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Inviscid stability analysis has been applied to the mixing layer profile of an axisymmetric jet and a coflowing stream. A collection of computer subprograms has been developed to solve the resulting eigenvalue problem. The effect of changing the velocity profile and its parameters can be easily assessed. Results for Gaussian profiles are included.

Key Words: Axisymmetric jet; eigenvalue problem; mixing layer; numerical methods for eigenvalue problems; stability analysis.

Introduction

The mixing region between a jet and a coflowing stream and the resulting large-scale structures are currently being studied at NBS using a time-dependent axisymmetric computer code. When these structures pair and merge, there is improved mixing between two streams, which is useful in practical applications. In order to better understand the flow development, controlled forcing can be applied and the effect of certain types of perturbations on the growth of the mixing layer observed.

Linear stability theory appears useful in predicting some behavior of the large-scale structures in a jet [1]. The evolution of vortices seems to be related to the unstable frequency range determined for the upstream velocity profile. Controlled perturbations applied at the most unstable frequency and its subharmonics can affect vortex merging and the growth of the mixing region.

In the present work, the jet's mixing layer is studied using stability analysis. The objective is to determine the spatial stability of the jet profile and use the results as the controlled perturbations at the inlet of the jet computation. Although the procedure developed applies for arbitrary profiles, the velocity ratio of the jet and freestream has been the main parameter varied. The analysis assumes that the flow is incompressible, inviscid, and axisymmetric, and that linearized equations are applicable. The validity of the results can be determined by comparisons of computations and experiments.

In the following section, the equations used in the stability analysis will be presented. The solution technique using software currently available on the NBS UNIVAC 1100/82 will then be described. Results for Gaussian profiles are also included.

Basic Equations

The method presented here parallels the solution procedure described in reference [2]. The axisymmetric Euler and continuity equations for an incompressible fluid, written in terms of the three variables, $\hat{u}$, $\hat{v}$, $\hat{p}$, are the basis of the formulation. $\hat{u}$ and $\hat{v}$ are velocity components in the axial (z) and radial (r) directions, respectively, in cylindrical coordinates, and $\hat{p}$ is the ratio of pressure to constant density. Small disturbances are added to a known solution: $\hat{u} = U(r) + u'$, $\hat{v} = v'$, $\hat{p} = p'$, where the primed quantities represent the disturbances. Substituting these into the equations and neglecting higher order terms involving u' and v' yield the linearized disturbance equations. Assuming parallel flow, the form of the disturbance is that of a traveling wave whose amplitude varies with r and which moves parallel to the z-axis.

$$\phi' = \phi(r)e^{i(\alpha z - \beta t)} \tag{1}$$

where ϕ represents u, v, or p, α is the wavenumber, and β, the frequency, of the disturbances. Substituting this form into the linearized disturbance equations leads to three ordinary differential equations with complex coefficients.

$$(\alpha U - \beta)v = i \frac{dp}{dr} \tag{2}$$

$$(\alpha U - \beta)u - i \frac{dU}{dr} v = - \alpha p \tag{3}$$

$$\frac{dv}{dr} + \frac{v}{r} + i\alpha u = 0 \tag{4}$$

For a spatially growing disturbance, α is complex ($\alpha_r + i\alpha_i$) and β real. Eliminating u and v from equations (2)-(4) gives an equation for the pressure disturbance.

$$\frac{d^2p}{dr^2} + [\frac{1}{r} - \frac{2\frac{dU}{dr}}{(U - \beta/\alpha)}] \frac{dp}{dr} - \alpha^2 p = 0 \tag{5}$$

The boundary conditions to be satisfied are that p remains finite at r = 0 and p becomes zero as $r \to \infty$. Equation (5) and the boundary conditions constitute an eigenvalue problem. The objective is to find the corresponding values of α and β and associated eigenfunction p(r) for a given profile U(r) and dU(r)/dr.

