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Recommended Practice for Measuring Net Benefits and Internal Rates of Return for Investments in Buildings and Building Systems

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RECOMMENDED PRACTICE FOR MEASURING NET BENEFITS AND INTERNAL RATES OF RETURN FOR INVESTMENTS IN BUILDINGS AND BUILDING SYSTEMS

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ABSTRACT

This report describes how to calculate net benefits (savings) and internal
rates of return and how to use them in selecting building designs and building
systems that will be cost effective over time. Net benefits are the
difference between positive benefits or cost reductions (savings) and project
costs, measured in present value or annual value dollars. The net benefits
method is used to decide if a given project is cost effective and which size
or design for a given purpose is most cost effective when no budget constraint
exists. The internal rate of return on an investment is the compound rate of
interest that, when used to discount the streams of dollar benefits (savings)
and dollar costs over a defined study period, will make the two equal. The
internal rate of return is used to determine if a given project is cost
effective, to compare the relative cost effectiveness of different purpose
projects competing for a limited budget, and, when calculated on incremental
changes in benefits and costs, to evaluate which size or design for a given
purpose is most cost effective. The report describes formulas for calculating
net benefits and the internal rate of return, their applications in selecting
cost-effective projects, and limitations in their use. This recommended
practice for measuring net benefits and internal rates of return will assist
the private and public building communities in making cost-effective decisions
in the design, operation, maintenance, and retrofit of buildings.

PREFACE

Decreasing productivity in the construction industry coupled with high costs
of construction loans, labor, material, and energy have prompted builders,
architects, engineers, building owners and operators, and code writers to look
more closely at economic evaluation methods to identify building designs
and systems that are most cost effective. Practical, standardized, economic
methods for evaluating building alternatives are needed. The net benefits
(NB) method and the internal rate of return (IRR) method are economic
evaluation tools that can be used to assess the cost effectiveness of a given
project and to compare the cost effectiveness of alternative projects. The NB
method (also called the net present value method) computes the difference of
discounted benefits (or savings) and discounted costs as a measure of the cost
effectiveness of a project. The IRR method computes the compound rate of
interest which equates the stream of dollar benefits or savings to dollar
costs over a defined study period. The NB method is used to decide if a given
project is cost effective (net benefits greater than zero) and which size
or design competing for a given purpose is most cost effective where no budget
constraint exists. The IRR is used to determine if a given project is cost
effective, to compare the relative cost effectiveness of different purpose
projects competing for a limited budget, and, when calculated on incremental
changes in benefits and costs, to evaluate which size or design for a given
purpose is most cost effective.

This report describes how to calculate NB and illustrates use of the NB method in measuring the cost effectiveness of building-related decisions. The calculation procedures for the IRR are also described and illustrated. The NB and IRR methods are compared, and guidelines for their use are given. Applying standardized methods will help assure that the cost effectiveness of alternative building designs and projects can be determined and compared in a consistent and technically correct manner.

This report was prepared by the National Bureau of Standards (NBS) in support of an ongoing standards development activity by the American Society of Testing and Materials (ASTM E.06, Performance of Building Constructions) and in response to requests from the building community for assistance in applying economic analysis in a uniform and practicable manner. The report has been submitted to ASTM E06.81, the Building Economics Subcommittee, to be used as the technical base for the development of an ASTM standard practice for measuring NB and IRR's for building investments. It is the third in a series of reports to be submitted by NBS to ASTM E06.81, and it builds in part upon both the first report (NBSIR 80-2040, on life-cycle costing) and the second report (NBSIR 81-2397, on benefit-to-cost and savings-to-investment ratios).[1] The series of NBS reports and ASTM recommended practices are aimed at producing a comprehensive set of measures of economic performance that will meet the needs of the private and public building sectors.

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[1]Rosalie T. Ruegg, Stephen R. Petersen, and Harold E. Marshall, Recommended Practice for Measuring Life-Cycle Costs of Buildings and Building Systems, National Bureau of Standards Interagency Report 80-2040, June 1980; and Harold E. Marshall and Rosalie T. Ruegg, Recommended Practice for Measuring Benefit/Cost and Savings-to-Investment Ratios for Buildings and Building Systems, National Bureau of Standards Interagency Report 81-2397, November 1981.

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1. INTRODUCTION

The net benefits (NB) method and internal rate-of-return (IRR) method are part
of a family of economic evaluation methods that provide measures of economic
performance of an investment over some period of time. Included in this
family of evaluation methods are life-cycle cost analysis, benefit-to-cost and
savings-to-investment ratios, and payback analysis.

This report is the third in a series of National Bureau of Standards (NBS)
reports on recommended practices for applying economic evaluation methods to
building decisions. The first report in the series was <u>Recommended Practice
for Measuring Life-Cycle Costs of Buildings and Building Systems</u>, NBSIR 80-
2040. The second was <u>Recommended Practice for Measuring Benefit/Cost and
Savings-to-Investment Ratios for Buildings and Building Systems</u>, NBSIR 81-
2397. A future publication on payback methods is planned to complete the
basic package of recommended practices.

The NB method (sometimes called the net present value method) calculates the
difference between discounted benefits (or savings) and discounted costs as a
measure of the cost effectiveness of a project. The NB method is used to
decide if a project is cost effective (net benefits greater than zero) or
which size or design competing for a given purpose is most cost effective
(the one with the greatest net benefits).

The IRR provides the compound rate of interest which equates the stream of dollar benefits or savings to dollar costs over some defined study period.[1] It is used to determine if a given project is cost effective, to compare the relative cost effectiveness of different purpose projects competing for a limited budget, and, when calculated on incremental changes in benefits and costs, to evaluate which size or design for a given purpose is most cost effective.

1.1 Scope and Organization

This report establishes a technical base for the development of a recommended practice for calculating and interpreting the NB and IRR's for building designs and systems.[2] Section 2 discusses the formulation of objectives, alternatives, and constraints for NB and IRR analyses. Examples are given of problems that might be evaluated with each of the economic methods. Section 3 lists assumptions that must be made to perform the economic calculations and cites sources of information on how to make those assumptions. Section 4 identifies data needs under benefit and cost categories typically used in calculating NB or the IRR. Section 5 presents an overview of "discounting" and "compounding" for converting cash flows spread over time to their equivalent values at a common time. Discounting formulas are also provided. Section 6 describes the calculation procedures for measuring NB.

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[1]The study period (sometimes called time horizon) is the length of time over which an investment is analyzed.

