardly (see [9]) and write

$$\begin{aligned} L = {} & \frac{1}{\beta}\,\ell n\left(\frac{\xi_{1,i} + \beta D_1^{\frac{1}{2}}(x_i)}{|x_i - 2\lambda|\,\zeta}\right) \\ & - \frac{1}{\beta}\,\ell n\left(\frac{\xi_{1,i-1} + \beta D_1^{\frac{1}{2}}(x_{i-1})}{|x_{i-1} - 2\lambda|\,\zeta}\right) \\ & - \frac{1}{\beta}\,\ell n\left(\frac{\xi_{2,i} + \beta D_2^{\frac{1}{2}}(x_i)}{|x_i - 2\lambda|\,\zeta}\right) \\ & + \frac{1}{\beta}\,\ell n\left(\frac{\xi_{2,i-1} + \beta D_2^{\frac{1}{2}}(x_{i-1})}{|x_{i-1} - 2\lambda|\,\zeta}\right) \end{aligned} \tag{A.50}$$

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[8]See the end of this appendix for a derivation of the value of β.

where

$$\xi_1(x) = (2\lambda A_1 - B_1)x - 2\lambda B_1 + C_1$$

$$\xi_2(x) = (2\lambda A_2 - B_2)x - 2\lambda B_2 + C_2 \tag{A.51}$$

$$\xi_{1,i} = \xi_1(x_i) \quad , \quad \xi_{1,i-1} = \xi_1(x_{i-1})$$

$$\xi_{2,i} = \xi_2(x_i) \quad , \quad \xi_{2,i-1} = \xi_2(x_{i-1}) \tag{A.52}$$

Then noting that

$$\frac{\xi_1(x) + \beta D_1^{\frac{1}{2}}(x)}{|x-2\lambda|\zeta} = \frac{|x-2\lambda|\zeta}{-\xi_1(x) + \beta D_1^{\frac{1}{2}}(x)} \tag{A.53}$$

(and a similar equation in which all subscripts 1 are replaced by the subscript 2), we start by writing L in a form similar to that just used for $J_{1,0}$ and $J_{1,-1}$, so that there is no cancellation when any of the terms $\xi_{1,i}$, $\xi_{2,i}$, $\xi_{1,i-1}$, or $\xi_{2,i-1}$ is negative:

$$L = \frac{1}{\beta}\,\mathrm{sgn}(\xi_{1,i})\,\ln\left(\frac{|\xi_{1,i}| + \beta D_1^{\frac{1}{2}}(x_i)}{|x_i-2\lambda|\zeta}\right)$$

$$- \frac{1}{\beta}\,\mathrm{sgn}(\xi_{1,i-1})\,\ln\left(\frac{|\xi_{1,i-1}| + \beta D_1^{\frac{1}{2}}(x_{i-1})}{|x_{i-1}-2\lambda|\zeta}\right)$$

$$- \frac{1}{\beta}\,\mathrm{sgn}(\xi_{2,i})\,\ln\left(\frac{|\xi_{2,i}| + \beta D_2^{\frac{1}{2}}(x_i)}{|x_i-2\lambda|\zeta}\right)$$

$$+ \frac{1}{\beta}\,\mathrm{sgn}(\xi_{2,i-1})\,\ln\left(\frac{|\xi_{2,i-1}| + \beta D_2^{\frac{1}{2}}(x_{i-1})}{|x_{i-1}-2\lambda|\zeta}\right). \tag{A.54}$$

However, in this form of the expression for L there can be a large cancellation for x_i or x_{i-1} sufficiently close to 2λ, since, as we have noted, the integrand in the integral for L, (A.44), has no singularity for $x = 2\lambda$. We therefore write L in a form such that this cancellation is manifest. Equation (A.54) may simply be written in the form

$$L = L_i - L_{i-1} \tag{A.55}$$

where

$$\begin{aligned} L_i = & \frac{1}{\beta}\,\mathrm{sgn}(\xi_{1,i})\;\ell n\left(\frac{|\xi_{1,i}| + \beta D_1^{\frac{1}{2}}(x_i)}{\zeta}\right) \\ & - \frac{1}{\beta}\,\mathrm{sgn}(\xi_{2,i})\;\ell n\left(\frac{|\xi_{2,i}| + \beta D_2^{\frac{1}{2}}(x_i)}{\zeta}\right) \\ & - \frac{1}{\beta}\Big(\mathrm{sgn}(\xi_{1,i}) - \mathrm{sgn}(\xi_{2,i})\Big)\;\ell n|x_i - 2\lambda| \end{aligned} \tag{A.56}$$

and a similar expression with i replaced by i-1. This is the result given in eq (67). Now from (A.51) and eqs (50a) and (50b) we have

$$\begin{aligned} \xi_1(2\lambda) &= D_1(2\lambda) \\ \xi_2(2\lambda) &= D_2(2\lambda) \end{aligned} \tag{A.57}$$

and hence from (A.48) and eqs (50a) and (50b)

$$\xi_1(2\lambda) = \xi_2(2\lambda) > 0 \quad . \tag{A.58}$$

Therefore, for x_i sufficiently close to 2λ we have $\mathrm{sgn}(\xi_{1,i}) = +1$, $\mathrm{sgn}(\xi_{2,i}) = +1$ and thus L_i in (A.56) can be written more simply as

$$L_i = \frac{1}{\beta} \ln \left(\frac{|\xi_{1,i}| + \beta D_1^{\frac{1}{2}}(x_i)}{|\xi_{2,i}| + \beta D_2^{\frac{1}{2}}(x_i)} \right) \quad . \tag{A.59}$$

Here (i.e., when $\mathrm{sgn}(\xi_{1,i}) = \mathrm{sgn}(\xi_{2,i})$) there is clearly no singularity for $x_i = 2\lambda$. These observations can be used to avoid any possible problem arising in a computer program from x_i or x_{i-1} being chosen too close to 2λ. We may simply define L by (A.55), with L_i (and L_{i-1}) given by (A.59) if both $\xi_{1,i} > 0$ and $\xi_{2,i} > 0$, and otherwise given by (A.56). (It can be shown that we can not have both $\xi_{1,i} < 0$ and $\xi_{2,i} < 0$.)

Finally, we require the value of β, defined in (A.49), given in the body of the report in eq (65), and appearing in the equations for L, (A.50) and (A.54)-(A.59). From the definition, (A.49),

$$\beta = D_1^{\frac{1}{2}}(2\lambda) = D_2^{\frac{1}{2}}(2\lambda) \quad ,$$

it is clear that one can evaluate β simply by substituting $x = 2\lambda$ in the equations defining $D_1(x)$ and $D_2(x)$, (50a), and (50b). This is a rather cumbersome procedure, however. We therefore give the following simpler derivation:

For $x = 2\lambda$ we have

$$a' = a$$

$$D_1(2\lambda) = D_2(2\lambda) \quad ,$$

from (A.46) and (A.47), and

$$\omega = \kappa$$

from eqs (15) and (A.16). Further, also for $x = 2\lambda$, we have

$$\cos\theta_k = \frac{\varepsilon_1 - \varepsilon_2}{|\underline{u}|}$$

from (A.32), from which

$$\sin\theta_k = \frac{\sqrt{\underline{u}^2 - (\varepsilon_1 - \varepsilon_2)^2}}{|\underline{u}|}$$

$$= \frac{\sqrt{2\lambda}}{|\underline{u}|} \quad . \qquad \text{(A.60)}$$

Substituting these values for ω, $\cos\theta_k$, and $\sin\theta_k$ in the expressions (24) for a' and b we have, for $x = 2\lambda$,

$$a' = \frac{\kappa}{\underline{u}^2}\left[\varepsilon_1 \underline{u}^2 - p_1(\varepsilon_1 - \varepsilon_2)(p_1 - p_2\cos\theta)\right] \qquad \text{(A.61)}$$

and

$$b = -\frac{\kappa p_1 p_2 \sin\theta \sqrt{2\lambda}}{\underline{u}^2} \quad . \qquad \text{(A.62)}$$

In writing (A.61) we have used the expression for $\cos\theta_s$ given in (24), and in writing (A.62) we have used the expression for $\sin\theta_p$ given in (A.6). Recalling that $\underline{u}^2 = p_1^2 - 2p_1p_2\cos\theta + p_2^2$, a little algebra then gives

$$a' = \frac{\kappa}{\underline{u}^2}\,(\varepsilon_1+\varepsilon_2)(\varepsilon_1\varepsilon_2 - p_1p_2\cos\theta - m^2)$$

$$= \frac{\kappa\lambda(\varepsilon_1+\varepsilon_2)}{\underline{u}^2} \quad . \tag{A.63}$$

Substituting (A.63) and (A.62) in (A.25) we then have

$$D_2(2\lambda) = \frac{4\kappa^2\lambda}{\underline{u}^2}\left[\lambda(\varepsilon_1+\varepsilon_2)^2 - 2p_1{}^2p_2{}^2\sin^2\theta\right] \quad .$$

Again recalling that $\lambda = \varepsilon_1\varepsilon_2 - p_1p_2\cos\theta - m^2$, a little algebra gives

$$\lambda(\varepsilon_1+\varepsilon_2)^2 - 2p_1{}^2p_2{}^2\sin^2\theta = (\varepsilon_1\varepsilon_2 - p_1p_2 + m^2)(p_1{}^2 - 2p_1p_2\cos\theta + p_2{}^2)$$

$$= (\lambda + 2m^2)\,\underline{u}^2 \quad ,$$

so that

$$D_2(2\lambda) = 4\kappa^2\lambda(\lambda + 2m^2) \tag{A.64}$$

and

$$\beta = D_1^{\frac{1}{2}}(2\lambda) = D_2^{\frac{1}{2}}(2\lambda) = 2\kappa\sqrt{\lambda(\lambda + 2m^2)} \quad . \tag{A.65}$$

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[8] H. Nguyen-Ngoc and J. P. Perez-y-Jorba, Phys. Rev. 136, B1036 (1964).

[9] W. Gröbner and N. Hofreiter, Integraltafel, Part I (Springer, Vienna, 1961).

The form factors used in these timing runs were derived from a DWBA calculation for ^{12}C at 500 MeV (see subroutine PWAFFI in the computer program listing, pp. 98-100).

PWA denotes the piece-wise analytic evaluation of the integral as presented in this report.

All times listed are in seconds. Numerical integrals were calculated with an accuracy of 0.0001%.

p_1=250 MeV

1 deg.

p_2 (MeV)	Simpson	Gauss (n=16)	Kronrod (n=61)	PWA
10	21.87	5.42	1.57	.43
125	62.03	16.12	2.36	.22
240	3.15	1.22	.85	.02

5 deg.

p_2 (MeV)	Simpson	Gauss (n=16)	Kronrod (n=61)	PWA
10	11.66	2.86	.96	.42
125	4.37	1.30	.96	.22
240	1.86	.59	.61	.03

45 deg.

p_2 (MeV)	Simpson	Gauss (n=16)	Kronrod (n=61)	PWA
10	2.46	.84	.87	.43
125	2.33	1.08	.78	.23
240	2.21	.87	.80	.02

90 deg.

p_2 (MeV)	Simpson	Gauss (n=16)	Kronrod (n=61)	PWA
10	3.01	1.24	.93	.44
125	2.61	1.17	.84	.23
240	2.46	1.23	.89	.02

135 deg.

p_2 (MeV)	Simpson	Gauss (n=16)	Kronrod (n=61)	PWA
10	5.35	1.78	1.17	.43
125	3.15	1.37	.92	.23
240	3.38	1.31	.97	.01

175 deg.

p_2 (MeV)	Simpson	Gauss (n=16)	Kronrod (n=61)	PWA
10	11.71	2.43	1.06	.44
125	4.47	1.73	1.18	.23

All times are in seconds. Numerical integrals were calculated with an accuracy of 0.0001%.

p_1=500 MeV

1 deg.

p_2 (MeV)	Simpson	Gauss (n=16)	Kronrod (n=61)	PWA
10	48.45	8.53	2.39	.99
250	70.24	16.29	2.90	.44
490	3.66	1.20	1.01	.03

5 deg.

p_2 (MeV)	Simpson	Gauss (n=16)	Kronrod (n=61)	PWA
10	33.19	3.06	2.21	.99
250	6.57	2.16	1.32	.44
490	3.35	1.24	1.04	.03

45 deg.

p_2 (MeV)	Simpson	Gauss (n=16)	Kronrod (n=61)	PWA
10	3.26	1.31	.75	.98
250	3.76	1.64	1.33	.46
490	5.01	2.32	1.96	.02

90 deg.

p_2 (MeV)	Simpson	Gauss (n=16)	Kronrod (n=61)	PWA
10	6.40	2.33	1.29	.99
250	4.31	1.75	1.55	.48

135 deg.

p_2 (MeV)	Simpson	Gauss (n=16)	Kronrod (n=61)	PWA
10	20.92	3.61	1.97	.99
250	6.40	2.58	1.44	.53

175 deg.

p_2 (MeV)	Simpson	Gauss (n=16)	Kronrod (n=61)	PWA
10	23.49	5.90	1.86	1.00
250	8.70	3.46	1.89	.56

All times listed are in seconds. Numerical integrals were calculated with an accuracy of 0.0001%.

p_1=750 MeV

1 deg.

p_2 (MeV)	Simpson	Gauss (n=16)	Kronrod (n=61)	PWA
10	93.26	18.85	2.52	1.63
375	46.09	18.82	2.96	.70
710	5.72	1.86	1.23	.08

5 deg.

p_2 (MeV)	Simpson	Gauss (n=16)	Kronrod (n=61)	PWA
10	59.96	15.38	3.94	1.63
375	7.26	2.33	1.36	.70
710	4.49	1.89	1.27	.08

45 deg.

p_2 (MeV)	Simpson	Gauss (n=16)	Kronrod (n=61)	PWA
10	5.66	2.33	1.02	1.62
375	5.00	2.09	1.75	.75
710	5.60	2.57	1.30	.07

90 deg.

p_2 (MeV)	Simpson	Gauss (n=16)	Kronrod (n=61)	PWA
10	11.47	4.40	1.92	1.62
375	6.68	3.44	1.94	1.21

135 deg.

p_2 (MeV)	Simpson	Gauss (n=16)	Kronrod (n=61)	PWA
10	26.18	7.16	2.36	1.62
375	9.84	2.39	2.14	.98

175 deg.

p_2 (MeV)	Simpson	Gauss (n=16)	Kronrod (n=61)	PWA
10	57.47	15.11	3.70	1.62
375	10.22	3.56	1.47	.88