Computational Procedure

Several computer subprograms are used in finding a solution. The velocity profile being used in the present study is

$$U = 1 - \frac{C}{\sqrt{2\pi}}\, e^{-\frac{(r-1)^2}{2\sigma^2}}, \quad r \le 1$$

$$U = U_o - (U_o + \frac{C}{\sqrt{2\pi}} - 1)\, e^{-\frac{(r-1)^2}{2\sigma^2}}, \quad r \ge 1 \tag{6}$$

where all lengths are nondimensionalized by the jet radius and all velocities by the jet centerline velocity. Equation (6) represents either a single or double Gaussian, depending on the choice of U_o and $C/\sqrt{2\pi}$. $U(r)$ and $dU(r)/dr$ are continuous at $r = 1$. The profile $U(r)$ is specified in the program as a function UINLET(R) and dU/dr as DUDR(R). Thus, the parameters or form of equation (6) can be easily changed.

Equation (5) is integrated using the subroutine CDRIV1 from CMLIB (NBS Core Math Library), which solves initial value problems for complex-valued ordinary differential equations, integrated with respect to a single, real, independent variable. Equation (5) is expressed as a pair of first-order equations in subroutine F.

$$\dot{y}_1 = y_2$$

$$\dot{y}_2 = [-\frac{1}{r} + 2\frac{dU}{dr} / (U - \beta/\alpha)]\, y_2 + \alpha^2 y_1 \tag{7}$$

where $y_1 = p$ and $y_2 = dp/dr$. Subroutines USERS, JACOBN, FA, and G, which are options for the CDRIV package, are included as dummy routines to avoid error messages during the MAP processing.

Boundary values are needed at $r = 0$ and at the largest value of r. At each of these locations, dU/dr vanishes and equation (5) becomes a modified Bessel equation. The solutions consistent with the boundary conditions are $p = a\, I_o(\alpha r)$ at $r = 0$ and $p = b\, K_o(\alpha r)$ as r becomes large. This leads to boundary values at $r = 0$ of $p = a\, I_o(0) = a$, $dp/dr = a\alpha\, I_1(0) = 0$. Although this result can be determined directly from axisymmetry, the Bessel function expression is used to obtain the limiting value of $1/r\; dp/dr$ at $r = 0$ needed for equation (7). Since a is arbitrary, set $y_1 = 1$, $y_2 = 0$, and integrate from $r = 0$ to $r = 1$ to get the inner solution $f_i(r)$. The integration is done in radial increments chosen to provide velocity values at the grid points used in the jet computation, with the internal step of the integrator adjusted to just reach each point.

The initial values at the largest value of r, r_{max}, are $p = bK_o(\alpha r_{max})$, $dp/dr = - b\alpha K_1(\alpha r_{max})$. Since r_{max} is large and evaluating the functions K_o and K_1 with complex arguments is not straightforward, the numerical values are determined from asymptotic expressions. With b also being arbitrary, set $y_1 = K_o(\alpha r_{max})$, $y_2 = - \alpha K_1(\alpha r_{max})$, and integrate from $r = r_{max}$ to $r = 1$, as described above, to get the outer solution $f_o(r)$.

The required matching conditions at r = 1 are that p and dp/dr are continuous. This reduces to

$$f_o(1) \frac{df_i(1)}{dr} - f_i(1) \frac{df_o(1)}{dr} = 0. \tag{8}$$

The expression on the left side of equation (8) is written as a complex function FF whose root, α, is to be found. The integration procedure is included in the function subprogram.