[2]Terminology in this report is consistent with American Society for Testing and Materials, "Standard Definitions of Terms Relating to Building Economics, E833-83a," Annual Book of ASTM Standards (Philadelphia, Pennsylvania: ASTM, 1983).

Section 7 describes alternative calculation procedures for computing IRR and adjusted IRR measures. Trial-and-error, graphical, direct solution, and discount-factor-table approaches to finding the rate of return are illustrated with hypothetical investment problems. Section 8 describes the controversy regarding which measure (IRR or adjusted IRR) and which method (NB or IRR) are appropriate for evaluating typical investment problems facing the building community. Recommendations of specific methods and measures are made for selected problems. The economic consequences of applying other methods or measures than those recommended are described. Limitations of the two methods are also discussed. Appendix A, a selected set of discount factor tables, and Appendix B, a bibliography, conclude the report.

1.2 Procedures for NB and IRR Evaluations

The recommended steps for applying the NB or IRR method to an investment decision are summarized as follows:

- o Identify Objectives, Alternatives, and Constraints
- o Establish Assumptions
- o Compile Data
- o Convert Cash Flows to a Common Time Basis
- o Compute NB or IRR and Compare Alternatives

Sections 2 through 7 treat these steps in detail.

2. OBJECTIVES, ALTERNATIVES, AND CONSTRAINTS

The problem to be addressed by the NB or IRR method should first be clearly specified. The objectives of the investor or decision maker must be made explicit. Alternative approaches for reaching the objective should be identified. Any constraints to reaching the objective should also be identified.

An example of a building investment problem that could be evaluated with the NB method is the sizing of a manufacturing plant in a given location. Constraints of building site and size of market would limit the choice to a range of size alternatives. The objective of the investor is to select that plant size which maximizes profits (i.e., net benefits) while at the same time satisfying all of the constraints. The NB method can be used to guide the investor to the most efficient size.

An example of a building investment problem that could be evaluated with the IRR is selecting among alternative energy conservation retrofit projects for a corporation or government agency with many buildings and a limited budget. The objective is to minimize the costs of owning and operating the buildings (i.e., maximize net benefits from the retrofits). Feasible technical alternatives might be roof and wall insulation, improvements in the efficiency of heating and cooling equipment, passive or active solar systems, and improved controls. Constraints might include a given level of thermal comfort and a minimum hot water temperature requirement in addition to a maximum first cost budget for the overall program of conservation retrofits. The IRR method

provides a ranking tool that can be used to select the set of projects which

maximizes net benefits given the constraints.[1]

Note that other economic evaluation methods are also appropriate for solving

the problems cited above and that the NB and IRR methods are also appropriate

for other types of building investment problems. For examples of problems and

their solutions by other methods, see the first two reports in this series.[2]

Other applications and limitations of the NB and IRR methods will be discussed

in section 8.

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[1]A sequential approach to project economic analysis is assumed here. That is,
the appropriate size or scale of each of the technical alternatives is first
determined, and then the individual projects are chosen in descending order
of their IRR's until the budget is exhausted.

[2]See Rosalie T. Ruegg, Stephen R. Petersen, and Harold E. Marshall,
Recommended Practice for Measuring Life-Cycle Costs, and Harold E. Marshall
and Rosalie T. Ruegg, Recommended Practice for Measuring Benefit/Cost and
Savings-to-Investment Ratios.

3. ASSUMPTIONS

Assumptions about the values of parameters generally are required to perform economic evaluations of design and investment problems. Because the economic measure of a project's worth varies considerably depending on the assumptions, it is important to use care in making them and to consider the effects of different parameter values. Sensitivity analysis can be used to identify critical parameters and to test the outcome for a range of values in any given problem.

Policy dictates parameter values in some economic evaluations. In others, parameter values are assumed on the basis of judgment or market analyses. Important values that must sometimes be assumed in applying the NB method are the discount rate, length of study period, price escalation rates, and salvage/resale values.[1] For the IRR method, these same values are significant, with one exception. A minimum acceptable rate of return (MARR) instead of a discount rate must be chosen.[2] If the IRR is larger than the MARR, the investment is considered economically attractive.

Guidelines for selecting these critical parameters will not be discussed here because they are provided elsewhere.[3]

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[1]In this report an asset's salvage value is defined as that value assigned for tax computation purposes that is expected to remain at the end of the depreciation period. Resale value is the monetary sum expected from the disposal of an asset at the end of the study period.

[2]The MARR is the minimum economic return an investor will accept for making an investment.

[3]Rosalie T. Ruegg, Stephen R. Petersen, and Harold E. Marshall, Recommended Practice for Measuring Life-Cycle Costs; Harold E. Marshall and Rosalie T. Ruegg, Simplified Energy Design Economics: Principles of Economics Applied to Energy Conservation and Solar Energy Investments in Buildings, National Bureau of Standards Report 544, January 1980; and Harold E. Marshall and Rosalie T. Ruegg, Energy Conservation in Buildings: An Economics Guidebook for Investment Decisions, National Bureau of Standards Handbook 132, May 1980.

4. DATA

Data must be collected before NB or IRR calculations can be made for an
investment or design alternative. Data can be collected from published and
unpublished sources, they can be estimated, or they can be assumed.[1] Both
engineering data (e.g., heating loads and equipment efficiencies) and economic
data (e.g., tax rates, discount rates, MARR's, depreciation rates and
periods, and financing terms) will be needed.

Data requirements for evaluating the NB of replacing a conventional oil
furnace with an electric heat pump in a given climate provide an example of
the types of data that might be needed for conducting economic analyses. The
annual heating requirements of the building and the efficiencies of the two
heating systems are needed to calculate annual purchased energy requirements
with the two systems. The annual purchased energy requirements, the present
and future prices of electricity and oil, the discount rate, and the study
period are needed to calculate the costs of energy with the two systems.
Investment, maintenance, and replacement costs, as well as the resale value
are needed both for the oil system and for the heat pump to complete the data
requirements to find the NB from the heat pump substitution. Financing and
taxes should be considered if the investment is made by the private sector.

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[1]See in Rosalie T. Ruegg, Life-Cycle Cost Manual for the Federal Energy
Management Program, National Bureau of Standards Handbook 135, December 1980,
the chapter on "Data and Assumptions" (pp. 37-48), and some projections of
energy prices in appendix C. See Carol Chapman Rawie, Estimating Benefits and
Costs of Building Regulations: A Step-by-Step Guide, National Bureau of
Standards Report 81-2223, June 1981, the end of chapters 1-3, for listings of
sources of information on data relating to economic analyses of buildings.