COMPUTER PROGRAM LISTING

In the pages which follow we give the listing of the computer program. However, the subroutines PROMPT, TOD, and CLK (which are called in the main program MSPLTIM, as well as in the timing subroutines STARTT and STOPT) are system information routines which are unique to the Perkin-Elmer system on which the program was run. They are not, therefore, included in the program listing.

```
$TITL MSPLTIM -- timing for Tsai and PWA
C       This program compares the formula for the radiative tail
C       given by Tsai and the Piece-Wise Analytic method (PWA)
C
        IMPLICIT DOUBLE PRECISION (A-H,O-Z)
C       Maximum number of angles
        PARAMETER (MNT=10)
C       Maximum number of momenta
        PARAMETER (MNP=10)
        LOGICAL OPENED,FAST
        DOUBLE PRECISION TIM(4),THETA(MNT),PFIN(MNP)
        INTEGER ICLKB(4)
        CHARACTER FILE*40,TIME*8,DATE*8,YESNO*80
        COMMON /EVICOM/ NSTEP,EPS
        COMMON /CONST/ EM,EM2,AM,HBARC,ALPH,PIE,ALPHA,RTD,DTR
        COMMON /KIN/ EEX,TH,AMO,S3,P3,CTH,STH,S32,ES,P32,EP,AMF,SDP3,SDP,
       + U32,U3,U0,U2,CTHS,CTHP,STHP,XMU
        COMMON /CORREC/IWTILD,WTIL1,WTIL2,ISOFT,SOFT1
        COMMON /TIME/ TIM,ICLKB
        DATA FAST/.FALSE./
        CALL TOD(TIME,DATE)
        CALL INICON
        INQUIRE (1,OPENED=OPENED)
        IF (.NOT.OPENED) OPEN (1,FILE='CON:')
2       IF (.NOT.OPENED) THEN
            WRITE (1,901) ' ',TIME,DATE
901         FORMAT (A,'MSPLTIM version 1A ',A,1X,A,
       +      ' (last edit 09/06/83)')
            CALL PROMPT(1,'Output file')
        ENDIF
        READ (1,80) FILE
        IF (FILE.EQ.' ') STOP 'All done'
80      FORMAT (A)
        OPEN (6,FILE=FILE,ACCESS='SEQUENTIAL')
        IF (.NOT.OPENED) CALL PROMPT(1,'Excitation energy')
        READ (1,*) EEX
        EEX=EEX/HBARC
        IF (.NOT.OPENED) CALL PROMPT(1,'Incident momentum')
        READ (1,*) S3
        S3=S3/HBARC
        IF (.NOT.OPENED) CALL PROMPT(1,'A of target')
        READ (1,*) AMO
        AMO=AMO*AM
        IF (.NOT.OPENED) WRITE (1,910)
910     FORMAT (' Enter values for final momentum (end with negative):')
        NP=0
99      READ (1,*) P3
        IF (P3.GE.0.) THEN
            NP=NP+1
            PFIN(NP)=P3/HBARC
            IF (NP.EQ.MNP) GO TO 88
            GO TO 99
        ENDIF
        IF (NP.EQ.0) STOP
88      IF (.NOT.OPENED) WRITE (1,909)
909     FORMAT (' Enter values for THETA (end with negative):')
        NT=0
```

```
8     READ (1,*) TH
      IF (TH.GE.0.) THEN
          NT=NT+1
          THETA(NT)=TH*DTR
          IF (NT.EQ.MNT) GO TO 9
          GO TO 8
      ENDIF
      IF (NT.EQ.0) STOP
9     IF (.NOT.OPENED) CALL PROMPT(1,
     + 'Starting number of steps for the integral')
      READ (1,*) NSTEP
      IF (.NOT.OPENED) CALL PROMPT(1,'Accuracy of the integral')
      READ (1,*) ACC
      IF (.NOT.OPENED) CALL PROMPT(1,'W-TILDA correction')
      READ (1,80) YESNO
      IF (INDX(YESNO,'Y').GT.0) THEN
          IWTILD=1
      ELSE
          IWTILD=0
      ENDIF
      ISOFT=0
      CALL PWAFFI
      DO 502 I=1,NT
          TH=THETA(I)
          WRITE (6,901) '1',TIME,DATE
          WRITE (6,902) TH*RTD,S3*HBARC,
     +     AMO/AM,EEX*HBARC,ACC,YESNO
902       FORMAT ('O Scattering angle =',F7.2,' degrees',
     +     T62,'Incident momentum =',F8.3,' MeV/c'/
     +     ' Mass of nucleus =',F8.3,' AMU',
     +     T61,'Excitation energy =',F8.3,' MeV'/
     +     ' Accuracy of numeric integral =',1PE8.1,
     +     T61,'W-Tilda correction made? ',A5)
          WRITE (6,903)
903       FORMAT ('O',T6,'Final',T17,'Tsai',T26,'Form factor',
     +     T39,'Integral',T52,'Total',T65,'PWA',T76,
     +     'Search',T87,'Integral',T100,'Total',T110,'% Diff'
     +     T126,'Speed'/
     +     T3,'momentum',T15,'result',T27,'time',T39,'time',
     +     T51,'time',T63,'result',T75,'time',T87,'time',
     +     T99,'time',T124,'increase')
          CALL CLK(ICLK,Y'28',.TRUE.)
          DO 503 J=1,NP
              P3=PFIN(J)
C             Initialize some commons
              CALL INIKIN
              TIM(1)=0.DO
              TIM(2)=0.DO
              TIM(3)=0.DO
              TIM(4)=0.DO
              IF (J.LE.1.OR..NOT.FAST) THEN
                  EPS=-ACC
              ELSE
                  EPS=ACC*SIGTS
              ENDIF
              CALL STARTT(2)
              CALL TSAINT(SIGTS)
```

```
                CALL STOPT(2)
                CALL STARTT(4)
                CALL PWASPL(SIGPWA)
                CALL STOPT(4)
                WRITE (6,905) P3*HBARC,SIGTS,TIM(1),TIM(2)-TIM(1),
     +           TIM(2),SIGPWA,TIM(3),TIM(4)-TIM(3),TIM(4),
     +           200.*(SIGPWA-SIGTS)/(SIGPWA+SIGTS),TIM(2)/TIM(4)
 905            FORMAT (1X,F8.2,T12,1PE12.5,T24,0PF9.3,T36,F9.3,T48,
     +           F9.3,T60,1PE12.5,T72,0PF9.3,T84,F9.3,T96,F9.3,T108,
     +           1PE10.3,T120,0PF10.2)
 503        CONTINUE
 502    CONTINUE
        CLOSE (6)
        GO TO 2
        END
```

```
$TITL STARTT -- Start a time interval
      SUBROUTINE STARTT(I)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DOUBLE PRECISION TIM(4)
      INTEGER ICLKB(4)
      COMMON./TIME/ TIM,ICLKB
      CALL CLK(ICLKB(I),Y'28',.FALSE.)
      RETURN
      END
```

```
$TITL STOPT -- Stop a time interval
      SUBROUTINE STOPT(I)
C     Note that the constant is for optimized Fortran.
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DOUBLE PRECISION TIM(4)
      INTEGER ICLKB(4)
      COMMON /TIME/ TIM,ICLKB
      CALL CLK(ICLKE,Y'28',.FALSE.)
      TIM(I)=TIM(I)+(ICLKE-ICLKB(I))/1000.-.0009374
      RETURN
      END
```

```
$TITL INICON -- Initialize constants
      SUBROUTINE INICON
C     Subroutine to initialize the common CONST. All energies in
C     units of fm-1. Constants from RMP 52, No. 2 (April 1980) as
C     reprinted by the CERN particle properties data booklet.
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      COMMON /CONST/ EM,EM2,AM,HBARC,ALPH,PIE,ALPHA,RTD,DTR
      HBARC=197.32858D0
      ALPH=137.03604D0
      AM=931.5016D0/HBARC
      EM=.5110034D0/HBARC
      EM2=EM**2
      PIE=3.1415926535897D0
      ALPHA=1.0D0/ALPH
      DTR=PIE/180.D0
      RTD=180.D0/PIE
      RETURN
      END
```

```
$TITL INIKIN -- Initialize kinematic variables
      SUBROUTINE INIKIN
C     Subroutine to initialize the commons KIN and CORREC.
C     In general this must be called if any of the first
C     five elements of KIN change.
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      COMMON /CONST/ EM,EM2,AM,HBARC,ALPH,PIE,ALPHA,RTD,DTR
      COMMON /KIN/ EEX,TH,AMO,S3,P3,CTH,STH,S32,ES,P32,EP,AMF,SDP3,SDP,
     + U32,U3,UO,U2,CTHS,CTHP,STHP,XMU
      COMMON /CORREC/IWTILD,WTIL1,WTIL2,ISOFT,SOFT1
C     sin's and cosin's of angles
      CTH=DCOS(TH)
      STH=DSIN(TH)
C     Incident momentum squared
      S32=S3*S3
C     Incident energy of the electron
      ES=DSQRT(S32+EM2)
C     Final momentum squared
      P32=P3*P3
C     Final energy of electron
      EP=DSQRT(P32+EM2)
C     Final rest mass of the nucleus
      AMF=AMO+EEX
C     This is (AMF**2 - AMO**2)/AMO
      XMU=EEX+.5DO*EEX**2/AMO
C     3-vector dot product of s and p
      SDP3=S3*P3*CTH
C     4-vector dot product of s and p
      SDP=ES*EP-SDP3
C     Momentum part of u=s+t-p (momentum loss of the electron)
      U32=S32+P32-2.DO*SDP3
      U3=DSQRT(U32)
C     Energy part of u=s+t-p
      UO=ES+AMO-EP
C     Length of u=s+t-p (squared) (missing mass squared)
      U2=UO**2-U32
C     cosin of angle between s and u
      CTHS=(S3-P3*CTH)/U3
C     cosin of angle between p and u
      CTHP=(S3*CTH-P3)/U3
C     sin of angle between p and u
      STHP=S3*STH/U3
C
C     Initialize coefficients for change of W to W-TILDA.
      WTIL1=1.DO+28.DO*ALPHA/(9.DO*PIE)-ALPHA*DLOG(ES/EP)**2/(2.DO*PIE)+
     + ALPHA*(PIE*PIE/6.DO-SPENCE(.5DO*(1.DO+CTH)))/PIE
      WTIL2=13.DO*ALPHA/(PIE*6.DO)
C
C     Initialize coefficients for soft photon correction.
      WS=ES-EP/(1.DO-EP*(1.DO-CTH)/AMO)
      WP=ES/(1.DO+ES*(1.DO-CTH)/AMO)-EP
      SOFT1=WS*WP/(ES*(EP+WP))
      RETURN
      END
```

```
$TITL TSAINT -- Evaluate Tsai by doing numerical integral
      SUBROUTINE TSAINT(VAL)
C     The formula given by Tsai in Slac-pub 848 (as well as in
C     Mo and Tsai RMP 41, 205 and Stein et. al. Phys Rev D 12,1884)
C     is evaluated by doing the integral numerically.  This
C     is done in up to 7 separate intervals. Call INIKIN before
C     calling this routine.
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      EXTERNAL TSAI
      LOGICAL DEBUG
      INTEGER NS(7)
      DOUBLE PRECISION EDGE(8)
      COMMON /CONST/ EM,EM2,AM,HBARC,ALPH,PIE,ALPHA,RTD,DTR
      COMMON /KIN/ EEX,TH,AMO,S3,P3,CTH,STH,S32,ES,P32,EP,AMF,SDP3,SDP,
     + U32,U3,U0,U2,CTHS,CTHP,STHP,XMU
      DATA DEBUG/.FALSE./
      DATA EDGE(1)/-1.D0/,EDGE(8)/1.D0/
      VAL=0.D0
      THLIM=AMO*(ES-EP)/(2.D0*ES*EP)
      IF (THLIM.LE.1.D0) THEN
          THLIM=2.D0*DASIN(DSQRT(THLIM))
          IF (TH.GT.THLIM) RETURN
      ENDIF
      THS=DATAN2(DSQRT(1.D0-CTHS**2),CTHS)
      DTHS=DSQRT(EM/ES)*2.0D0
      THP=DATAN2(STHP,CTHP)
      DTHP=DSQRT(EM/EP)*2.0D0
      EDGE(2)=DCOS(DMIN1(PIE,THP+DTHP))
      EDGE(3)=CTHP
      EDGE(4)=DCOS(DMAX1(0.D0,THP-DTHP))
      EDGE(5)=DCOS(DMIN1(PIE,THS+DTHS))
      EDGE(6)=CTHS
      EDGE(7)=DCOS(DMAX1(0.D0,THS-DTHS))
      IF (EDGE(4).GT.EDGE(5)) THEN
          EDGE(4)=(EDGE(3)+EDGE(6))/2.D0
          EDGE(5)=EDGE(4)
      ENDIF
      DO 504 I=1,7
          NS(I)=0
          IF (EDGE(I).LT.EDGE(I+1)) THEN
              DSIG=EVALI(EDGE(I),EDGE(I+1),TSAI,NS(I))
              IF (DEBUG) WRITE (1,*) CHAR(ICHAR('A')+I-1),
     +         EDGE(I),EDGE(I+1),DSIG
              VAL=VAL+DSIG
          ENDIF
504   CONTINUE
      VAL=VAL*ALPHA**3*P32*AMO/(PIE*EP*S3)
      IF (DEBUG) WRITE (1,31) NS,VAL
31    FORMAT (1X,7I5,E10.3)
      RETURN
      END
```

```
$TITL TSAI -- Evaluate the integrand of Tsai's formula
         SUBROUTINE TSAI(CTHK,SIG)
C        Subroutine to calculate the integrand of the formula for
C        the brems. cross section given by Tsai in SLAC-PUB-848 (Jan
C        1971) (Formula A.24).  The constants in the commons CONST
C        and KIN must be set.  Angles in radians, energies and
C        momenta in fm-1
         IMPLICIT DOUBLE PRECISION (A-H,O-Z)
         COMMON /CONST/ EM,EM2,AM,HBARC,ALPH,PIE,ALPHA,RTD,DTR
         COMMON /KIN/ EEX,TH,AMO,S3,P3,CTH,STH,S32,ES,P32,EP,AMF,SDP3,SDP,
        + U32,U3,UO,U2,CTHS,CTHP,STHP,XMU
         COMMON /CORREC/IWTILD,WTIL1,WTIL2,ISOFT,SOFT1
C        sin of photon angle
         STHK=DSQRT(1.DO-CTHK**2)
C        Energy of the real photon (avoid the cancelation of mass of target
C        squared)
         W=(EM2-SDP+AMO*(ES-EP-XMU))/(UO-U3*CTHK)
C        Momentum part of s-p-k
         Q32=U32+W**2-2.DO*U3*W*CTHK
         Q3=DSQRT(Q32)
C        Length of q=s-p-k (4-momentum transfer to nucleus)
         Q2=(ES-EP-W)**2-Q32
C
         A=W*(EP-P3*CTHP*CTHK)
         AA=W*(ES-S3*CTHS*CTHK)
         B2=(W*P3*STHP*STHK)**2
         X=DSQRT(A**2-B2)
         Y=DSQRT(AA**2-B2)
C        The constant NU*(1/x-1/y) of Tsai
         XYNU=(AA+A)/(X*Y*(X+Y))
C        Calculate the integrand
         SIG=W*(FORM2(Q2)*(-A*EM2*(2.DO*ES*(EP+W)+.5DO*Q2)/X**3 -
        + AA*EM2*(2.DO*EP*(ES-W)+.5DO*Q2)/Y**3 - 2.DO +
        + 2.DO*XYNU*(EM2*(SDP-W**2) + SDP*(2.DO*ES*EP-SDP+W*(ES-EP))) +
        + (2.DO*(ES*EP+ES*W+EP**2)+.5DO*Q2-SDP-EM2)/X -
        + (2.DO*(ES*EP-EP*W+ES**2)+.5DO*Q2-SDP-EM2)/Y) +