The values of α are determined by using the subroutine ZANLYT from IMSL (International Mathematical and Statistical Libraries), which finds zeros of a single nonlinear complex function. For each value of β chosen, ZANLYT is called to find one root, with one guess and no known roots input. The first guess for α found most appropriate is $\beta(1 - i/2)$. When the root is returned, several tests are made before it is accepted. The error parameter, IER, must be zero, indicating convergence was obtained. For the problem considered here, other requirements are that α_r is positive, α_i is negative, and the phase velocity, $c_{ph} = \beta/\alpha_r$, converges to one at $\beta = 0$. The last condition is checked by finding α for a slightly smaller value of $\beta(\Delta\beta = 0.01)$, using the first α as a guess, and comparing c_{ph} values. If c_{ph} is less than one and decreases, or c_{ph} is greater than one but less than 1.5 and increases, the value of β is further decreased and α determined, to see if the trend reverses in a short interval (20 $\Delta\beta$, $\Delta\beta = 0.04$). If the first root is rejected, a second guess is tried. When convergence was obtained on the first guess with α_r positive but α_i not negative, the complex conjugate of α is guessed for the second trial. Otherwise the guess used is $\beta + 1/4 + i/2(1 - \beta)$ for $\beta \geq 1$ or $\beta(1 - i/10)$ for $\beta < 1$. For the cases tested, a root satisfying the conditions described above was found without additional trials needed.

When determining α as a function of β, the testing is done only for the initial value of β, with $\Delta\beta = 0.04$ in the c_{ph} comparison. The guess for successive points is the value of α from the preceding β value. This case is of interest for finding the most unstable frequency, the value of β for which α_i has the largest negative value. The curve is started from $\beta = 4$ and proceeds to smaller values of β, with $\Delta\beta = 0.04$. If the value of α_i is decreasing as β decreases, the curve is continued in the same direction until the minimum value of α_i has been passed. If α_i increases in the first set of intervals, the curve is continued similarly in positive increments from $\beta = 4$ until the minimum is located.

Once α has been found for a particular value of β, the corresponding velocity eigenfunctions can be determined. The outer pressure solution $f_o(r)$ and $df_o(r)/dr$ are multiplied by the ratio of the constants $b/a = f_i(1)/f_o(1)$ or $df_i(1)/dr$ / $df_o(1)/dr$, derived from the matching conditions. One arbitrary constant remains, multiplying the entire solution. The velocity v can be computed at each value of r where the integration was done from the values of dp/dr and equation (2). The u velocity is then calculated from equation (3) using v and the p solution. In the jet computation, v is needed at points halfway between those used in the integration, so averaging is done there. The symmetry conditions at $r = 0$ give $u(-\Delta r) = u(\Delta r)$, $v(0) = 0$, $v(-\Delta r) = -v(\Delta r)$.

From equation (1), the real velocity disturbance at $z = 0$ is $u' = u_r(r) \cos \beta t + u_i(r) \sin \beta t$, where $u = u_r + iu_i$. At any value of r, u' is a maximum when $\beta t = \tan^{-1} (u_i/u_r)$, giving $u'_{max} = \sqrt{u_r^2 + u_i^2}$. The u_r and u_i arrays are searched to determine the radial locations where each has the largest magnitude. At these two values of r, the quantity $0.1\ U(r)/u'_{max}$ is computed and the smaller one selected for the constant a. The complete u and v solutions are then multiplied by this factor.

When the procedure is called from the program doing the jet computation, subroutine PERTRB is used. Inputs to this subroutine are the array of r values where the u velocities are needed and the number of points in the array, the matching location ($r = 1$), and β. When $\beta = 0$ is input, the most unstable frequency is found and that value of β is returned. Additional outputs are α_r and arrays of u_r, u_i, v_r, and v_i. The perturbations applied to the jet velocities are:

$$u' = \sum_{k=1}^{N} a_k[u_{k_r}(r) \cos (\beta_k t - \alpha_{k_r} z) + u_{k_i}(r) \sin (\beta_k t - \alpha_{k_r} z)],$$

$$v' = \sum_{k=1}^{N} a_k[v_{k_r}(r) \cos (\beta_k t - \alpha_{k_r} z) + v_{k_i}(r) \sin (\beta_k t - \alpha_{k_r} z)],$$

where N frequencies have been included.