Some of the benefits and costs that may be relevant when calculating NB and
IRR measures are listed in table 4.1. The time schedule of their occurrence
is also needed to convert them to a common time basis. These benefits and
costs may be available directly or only after computations are performed using
engineering and economic data, as would be required for determining the dollar
value of energy savings in the heat pump example given above. If private
sector investments subject to taxes are evaluated, then benefit and cost data
should be adjusted for taxes so that the NB and IRR methods describe "after
tax" effects.

Table 4.1   Benefits and Costs

Benefits (Savings)	Costs
Revenues	Investment
Resale/Salvage	Planning
Tax Credits	Design
Subsidies (e.g., grants)	Engineering
Energy Cost Reductions	Construction
Operating Cost Reductions	Purchase
Replacement Cost Reductions	Installation
	Financing
	Operation and Maintenance
	Fuel
	Repairs and Replacement
	Property and Capital Gains Taxes

Risky and uncertain investments may require additional data and analysis to evaluate their worth. One approach is to calculate expected values of benefits and costs when probability data exist. However, since probability data are often difficult to acquire, two common approaches to account for more risk are to reduce the time period (study period) over which a project is evaluated and to raise the discount rate or MARR.[1] Both of these common approaches have shortcomings. Reducing the study period ignores net benefits late in the project life and fails to account for risk early in the project life. Raising the discount rate biases project choice against projects yielding most of their benefits late in life and fails to account for risk early in the project life.

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[1]For a description of how to apply these and other techniques, and of what data are needed, see Lee G. Anderson and Russell F. Settle, Benefit-Cost Analysis: A Practical Guide (Lexington, Massachusetts: Lexington Books, 1977), pp. 99-106.

5. CONVERSION OF CASH AMOUNTS TO A COMMON TIME BASIS

To make economic evaluations, all benefits and costs that accrue over the study period must be converted to equivalent values on a common time basis, usually to present values or annual values. Converting these cash amounts to equivalent values (generally called discounting) is performed by applying discount formulas, or corresponding discount factors, to benefit and cost data associated with a given investment or design alternative.[1] Table 5.1 presents the most commonly used discount formulas, how they are used, and their algebraic form.[2]

In calculating NB, all cash amounts are converted to equivalent values at the present, at some future or past date, or on an annual basis. Cash amounts are converted typically to a present value (i.e., an equivalent value occurring now) or an annual value (i.e., a time-equivalent value occurring in a uniform amount each year over the study period).[3] The resulting discounted costs are subtracted from discounted benefits to arrive at the NB measure in present value or annual value terms. Some examples of the discounting procedure are shown in section 6.

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[1]Cash amounts refer to both the annually recurring benefits and costs as well as the non-annually recurring benefits and costs that make up the cash flow associated with an investment.

[2]For an explanation of discounting and a complete set of discount factor tables, see the ASTM Adjunct Discount Factor Tables, to be published with "Recommended Practice for Measuring Life-Cycle Costs of Buildings and Building Systems" (Philadelphia, Pennsylvania: ASTM, 1983).

[3]Cash amounts converted to equivalent values at some future time are called future values. The process of calculating future values is often called compounding because present values or annual values are compounded by some rate of interest to determine future values.

Table 5.1 Discount Formulas

Formula Name	Illustration	Use	Algebraic Form [a, b]
Single Compound Amount (SCA) Formula	$\boxed{P} \longrightarrow \boxed{F?}$	To find F when P is known	$F = P \cdot (1 + i)^N$
Single Present Value (SPV) Formula	$\boxed{P?} \longleftarrow \boxed{F}$	To find P when F is known	$P = F \cdot \dfrac{1}{(1 + i)^N}$
Uniform Sinking Fund (USF) Formula	$\boxed{A?} + \boxed{A?} \cdots + \boxed{A?} \longrightarrow \boxed{F}$	To find A when F is known	$A = F \cdot \dfrac{1}{(1 + i)^N - 1}$
Uniform Capital Recovery (UCR) Formula	$\boxed{P} \longrightarrow \boxed{A?} + \boxed{A?} \cdots + \boxed{A?}$	To find A when P is known	$A = P \cdot \dfrac{i(1 + i)^N}{(1 + i)^N - 1}$
Uniform Compound Amount (UCA) Formula	$\boxed{A} + \boxed{A} \cdots + \boxed{A} \longrightarrow \boxed{F?}$	To find F when A is known	$F = A \cdot \dfrac{(1 + i)^N - 1}{i}$
Uniform Present (UPV) Value	$\boxed{P?} \longleftarrow \boxed{A} + \boxed{A} \cdots + \boxed{A}$	To find P when A is known	$P = A \cdot \dfrac{(1 + i)^N - 1}{i(1 + i)^N}$

where:

P = a present sum of money.

F = a future sum of money.

i = an interest or discount rate for the study period being considered.

N = number of interest or discounting intervals in the study period.

A = an end-of-period payment (or receipt) in a uniform series of payments (or receipts) over N periods at i interest or discount rate.

F? = a future value to be found; P?, a present value to be found; and A?, an annual value to be found.

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[a] Note that the USF, UCR, UCA, and UPV formulas yield undefined answers when i=0. Alternative algebraic forms for this special case would be as follows: USF formula, A = F/N; UCR formula, A = P/N; UCA formula, F = A · N; and UPV formula, P = A · N.

[b] The terms by which the known values are multiplied in these equations are the formulas for the factors calculated in Appendix A. Using acronyms to represent the factor formulas, the discount formulas can also be written, in the order as listed above, as F = P · SCA, P = F · SPV, A = F · USF, A = P · UCR, F = A · UCA, and P = A · UPV.

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Discounting is also a procedure used in calculating the IRR. For example, to find the IRR with the trial-and-error method, trial interest rates are used to discount benefit and cost cash flows to a base time until a rate is found which will make the two streams equal. Thus the discounting process must be repeated with trial rates until the IRR is found for which net benefits are zero, or until interpolation can be used to find the IRR between two trial rates for which NB are above and below zero. Examples of this and related procedures are given in section 7.