        + FORM1(Q2)*((A/X**3+AA/Y**3)*EM2*(2.DO*EM2+Q2) + 4.DO +
        + 4.DO*XYNU*SDP*(SDP-2.DO*EM2) +
        + (1.DO/X-1.DO/Y)*(2.DO*SDP+2.DO*EM2-Q2)))/(Q2*Q2*(UO-U3*CTHK))
         IF (ISOFT.EQ.O) RETURN
         SIG=SIG*SOFT1**(ALPHA*(DLOG(-Q2/EM2)-1.DO)/PIE)
         RETURN
         END
```

```
$TITL PWASPL -- Piece-wise analytic integration of Tsai
      SUBROUTINE PWASPL(VAL)
C     We assume that the COMMON KIN has been setup already
C     by a call to INIKIN and that the common PWAFF has been
C     initialized with a call to PWAFFI.
C     Size of arrays in PWAFF
      PARAMETER (MNPT=200)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DOUBLE PRECISION X(MNPT),A(MNPT),B(MNPT),C(MNPT),D(MNPT)
      DOUBLE PRECISION AP(MNPT),BP(MNPT),CP(MNPT),DP(MNPT)
      COMMON /PWAFF/ NPT,X,A,B,C,D,AP,BP,CP,DP
      COMMON /KIN/ EEX,TH,AMO,P1,P2,CTH,STH,P12,E1,P22,E2,AMF,SDP3,SDP,
     + U32,U3,U0,U2,CTHS,CTHP,STHP,XMU
      COMMON /CONST/ EM,EM2,AM,HBARC,ALPH,PIE,ALPHA,RTD,DTR
      VAL=0.D0
      CALL PWAINI(X1,XMAX)
      CALL STARTT(3)
      DO 30 I=1,NPT-1
          IF (X1.LT.X(I)) GO TO 40
30    CONTINUE
      I=NPT
40    CALL STOPT(3)
50    X2=DMIN1(X(I),XMAX)
      CALL PWAEVA(X1,X2,A(I),B(I),C(I),D(I),
     + AP(I),BP(I),CP(I),DP(I),XINT)
      VAL=VAL+XINT
      I=I+1
      X1=X2
      IF (X2.LT.XMAX) GO TO 50
      VAL=VAL*ALPHA**3*P22/(PIE*E2*P1)
      RETURN
      END
```

```
$TITL PWAINI, PWAEVA -- Analytic Tsai, cubic form factor
      SUBROUTINE PWAINI(XMIN,XMAX)
C     PWAINI should be called once, after the common KIN has
C     been initialized in order to calculate constants which
C     do not change when only THETA-k (or q-MU) changes.
      IMPLICIT DOUBLE PRECISION (A-Z)
      COMMON /PWACOM/K1M2,K1M1,K10,K1P1,K1P2,K1P3,K2M2,K2M1,K20,
     + K2P1,K2P2,K2P3,J1M2,J1M1,J10,J1P1,J1P2,J2M2,J2M1,J20,J2P1,
     + J2P2,L,LAMBDA
      COMMON /CONST/ EM,EM2,AM,HBARC,ALPH,PIE,ALPHA,RTD,DTR
      COMMON /KIN/ EEX,TH,AMO,P1,P2,CTH,STH,P12,E1,P22,E2,AMF,SDP3,SDP,
     + U32,U3,U0,U2,CTHS,CTHP,STHP,XMU
      LAMBDA=SDP-EM2
      KAPPA=E1-E2-XMU-LAMBDA/AMO
      CALL PWACOE(LAMBDA,KAPPA,P1,E1,P2,E2,1.DO,A1,B1,C1,ETA1,
     + XMIN,BETO,BET1,BET2,TAU1,RHO1,BETOP,BET1P,BET2P)
      CALL PWACOE(LAMBDA,KAPPA,P2,E2,P1,E1,-1.DO,A2,B2,C2,ETA2,
     + XMAX,DELO,DEL1,DEL2,TAU2,RHO2,DELOP,DEL1P,DEL2P)
      GAMMA=((LAMBDA+EM2)*(2.DO*E1*E2-LAMBDA)+LAMBDA*KAPPA**2+
     + (LAMBDA+EM2)*(LAMBDA/AMO+XMU)*KAPPA)/LAMBDA**2
      ETA=1.DO
      TAUP=-2.DO*(LAMBDA**2-2.DO*EM2*(LAMBDA+EM2))/LAMBDA
      GAMMAP=2.DO*(LAMBDA**2-EM2**2)/LAMBDA**2
      ETAP=-2.DO
      RHOP=(LAMBDA**2-2.DO*EM2**2)/LAMBDA**2
      RETURN
C
C     PWAEVA calculates the integral of Tsai from XI to XF assuming
C     that the longitudinal form factor (W2) is a cubic defined
C     by A, B, C, and D and that the transverse form factor (W1)
C     is a cubic given by AP, BP, CP, and DP.
      ENTRY PWAEVA(XI,XF,A,B,C,D,AP,BP,CP,DP,VAL)
      CALL PWAKIN(XI,XF,LAMBDA,KAPPA,P1,E1,P2,E2,ETA1,A1,B1,C1,
     + D1I,D1F,J10,J1M1,J1P1,J1M2,J1P2,K10,K1M1,K1P1,K1M2,K1P2,K1P3)
      CALL PWAKIN(XI,XF,LAMBDA,KAPPA,P2,E2,P1,E1,ETA2,A2,B2,C2,
     + D2I,D2F,J20,J2M1,J2P1,J2M2,J2P2,K20,K2M1,K2P1,K2M2,K2P2,K2P3)
      BETA=2.DO*KAPPA*DSQRT(LAMBDA*(LAMBDA+2.DO*EM2))
      ZETA1I=(2.0DO*LAMBDA*A1-B1)*XI-2.DO*LAMBDA*B1+C1
      ZETA1F=(2.0DO*LAMBDA*A1-B1)*XF-2.DO*LAMBDA*B1+C1
      ZETA2I=(2.0DO*LAMBDA*A2-B2)*XI-2.DO*LAMBDA*B2+C2
      ZETA2F=(2.0DO*LAMBDA*A2-B2)*XF-2.DO*LAMBDA*B2+C2
C     This is the formula as originally written in Maximon's notes
C     L=-DLOG((ZETA2F+BETA*D2F)/(ZETA1F+BETA*D1F)*
C    + (ZETA1I+BETA*D1I)/(ZETA2I+BETA*D2I))/BETA
C     The following version of L is rewritten to avoid cancelation
      S1I=1.DO
      EPS1I=0.DO
      IF (ZETA1I.LT.0.DO) THEN
          S1I=-1.DO
          EPS1I=2.DO
      ENDIF
      S2I=1.DO
      EPS2I=0.DO
      IF (ZETA2I.LT.0.DO) THEN
          S2I=-1.DO
          EPS2I=2.DO
      ENDIF
```

```
      S1F=1.D0
      EPS1F=0.D0
      IF (ZETA1F.LT.0.D0) THEN
          S1F=-1.D0
          EPS1F=2.D0
      ENDIF
      S2F=1.D0
      EPS2F=0.D0
      IF (ZETA2F.LT.0.D0) THEN
          S2F=-1.D0
          EPS2F=2.D0
      ENDIF
      DEN=2.D0*EM*P1*P2*KAPPA*STH
      L=(S1F*DLOG((DABS(ZETA1F)+BETA*D1F)/DEN)-
     + S1I*DLOG((DABS(ZETA1I)+BETA*D1I)/DEN)-
     + S2F*DLOG((DABS(ZETA2F)+BETA*D2F)/DEN)+
     + S2I*DLOG((DABS(ZETA2I)+BETA*D2I)/DEN)+
     + EPS1F*DLOG(DABS(XF-2.D0*LAMBDA))-
     + EPS1I*DLOG(DABS(XI-2.D0*LAMBDA))-
     + EPS2F*DLOG(DABS(XF-2.D0*LAMBDA))+
     + EPS2I*DLOG(DABS(XI-2.D0*LAMBDA)))/BETA
      CALL PWAINT(XI,XF,BET0,BET1,BET2,DEL0,DEL1,DEL2,ETA,
     + TAU1,TAU2,RHO1,RHO2,GAMMA,A,B,C,D,I)
      CALL PWAINT(XI,XF,BET0P,BET1P,BET2P,DEL0P,DEL1P,DEL2P,ETAP,
     + TAUP,TAUP,RHOP,RHOP,GAMMAP,AP,BP,CP,DP,IP)
      VAL=I+IP
      RETURN
      END
```

```
$TITL PWACOE -- Initialize constants constant with q-MU
      SUBROUTINE PWACOE(LAMBDA,KAPPA,PA,EA,PB,EB,SGN,A,B,C,ETA,
     + X,BETO,BET1,BET2,TAU,RHO,BETOP,BET1P,BET2P)
C     Note that the argument SGN will be either +1 or -1.  This
C     permits one to use the same formulae to calculate two sets
C     of variables necessary for the PWA calculation.
      IMPLICIT DOUBLE PRECISION (A-Z)
      COMMON /CONST/ EM,EM2,AM,HBARC,ALPH,PIE,ALPHA,RTD,DTR
      COMMON /KIN/ EEX,TH,AMO,P1,P2,CTH,STH,P12,E1,P22,E2,AMF,SDP3,SDP,
     + U32,U3,UO,U2,CTHS,CTHP,STHP,XMU
      A=(PB+SGN*(EA*PB-EB*PA*CTH)/AMO)**2+(EM*PA*STH/AMO)**2
      B=2.DO*PA*(PA*PB-(EA*EB-EM2)*CTH)*(PB+SGN*(EA*PB-EB*PA*CTH)/
     + AMO)+SGN*2.DO*EM2*PA**2*(EA-EB)*STH**2/AMO+2.DO*XMU*
     + (-SGN*PB*(EA*PB-EB*PA*CTH)-LAMBDA*(LAMBDA+2.DO*EM2)/AMO)
      C=4.DO*(PA*LAMBDA-SGN*XMU*(PA*EB-EA*PB*CTH))**2+
     + (2.DO*EM*XMU*PB*STH)**2
      ETA=((2.DO*EM*PA*PB*KAPPA*STH)/A)**2
      X=2.DO*(LAMBDA+(EA-EB-SGN*XMU)*(EA-EB-U3))/
     + (1.DO+SGN*(EA-EB-U3)/AMO)
      PAB=SGN*PA*(PA-PB*CTH)
      EAMU=EA*(EA-SGN*XMU)
      EAAMO=1.DO+SGN*2.DO*EA/AMO
      BETO=8.DO*EM2*EAMU*LAMBDA*(PAB-(EA+EB)*XMU)
      BET1=-EM2*(4.DO*EAMU*(PAB-(EA+EB)*(KAPPA+XMU))+
     + 2.DO*LAMBDA*(PAB-(EA+EB)*XMU)*EAAMO)
      BET2=EM2*(PAB-(EA+EB)*(KAPPA+XMU))*EAAMO
      TAU=(LAMBDA**2-4.DO*EM2*EA*EB-SGN*2.DO*XMU*
     + (EM2*(EA-EB-SGN*XMU)+EB*LAMBDA))/LAMBDA
      RHO=((LAMBDA+EM2)*(2.DO*EA*EB-LAMBDA)+LAMBDA*KAPPA**2+
     + (LAMBDA+EM2)*(LAMBDA/AMO+XMU)*KAPPA)/LAMBDA**2+
     + EM2/AMO**2+.5DO+SGN*EA/AMO
      BETOP=-8.DO*EM2**2*LAMBDA*(PAB-(EA+EB)*XMU)
      BET1P=4.DO*EM2*(EM2*(PAB-(EA+EB)*(KAPPA+XMU))+
     + LAMBDA*(PAB-(EA+EB)*XMU))
      BET2P=-2.DO*EM2*(PAB-(EA+EB)*(KAPPA+XMU))
      RETURN
      END
```

```
$TITL PWAKIN -- Evaluate constants which change with q-MU
      SUBROUTINE PWAKIN(XI,XF,LAMBDA,KAPPA,PA,EA,PB,EB,ETA,A,B,C,
     + DI,DF,JO,JM1,JP1,JM2,JP2,KO,KM1,KP1,KM2,KP2,KP3)
      IMPLICIT DOUBLE PRECISION (A-Z)
      COMMON /CONST/ EM,EM2,AM,HBARC,ALPH,PIE,ALPHA,RTD,DTR
      COMMON /KIN/ EEX,TH,AMO,P1,P2,CTH,STH,P12,E1,P22,E2,AMF,SDP3,SDP,
     + U32,U3,UO,U2,CTHS,CTHP,STHP,XMU
      CHI=B/A
      DI=DSQRT(A*((XI-CHI)**2+ETA))
      DF=DSQRT(A*((XF-CHI)**2+ETA))
      SQTA=DSQRT(A)
      SQTC=DSQRT(C)
C     The following is the formula as originally written
C     JO=DLOG((A*XF-B+DF*SQTA)/(A*XI-B+DI*SQTA))/SQTA
C     This is the version as corrected to reduce "cancelation"
      SI=1..DO
      IF (A*XI-B.LT.O.DO) SI=-1.DO
      SF=1.DO
      IF (A*XF-B.LT.O.DO) SF=-1.DO
      DEN=2.DO*EM*PA*PB*KAPPA*STH
      JO=(SF*DLOG((DABS(A*XF-B)+DF*SQTA)/DEN)-
     + SI*DLOG((DABS(A*XI-B)+DI*SQTA)/DEN))/SQTA
C     The following is the formula as originally written
C     JM1=-DLOG(XI/XF*(-B*XF+C+SQTC*DF)/
C    + (-B*XI+C+SQTC*DI))/SQTC
C     This version is corrected to avoid cancelation
      SI=1.DO
      IF (B*XI-C.LT.O.DO) SI=-1.DO
      SF=1.DO
      IF (B*XF-C.LT.O.DO) SF=-1.DO
      JM1=(SF*DLOG((DABS(B*XF-C)+SQTC*DF)/(XF*DEN))-
     + SI*DLOG((DABS(B*XI-C)+SQTC*DI)/(XI*DEN)))/SQTC
      JP1=(DF-DI)/A+CHI*JO
      JM2=(-DF/XF+DI/XI+B*JM1)/C
      JP2=(XF*DF-XI*DI+3.DO*B*JP1-C*JO)/(2.DO*A)
      KO=((A*XF-B)/DF-(A*XI-B)/DI)/(ETA*A**2)
      KP1=-((-B*XF+C)/DF-(-B*XI+C)/DI)/(ETA*A**2)
      KP2=(JO+2.DO*B*KP1-C*KO)/A
      KP3=(XF**2/DF-XI**2/DI+3.DO*B*KP2-2.DO*C*KP1)/A
      KM1=(1.DO/DF-1.DO/DI+JM1+B*KO)/C
      KM2=(-1.DO/(XF*DF)+1.DO/(XI*DI)+3.DO*B*KM1-2.DO*A*KO)/C
      RETURN
      END
```

```
$TITL PWAINT -- Determine integral of long. or trans term.
      SUBROUTINE PWAINT(XI,XF,BETO,BET1,BET2,DELO,DEL1,DEL2,ETA,
     + TAU1,TAU2,RHO1,RHO2,GAMMA,A,B,C,D,I)
      IMPLICIT DOUBLE PRECISION (A-Z)
      COMMON /PWACOM/K1M2,K1M1,K10,K1P1,K1P2,K1P3,K2M2,K2M1,K20,
     + K2P1,K2P2,K2P3,J1M2,J1M1,J10,J1P1,J1P2,J2M2,J2M1,J20,J2P1,
     + J2P2,L,LAMBDA
      COMMON /KIN/ EEX,TH,AMO,P1,P2,CTH,STH,P12,E1,P22,E2,AMF,SDP3,SDP,
     + U32,U3,UO,U2,CTHS,CTHP,STHP,XMU
      CALL PWAEIN(BETO,BET1,BET2,K1M2,K1M1,K10,K1P1,K1P2,K1P3,
     + J1M2,J1M1,J10,J1P1,J1P2,A,B,C,D,I11,I31,I41)
      CALL PWAEIN(DELO,DEL1,DEL2,K2M2,K2M1,K20,K2P1,K2P2,K2P3,
     + J2M2,J2M1,J20,J2P1,J2P2,A,B,C,D,I12,I32,I42)
      I2=A*(1.DO/XI-1.DO/XF)+B*DLOG(XF/XI)+C*(XF-XI)+
     + .5DO*D*(XF**2-XI**2)
      I5=(A+B*2.DO*LAMBDA+C*4.DO*LAMBDA**2+D*8.DO*LAMBDA**3)*L-
     + (B+2.DO*LAMBDA*C+4.DO*LAMBDA**2*D)*(J10-J20)-
     + (C+2.DO*LAMBDA*D)*(J1P1-J2P1)-
     + D*(J1P2-J2P2)
      I=-I11-I12-ETA*I2/U3+TAU1*I31-TAU2*I32-
     + RHO1*I41+RHO2*I42-GAMMA*I5
      RETURN
      END
```

```
$TITL PWAEIN -- Evaluate some terms in the analytic integral
      SUBROUTINE PWAEIN(BETO,BET1,BET2,KM2,KM1,KO,KP1,KP2,KP3,
     + JM2,JM1,JO,JP1,JP2,A,B,C,D,I1,I3,I4)
      IMPLICIT DOUBLE PRECISION (A-Z)
      I1=BETO*A*KM2+(BETO*B+BET1*A)*KM1+
     + (BETO*C+BET1*B+BET2*A)*KO+(BETO*D+BET1*C+BET2*B)*KP1+
     + (BET1*D+BET2*C)*KP2+BET2*D*KP3
      I3=A*JM2+B*JM1+C*JO+D*JP1
      I4=A*JM1+B*JO+C*JP1+D*JP2
      RETURN
      END
```

```
$TITL SPCE -- Series expansion of Spence function
      FUNCTION SPCE(X)
C     Series expansion of the Spence function for range 0 to 0.5
C     Note that the number of terms used in the series is chosen
C     to avoid floating underflow.
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DOUBLE PRECISION SPCE
      SPCE=0.0D0
      IF (X.LE.0.D0) RETURN
      TOP=1.0D0
      N=1.D0-25.D0/DLOG(X)
      DO 10 I=1,N
          TOP=TOP*X
          SPCE=SPCE+TOP/I**2
10    CONTINUE
      RETURN
      END
```

```
$TITL PWAEIN -- Evaluate some terms in the analytic integral
      SUBROUTINE PWAEIN(BETO,BET1,BET2,KM2,KM1,KO,KP1,KP2,KP3,
     + JM2,JM1,JO,JP1,JP2,A,B,C,D,I1,I3,I4)
      IMPLICIT DOUBLE PRECISION (A-Z)
      I1=BETO*A*KM2+(BETO*B+BET1*A)*KM1+
     + (BETO*C+BET1*B+BET2*A)*KO+(BETO*D+BET1*C+BET2*B)*KP1+
     + (BET1*D+BET2*C)*KP2+BET2*D*KP3
      I3=A*JM2+B*JM1+C*JO+D*JP1
      I4=A*JM1+B*JO+C*JP1+D*JP2
      RETURN
      END
```

```
$TITL SPCE -- Series expansion of Spence function
      FUNCTION SPCE(X)
C     Series expansion of the Spence function for range 0 to 0.5
C     Note that the number of terms used in the series is chosen
C     to avoid floating underflow.
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DOUBLE PRECISION SPCE
      SPCE=0.0D0
      IF (X.LE.0.D0) RETURN
      TOP=1.0D0
      N=1.D0-25.D0/DLOG(X)
      DO 10 I=1,N
         TOP=TOP*X
         SPCE=SPCE+TOP/I**2
10    CONTINUE
      RETURN
      END
```

```
$TITL SPENCE -- The Spence function
      FUNCTION SPENCE(X)