Results

Results will now be presented for the inlet velocity profile of equation (6) for three choices of the constants: (a) a single Gaussian, $U_o = 0.65$, $C/\sqrt{2\pi} = 0.35$, (b) a double Gaussian, $U_o = 0.3$, $C/\sqrt{2\pi} = 0.8$, and (c) a single Gaussian, $U_o = 0.15$, $C/\sqrt{2\pi} = 0.85$, where $\sigma = 0.1$ in each case. These profiles are shown in figure 1. For each of these, α was determined over a range of β's. For these runs, the stopping criteria in ZANLYT were (1) the magnitude of the function FF reaching or falling below 10^{-6}, (2) two successive approximations agreeing in the first six digits, or (3) the number of iterations exceeding the maximum of 50. The most iterations actually used in finding any one root was 38. The requested relative accuracy in CDRIV1 was set at 10^{-6}. The value of r_{max} was 5.2.

In figure 2, $-\alpha_i$ is plotted against β, where the variation of the peak value and most unstable frequency can be seen. The latter ranges from 3.2 to 4.32 for these cases. In figure 3, c_{ph} is plotted against β. The trend toward $c_{ph} = 1$ at $\beta = 0$ is evident here. The velocity ratio listed in figures 2 and 3 is defined as $1/U_o$.

Conclusion

A collection of subroutines has been developed to solve the eigenvalue problem resulting from applying stability analysis to the axisymmetric continuity and Euler equations. The velocity eigenfunctions are directly available for use as perturbations to the inlet velocity profile in the jet computation.

References

1. Ho, Chih-Ming and Huang, Lein-Saing, Subharmonics and vortex merging in mixing layers, J. Fluid Mech., Vol. 119, p. 443, 1982.
2. Michalke, Alfons and Hermann, Gunter, On the inviscid instability of a circular jet with external flow, J. Fluid Mech., Vol. 114, p. 343, 1982.

(a)

(b)

(c)