6. COMPUTE NB

The NB method finds the difference between benefits and costs, where both are discounted to present or annual value dollars. Equation 6.1 is the formula for finding the present value of net benefits for a project with benefits and costs accruing in the future. (The discounting operation is performed by using the SPV formula in table 5.1.)

$$\text{PVNB} = \sum_{t=1}^{N} \left[\frac{B_t - C_t}{(1 + i)^t} \right] - C_o , \qquad (6.1)$$

where   PVNB = present value of net benefits,

B_t = the dollar value of benefits (including savings and resale values) in time period t,

C_t = the dollar value of costs in time period t,

$B_t - C_t$ = net cash flows in time period t,[1]

i = the discount rate,

t = a discounting period,

N = the total number of discounting periods in the study period, and

C_o = initial project costs as of the base time.

Note that the NB method assumes implicitly that all net cash flows over the study period are reinvested at rate i.[2]

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[1]Cash flows are assumed to accrue at the end of each period.

[2]An adjusted PVNB could be calculated where the net cash flows are compounded to a terminal value at some reinvestment rate (r) different from the discount rate and then are brought back to the present with the discount rate. The formula would appear as:

$$\text{Adjusted PVNB} = \frac{\sum_{t=1}^{N} (B_t - C_t) (1 + r_t)^{N-t}}{(1 + i)^N} - C_o.$$

Since, in a profit-maximizing context, reinvestment is likely made at the highest return available (the rate represented by i), r_t will probably equal i, and equation 6.1 applies. Thus the adjusted PVNB is not discussed further.

Equation 6.2 (the UCR formula in table 5.1) can be used to convert the present

value of net benefits to annual value terms, where N is the number of years in

the study period.

$$AVNB = PVNB \cdot \frac{i(1 + i)^N}{(1 + i)^N - 1} , \qquad (6.2)$$

where AVNB = annual value of net benefits,

$\dfrac{i(1 + i)^N}{(1 + i)^N - 1}$ = the formula for the uniform capital recovery (UCR) factor.[1]

Equations 6.1 and 6.2 provide no particular computational difficulties.

Furthermore, for a given problem and data set, solutions in present value

and annual value terms will be time equivalent values (although different in

nominal dollar values) and will support the same investment or design

decisions.

A simple application of equation 6.1 is presented in table 6.1 for an initial

investment of $10,000 that yields an uneven yearly cash flow over four years.[2]

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[1]The UCR factor is the term by which a present value is multiplied to compute
an annual value. Using precalculated multiplicative factors instead of the
whole discount formulas is often more convenient in discounting operations.
Appendix A shows how to calculate these factors and has tables of values for
the UCR and other factors. Only those tables whose values are needed for
problems described in the report are included in Appendix A.

[2]It is assumed in all illustrations of discounting in this report that net
cash flows accrue annually at the end of the year.

Table 6.1 Calculation of Net Benefits

Year (t) (1)	Benefits (B_t) (2)	Costs (C_t) (3)	Net Cash Flow $(B_t - C_t)$ (4)=(2)-(3)	SPV Factor[a] for i=15% (5)	PVNB (6)=(4) · (5)
0	0	$10,000	$-10,000	1.000	$-10,000
1	$4,000	3,000	+1,000	.8696	+870
2	11,500	4,500	+7,000	.7561	+5,293
3	10,000	4,000	+6,000	.6575	+3,945
4	8,000	5,000	+3,000	.5718	+1,715
TOTAL	$33,500	$26,500	$+7,000		$1,823

[a]To find the PVNB of the net cash flow for each discounting period, the single present value (SPV) factor in column 5 is multiplied times the net cash flow in column 4. The SPV factor, calculated as $\frac{1}{(1 + i)^N}$, is found for each of various i and N combinations in Table 15 in Appendix A.

Assuming a discount rate of 15%, the discounted cash flows yield a PVNB of $1,823. The larger the PVNB for a given project, the more economically attractive it will be, other things being equal. (Note that the sum of net cash flows, $7,000, does not reflect the time value of money.)

To find the AVNB that is time equivalent to $1,823, equation 6.2 can be used. The UCR factor for four years and 15% from table 15 in Appendix A is .3503. Multiplying it times the PVNB of $1,823 yields an AVNB of $639.

7. COMPUTE IRR AND ADJUSTED IRR

Two measures of the rate of return on investment are discussed in detail in this section. One is the IRR and the other is the adjusted IRR. The IRR method is used throughout this report as a general term to include both the unadjusted and adjusted rate of return measures. Several approaches are described for calculating the two measures. The section concludes with a brief listing of other rate-of-return measures used in making economic evaluations of investments.

7.1 IRR

The IRR is the compound rate of interest that, when used to discount a project's cash flows, will reduce the PVNB to zero. The solution value of i in equation 7.1 is the IRR.[1]

$$\sum_{t=1}^{N} \left[\frac{B_t - C_t}{(1+i)^t} \right] - C_0 = 0 , \qquad (7.1)$$

where B_t = the dollar value of benefits (including savings and resale values) in time t,

C_t = the dollar value of costs in time t,

C_0 = initial project costs as of the beginning of the base time, and

i = the rate of interest which discounts net cash flows to zero.

An implicit assumption in solving for the IRR is that net cash flows are reinvested over the remainder of the study period at the derived IRR. Allowing for different rates of reinvestment requires an adjustment to equation 7.1 that is presented in section 7.2.

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[1]Note that the IRR will be equal to the user's discount rate as used in equation 6.1 only when the PVNB is zero at that discount rate.

Direct solution of i is not possible with equation 7.1, so calculation methods have been developed to approximate the IRR. Three commonly used methods to be examined in this section are the trial-and-error, graphical, and target UPV factor approaches.

7.1.1 Trial and Error

One technique for calculating the IRR is the trial-and-error approach. A trial rate of return is chosen that would be expected to approximately balance benefits and costs over the project study period. Then present value calculations are made for that discount rate. If the PVNB is zero, then the trial rate is the internal rate of return. If the PVNB is negative, the trial rate is too high, and a second, lower trial rate should be used. If the PVNB is positive for the original trial rate, then that rate is lower than the IRR, and a second, higher trial rate should be used. When two trial rates are found such that one yields a PVNB greater than zero and the other a PVNB less than zero, the IRR will lie between those rates and can be approximated by linear interpolation using the PVNB figures.

Considerable time can be saved in the trial-and-error approach if the first trial rate is close to the true rate. One approach is to start with the MARR. If the PVNB is negative with the MARR, then the project will not be economically feasible, and no further calculations are necessary. If the PVNB is positive, then higher trial rates are selected in an attempt to bound the true rate.