C     Program for the computation of the Spence function for all real
C     values of the argument x
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DOUBLE PRECISION SPENCE
      DATA PI23,PI26/3.289868134D0,1.644934067D0/
      IF (X.GT.2.0D0) GO TO 1
      IF (X.GT.1.0D0) GO TO 2
      IF (X.GT.0.5D0) GO TO 3
      IF (X.LT.-1.D0) GO TO 4
      IF (X.LT.0.0D0) GO TO 5
      SPENCE=SPCE(X)
      RETURN
1     Y=1.0D0/X
      SPENCE=PI23-SPCE(Y)-.5D0*DLOG(X)**2
      RETURN
2     Y=(X-1.0D0)/X
      Y1=(X-1.0D0)*Y
      SPENCE=PI26+SPCE(Y)-0.5D0*DLOG(X)*DLOG(Y1)
      RETURN
3     Y=1.0D0-X
      SPENCE=PI26-SPCE(Y)-DLOG(X)*DLOG(Y)
      RETURN
4     Y=1.0D0-X
      Y1=1.0D0/Y
      SPENCE=-PI26+SPCE(Y1)-0.5D0*DLOG(Y)*DLOG(X*X/Y)
      RETURN
5     Y=X/(X-1.0D0)
      Y1=-X+1.0D0
      SPENCE=-SPCE(Y)-0.5D0*DLOG(Y1)**2
      RETURN
      END
```

```
$TITL EVALI -- Simpson's rule integration
      FUNCTION EVALI(A,B,FUNC,NS)
C     Function to evaluate the integral of the function FUNC
C     between A and B.  The number of steps starts at NSTEP
C     as defined in EVALI and then goes up by a factor of two
C     until the integral stops changing by more than a factor
C     of EPS.  In this version, which uses Simpson's rule, the
C     new values needed when NSTEP increases are not all re-
C     calculated.  Values computed during the calculation of
C     the old (inaccurate) value of the integral are re-used.
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      EXTERNAL FUNC
      DOUBLE PRECISION EVALI
      COMMON /EVICOM/ NSTEP,EPS
      CALL FUNC(A,EVALA)
      CALL FUNC(B,EVALB)
      NS=NSTEP-1
      STEPO=(B-A)/NSTEP
      EVAL2=0.D0
      DO 900 I=1,NS
          CALL FUNC(A+I*STEPO,V)
          EVAL2=EVAL2+V
900   CONTINUE
      NS=NSTEP
C
C     Start of loop
4     STEP=.5D0*STEPO
      EVAL4=0.D0
      DO 901 I=1,NS
          CALL FUNC(A+STEP+(I-1)*STEPO,V)
          EVAL4=EVAL4+V
901   CONTINUE
      EV=(EVALA+2.D0*EVAL2+4.D0*EVAL4+EVALB)*STEPO/6.D0
      IF (NS.NE.NSTEP) THEN
          IF (EPS.LE.0.D0) THEN
              IF (DABS(EV-EVALIO).LE.DABS(EV*EPS)) GO TO 6
CD            WRITE (1,*) A,B,DABS(EV-EVALIO),DABS(EV*EPS)
          ELSE
              IF (DABS(EV-EVALIO).LE.EPS) GO TO 6
CD            WRITE (1,*) A,B,DABS(EV-EVALIO),EPS
          ENDIF
      ENDIF
      EVALIO=EV
      EVAL2=EVAL4+EVAL2
      NS=2*NS
      STEPO=STEP
      GO TO 4
6     EVALI=EV
      RETURN
      END
```

```
$TITL EVALI -- Gaussian integration
      FUNCTION EVALI(A,B,FUNC,NS)
C     Function to evaluate the integral of the function FUNC
C     between A and B.  The number of intervals starts at NSTEP
C     as defined in EVICOM and then goes up by a factor of two
C     until the integral stops changing by more than a factor
C     of EPS.  In this version, NGAUSS-point Gaussian integration
C     is used (for particularly tough functions).
      PARAMETER (NGAUSS=16)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      EXTERNAL FUNC
      DOUBLE PRECISION EVALI,Z(NGAUSS),W(NGAUSS)
      COMMON /EVICOM/ NSTEP,EPS
      NS=NSTEP
C
C     Start of loop
4     STEP=(B-A)/NS
      EV=0.D0
      DO 902 I=1,NS
          AX=A+(I-1)*STEP
          BX=AX+STEP
          CALL GSET (AX,BX,NGAUSS,Z,W)
          DO 903 J=1,NGAUSS
              CALL FUNC(Z(J),V)
              EV=EV+V*W(J)
903       CONTINUE
902   CONTINUE
      IF (NS.NE.NSTEP) THEN
          IF (EPS.LE.0.D0) THEN
              IF (DABS(EV-EVALIO).LE.DABS(EV*EPS)) GO TO 6
CD            WRITE (1,*) A,B,DABS(EV-EVALIO),DABS(EV*EPS)
          ELSE
              IF (DABS(EV-EVALIO).LE.EPS) GO TO 6
CD            WRITE (1,*) A,B,DABS(EV-EVALIO),EPS
          ENDIF
      ENDIF
      EVALIO=EV
      NS=2*NS
      GO TO 4
6     EVALI=EV
      RETURN
      END
```

```
$TITL GSET -- N-point Gauss zeros and weights
      SUBROUTINE GSET (AX,BX,NX,Z,W)
C     N-point Gauss zeros and weights for the interval (AX,BX) are
C     stored in arrays Z and W respectively.
C
C     This version converted from the version in the CERN library
C     for the CDC-7600 to a double precision version for a 32-bit
C     machine.
C
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DOUBLE PRECISION Z(2),W(2),A(273),X(273)
      INTEGER KTAB(96)
C
C     Table of initial subscripts for N=2(1)16(4)96
      DATA KTAB(2)/1/
      DATA KTAB(3)/2/
      DATA KTAB(4)/4/
      DATA KTAB(5)/6/
      DATA KTAB(6)/9/
      DATA KTAB(7)/12/
      DATA KTAB(8)/16/
      DATA KTAB(9)/20/
      DATA KTAB(10)/25/
      DATA KTAB(11)/30/
      DATA KTAB(12)/36/
      DATA KTAB(13)/42/
      DATA KTAB(14)/49/
      DATA KTAB(15)/56/
      DATA KTAB(16)/64/
      DATA KTAB(20)/72/
      DATA KTAB(24)/82/
      DATA KTAB(28)/82/
      DATA KTAB(32)/94/
      DATA KTAB(36)/94/
      DATA KTAB(40)/110/
      DATA KTAB(44)/110/
      DATA KTAB(48)/130/
      DATA KTAB(52)/130/
      DATA KTAB(56)/130/
      DATA KTAB(60)/130/
      DATA KTAB(64)/154/
      DATA KTAB(68)/154/
      DATA KTAB(72)/154/
      DATA KTAB(76)/154/
      DATA KTAB(80)/186/
      DATA KTAB(84)/186/
      DATA KTAB(88)/186/
      DATA KTAB(92)/186/
      DATA KTAB(96)/226/
C
C     Table of abscissae (X) and weights (A) for interval (-1,+1).
C
C     N=2
      DATA X(1)/0.577350269189626D0  /, A(1)/1.000000000000000D0  /
C     N=3
      DATA X(2)/0.774596669241483D0  /, A(2)/0.555555555555556D0  /
      DATA X(3)/0.000000000000000D0  /, A(3)/0.888888888888889D0  /
```

```
C     N=4
      DATA X(4)/0.861136311594053D0  /, A(4)/0.347854845137454D0  /
      DATA X(5)/0.339981043584856D0  /, A(5)/0.652145154862546D0  /
C     N=5
      DATA X(6)/0.906179845938664D0  /, A(6)/0.236926885056189D0  /
      DATA X(7)/0.538469310105683D0  /, A(7)/0.478628670499366D0  /
      DATA X(8)/0.000000000000000D0  /, A(8)/0.568888888888889D0  /
C     N=6
      DATA X(9)/0.932469514203152D0  /, A(9)/0.171324492379170D0  /
      DATA X(10)/0.661209386466265D0 /, A(10)/0.360761573048139D0 /
      DATA X(11)/0.238619186083197D0 /, A(11)/0.467913934572691D0 /
C     N=7
      DATA X(12)/0.949107912342759D0 /, A(12)/0.129484966168870D0 /
      DATA X(13)/0.741531185599394D0 /, A(13)/0.279705391489277D0 /
      DATA X(14)/0.405845151377397D0 /, A(14)/0.381830050505119D0 /
      DATA X(15)/0.000000000000000D0 /, A(15)/0.417959183673469D0 /
C     N=8
      DATA X(16)/0.960289856497536D0 /, A(16)/0.101228536290376D0 /
      DATA X(17)/0.796666477413627D0 /, A(17)/0.222381034453374D0 /
      DATA X(18)/0.525532409916329D0 /, A(18)/0.313706645877887D0 /
      DATA X(19)/0.183434642495650D0 /, A(19)/0.362683783378362D0 /
C     N=9
      DATA X(20)/0.968160239507626D0 /, A(20)/0.081274388361574D0 /
      DATA X(21)/0.836031107326636D0 /, A(21)/0.180648160694857D0 /
      DATA X(22)/0.613371432700590D0 /, A(22)/0.260610696402935D0 /
      DATA X(23)/0.324253423403809D0 /, A(23)/0.312347077040003D0 /
      DATA X(24)/0.000000000000000D0 /, A(24)/0.330239355001260D0 /
C     N=10
      DATA X(25)/0.973906528517172D0 /, A(25)/0.066671344308688D0 /
      DATA X(26)/0.865063366688985D0 /, A(26)/0.149451349150581D0 /
      DATA X(27)/0.679409568299024D0 /, A(27)/0.219086362515982D0 /
      DATA X(28)/0.433395394129247D0 /, A(28)/0.269266719309996D0 /
      DATA X(29)/0.148874338981631D0 /, A(29)/0.295524224714753D0 /
C     N=11
      DATA X(30)/0.978228658146057D0 /, A(30)/0.055668567116174D0 /
      DATA X(31)/0.887062599768095D0 /, A(31)/0.125580369464905D0 /
      DATA X(32)/0.730152005574049D0 /, A(32)/0.186290210927734D0 /
      DATA X(33)/0.519096129206812D0 /, A(33)/0.233193764591990D0 /
      DATA X(34)/0.269543155952345D0 /, A(34)/0.262804544510247D0 /
      DATA X(35)/0.000000000000000D0 /, A(35)/0.272925086777901D0 /
C     N=12
      DATA X(36)/0.981560634246719D0 /, A(36)/0.047175336386512D0 /
      DATA X(37)/0.904117256370475D0 /, A(37)/0.106939325995318D0 /
      DATA X(38)/0.769902674194305D0 /, A(38)/0.160078328543346D0 /
      DATA X(39)/0.587317954286617D0 /, A(39)/0.203167426723066D0 /
      DATA X(40)/0.367831498998180D0 /, A(40)/0.233492536538355D0 /
      DATA X(41)/0.125233408511469D0 /, A(41)/0.249147045813403D0 /
C     N=13
      DATA X(42)/0.984183054718588D0 /, A(42)/0.040484004765316D0 /
      DATA X(43)/0.917598399222978D0 /, A(43)/0.092121499837728D0 /
      DATA X(44)/0.801578090733310D0 /, A(44)/0.138873510219787D0 /
      DATA X(45)/0.642349339440340D0 /, A(45)/0.178145980761946D0 /
      DATA X(46)/0.448492751036447D0 /, A(46)/0.207816047536889D0 /
      DATA X(47)/0.230458315955135D0 /, A(47)/0.226283180262897D0 /
      DATA X(48)/0.000000000000000D0 /, A(48)/0.232551553230874D0 /
C     N=14
      DATA X(49)/0.986283808696812D0 /, A(49)/0.035119460331752D0 /
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      DATA X(50)/0.928434883663574D0 /, A(50)/0.080158087159760D0 /
      DATA X(51)/0.827201315069765D0 /, A(51)/0.121518570687903D0 /
      DATA X(52)/0.687292904811685D0 /, A(52)/0.157203167158194D0 /
      DATA X(53)/0.515248636358154D0 /, A(53)/0.185538397477938D0 /
      DATA X(54)/0.319112368927890D0 /, A(54)/0.205198463721296D0 /
      DATA X(55)/0.108054948707344D0 /, A(55)/0.215263853463158D0 /
C     N=15
      DATA X(56)/0.987992518020485D0 /, A(56)/0.030753241996117D0 /
      DATA X(57)/0.937273392400706D0 /, A(57)/0.070366047488108D0 /
      DATA X(58)/0.848206583410427D0 /, A(58)/0.107159220467172D0 /
      DATA X(59)/0.724417731360170D0 /, A(59)/0.139570677926154D0 /
      DATA X(60)/0.570972172608539D0 /, A(60)/0.166269205816994D0 /
      DATA X(61)/0.394151347077563D0 /, A(61)/0.186161000015562D0 /
      DATA X(62)/0.201194093997435D0 /, A(62)/0.198431485327111D0 /
      DATA X(63)/0.000000000000000D0 /, A(63)/0.202578241925561D0 /
C     N=16
      DATA X(64)/0.989400934991650D0 /, A(64)/0.027152459411754D0 /
      DATA X(65)/0.944575023073233D0 /, A(65)/0.062253523938648D0 /
      DATA X(66)/0.865631202387832D0 /, A(66)/0.095158511682493D0 /
      DATA X(67)/0.755404408355003D0 /, A(67)/0.124628971255534D0 /
      DATA X(68)/0.617876244402644D0 /, A(68)/0.149595988816577D0 /
      DATA X(69)/0.458016777657227D0 /, A(69)/0.169156519395003D0 /
      DATA X(70)/0.281603550779259D0 /, A(70)/0.182603415044924D0 /
      DATA X(71)/0.095012509837637D0 /, A(71)/0.189450610455069D0 /
C     N=20
      DATA X(72)/0.993128599185094D0 /, A(72)/0.017614007139152D0 /
      DATA X(73)/0.963971927277913D0 /, A(73)/0.040601429800386D0 /
      DATA X(74)/0.912234428251325D0 /, A(74)/0.062672048334109D0 /
      DATA X(75)/0.839116971822218D0 /, A(75)/0.083276741576704D0 /
      DATA X(76)/0.746331906460150D0 /, A(76)/0.101930119817240D0 /
      DATA X(77)/0.636053680726515D0 /, A(77)/0.118194531961518D0 /
      DATA X(78)/0.510867001950827D0 /, A(78)/0.131688638449176D0 /
      DATA X(79)/0.373706088715419D0 /, A(79)/0.142096109318382D0 /
      DATA X(80)/0.227785851141645D0 /, A(80)/0.149172986472603D0 /
      DATA X(81)/0.076526521133497D0 /, A(81)/0.152753387130725D0 /
C     N=24
      DATA X(82)/0.995187219997021D0 /, A(82)/0.012341229799987D0 /
      DATA X(83)/0.974728555971309D0 /, A(83)/0.028531388628933D0 /
      DATA X(84)/0.938274552002732D0 /, A(84)/0.044277438817419D0 /
      DATA X(85)/0.886415527004401D0 /, A(85)/0.059298584915436D0 /
      DATA X(86)/0.820001985973902D0 /, A(86)/0.073346481411080D0 /
      DATA X(87)/0.740124191578554D0 /, A(87)/0.086190161531953D0 /
      DATA X(88)/0.648093651936975D0 /, A(88)/0.097618652104113D0 /