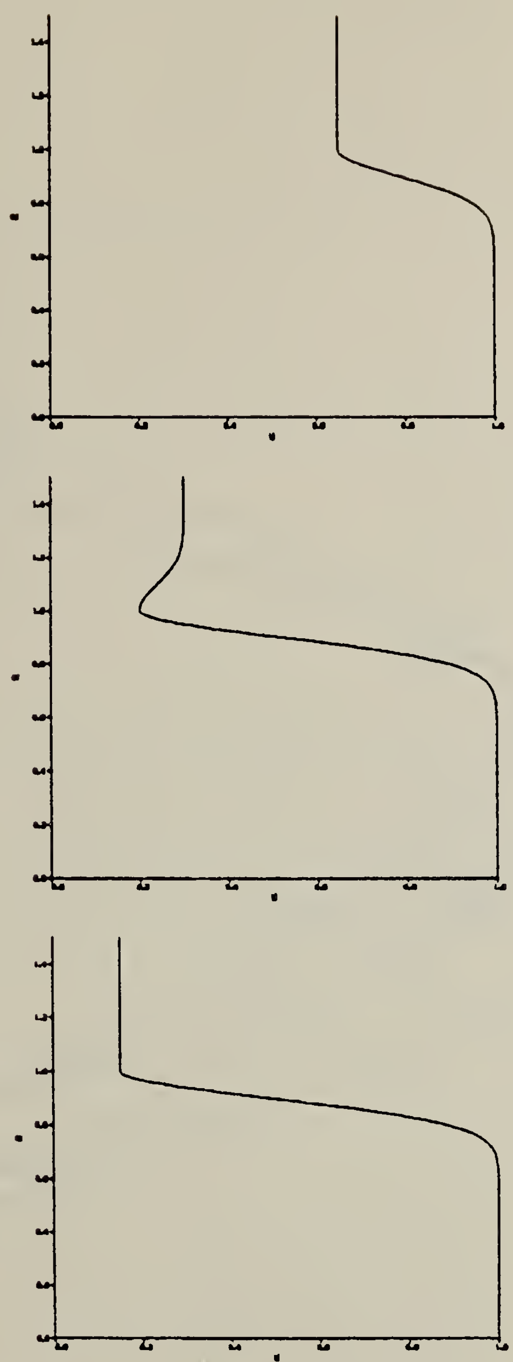

Fig. 1. Velocity profiles

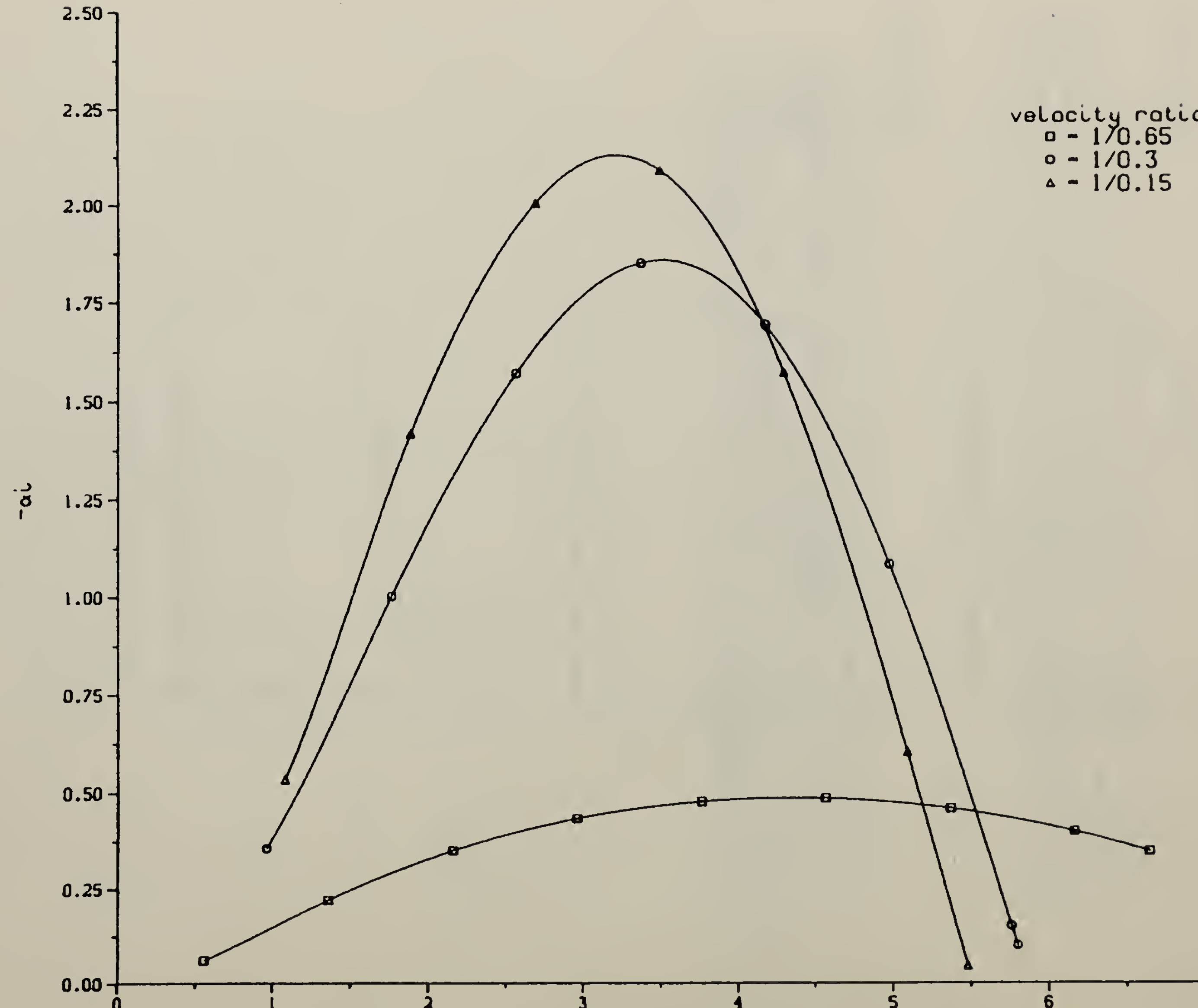

Fig. 2. Spatial growth rate vs. frequency for three velocity profiles