If by inspection of project cash flows it is clear that the IRR will exceed the MARR, then other approaches may be more appropriate for finding the

first trial rate. For example, when cash flows are relatively uniform throughout the study period, the UPV factor tables can be used to find the first trial rate. The first step is to add the undiscounted net cash flows, divide the sum by the number of years in the study period, and then divide the initial project cost by this average annual net cash flow to obtain a rough estimate of the number of years the project would take to pay off the investment. This crude approximation of "simple payback" is also an approximation of the UPV factor, P/A, as shown in footnote b of table 5.1. The second step is to find in the UPV factor tables of Appendix A the UPV factor value at that project's study period which is closest to the calculated payback figure. The corresponding discount rate then becomes the first trial rate. The more uniform are annual net cash flows, the better this approach generally works in finding a first trial rate close to the true rate.[1]

A third approach is to calculate simple payback directly by adding annual net cash flows until they just cover initial costs. The first trial rate will be the discount factor that corresponds to the UPV factor closest to that payback value.[2]

Table 7.1 illustrates the trial-and-error approach for calculating the IRR for the same investment problem that was described in table 6.1. Columns 2 and 3

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[1]Section 7.3 describes how the IRR can be approximated directly through interpolation of UPV factors when annual net cash flows are constant.

[2]For extended discussions of how to select trial rates of return that bracket the true rate, see Eugene L. Grant and W. Grant Ireson, Principles of Engineering Economy, 5th ed. (New York: The Ronald Press Co., 1970), pp. 122-123; and Maurice G. Wright, Discounted Cash Flow (New York: McGraw Hill, 1967), pp. 29-32. For a numerical procedure that converges on the IRR within .001 percent in two or three trials, see Dennis Eisen, "Computer Connection: Program Description of a Fast, Efficient, Algorithm for the IRR," The Appraisal Journal (January 1982), pp. 127-134.

Table 7.1 Trial-and-Error Solution for Internal Rate of Return

Year (t)	Benefits (B_t)	Costs (C_t)	Net Cash Flow ($B_t - C_t$)	SPV Factor for i=25%	PVNB at 25%	SPV Factor for i=22%	PVNB at 22%
(1)	(2)	(3)	(4)=(2)-(3)	(5)	(6)=(4)x(5)	(7)	(8)=(4)x(7)
0	0	$10,000	$-10,000	1.000	$-10,000	1.000	$-10,000
1	$4,000	3,000	+1,000	0.8000	+800	0.8197	+820
2	11,500	4,500	+7,000	0.6400	+4,480	0.6719	+4,703
3	10,000	4,000	+6,000	0.5126	+3,076	0.5507	+3,304
4	8,000	5,000	+3,000	0.4096	+1,229	0.4514	+1,354
Total			$+7,000		$-415		$+181

list the dollar values of benefits and costs that accrue in each of the four years, and column 4 shows the net cash flow for each of those years, including the initial investment. By inspection of column 4, the analyst might expect a relatively high return given the net cash flow beyond the $10,000 investment of $7,000 over four years. Using the second or third approaches described above for selecting a trial rate, the calculated UPV value for four years would correspond in Appendix A most closely to a discount rate of 25%. Multiplying yearly net cash flows by the corresponding single present value (SPV) factor for a 25% discount rate (column 5) converts them to equivalent present values (column 6). Summing the values in column 6 produces a PVNB of $-415.

Since the PVNB is less than zero, the IRR must be lower than the 25% trial rate. Thus another, lower rate is chosen, and the corresponding SPV factors

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are multiplied times yearly net cash flows. Using a trial rate of 22% (as illustrated in columns 7 and 8) yields a positive PVNB of $181. Therefore the IRR must lie between 22% and 25%. Linear interpolation yields an IRR of 22.9% as follows:

$$\text{IRR} = 22\% + \frac{\$181 - \$0}{\$181 + \$415} \ (25\% - 22\%) = 22.9\% \ .$$

7.1.2 Graph

Another approach in approximating the IRR is to use a graphical technique. The profile of the PVNB for a given investment is plotted for different discount rates. The IRR will be that rate where the PVNB curve intersects with the discount rate axis (i.e., where the PVNB is zero). Figure 7.1 illustrates such a graph for the project described in table 7.2.

Given a discount rate of zero, the PVNB would be the arithmetic sum over time of the net cash flows (i.e., $1,300). Thus the function intersects the vertical axis at $1,300. Using each discount rate on the horizontal axis of figure 7.1 as a trial discount rate, and substituting into equation 6.1, the PVNB values are calculated as shown in table 7.2 and then plotted in figure 7.1.[1]

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[1]For the function shown in figure 7.1, only a few of the PVNB points about the horizontal axis would be required to approximate reasonably well the IRR. For other curvilinear functions, more points are needed.