      DATA X(89)/0.545421471388839D0 /, A(89)/0.107444270115965D0 /
      DATA X(90)/0.433793507626045D0 /, A(90)/0.115505668053725D0 /
      DATA X(91)/0.315042679696163D0 /, A(91)/0.121670472927803D0 /
      DATA X(92)/0.191118867473616D0 /, A(92)/0.125837456346828D0 /
      DATA X(93)/0.064056892862605D0 /, A(93)/0.127938195346752D0 /
C     N=32
      DATA X(94)/0.997263861849481D0 /, A(94)/0.007018610009470D0 /
      DATA X(95)/0.985611511545268D0 /, A(95)/0.016274394730905D0 /
      DATA X(96)/0.964762255587506D0 /, A(96)/0.025392065309262D0 /
      DATA X(97)/0.934906075937739D0 /, A(97)/0.034273862913021D0 /
      DATA X(98)/0.896321155766052D0 /, A(98)/0.042835898022226D0 /
      DATA X(99)/0.849367613732569D0 /, A(99)/0.050998059262376D0 /
      DATA X(100)/0.794483795967942D0/, A(100)/0.058684093478535D0/
      DATA X(101)/0.732182118740289D0/, A(101)/0.065822222776361D0/
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      DATA X(102)/0.663044266930215D0/, A(102)/0.072345794108848D0/
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      DATA X(104)/0.506899908932229D0/, A(104)/0.083311924226946D0/
      DATA X(105)/0.421351276130635D0/, A(105)/0.087652093004403D0/
      DATA X(106)/0.331868602282127D0/, A(106)/0.091173878695763D0/
      DATA X(107)/0.239287362252137D0/, A(107)/0.093844399080804D0/
      DATA X(108)/0.144471961582796D0/, A(108)/0.095638720079274D0/
      DATA X(109)/0.048307665687738D0/, A(109)/0.096540088514727D0/
C     N=40
      DATA X(110)/0.998237709710559D0/, A(110)/0.004521277098533D0/
      DATA X(111)/0.990726238699457D0/, A(111)/0.010498284531152D0/
      DATA X(112)/0.977259949983774D0/, A(112)/0.016421058381907D0/
      DATA X(113)/0.957916819213791D0/, A(113)/0.022245849194166D0/
      DATA X(114)/0.932812808278676D0/, A(114)/0.027937006980023D0/
      DATA X(115)/0.902098806968874D0/, A(115)/0.033460195282547D0/
      DATA X(116)/0.865959503212259D0/, A(116)/0.038782167974472D0/
      DATA X(117)/0.824612230833311D0/, A(117)/0.043870908185673D0/
      DATA X(118)/0.778305651426519D0/, A(118)/0.048695807635072D0/
      DATA X(119)/0.727318255189927D0/, A(119)/0.053227846983936D0/
      DATA X(120)/0.671956684614179D0/, A(120)/0.057439769099391D0/
      DATA X(121)/0.612553889667980D0/, A(121)/0.061306242492928D0/
      DATA X(122)/0.549467125095128D0/, A(122)/0.064804013456601D0/
      DATA X(123)/0.483075801686178D0/, A(123)/0.067912045815233D0/
      DATA X(124)/0.413779204371605D0/, A(124)/0.070611647391286D0/
      DATA X(125)/0.341994090825758D0/, A(125)/0.072886582395804D0/
      DATA X(126)/0.268152185007253D0/, A(126)/0.074723169057968D0/
      DATA X(127)/0.192697580701371D0/, A(127)/0.076110361900626D0/
      DATA X(128)/0.116084070675255D0/, A(128)/0.077039818164247D0/
      DATA X(129)/0.038772417506050D0/, A(129)/0.077505947978424D0/
C     N=48
      DATA X(130)/0.998771007252426D0/, A(130)/0.003153346052305D0/
      DATA X(131)/0.993530172266350D0/, A(131)/0.007327553901276D0/
      DATA X(132)/0.984124583722826D0/, A(132)/0.011477234579234D0/
      DATA X(133)/0.970591592546247D0/, A(133)/0.015579315722943D0/
      DATA X(134)/0.952987703160430D0/, A(134)/0.019616160457355D0/
      DATA X(135)/0.931386690706554D0/, A(135)/0.023570760839324D0/
      DATA X(136)/0.905879136715569D0/, A(136)/0.027426509708356D0/
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      DATA X(139)/0.807066204029442D0/, A(139)/0.038241351065830D0/
      DATA X(140)/0.767159032515740D0/, A(140)/0.041545082943464D0/
      DATA X(141)/0.724034130923814D0/, A(141)/0.044674560856694D0/
      DATA X(142)/0.677872379632663D0/, A(142)/0.047616658492490D0/
      DATA X(143)/0.628867396776513D0/, A(143)/0.050359035553854D0/
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      DATA X(147)/0.408686481990716D0/, A(147)/0.059114839698395D0/
      DATA X(148)/0.348755886292160D0/, A(148)/0.060704439165893D0/
      DATA X(149)/0.287362487355455D0/, A(149)/0.062039423159892D0/
      DATA X(150)/0.224763790394689D0/, A(150)/0.063114192286254D0/
      DATA X(151)/0.161222356068891D0/, A(151)/0.063924238584648D0/
      DATA X(152)/0.097004699209462D0/, A(152)/0.064466164435950D0/
      DATA X(153)/0.032380170962869D0/, A(153)/0.064737696812683D0/
C     N=64
      DATA X(154)/0.999305041735772D0/, A(154)/0.001783280721696D0/
      DATA X(155)/0.996340116771955D0/, A(155)/0.004147033260562D0/
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      DATA X(156)/0.991013371476744D0/, A(156)/0.006504457968978D0/
      DATA X(157)/0.983336253884625D0/, A(157)/0.008846759826363D0/
      DATA X(158)/0.973326827789910D0/, A(158)/0.011168139460131D0/
      DATA X(159)/0.961008799652053D0/, A(159)/0.013463047896718D0/
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      DATA X(161)/0.929569172131939D0/, A(161)/0.017951715775697D0/
      DATA X(162)/0.910522137078502D0/, A(162)/0.020134823153530D0/
      DATA X(163)/0.889315445995114D0/, A(163)/0.022270173808383D0/
      DATA X(164)/0.865999398154092D0/, A(164)/0.024352702568710D0/
      DATA X(165)/0.840629296252580D0/, A(165)/0.026377469715054D0/
      DATA X(166)/0.813265315122797D0/, A(166)/0.028339672614259D0/
      DATA X(167)/0.783972358943341D0/, A(167)/0.030234657072402D0/
      DATA X(168)/0.752819907260531D0/, A(168)/0.032057928354851D0/
      DATA X(169)/0.719881850171610D0/, A(169)/0.033805161837141D0/
      DATA X(170)/0.685236313054233D0/, A(170)/0.035472213256882D0/
      DATA X(171)/0.648965471254657D0/, A(171)/0.037055128540240D0/
      DATA X(172)/0.611155355172393D0/, A(172)/0.038550153178615D0/
      DATA X(173)/0.571895646202634D0/, A(173)/0.039953741132720D0/
      DATA X(174)/0.531279464019894D0/, A(174)/0.041262563242623D0/
      DATA X(175)/0.489403145707052D0/, A(175)/0.042473515123653D0/
      DATA X(176)/0.446366017253464D0/, A(176)/0.043583724529323D0/
      DATA X(177)/0.402270157963991D0/, A(177)/0.044590558163756D0/
      DATA X(178)/0.357220158337668D0/, A(178)/0.045491627927418D0/
      DATA X(179)/0.311322871990210D0/, A(179)/0.046284796581314D0/
      DATA X(180)/0.264687162208767D0/, A(180)/0.046968182816210D0/
      DATA X(181)/0.217423643740007D0/, A(181)/0.047540165714830D0/
      DATA X(182)/0.169644420423992D0/, A(182)/0.047999388596458D0/
      DATA X(183)/0.121462819296120D0/, A(183)/0.048344762234802D0/
      DATA X(184)/0.072993121787799D0/, A(184)/0.048575467441503D0/
      DATA X(185)/0.024350292663424D0/, A(185)/0.048690957009139D0/
C     N=80
      DATA X(186)/0.999553822651630D0/, A(186)/0.001144950003186D0/
      DATA X(187)/0.997649864398237D0/, A(187)/0.002663533589512D0/
      DATA X(188)/0.994227540965688D0/, A(188)/0.004180313124694D0/
      DATA X(189)/0.989291302499755D0/, A(189)/0.005690922451403D0/
      DATA X(190)/0.982848572738629D0/, A(190)/0.007192904768117D0/
      DATA X(191)/0.974909140585727D0/, A(191)/0.008683945269260D0/
      DATA X(192)/0.965485089043799D0/, A(192)/0.010161766041103D0/
      DATA X(193)/0.954590766343634D0/, A(193)/0.011624114120797D0/
      DATA X(194)/0.942242761309872D0/, A(194)/0.013068761592401D0/
      DATA X(195)/0.928459877172445D0/, A(195)/0.014493508040509D0/
      DATA X(196)/0.913263102571757D0/, A(196)/0.015896183583725D0/
      DATA X(197)/0.896675579438770D0/, A(197)/0.017274652056269D0/
      DATA X(198)/0.878722567678213D0/, A(198)/0.018626814208299D0/
      DATA X(199)/0.859431406663111D0/, A(199)/0.019950610878141D0/
      DATA X(200)/0.838831473580255D0/, A(200)/0.021244026115782D0/
      DATA X(201)/0.816954138681463D0/, A(201)/0.022505090246332D0/
      DATA X(202)/0.793832717504605D0/, A(202)/0.023731882865930D0/
      DATA X(203)/0.769502420135041D0/, A(203)/0.024922535764115D0/
      DATA X(204)/0.744000297583597D0/, A(204)/0.026075235767565D0/
      DATA X(205)/0.717365185362099D0/, A(205)/0.027188227500486D0/
      DATA X(206)/0.689637644342027D0/, A(206)/0.028259816057276D0/
      DATA X(207)/0.660859898986119D0/, A(207)/0.029288369583267D0/
      DATA X(208)/0.631075773046871D0/, A(208)/0.030272321759557D0/
      DATA X(209)/0.600330622829751D0/, A(209)/0.031210174188114D0/
      DATA X(210)/0.568671268122709D0/, A(210)/0.032100498673487D0/
      DATA X(211)/0.536145920897131D0/, A(211)/0.032941939397645D0/
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      DATA X(212)/0.502804111888784D0/, A(212)/0.033733214984611D0/
      DATA X(213)/0.468696615170544D0/, A(213)/0.034473120451753D0/
      DATA X(214)/0.433875370831756D0/, A(214)/0.035160529044747D0/
      DATA X(215)/0.398393405881969D0/, A(215)/0.035794393953416D0/
      DATA X(216)/0.362304753499487D0/, A(216)/0.036373749905835D0/
      DATA X(217)/0.325664370747701D0/, A(217)/0.036897714638276D0/
      DATA X(218)/0.288528054884511D0/, A(218)/0.037365490238730D0/
      DATA X(219)/0.250952358392272D0/, A(219)/0.037776364362001D0/
      DATA X(220)/0.212994502857666D0/, A(220)/0.038129711314477D0/
      DATA X(221)/0.174712291832646D0/, A(221)/0.038424993006959D0/
      DATA X(222)/0.136164022809143D0/, A(222)/0.038661759774076D0/
      DATA X(223)/0.097408398441584D0/, A(223)/0.038839651059051D0/
      DATA X(224)/0.058504437152420D0/, A(224)/0.038958395962769D0/
      DATA X(225)/0.019511383256793D0/, A(225)/0.039017813656306D0/
C     N=96 .
      DATA X(226)/0.999689503883230D0/, A(226)/0.000796792065552D0/
      DATA X(227)/0.998364375863181D0/, A(227)/0.001853960788946D0/
      DATA X(228)/0.995981842987209D0/, A(228)/0.002910731817934D0/
      DATA X(229)/0.992543900323762D0/, A(229)/0.003964554338444D0/
      DATA X(230)/0.988054126329623D0/, A(230)/0.005014202742927D0/
      DATA X(231)/0.982517263563014D0/, A(231)/0.006058545504235D0/
      DATA X(232)/0.975939174585136D0/, A(232)/0.007096470791153D0/
      DATA X(233)/0.968326828463264D0/, A(233)/0.008126876925698D0/
      DATA X(234)/0.959688291448742D0/, A(234)/0.009148671230783D0/
      DATA X(235)/0.950032717784437D0/, A(235)/0.010160770535008D0/
      DATA X(236)/0.939370339752755D0/, A(236)/0.011162102099838D0/
      DATA X(237)/0.927712456722308D0/, A(237)/0.012151604671088D0/
      DATA X(238)/0.915071423120898D0/, A(238)/0.013128229566961D0/
      DATA X(239)/0.901460635315852D0/, A(239)/0.014090941772314D0/
      DATA X(240)/0.886894517402420D0/, A(240)/0.015038721026994D0/
      DATA X(241)/0.871388505909296D0/, A(241)/0.015970562902562D0/
      DATA X(242)/0.854959033434601D0/, A(242)/0.016885479864245D0/
      DATA X(243)/0.837623511228187D0/, A(243)/0.017782502316045D0/
      DATA X(244)/0.819400310737931D0/, A(244)/0.018660679627411D0/
      DATA X(245)/0.800308744139140D0/, A(245)/0.019519081140145D0/
      DATA X(246)/0.780369043867433D0/, A(246)/0.020356797154333D0/
      DATA X(247)/0.759602341176647D0/, A(247)/0.021172939892191D0/
      DATA X(248)/0.738030643744400D0/, A(248)/0.021966644438744D0/
      DATA X(249)/0.715676812348967D0/, A(249)/0.022737069658329D0/
      DATA X(250)/0.692564536642171D0/, A(250)/0.023483399085926D0/
      DATA X(251)/0.668718310043916D0/, A(251)/0.024204841792364D0/
      DATA X(252)/0.644163403784967D0/, A(252)/0.024900633222483D0/
      DATA X(253)/0.618925840125468D0/, A(253)/0.025570036005349D0/
      DATA X(254)/0.593032364777572D0/, A(254)/0.026212340735672D0/
      DATA X(255)/0.566510418561397D0/, A(255)/0.026826866725591D0/
      DATA X(256)/0.539388108324357D0/, A(256)/0.027412962726029D0/
      DATA X(257)/0.511694177154667D0/, A(257)/0.027970007616848D0/