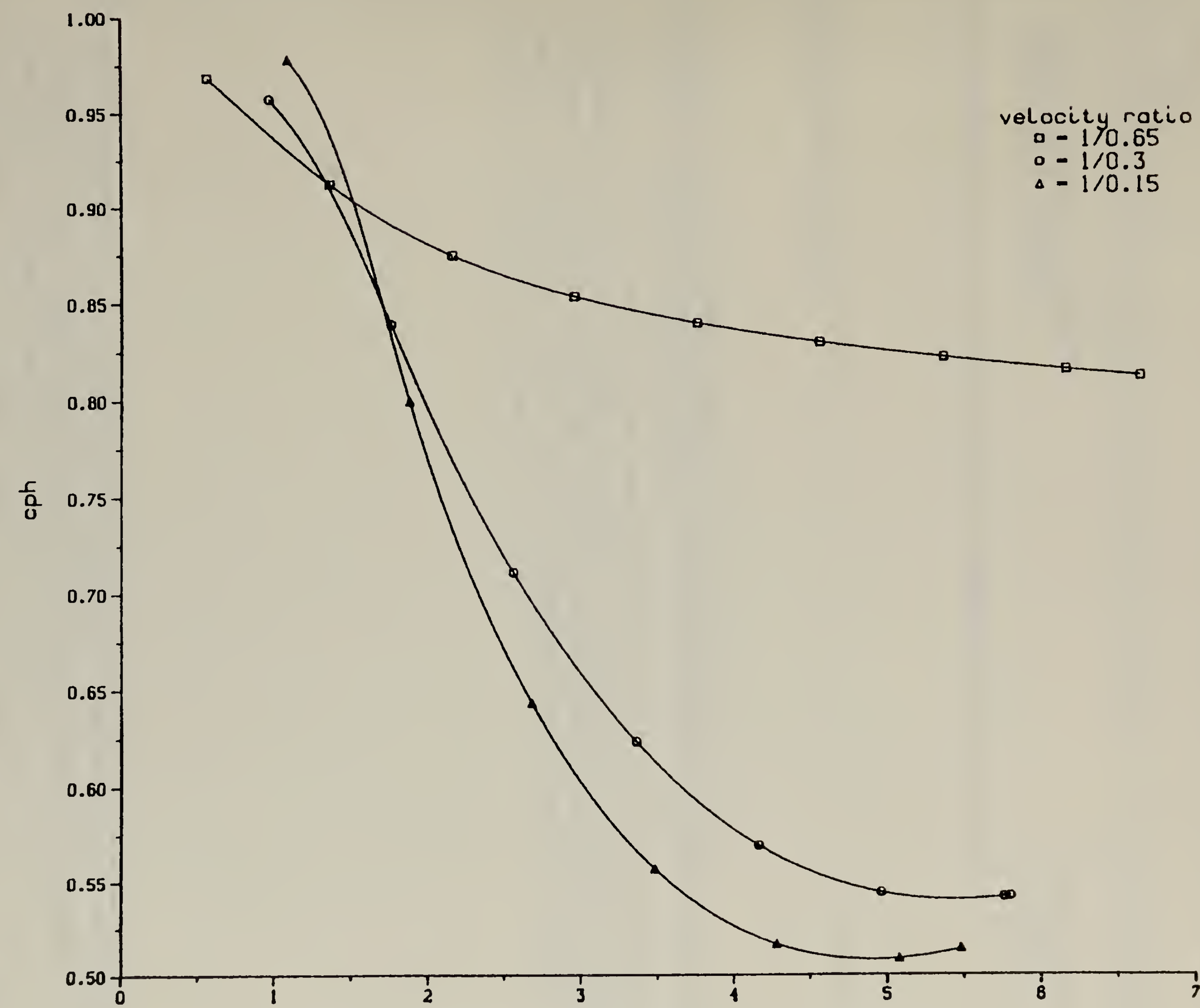

Fig. 3. Phase velocity vs. frequency for three velocity profiles

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