Figure 7.1 Graphical Solution of IRR
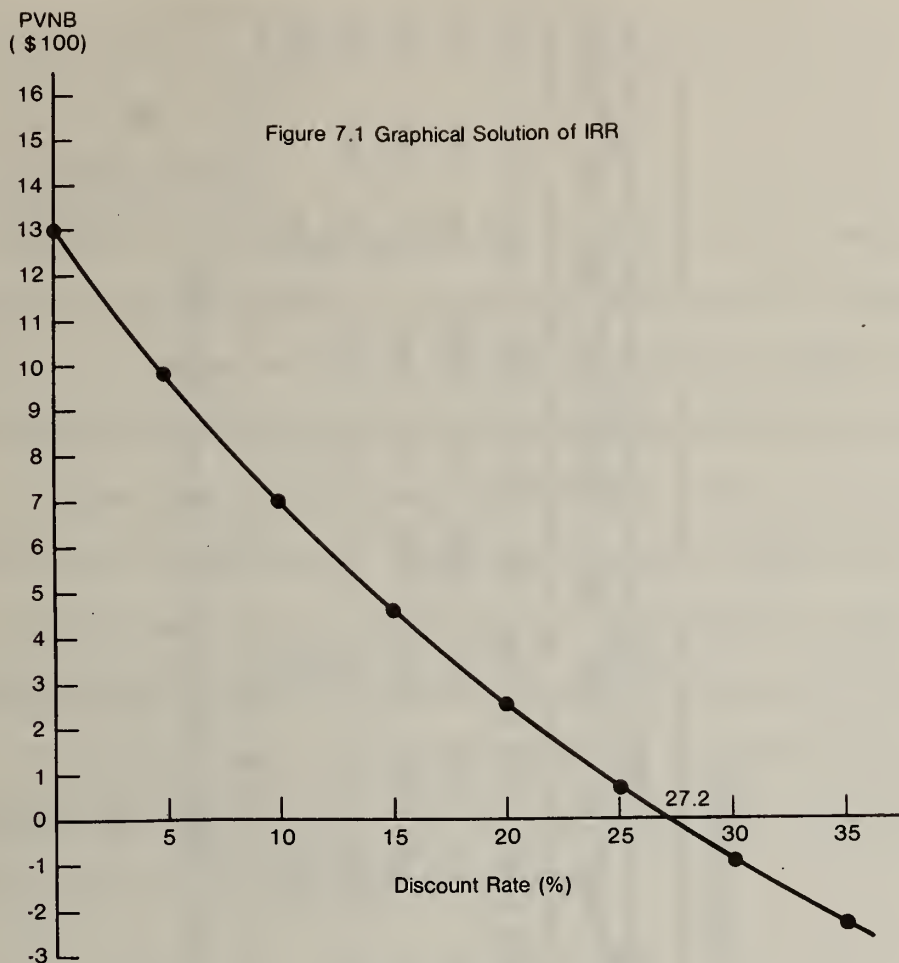
An IRR of 27.2% is found by visual inspection of the intersection of the profile of PVNB with the horizontal axis. Note that the graphical method can be expected to give approximations closer to the true IRR than the linear interpolation illustrated earlier with the trial-and-error approach, assuming that enough PVNB points above and below the horizontal axis are plotted to accurately form the curve. However, if two trial rates are chosen very close to the true value, the linear interpolation approach may approximate the true value closely enough without having to calculate the PVNB for many discount rates. In this problem, for example, the trial-and-error approximation for trial rates 27% and 28% is 27.2%, the same as the graphical solution. The true value of the IRR is in fact 27.2%.[1] The trial-and-error approximation for trial rates of 25% and 30% is 27.3%, being only .1% off.

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[1]The 27.2% value of the IRR can be confirmed by substituting 27.2% for i in equation 7.1.

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Table 7.2 Data for Graphic Solution of IRR

Year (t)	Benefits (B$_t$)	Costs (C$_t$)	Net Cash Flow (B$_t$ - C$_t$)	Discount Rates													
				5%		10%		15%		20%		25%		30%		35%	
				SPV	PVNB	SPV	PVNB	SPV	PVNB	SPV	PVNB	SPV	PVNB	SPV^a	PVNB	SPV	PVNB
0	0	$-2,200	$-2,200	1.000	$-2,200	1.000	$-2,200	1.000	$-2,200	1.000	$-2,200	1.000	$-2,200	1.000	$-2,200	1.000	$-2,200
1	$1,000	0	+1,000	.9524	952	.9091	909	.8696	870	.8333	833	.8000	800	.7692	769	.7407	741
2	1,500	0	+1,500	.9070	1,361	.8264	1,240	.7561	1,134	.6944	1,042	.6400	960	.5917	888	.5487	823
3	1,000	0	+1,000	.8638	864	.7513	751	.6575	658	.5787	579	.5126	513	.4552	455	.4064	406
			$+1,300		$977		$700		$462		$254		$73		$-88		$-230

^{a}The SPV factors for 30% and 35% were calculated from the formula $SPV = \dfrac{1}{(1 + 1)}$ since factor tables for 30% and 35% were unavailable.

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7.1.3 Target UPV Factor

A third technique for calculating the IRR is to use the UPV factor tables. This approach works only when the annual net cash flows are constant. The procedure is to compute the target Uniform Present Value (UPV) factor that equates benefits and costs, and then to search the UPV tables for the values closest on either side to the computed UPV target value to see what discount rates those factors represent.[1] Linear interpolation using the UPV figures yields an approximation of the IRR between the two discount rates. Note in this approach that the IRR is interpolated directly from the UPV factor values. In the trial-and-error method described in section 7.1.1, the IRR is interpolated from PVNB figures. The UPV tables are used there only to find the first trial rate for calculating the PVNB.

Equation 7.2 shows how to calculate the target UPV factor that embodies the IRR (i.e., the UPV factor that makes the present value of net benefits equal zero).[1]

---

[1]The term $\dfrac{C_O}{(\overline{B-C})}$ is derived from the UPV formula $P = A \cdot UPV$ in footnote b of table 5.1 for the special case where the present value of net benefits is equal to zero; that is, $(\overline{B-C})(UPV) - C_O = 0$, and therefore $UPV = \dfrac{C_O}{(\overline{B-C})}$, where $P = C_O$ and $A = (\overline{B-C})$.

---

[1]The UPV factor in this case of constant annual cash flows is identical to simple payback.

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$$\text{target UPV factor} = \frac{C_o}{(\overline{B-C})} \quad , \tag{7.2}$$

where target UPV factor = uniform present value factor where the present
value of net benefits equals zero,

C_o = initial project costs as of the base time, and

$(\overline{B-C})$ = constant annual net cash flow.

This approach can be illustrated with the following problem. Find the IRR for
a project with a one-time initial cost of $1,000 and benefits that accrue in a
uniform stream of $200 per year for 16 years. Dividing the one-time initial
cost of $1,000 by the constant annual net cash flow of $200 yields a target
UPV factor of 5. Searching the rows where N = 16 in the UPV factor tables of
Appendix A, two UPV factors are found that most closely bracket the value 5.
One factor is 5.162 for a discount rate of 18%, and the other is 4.938 for a
discount rate of 19%. Having bracketed the IRR between 18% and 19%, a single
value of 18.7% can be approximated through linear interpolation as follows:

$$\text{IRR} = 18\% + \frac{5.162-5.000}{5.162-4.938} \ (19\% - 18\%) = 18.7\%. \ [1]$$

7.2 Adjusted IRR

The approaches to approximating the IRR described above assume that returns or
benefits from the investment are reinvested at a rate of return equal to the
IRR. If the net cash generated by the investment is not reinvested at a rate
equal to the IRR, equation 7.1 will not give the true IRR. An _adjusted_ IRR

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[1]Although the relationship between the table values is not linear, a solution
based on a linear relationship will give a reasonable approximation. The
closer the table values are to the true value, the closer will be the
interpolated IRR to the true IRR.