      DATA X(258)/0.483457973920596D0/, A(258)/0.028497411065085D0/
      DATA X(259)/0.454709422167743D0/, A(259)/0.028994614150555D0/
      DATA X(260)/0.425478988407300D0/, A(260)/0.029461089958167D0/
      DATA X(261)/0.395797649828908D0/, A(261)/0.029896344136328D0/
      DATA X(262)/0.365696861472313D0/, A(262)/0.030299915420827D0/
      DATA X(263)/0.335208522892625D0/, A(263)/0.030671376123669D0/
      DATA X(264)/0.304364944354496D0/, A(264)/0.031010332586313D0/
      DATA X(265)/0.273198812591049D0/, A(265)/0.031316425596861D0/
      DATA X(266)/0.241743156163840D0/, A(266)/0.031589330770727D0/
      DATA X(267)/0.210031310460567D0/, A(267)/0.031828758894411D0/
```

```
      DATA X(268)/0.178096882367618D0/, A(268)/0.032034456231992D0/
      DATA X(269)/0.145973714654896D0/, A(269)/0.032206204794030D0/
      DATA X(270)/0.113695850110665D0/, A(270)/0.032343822568575D0/
      DATA X(271)/0.081297495464425D0/, A(271)/0.032447163714064D0/
      DATA X(272)/0.048812985136049D0/, A(272)/0.032516118713868D0/
      DATA X(273)/0.016276744849602D0/, A(273)/0.032550614492363D0/
C
C     Test N
      N=NX
      ALPHA=0.5D0*(BX+AX)
      BETA=0.5D0*(BX-AX)
      IF (N.EQ.1) THEN
          Z(1)=ALPHA
          W(1)=BX-AX
      ELSEIF ((N.GT.1.AND.N.LE.16).OR.N.EQ.20.OR.N.EQ.24.OR.N.EQ.32.OR.
     + N.EQ.40.OR.N.EQ.48.OR.N.EQ.64.OR.N.EQ.80.OR.N.EQ.96) THEN
C         Set K equal to initial subscript and store results
          K=KTAB(N)
          M=N/2
          DO 904 J=1,M
              JTAB=K-1+J
              WTEMP=BETA*A(JTAB)
              DELTA=BETA*X(JTAB)
              Z(J)=ALPHA-DELTA
              W(J)=WTEMP
              JP=N+1-J
              Z(JP)=ALPHA+DELTA
              W(JP)=WTEMP
  904     CONTINUE
          IF ((N-M-M).EQ.0) RETURN
          Z(M+1)=ALFHA
          JMID=K+M
          W(M+1)=BETA*A(JMID)
      ELSE
          ZN=N
          WRITE (1,50) ZN
   50     FORMAT('0Error in GSET. N has the nonpermissable value',
     +     1PE11.3)
      ENDIF
      RETURN
      END
```

```
$TITL EVALI -- 61-point Gauss-Kronrod integration
      FUNCTION EVALI(A,B,FUNC,NS)
C     Function to evaluate the integral of the function FUNC
C     between A and B.  The number of intervals starts at NSTEP
C     as defined in EVICOM and then goes up by a factor of two
C     until the error in the integral is smaller than EPS.
C     In this version, 61-point Gauss-Kronrod integration
C     using the subroutine DQK61 from the NBS library CMLIB
C     is done.
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
C     The maximum number of times that the integral is evaluated
      PARAMETER (IDIVMX=10)
      EXTERNAL FUNC
      COMMON /EVICOM/ NSTEP,EPS
      NS=NSTEP
      IDIV=0
C
C     Start of loop
4     STEP=(B-A)/NS
      EV=0.D0
      ERR=0.D0
      DO 902 I=1,NS
          AX=A+(I-1)*STEP
          BX=AX+STEP
          CALL DQK61(FUNC,AX,BX,RESULT,ABSERR,RESABS,RESASC)
          EV=EV+RESULT
          ERR=ERR+ABSERR
902   CONTINUE
      IF (EPS.LE.0.D0) THEN
CD        WRITE (1,*) A,B,ERR,DABS(EV*EPS)
          IF (ERR.LE.DABS(EV*EPS)) GO TO 6
      ELSE
CD        WRITE (1,*) A,B,ERR,EPS
          IF (ERR.LE.EPS) GO TO 6
      ENDIF
      IF (NS.NE.NSTEP) THEN
          IF (EPS.LE.0.D0) THEN
              IF (DABS(EV-EVALIO).LE.DABS(EV*EPS)) GO TO 6
CD            WRITE (1,*) A,B,DABS(EV-EVALIO),DABS(EV*EPS)
          ELSE
              IF (DABS(EV-EVALIO).LE.EPS) GO TO 6
CD            WRITE (1,*) A,B,DABS(EV-EVALIO),EPS
          ENDIF
      ENDIF
      EVALIO=EV
      NS=2*NS
      IDIV=IDIV+1
      IF (IDIV.LE.IDIVMX) GO TO 4
      NS=-1
6     EVALI=EV
      RETURN
      END
```

```
      SUBROUTINE DQK61(F,A,B,RESULT,ABSERR,RESABS,RESASC)
C***BEGIN PROLOGUE  DQK61
C***DATE WRITTEN   800101   (YYMMDD)
C***REVISION DATE  810101   (YYMMDD)
C***CATEGORY NO.  D1
C***KEYWORDS  61-POINT GAUSS-KRONROD RULES
C***AUTHOR  PIESSENS,ROBERT,APPL. MATH. & PROGR. DIV. - K.U.LEUVEN
C           DE DONCKER,ELISE,APPL. MATH. & PROGR. DIV. - K.U.LEUVEN
C           D.K. KAHANER (NBS)
C***PURPOSE  TO COMPUTE I = INTEGRAL OF F OVER (A,B) WITH ERROR
C                           ESTIMATE
C                       J = INTEGRAL OF DABS(F) OVER (A,B)
C***DESCRIPTION
C.......................................................................
C
C        INTEGRATION RULE
C        STANDARD FORTRAN SUBROUTINE
C        DOUBLE PRECISION VERSION
C
C
C        PARAMETERS
C         ON ENTRY
C           F      - DOUBLE PRECISION
C                    SUBROUTINE DEFINING THE INTEGRAND F(X) AT X:
C                    CALL F(X,F(X)). THE ACTUAL NAME FOR F NEEDS TO BE
C                    DECLARED E X T E R N A L IN THE CALLING PROGRAM.
C
C           A      - DOUBLE PRECISION
C                    LOWER LIMIT OF INTEGRATION
C
C           B      - DOUBLE PRECISION
C                    UPPER LIMIT OF INTEGRATION
C
C         ON RETURN
C           RESULT - DOUBLE PRECISION
C                    APPROXIMATION TO THE INTEGRAL I
C                    RESULT IS COMPUTED BY APPLYING THE 61-POINT
C                    KRONROD RULE (RESK) OBTAINED BY OPTIMAL ADDITION OF
C                    ABSCISSAE TO THE 30-POINT GAUSS RULE (RESG).
C
C           ABSERR - DOUBLE PRECISION
C                    ESTIMATE OF THE MODULUS OF THE ABSOLUTE ERROR,
C                    WHICH SHOULD EQUAL OR EXCEED DABS(I-RESULT)
C
C           RESABS - DOUBLE PRECISION
C                    APPROXIMATION TO THE INTEGRAL J
C
C           RESASC - DOUBLE PRECISION
C                    APPROXIMATION TO THE INTEGRAL OF DABS(F-I/(B-A))
C
C
C***REFERENCES  (NONE)
C***ROUTINES CALLED  D1MACH,DABS,DMAX1,DMIN1,F(USER-PROVIDED FUNCTION)
C
C.......................................................................
C***END PROLOGUE  DQK61
C
```

```
      DOUBLE PRECISION A,DABSC,ABSERR,B,CENTR,DABS,DHLGTH,DMAX1,DMIN1,
     *  D1MACH,EPMACH,FC,FSUM,FVAL1,FVAL2,FV1,FV2,HLGTH,RESABS,RESASC,
     *  RESG,RESK,RESKH,RESULT,UFLOW,WG,WGK,XGK
      INTEGER J,JTW,JTWM1
C
      DIMENSION FV1(30),FV2(30),XGK(31),WGK(31),WG(15)
C
C           THE ABSCISSAE AND WEIGHTS ARE GIVEN FOR THE
C           INTERVAL (-1,1). BECAUSE OF SYMMETRY ONLY THE POSITIVE
C           ABSCISSAE AND THEIR CORRESPONDING WEIGHTS ARE GIVEN.
C
C           XGK    - ABSCISSAE OF THE 61-POINT KRONROD RULE
C                    XGK(2), XGK(4)  ... ABSCISSAE OF THE 30-POINT
C                    GAUSS RULE
C                    XGK(1), XGK(3)  ... OPTIMALLY ADDED ABSCISSAE
C                    TO THE 30-POINT GAUSS RULE
C
C           WGK    - WEIGHTS OF THE 61-POINT KRONROD RULE
C
C           WG     - WEIGTHS OF THE 30-POINT GAUSS RULE
C
C
C GAUSS QUADRATURE WEIGHTS AND KRONRON QUADRATURE ABSCISSAE AND WEIGHTS
C AS EVALUATED WITH 80 DECIMAL DIGIT ARITHMETIC BY L. W. FULLERTON,
C BELL LABS, NOV. 1981.
C
      DATA WG  (  1) / 0.0079681924 9616660561 5465883474 674 DO /
      DATA WG  (  2) / 0.0184664683 1109095914 2302131912 047 DO /
      DATA WG  (  3) / 0.0287847078 8332336934 9719179611 292 DO /
      DATA WG  (  4) / 0.0387991925 6962704959 6801936446 348 DO /
      DATA WG  (  5) / 0.0484026728 3059405290 2938140422 808 DO /
      DATA WG  (  6) / 0.0574931562 1761906648 1721689402 056 DO /
      DATA WG  (  7) / 0.0659742298 8218049512 8128515115 962 DO /
      DATA WG  (  8) / 0.0737559747 3770520626 8243850022 191 DO /
      DATA WG  (  9) / 0.0807558952 2942021535 4694938460 530 DO /
      DATA WG  ( 10) / 0.0868997872 0108297980 2387530715 126 DO /
      DATA WG  ( 11) / 0.0921225222 3778612871 7632707087 619 DO /
      DATA WG  ( 12) / 0.0963687371 7464425963 9468626351 810 DO /
      DATA WG  ( 13) / 0.0995934205 8679526706 2780282103 569 DO /
      DATA WG  ( 14) / 0.1017623897 4840550459 6428952168 554 DO /
      DATA WG  ( 15) / 0.1028526528 9355884034 1285636705 415 DO /
C
      DATA XGK (  1) / 0.9994844100 5049063757 1325895705 811 DO /
      DATA XGK (  2) / 0.9968934840 7464954027 1630050918 695 DO /
      DATA XGK (  3) / 0.9916309968 7040459485 8628366109 486 DO /
      DATA XGK (  4) / 0.9836681232 7974720997 0032581605 663 DO /
      DATA XGK (  5) / 0.9731163225 0112626837 4693868423 707 DO /
      DATA XGK (  6) / 0.9600218649 6830751221 6871025581 798 DO /
      DATA XGK (  7) / 0.9443744447 4855997941 5831324037 439 DO /
      DATA XGK (  8) / 0.9262000474 2927432587 9324277080 474 DO /
      DATA XGK (  9) / 0.9055733076 9990779854 6522558925 958 DO /
      DATA XGK ( 10) / 0.8825605357 9205268154 3116462530 226 DO /
      DATA XGK ( 11) / 0.8572052335 4606109895 8658510658 944 DO /
      DATA XGK ( 12) / 0.8295657623 8276839744 2898119732 502 DO /
      DATA XGK ( 13) / 0.7997278358 2183908301 3668942322 683 DO /
      DATA XGK ( 14) / 0.7677774321 0482619491 7977340974 503 DO /
      DATA XGK ( 15) / 0.7337900624 5322680472 6171131369 528 DO /
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      DATA XGK ( 16) / 0.6978504947 9331579693 2292388026 640 DO /
      DATA XGK ( 17) / 0.6600610641 2662696137 0053668149 271 DO /
      DATA XGK ( 18) / 0.6205261829 8924286114 0477556431 189 DO /
      DATA XGK ( 19) / 0.5793452358 2636169175 6024932172 540 DO /
      DATA XGK ( 20) / 0.5366241481 4201989926 4169793311 073 DO /
      DATA XGK ( 21) / 0.4924804678 6177857499 3693061207 709 DO /
      DATA XGK ( 22) / 0.4470337695 3808917678 0609900322 854 DO /
      DATA XGK ( 23) / 0.4004012548 3039439253 5476211542 661 DO /
      DATA XGK ( 24) / 0.3527047255 3087811347 1037207089 374 DO /
      DATA XGK ( 25) / 0.3040732022 7362507737 2677107199 257 DO /
      DATA XGK ( 26) / 0.2546369261 6788984643 9805129817 805 DO /
      DATA XGK ( 27) / 0.2045251166 8230989143 8957671002 025 DO /
      DATA XGK ( 28) / 0.1538699136 0858354696 3794672743 256 DO /
      DATA XGK ( 29) / 0.1028069379 6673703014 7096751318 001 DO /
      DATA XGK ( 30) / 0.0514718425 5531769583 3025213166 723 DO /
      DATA XGK ( 31) / 0.0000000000 0000000000 0000000000 000 DO /
C
      DATA WGK (  1) / 0.0013890136 9867700762 4551591226 760 DO /
      DATA WGK (  2) / 0.0038904611 2709988405 1267201844 516 DO /
      DATA WGK (  3) / 0.0066307039 1593129217 3319826369 750 DO /
      DATA WGK (  4) / 0.0092732796 5951776342 8441146892 024 DO /
      DATA WGK (  5) / 0.0118230152 5349634174 2232898853 251 DO /
      DATA WGK (  6) / 0.0143697295 0704580481 2451432443 580 DO /
      DATA WGK (  7) / 0.0169208891 8905327262 7572289420 322 DO /
      DATA WGK (  8) / 0.0194141411 9394238117 3408951050 128 DO /
      DATA WGK (  9) / 0.0218280358 2160919229 7167485738 339 DO /
      DATA WGK ( 10) / 0.0241911620 7808060136 5686370725 232 DO /
      DATA WGK ( 11) / 0.0265099548 8233310161 0601709335 075 DO /
      DATA WGK ( 12) / 0.0287540487 6504129284 3978785354 334 DO /
      DATA WGK ( 13) / 0.0309072575 6238776247 2884252943 092 DO /
      DATA WGK ( 14) / 0.0329814470 5748372603 1814191016 854 DO /
      DATA WGK ( 15) / 0.0349793380 2806002413 7499670731 468 DO /
      DATA WGK ( 16) / 0.0368823646 5182122922 3911065617 136 DO /
      DATA WGK ( 17) / 0.0386789456 2472759295 0348651532 281 DO /
      DATA WGK ( 18) / 0.0403745389 5153595911 1995279752 468 DO /
      DATA WGK ( 19) / 0.0419698102 1516424614 7147541285 970 DO /
      DATA WGK ( 20) / 0.0434525397 0135606931 6831728117 073 DO /
      DATA WGK ( 21) / 0.0448148001 3316266319 2355551616 723 DO /
      DATA WGK ( 22) / 0.0460592382 7100698811 6271735559 374 DO /