(AIRR) is needed to find the true return. The solution value of i in equation 7.3 is the AIRR.

$$\frac{\displaystyle\sum_{t=1}^{N} (B_t - C_t)(1 + r_t)^{N-t}}{(1 + i)^N} - C_o = 0, \qquad (7.3)$$

where r_t = the rate of return on reinvestment of cash flows realized in year t.

Net cash flows $(B_t - C_t)$ are carried forward (compounded) at the appropriate reinvestment rates (r_t) to equivalent values at the end of the study period (end of the Nth period) and summed. The value of i that discounts the resulting future value to a present value equal to C_o is the AIRR. The higher the assumed reinvestment rate, the higher will be the calculated value of the AIRR.

Equation 7.3 is the same as equation 7.1 if the assumed reinvestment rate is the same for each reinvestment opportunity and equal to the unadjusted IRR (i.e., when r_t = the value of i found in equation 7.1)[1]. Thus, given a constant reinvestment rate, the AIRR is equal to the IRR when r_t = i. If r_t < i, then the AIRR < IRR. If r_t > i, on the other hand, then AIRR > IRR. If r_t changes from one period to the next, and if some values are larger and others smaller than i, then the AIRR and IRR measures would have to be computed for each project to determine which measure was larger.

---

[1]It can be proved that equations 7.1 and 7.3 are the same when r_t = i by showing that the term $\dfrac{(1 + r_t)^{N-t}}{(1 + i)^N}$ in equation 7.3 becomes $\dfrac{1}{(1 + i)^t}$ as shown in equation 7.1 when r_t = i. Substituting i for r_t into $\dfrac{(1 + r_t)^{N-t}}{(1 + i)^N}$ yields $\dfrac{(1+i)^{N-t}}{(1+i)^N} = \dfrac{(1 + i)^N (1 + i)^{-t}}{(1 + i)^N}$, which upon simplification reduces to $\dfrac{1}{(1 + i)^t}$.

The i in equation 7.3 can be solved for directly if the equation is rearranged as shown in equation 7.4.[1]

$$AIRR = -1.0 + (TV/C_o)^{1/N} \ ,\tag{7.4}$$

where $TV = \sum_{t=1}^{N} (B_t - C_t)(1 + r)^{N-t}$, which is the terminal (future) value

at the end of the study period of a net cash flow in time t.

Using equation 7.4, the TV must first be computed by applying the single compound amount factor to each net cash amount and then summing them. Substitution of known values for C_o and N is then all that is required to solve directly for the AIRR.

Once the terminal value is computed, an alternative approach to equation 7.4, utilizing the discount factor tables, can be used if calculation aids are not available for taking roots. Since the single compound amount (SCA) factor (as shown in table 5.1) is multiplied times a present value to calculate a terminal (future) value, we know that $TV = C_o \cdot SCA$ when the terminal value of net cash flows equals the future value of initial costs. (This is equivalent to the condition that the PVNB equals zero.) Thus we know that the value of i that equals the AIRR will be the rate that corresponds to the SCA factor equal to TV/C_o. To find the AIRR, the SCA factor is computed as TV/C_o. Then the SCA

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[1] If we let $TV = \sum_{t=1}^{N} (B_t - C_t)(1 + r_t)^{N-t}$ in equation 7.3, then $\dfrac{TV}{(1 + i)^N} - C_o = 0$.

Rearrangement of terms yields $(1 + i)^N = \dfrac{TV}{C_o}$, and therefore $i = -1 + (TV/C_o)^{1/N}$

= AIRR.

factor tables are searched for the values on either side that are closest to the computed SCA factor. The AIRR is interpolated by using those SCA values and their corresponding discount rates.

To illustrate the calculation of the AIRR when different reinvestment rates apply, equation 7.4 will be used. In section 7.1.2, an approximate IRR of 27.2% was calculated for the investment data in table 7.2. Now an AIRR will be computed for the modified case where reinvestment rates are lower than 27%, and where those reinvestment rates change during the project life.[1]

Table 7.3 illustrates the calculations of terminal values from a reinvestment of the first year's net cash flow at 20% and a reinvestment of the second year's cash flow at 15%. Note that net cash flows are identical to those for the investment described in table 7.2.

---

[1]Reinvestment rates that change over the project's life are quite likely with changing economic conditions. Although an analyst might prefer to use a single rate for an ex ante analysis due to the difficulty in predicting how rates are going to change, an AIRR computed with changing rates seems appropriate when an analyst is making an ex post evaluation and knows the different rates at which net cash flows were reinvested.

Table 7.3 Investment Project With Different Reinvestment Rates

Year (t)	Net Cash Flow $(B_t - C_t)$	Reinvestment Rate (r_t)	SCA Factor	Terminal Value
0	$-2,200[a]			
1	+1,000	20%	1.440	$1,440
2	+1,500	15%	1.150	1,725
3	+1,000			1,000
	$+1,300			$4,165

[a]This is equal to $-C_0$ in equation 7.3.

The future value of $1,000 earning 20% for two years is $1,440, the future value of $1,500 earning 15% for one year is $1,725, and the future value of $1,000 earned in that same year is $1,000. The terminal value (combined future values of the positive net cash flows) is $4,165. Solving the problem in table 7.3 by using equation 7.4 would yield the following:

$$AIRR = -1.0 + \left[\frac{\$4,165}{\$2,200}\right]^{1/3} = .237 \text{ or } 23.7\%.$$

As would be expected, reinvestment rates lower than the earlier computed IRR result in an AIRR lower than the IRR. This example shows that two investments with identical net cash flows may yield an AIRR that differs from the IRR, depending on how those net cash flows are reinvested over the project life. The more concentrated the net cash receipts are early in the study period, and the greater the difference between reinvestment rates and the IRR, the larger will be the difference between the AIRR and the IRR for any given investment, and the more important it will be to use the AIRR to find the true return on an investment.

Equation 7.5, which is equivalent to equation 7.4, can be used when the reinvestment rate is constant from year to year and the PVNB has already been computed.[1]

$$AIRR = -1.0 + (1 + r)\left[1 + \frac{PVNB}{C_o}\right]^{1/N} . \qquad (7.5)$$

Not having to derive the TV simplifies the calculation of the AIRR. (Note that r is a constant reinvestment rate and that the PVNB has been discounted at r.)

A major computational advantage of the AIRR over the IRR is that the AIRR can be computed directly without using the awkward approximation techniques required for the IRR.