      DATA WGK ( 23) / 0.0471855465 6929915394 5261478181 099 DO /
      DATA WGK ( 24) / 0.0481858617 5708712914 0779492298 305 DO /
      DATA WGK ( 25) / 0.0490554345 5502977888 7528165367 238 DO /
      DATA WGK ( 26) / 0.0497956834 2707420635 7811569379 942 DO /
      DATA WGK ( 27) / 0.0504059214 0278234684 0893085653 585 DO /
      DATA WGK ( 28) / 0.0508817958 9874960649 2297473049 805 DO /
      DATA WGK ( 29) / 0.0512215478 4925877217 0656282604 944 DO /
      DATA WGK ( 30) / 0.0514261285 3745902593 3862879215 781 DO /
      DATA WGK ( 31) / 0.0514947294 2945156755 8340433647 099 DO /
C
C           LIST OF MAJOR VARIABLES
C           -----------------------
C
C           CENTR  - MID POINT OF THE INTERVAL
C           HLGTH  - HALF-LENGTH OF THE INTERVAL
C           DABSC  - ABSCISSA
C           FVAL*  - FUNCTION VALUE
C           RESG   - RESULT OF THE 30-POINT GAUSS RULE
```

```
C            RESK   - RESULT OF THE 61-POINT KRONROD RULE
C            RESKH  - APPROXIMATION TO THE MEAN VALUE OF F
C                     OVER (A,B), I.E. TO I/(B-A)
C
C            MACHINE DEPENDENT CONSTANTS
C            ---------------------------
C
C            EPMACH IS THE LARGEST RELATIVE SPACING.
C            UFLOW IS THE SMALLEST POSITIVE MAGNITUDE.
C
      EPMACH = D1MACH(4)
      UFLOW = D1MACH(1)
C
      CENTR = 0.5D+00*(B+A)
      HLGTH = 0.5D+00*(B-A)
      DHLGTH = DABS(HLGTH)
C
C            COMPUTE THE 61-POINT KRONROD APPROXIMATION TO THE
C            INTEGRAL, AND ESTIMATE THE ABSOLUTE ERROR.
C
C***FIRST EXECUTABLE STATEMENT  DQK61
      RESG = 0.0D+00
      CALL F(CENTR,FC)
      RESK = WGK(31)*FC
      RESABS = DABS(RESK)
      DO 10 J=1,15
        JTW = J*2
        DABSC = HLGTH*XGK(JTW)
        CALL F(CENTR-DABSC,FVAL1)
        CALL F(CENTR+DABSC,FVAL2)
        FV1(JTW) = FVAL1
        FV2(JTW) = FVAL2
        FSUM = FVAL1+FVAL2
        RESG = RESG+WG(J)*FSUM
        RESK = RESK+WGK(JTW)*FSUM
        RESABS = RESABS+WGK(JTW)*(DABS(FVAL1)+DABS(FVAL2))
   10 CONTINUE
      DO 15 J=1,15
        JTWM1 = J*2-1
        DABSC = HLGTH*XGK(JTWM1)
        CALL F(CENTR-DABSC,FVAL1)
        CALL F(CENTR+DABSC,FVAL2)
        FV1(JTWM1) = FVAL1
        FV2(JTWM1) = FVAL2
        FSUM = FVAL1+FVAL2
        RESK = RESK+WGK(JTWM1)*FSUM
        RESABS = RESABS+WGK(JTWM1)*(DABS(FVAL1)+DABS(FVAL2))
  15    CONTINUE
      RESKH = RESK*0.5D+00
      RESASC = WGK(31)*DABS(FC-RESKH)
      DO 20 J=1,30
        RESASC = RESASC+WGK(J)*(DABS(FV1(J)-RESKH)+DABS(FV2(J)-RESKH))
   20 CONTINUE
      RESULT = RESK*HLGTH
      RESABS = RESABS*DHLGTH
      RESASC = RESASC*DHLGTH
      ABSERR = DABS((RESK-RESG)*HLGTH)
```

```
      IF(RESASC.NE.0.0D+00.AND.ABSERR.NE.0.0D+00)
     *  ABSERR = RESASC*DMIN1(0.1D+01,(0.2D+03*ABSERR/RESASC)**1.5D+00)
      IF(RESABS.GT.UFLOW/(0.5D+02*EPMACH)) ABSERR = DMAX1
     *  ((EPMACH*0.5D+02)*RESABS,ABSERR)
      RETURN
      END
```

```
      DOUBLE PRECISION FUNCTION D1MACH(I)
C
C  DOUBLE-PRECISION MACHINE CONSTANTS
C
C  D1MACH( 1) = B**(EMIN-1), THE SMALLEST POSITIVE MAGNITUDE.
C
C  D1MACH( 2) = B**EMAX*(1 - B**(-T)), THE LARGEST MAGNITUDE.
C
C  D1MACH( 3) = B**(-T), THE SMALLEST RELATIVE SPACING.
C
C  D1MACH( 4) = B**(1-T), THE LARGEST RELATIVE SPACING.
C
C  D1MACH( 5) = LOG10(B)
C
C  TO ALTER THIS FUNCTION FOR A PARTICULAR ENVIRONMENT,
C  THE DESIRED SET OF DATA STATEMENTS SHOULD BE ACTIVATED BY
C  REMOVING THE C FROM COLUMN 1.
C
C  WHERE POSSIBLE, OCTAL OR HEXADECIMAL CONSTANTS HAVE BEEN USED
C  TO SPECIFY THE CONSTANTS EXACTLY WHICH HAS IN SOME CASES
C  REQUIRED THE USE OF EQUIVALENT INTEGER ARRAYS.
C
      INTEGER SMALL(4)
      INTEGER LARGE(4)
      INTEGER RIGHT(4)
      INTEGER DIVER(4)
      INTEGER LOG10(4)
C
      DOUBLE PRECISION DMACH(5)
C
      EQUIVALENCE (DMACH(1),SMALL(1))
      EQUIVALENCE (DMACH(2),LARGE(1))
      EQUIVALENCE (DMACH(3),RIGHT(1))
      EQUIVALENCE (DMACH(4),DIVER(1))
      EQUIVALENCE (DMACH(5),LOG10(1))
C
C     MACHINE CONSTANTS FOR THE BURROUGHS 1700 SYSTEM.
C
C     DATA SMALL(1) / ZC00800000 /
C     DATA SMALL(2) / Z000000000 /
C
C     DATA LARGE(1) / ZDFFFFFFFF /
C     DATA LARGE(2) / ZFFFFFFFFF /
C
C     DATA RIGHT(1) / ZCC5800000 /
C     DATA RIGHT(2) / Z000000000 /
C
C     DATA DIVER(1) / ZCC6800000 /
C     DATA DIVER(2) / Z000000000 /
C
C     DATA LOG10(1) / ZD00E730E7 /
C     DATA LOG10(2) / ZC77800DC0 /
C
C     MACHINE CONSTANTS FOR THE BURROUGHS 5700 SYSTEM.
C
C     DATA SMALL(1) / 01771000000000000 /
C     DATA SMALL(2) / 00000000000000000 /
```

```
C
C     DATA LARGE(1) / 00777777777777777 /
C     DATA LARGE(2) / 00007777777777777 /
C
C     DATA RIGHT(1) / 01461000000000000 /
C     DATA RIGHT(2) / 00000000000000000 /
C
C     DATA DIVER(1) / 01451000000000000 /
C     DATA DIVER(2) / 00000000000000000 /
C
C     DATA LOG10(1) / 01157163034761674 /
C     DATA LOG10(2) / 00006677466732724 /
C
C     MACHINE CONSTANTS FOR THE BURROUGHS 6700/7700 SYSTEMS.
C
C     DATA SMALL(1) / 01771000000000000 /
C     DATA SMALL(2) / 07770000000000000 /
C
C     DATA LARGE(1) / 00777777777777777 /
C     DATA LARGE(2) / 07777777777777777 /
C
C     DATA RIGHT(1) / 01461000000000000 /
C     DATA RIGHT(2) / 00000000000000000 /
C
C     DATA DIVER(1) / 01451000000000000 /
C     DATA DIVER(2) / 00000000000000000 /
C
C     DATA LOG10(1) / 01157163034761674 /
C     DATA LOG10(2) / 00006677466732724 /
C
C     MACHINE CONSTANTS FOR THE CDC 6000/7000 SERIES.
C
C     DATA SMALL(1) / 00604000000000000000B /
C     DATA SMALL(2) / 00000000000000000000B /
C
C     DATA LARGE(1) / 37767777777777777777B /
C     DATA LARGE(2) / 37167777777777777777B /
C
C     DATA RIGHT(1) / 15604000000000000000B /
C     DATA RIGHT(2) / 15000000000000000000B /
C
C     DATA DIVER(1) / 15614000000000000000B /
C     DATA DIVER(2) / 15010000000000000000B /
C
C     DATA LOG10(1) / 17164642023241175717B /
C     DATA LOG10(2) / 16367571421742254654B /
C
C     MACHINE CONSTANTS FOR THE CRAY 1
C
C     DATA SMALL(1) / 200004000000000000000B /
C     DATA SMALL(2) / 00000000000000000000B /
C
C     DATA LARGE(1) / 577777777777777777777B /
C     DATA LARGE(2) / 000007777777777777777B /
C
C     DATA RIGHT(1) / 377214000000000000000B /
C     DATA RIGHT(2) / 000000000000000000000B /
```

```
C
C     DATA DIVER(1) / 377224000000000000000B /
C     DATA DIVER(2) / 000000000000000000000B /
C
C     DATA LOG10(1) / 377774642023241175717B /
C     DATA LOG10(2) / 000007571421742254654B /
C
C     MACHINE CONSTANTS FOR THE DATA GENERAL ECLIPSE S/200
C
C     NOTE - IT MAY BE APPROPRIATE TO INCLUDE THE FOLLOWING CARD -
C     STATIC DMACH(5)
C
C     DATA SMALL/20K,3*0/,LARGE/77777K,3*177777K/
C     DATA RIGHT/31420K,3*0/,DIVER/32020K,3*0/
C     DATA LOG10/40423K,42023K,50237K,74776K/
C
C     MACHINE CONSTANTS FOR THE HARRIS 220
C
C     DATA SMALL(1),SMALL(2) / '20000000, '00000201 /
C     DATA LARGE(1),LARGE(2) / '37777777, '37777577 /
C     DATA RIGHT(1),RIGHT(2) / '20000000, '00000333 /
C     DATA DIVER(1),DIVER(2) / '20000000, '00000334 /
C     DATA LOG10(1),LOG10(2) / '23210115, '10237777 /
C
C     MACHINE CONSTANTS FOR THE HONEYWELL 600/6000 SERIES.
C
C     DATA SMALL(1),SMALL(2) / 0402400000000, 0000000000000 /
C     DATA LARGE(1),LARGE(2) / 0376777777777, 0777777777777 /
C     DATA RIGHT(1),RIGHT(2) / 0604400000000, 0000000000000 /
C     DATA DIVER(1),DIVER(2) / 0606400000000, 0000000000000 /
C     DATA LOG10(1),LOG10(2) / 0776464202324, 0117571775714 /
C
C     MACHINE CONSTANTS FOR THE HP 2100
C     3 WORD DOUBLE PRECISION OPTION WITH FTN4
C
C     DATA SMALL(1), SMALL(2), SMALL(3) / 40000B,       0,       1 /
C     DATA LARGE(1), LARGE(2), LARGE(3) / 77777B, 177777B, 177776B /
C     DATA RIGHT(1), RIGHT(2), RIGHT(3) / 40000B,       0,    265B /
C     DATA DIVER(1), DIVER(2), DIVER(3) / 40000B,       0,    276B /
C     DATA LOG10(1), LOG10(2), LOG10(3) / 46420B,  46502B,  77777B /
C
C
C     MACHINE CONSTANTS FOR THE HP 2100
C     4 WORD DOUBLE PRECISION OPTION WITH FTN4
C
C     DATA SMALL(1), SMALL(2) /  40000B,       0 /
C     DATA SMALL(3), SMALL(4) /       0,       1 /
C     DATA LARGE(1), LARGE(2) /  77777B, 177777B /
C     DATA LARGE(3), LARGE(4) / 177777B, 177776B /
C     DATA RIGHT(1), RIGHT(2) /  40000B,       0 /
C     DATA RIGHT(3), RIGHT(4) /       0,    225B /
C     DATA DIVER(1), DIVER(2) /  40000B,       0 /
C     DATA DIVER(3), DIVER(4) /       0,    227B /
C     DATA LOG10(1), LOG10(2) /  46420B,  46502B /
C     DATA LOG10(3), LOG10(4) /  76747B, 176377B /
C
C     MACHINE CONSTANTS FOR THE IBM 360/370 SERIES,
```

```
C     THE XEROX SIGMA 5/7/9, THE SEL SYSTEMS 85/86 AND
C     THE PERKIN ELMER (INTERDATA) 7/32.
C
      DATA SMALL(1),SMALL(2) / Z00100000, Z00000000 /
      DATA LARGE(1),LARGE(2) / Z7FFFFFFF, ZFFFFFFFF /
      DATA RIGHT(1),RIGHT(2) / Z33100000, Z00000000 /
      DATA DIVER(1),DIVER(2) / Z34100000, Z00000000 /
      DATA LOG10(1),LOG10(2) / Z41134413, Z509F79FF /
C
C     MACHINE CONSTANTS FOR THE PDP-10 (KA PROCESSOR).
C
C     DATA SMALL(1),SMALL(2) / "033400000000, "000000000000 /
C     DATA LARGE(1),LARGE(2) / "377777777777, "344777777777 /
C     DATA RIGHT(1),RIGHT(2) / "113400000000, "000000000000 /
C     DATA DIVER(1),DIVER(2) / "114400000000, "000000000000 /
C     DATA LOG10(1),LOG10(2) / "177464202324, "144117571776 /
C
C     MACHINE CONSTANTS FOR THE PDP-10 (KI PROCESSOR).
C
C     DATA SMALL(1),SMALL(2) / "000400000000, "000000000000 /
C     DATA LARGE(1),LARGE(2) / "377777777777, "377777777777 /
C     DATA RIGHT(1),RIGHT(2) / "103400000000, "000000000000 /
C     DATA DIVER(1),DIVER(2) / "104400000000, "000000000000 /
C     DATA LOG10(1),LOG10(2) / "177464202324, "476747767461 /
C
C     MACHINE CONSTANTS FOR PDP-11 FORTRAN'S SUPPORTING
C     32-BIT INTEGERS (EXPRESSED IN INTEGER AND OCTAL).
C
C     DATA SMALL(1),SMALL(2) /    8388608,           0 /
C     DATA LARGE(1),LARGE(2) / 2147483647,          -1 /
C     DATA RIGHT(1),RIGHT(2) /  612368384,           0 /
C     DATA DIVER(1),DIVER(2) /  620756992,           0 /
C     DATA LOG10(1),LOG10(2) / 1067065498, -2063872008 /
C
C     DATA SMALL(1),SMALL(2) / O00040000000, O00000000000 /
C     DATA LARGE(1),LARGE(2) / O17777777777, O37777777777 /
C     DATA RIGHT(1),RIGHT(2) / O04440000000, O00000000000 /
C     DATA DIVER(1),DIVER(2) / O04500000000, O00000000000 /
C     DATA LOG10(1),LOG10(2) / O07746420232, O20476747770 /
C
C     MACHINE CONSTANTS FOR PDP-11 FORTRAN'S SUPPORTING
C     16-BIT INTEGERS (EXPRESSED IN INTEGER AND OCTAL).
C
C     DATA SMALL(1),SMALL(2) /    128,      0 /
C     DATA SMALL(3),SMALL(4) /      0,      0 /
C
C     DATA LARGE(1),LARGE(2) /  32767,     -1 /
C     DATA LARGE(3),LARGE(4) /     -1,     -1 /
C
C     DATA RIGHT(1),RIGHT(2) /   9344,      0 /
C     DATA RIGHT(3),RIGHT(4) /      0,      0 /
C
C     DATA DIVER(1),DIVER(2) /   9472,      0 /
C     DATA DIVER(3),DIVER(4) /      0,      0 /
C
C     DATA LOG10(1),LOG10(2) /  16282,   8346 /
C     DATA LOG10(3),LOG10(4) / -31493, -12296 /
```

```
C
C     DATA SMALL(1),SMALL(2) / 0000200, 0000000 /
C     DATA SMALL(3),SMALL(4) / 0000000, 0000000 /
C
C     DATA LARGE(1),LARGE(2) / 0077777, 0177777 /
C     DATA LARGE(3),LARGE(4) / 0177777, 0177777 /
C
C     DATA RIGHT(1),RIGHT(2) / 0022200, 0000000 /
C     DATA RIGHT(3),RIGHT(4) / 0000000, 0000000 /
C
C     DATA DIVER(1),DIVER(2) / 0022400, 0000000 /
C     DATA DIVER(3),DIVER(4) / 0000000, 0000000 /
C
C     DATA LOG10(1),LOG10(2) / 0037632, 0020232 /
C     DATA LOG10(3),LOG10(4) / 0102373, 0147770 /
C
C     MACHINE CONSTANTS FOR THE UNIVAC 1100 SERIES. FTN COMPILER
C
C     DATA SMALL(1),SMALL(2) / O000040000000, O000000000000 /
C     DATA LARGE(1),LARGE(2) / O377777777777, O777777777777 /
C     DATA RIGHT(1),RIGHT(2) / O170540000000, O000000000000 /
C     DATA DIVER(1),DIVER(2) / O170640000000, O000000000000 /
C     DATA LOG10(1),LOG10(2) / O177746420232, O411757177572 /
C
C
C     MACHINE CONSTANTS FOR THE UNIVAC 1100 SERIES. FOR COMPILER
C
C     DATA SMALL(1),SMALL(2) / O000040000000, O000000000000 /
C     DATA LARGE(1),LARGE(2) / O377777777777, O777777777777 /
C     DATA RIGHT(1),RIGHT(2) / O170540000000, O000000000000 /
C     DATA DIVER(1),DIVER(2) / O170640000000, O000000000000 /
C     DATA LOG10(1),LOG10(2) / O177746420232, O411757177572 /
C
C     MACHINE CONSTANTS FOR VAX-11/780
C     (EXPRESSED IN INTEGER AND OCTAL).
C
C     DATA SMALL(1),SMALL(2) /        128,           0 /
C     DATA LARGE(1),LARGE(2) /     -32769,          -1 /
C     DATA RIGHT(1),RIGHT(2) /       9344,           0 /
C     DATA DIVER(1),DIVER(2) /       9472,           0 /
C     DATA LOG10(1),LOG10(2) /  546979738,  -805665541 /
C
C     DATA SMALL(1),SMALL(2) / O00000000200, O00000000000 /
C     DATA LARGE(1),LARGE(2) / O37777677777, O37777777777 /
C     DATA RIGHT(1),RIGHT(2) / O00000022200, O00000000000 /
C     DATA DIVER(1),DIVER(2) / O00000022400, O00000000000 /
C     DATA LOG10(1),LOG10(2) / O04046437632, O31776502373 /
C
C     IF (I .LT. 1  .OR.  I .GT. 5)
C    1   CALL SETERR(24HD1MACH - I OUT OF BOUNDS,24,1,2)
C
      D1MACH = DMACH(I)
      RETURN
C
      END
```

```
$TITL PWAFFI -- Initialize form-factor table for C-12
      SUBROUTINE PWAFFI
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
C     The size of the arrays in the common PWAFF
      PARAMETER (MNPT=200)
C     The number of points in the interpolation table
      PARAMETER (NTBM1=179)
C     The number of points + 1
      PARAMETER (NTB=NTBM1+1)
      DOUBLE PRECISION XC1(100),XC2(79),FC1(100),FC2(79),X(MNPT)
      DOUBLE PRECISION F(NTBM1),DF(NTBM1),A(MNPT),B(MNPT),C(MNPT)
      DOUBLE PRECISION D(MNPT),AP(MNPT),BP(MNPT),CP(MNPT),DP(MNPT)
      DOUBLE PRECISION XC(NTBM1),FC(NTBM1)
      COMMON /CONST/ EM,EM2,AM,HBARC,ALPH,PIE,ALPHA,RTD,DTR
      COMMON /PWAFF/ NPT,X,A,B,C,D,AP,BP,CP,DP
      COMMON /CORREC/IWTILD,WTIL1,WTIL2,ISOFT,SOFT1
      EQUIVALENCE (FC(1),FC1(1)),(FC(101),FC2(1))
      EQUIVALENCE (XC(1),XC1(1)),(XC(101),XC2(1))
C
C     Table of q and corresponding form factor from DWBA code.  In
C     principle this table could be read in rather than defined in
C     data statements.
      DATA XC1/5.3068D-02,1.0613D-01,1.5918D-01,2.1222D-01,2.6523D-01,
     + 3.1822D-01,3.7118D-01,4.2410D-01,4.7697D-01,5.2980D-01,
     + 5.8258D-01,6.3530D-01,6.8795D-01,7.4054D-01,7.9305D-01,
     + 8.4549D-01,8.9784D-01,9.5010D-01,1.0023D+00,1.0543D+00,
     + 1.1063D+00,1.1582D+00,1.2099D+00,1.2615D+00,1.3130D+00,
     + 1.3644D+00,1.4156D+00,1.4667D+00,1.5177D+00,1.5685D+00,
     + 1.6191D+00,1.6696D+00,1.7199D+00,1.7701D+00,1.8201D+00,
     + 1.8699D+00,1.9195D+00,1.9690D+00,2.0182D+00,2.0673D+00,
     + 2.1162D+00,2.1648D+00,2.2133D+00,2.2615D+00,2.3096D+00,
     + 2.3574D+00,2.4050D+00,2.4524D+00,2.4995D+00,2.5464D+00,
     + 2.5931D+00,2.6395D+00,2.6856D+00,2.7316D+00,2.7772D+00,
     + 2.8226D+00,2.8678D+00,2.9127D+00,2.9573D+00,3.0017D+00,
     + 3.0457D+00,3.0895D+00,3.1331D+00,3.1763D+00,3.2193D+00,
     + 3.2619D+00,3.3043D+00,3.3464D+00,3.3881D+00,3.4296D+00,
     + 3.4708D+00,3.5116D+00,3.5522D+00,3.5924D+00,3.6324D+00,
     + 3.6720D+00,3.7113D+00,3.7502D+00,3.7889D+00,3.8272D+00,
     + 3.8652D+00,3.9029D+00,3.9402D+00,3.9772D+00,4.0138D+00,
     + 4.0501D+00,4.0861D+00,4.1218D+00,4.1570D+00,4.1920D+00,
     + 4.2266D+00,4.2608D+00,4.2947D+00,4.3282D+00,4.3614D+00,
     + 4.3942D+00,4.4267D+00,4.4588D+00,4.4906D+00,4.5219D+00/
      DATA XC2/4.5529D+00,4.5836D+00,4.6139D+00,4.6438D+00,4.6733D+00,
     + 4.7025D+00,4.7313D+00,4.7598D+00,4.7878D+00,4.8155D+00,
     + 4.8428D+00,4.8698D+00,4.8963D+00,4.9225D+00,4.9483D+00,
     + 4.9737D+00,4.9987D+00,5.0234D+00,5.0477D+00,5.0715D+00,
     + 5.0950D+00,5.1182D+00,5.1409D+00,5.1632D+00,5.1852D+00,
     + 5.2068D+00,5.2280D+00,5.2488D+00,5.2692D+00,5.2892D+00,
     + 5.3088D+00,5.3280D+00,5.3469D+00,5.3653D+00,5.3834D+00,
     + 5.4011D+00,5.4183D+00,5.4352D+00,5.4517D+00,5.4678D+00,
     + 5.4835D+00,5.4988D+00,5.5137D+00,5.5283D+00,5.5424D+00,
     + 5.5561D+00,5.5695D+00,5.5824D+00,5.5949D+00,5.6071D+00,
     + 5.6188D+00,5.6302D+00,5.6411D+00,5.6517D+00,5.6618D+00,
     + 5.6716D+00,5.6810D+00,5.6899D+00,5.6985D+00,5.7067D+00,
     + 5.7145D+00,5.7218D+00,5.7288D+00,5.7354D+00,5.7416D+00,
     + 5.7474D+00,5.7527D+00,5.7577D+00,5.7623D+00,5.7665D+00,
     + 5.7703D+00,5.7737D+00,5.7767D+00,5.7793D+00,5.7815D+00,
```

```
     + 5.7833D+00,5.7847D+00,5.7857D+00,5.7862D+00/
      DATA FC1/1.0000D+00,9.8087D-01,9.5239D-01,9.1556D-01,8.6975D-01,
     + 8.1646D-01,7.5736D-01,6.9420D-01,6.2873D-01,5.6261D-01,
     + 4.9734D-01,4.3427D-01,3.7450D-01,3.1890D-01,2.6807D-01,
     + 2.2240D-01,1.8203D-01,1.4692D-01,1.1688D-01,9.1572D-02,
     + 7.0602D-02,5.3507D-02,3.9805D-02,2.9015D-02,2.0675D-02,
     + 1.4357D-02,9.6756D-03,6.2920D-03,3.9161D-03,2.3048D-03,
     + 1.2588D-03,6.1895D-04,2.6065D-04,8.9112D-05,3.4238D-05,
     + 4.5899D-05,8.9708D-05,1.4332D-04,1.9333D-04,2.3265D-04,
     + 2.5852D-04,2.7093D-04,2.7143D-04,2.6234D-04,2.4618D-04,
     + 2.2533D-04,2.0186D-04,1.7744D-04,1.5334D-04,1.3047D-04,
     + 1.0941D-04,9.0497D-05,7.3864D-05,5.9512D-05,4.7338D-05,
     + 3.7177D-05,2.8825D-05,2.2064D-05,1.6672D-05,1.2435D-05,
     + 9.1554D-06,6.6544D-06,4.7753D-06,3.3842D-06,2.3691D-06,
     + 1.6387D-06,1.1202D-06,7.5662D-07,5.0471D-07,3.3205D-07,
     + 2.1493D-07,1.3630D-07,8.4139D-08,5.0045D-08,2.8251D-08,
     + 1.4815D-08,7.0487D-09,3.1135D-09,1.7327D-09,2.0021D-09,
     + 3.2652D-09,5.0351D-09,6.9478D-09,8.7363D-09,1.0215D-08,
     + 1.1272D-08,1.1855D-08,1.1969D-08,1.1660D-08,1.1001D-08,
     + 1.0085D-08,9.0060D-09,7.8529D-09,6.7016D-09,5.6109D-09,
     + 4.6210D-09,3.7547D-09,3.0203D-09,2.4145D-09,1.9268D-09/
      DATA FC2/1.5422D-09,1.2443D-09,1.0168D-09,8.4509D-10,7.1668D-10,
     + 6.2124D-10,5.5047D-10,4.9786D-10,4.5831D-10,4.2782D-10,
     + 4.0331D-10,3.8236D-10,3.6319D-10,3.4448D-10,3.2538D-10,
     + 3.0541D-10,2.8440D-10,2.6246D-10,2.3986D-10,2.1698D-10,
     + 1.9427D-10,1.7217D-10,1.5108D-10,1.3133D-10,1.1315D-10,
     + 9.6708D-11,8.2061D-11,6.9205D-11,5.8070D-11,4.8544D-11,
     + 4.0484D-11,3.3732D-11,2.8123D-11,2.3498D-11,1.9707D-11,
     + 1.6614D-11,1.4098D-11,1.2055D-11,1.0398D-11,9.0514D-12,
     + 7.9547D-12,7.0577D-12,6.3195D-12,5.7071D-12,5.1943D-12,
     + 4.7601D-12,4.3877D-12,4.0639D-12,3.7782D-12,3.5222D-12,
     + 3.2896D-12,3.0756D-12,2.8765D-12,2.6900D-12,2.5143D-12,
     + 2.3483D-12,2.1911D-12,2.0422D-12,1.9009D-12,1.7673D-12,
     + 1.6412D-12,1.5229D-12,1.4126D-12,1.3105D-12,1.2164D-12,
     + 1.1302D-12,1.0513D-12,9.7921D-13,9.1362D-13,8.5462D-13,
     + 8.0261D-13,7.5814D-13,7.2158D-13,6.9264D-13,6.6982D-13,
     + 6.4928D-13,6.2152D-13,5.5475D-13,3.2798D-13/
C
C     Convert table from form factor versus q to form factor
C     times Z**2 versus q**2
      DO 10 I=1,NTBM1
          X(I)=XC(I)**2
          F(I)=FC(I)*36.DO
          IF (IWTILD.NE.O) F(I)=F(I)*(WTIL1+WTIL2*DLOG(X(I)/EM2))
   10 CONTINUE
C
C     Find the cubic coefficients for spline interpolation of
C     the table.  A good reference for spline interpolation is
C     Elementary numerical analysis by S.D. Conte and Carl deBoor.
      A(1)=36.DO
      B(1)=O.DO
      C(1)=O.DO
      D(1)=O.DO
      X(NTB)=1.D74
      A(NTB)=O.DO
      B(NTB)=O.DO
      C(NTB)=O.DO
```

```
      D(NTB)=0.D0
C     Calculate the derivatives of the form factor (DF) according
C     to the spline method at each of the tabulated points.
      P=0.D0
      Q=0.D0
      H2=1.D0/(X(2)-X(1))
      DO 908 I=2,NTBM1
          H1=H2
          H12=H1*H1
          H2=1.D0/(X(I+1)-X(I))
          H22=H2*H2
          R=2.D0*(H1+H2)-P
          B(I)=H2/R
          DF(I)=-3.D0*(F(I-1)*H12+F(I)*(H22-H12)-F(I+1)*H22)
          DF(I)=(DF(I)-Q)/R
          P=B(I)*H2
          Q=DF(I)*H2
908   CONTINUE
C     Now back substitution
      P=0.D0
      DO 909 I=1,NTBM1
          J=NTBM1+1-I
          DF(J)=DF(J)-P*B(J)
          P=DF(J)
909   CONTINUE
C     Using the spline approximation values for the derivatives
C     at the points in the table, calculate the coefficients
C     for a cubic which exactly fits these derivatives and the
C     values of the function at the end points of each interval.
      DO 20 I=2,NTBM1
          H=1.D0/(X(I)-X(I-1))
          H2=H*H
          A1=DF(I)-2.D0*F(I)*H
          C1=F(I)-A1*X(I)
          A2=DF(I-1)+2.D0*F(I-1)*H
          C2=F(I-1)-A2*X(I-1)
          D(I)=(A1+A2)*H2
          C(I)=(C1-2.D0*A1*X(I-1)+C2-2.D0*A2*X(I))*H2
          B(I)=(A1*X(I-1)**2-2.D0*C1*X(I-1)+A2*X(I)**2-
     +     2.D0*C2*X(I))*H2
          A(I)=(C1*X(I-1)**2+C2*X(I)**2)*H2
20    CONTINUE
C     For this model, we assume that W1 (the transverse form factor)
C     is 0.
      DO 30 I=1,NTB
          AP(I)=0.D0
          BP(I)=0.D0
          CP(I)=0.D0
          DP(I)=0.D0
30    CONTINUE
      NPT=NTB
      RETURN
      END
```

```
$TITL FORM1 -- Transverse form factor by interpolation
      FUNCTION FORM1(Q2)
C     Size of arrays in PWAFF
      PARAMETER (MNPT=200)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DOUBLE PRECISION X(MNPT),A(MNPT),B(MNPT),C(MNPT),D(MNPT)
      DOUBLE PRECISION AP(MNPT),BP(MNPT),CP(MNPT),DP(MNPT)
      COMMON /PWAFF/ NPT,X,A,B,C,D,AP,BP,CP,DP
C     CALL STARTT(1)
      QQ=-Q2
      DO 10 I=1,NPT-1
          IF (QQ.LT.X(I)) GO TO 20
10    CONTINUE
      I=NPT
20    FORM1=((DP(I)*QQ+CP(I))*QQ+BP(I))*QQ+AP(I)
C     CALL STOPT(1)
      RETURN
      END
```

```
$TITL FORM2 -- Longitudinal form factor by interpolation
      FUNCTION FORM2(Q2)
C     Size of arrays in PWAFF
      PARAMETER (MNPT=200)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DOUBLE PRECISION X(MNPT),A(MNPT),B(MNPT),C(MNPT),D(MNPT)
      DOUBLE PRECISION AP(MNPT),BP(MNPT),CP(MNPT),DP(MNPT)
      COMMON /PWAFF/ NPT,X,A,B,C,D,AP,BP,CP,DP
C     CALL STARTT(1)
      QQ=-Q2
      DO 10 I=1,NPT-1
         IF (QQ.LT.X(I)) GO TO 20
10    CONTINUE
      I=NPT
20    FORM2=((D(I)*QQ+C(I))*QQ+B(I))*QQ+A(I)
C     CALL STOPT(1)
      RETURN
      END
```

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We consider here the calculation of the radiative tail from the elastic peak in medium and high energy electron scattering as well as from a discrete inelastic level of the recoiling nucleus. We examine the method generally used for this calculation, viz., a numerical integration of the differential cross section over the angles of the unobserved photon, and discuss the difficulties inherent in this numerical integration due to the sharp peaking of the integrand. We present an alternative method for calculating the radiative tail, in which the region of integration is divided into an arbitrary number of subintervals, the structure functions are fitted by cubic spline functions in each subinterval, and the integrations are then performed analytically in closed form. This method has the advantages of greatly increased accuracy and a reduction of the computation time by a factor which can vary between 10 and 10^3, depending on the kinematics.

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