7.3 Other Return-on-Investment Measures

Numerous names exist for measures of the percentage return on investments. Some are equivalent to the IRR and AIRR as defined above, whereas others are different measures. Two reasons that account for many of the differences are that some measures use discounting and others do not, and some use different

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[1]Equation 7.5 is derived as follows: To have the PVNB term replace TV, we define TV in relation to PVNB as $TV = (PVNB + C_o) (1 + r)^N$. Since PVNB has subtracted C_o from the present value of the net cash flows, C_o has to be added back to PVNB to find the present value of net cash flows which are then compounded at $(1 + r)$ for N periods to find the terminal value. Thus TV as defined above is substituted for TV in equation 7.4, yielding

$$AIRR = -1.0 + \left[\frac{(PVNB + C_o) (1 + r)^N}{C_o}\right]^{1/N}$$, which with rearrangement

of terms yields equation 7.5.

categories of costs and earnings in computing the percentage returns. Table 7.4 lists some of the common names of measures of the percentage return on investments.[1] These terms are not always used consistently. This is due in part to their evolving from several different fields, including economics, business finance, management science, and real estate.[2] The IRR method discussed in this paper deals exclusively with the IRR as found using equation 7.1 and with the AIRR discussed in section 7.2.

Table 7.4 Measures of Percentage Return on Investments

accounting rate of return	internal rate of interest
after-tax equity yield	investor's method
discounted cash flow (DCF) rate of return	project rate of return
	rate of return
discounted internal rate of return	rate of return over cost
discounted rate of return	return on investment (ROI)
external rate of return	time-adjusted rate of return
financial-management rate of return	unadjusted rate of return
interest rate of return	yield
internal rate of discount	yield to maturity

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[1]It is recognized that measures included in table 7.4 that are computed with other formulas than equations 7.1 and 7.4 are also helpful in making certain types of decisions not addressed in this paper.

[2]For a comprehensive survey of the IRR literature evolving from these fields, see Austin J. Jaffe, "Is There a New Internal Rate of Return Literature?," Journal of the American Real Estate and Urban Economics Association (Vol. 5, No. 4, Winter 1977).

8. RECOMMENDED APPLICATIONS AND LIMITATIONS

This section makes a case for using the AIRR measure instead of the IRR
measure for those economic evaluations where an IRR method is appropriate.
Recommendations for the application of the NB and IRR methods are made for
cases where there is some controversy as to the appropriate use of the two
methods. Calculation limitations of the two methods are also described.

8.1 IRR or AIRR Measure

Controversy persists over whether the IRR measure or the AIRR measure is a
more accurate index of an investment's return. This question is complicated by
the conceptual question of what analysts should be measuring; namely, the
return solely on the original investment (which implicitly suggests the same
return on reinvestments as the original investment), or the combined return
on the original investment and on the reinvested earnings (at possibly
differential rates).

Some authors accept the unadjusted IRR as technically correct.[1] Among them is
Kerr who rejects reinvestment models (which include the AIRR) in general with
the following statement:

> Reinvestment rate models ..., which have been developed to avoid
> the ambiguities of the IRR yet retain the flavor of a yield, are
> incorrect because the measure of the desirability of a proposed
> investment is contaminated by expected yields on other invest-
> ments; ....[2]

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[1]See Geraldine Montag, "Commercial Building Ownership Energy Conservation
Cost Analysis Model--A Follow-Up," ASHRAE Journal (September 1981), p. 45, and
William R. Park, Cost Engineering Analysis (New York, New York: John Wiley and
Sons, 1973), pp. 42-48.

[2]H. S. Kerr, "A Final Word on FMRR," The Appraisal Journal (January 1980), p.
102.

Other authors acknowledge that both the AIRR and IRR measures have certain advantages and disadvantages, and that each measure might be appropriate under certain circumstances.[1]

The finding of this paper is that the AIRR measure may be preferred over the IRR measure for the following three reasons: the AIRR is a more accurate measure of project returns; it is generally easier to compute; and it does not present the problem of yielding multiple solutions as does the IRR measure.

The AIRR is a more accurate measure because a direct solution is possible, whereas approximating techniques are generally used for the IRR. It is also more accurate in the sense that the AIRR incorporates the expected real earnings of reinvested net cash flows whereas the IRR measure assumes reinvestment at the IRR rate. There is no reason to expect in general that each project's reinvestment rates will be the same as the IRR for that project.[2] Furthermore, when comparing two competing projects with different IRR's, different reinvestment rates are implicitly assumed when in fact one would expect the same reinvestment rates to apply to earnings on those two projects.

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[1]See Eisen, "Computer Connection," p. 127, and J. Thomas Montgomery and Charles F. Raper, "Equity Yields and the Reinvestment Issue," The Appraisal Journal (October 1981), pp. 519-521.

[2]If r = IRR, then the IRR gives the same answer as the AIRR as long as a unique IRR can be computed.

The reinvestment rate r needed in solving equation 7.3 for the AIRR might be specified as part of the investment project, or, if the project were already built and operating, the return on reinvestments would have been established historically. If r is not known, however, the discount rate (i.e., the rate of return available on the next best investment) would appear to be the best approximation for r. Having an r higher than the discount rate would suggest that the discount rate is too low; that is, it does not reflect the next best investment opportunity. If r were lower than the discount rate, it would suggest that the discount rate is not high enough to reflect the next best investment opportunity. Thus in absence of information on the rate r, the discount rate seems the appropriate value to substitute for r in using equations 7.3, 7.4, and 7.5.

So far the AIRR has been recommended over the IRR as the more accurate economic measure when evaluating one investment. However, the AIRR is also preferred when choosing among a group of non-mutually-exclusive projects, since the AIRR and IRR will not always yield the same rankings of a set of projects. That is, the AIRR will be a more accurate guide than the IRR to the set of projects that will maximize the aggregate PVNB when a budget constraint exists.

The AIRR is easier to compute than the IRR because a direct solution of the AIRR is possible, using equations 7.4 and 7.5, whereas iterative procedures such as the trial-and-error approach are required to arrive at an approximation for the IRR. (Computers, however, make the computational differences of the two measures less important.)

A limitation to the IRR is that multiple measures may result. That is, there may be no unique IRR value that serves as a solution value for i in equation 7.1. This situation is likely to occur when the cash flow shows reversals in sign over the study period.[1] Finding more than one solution value results in confusion, for example, when the MARR is between two solution values of the IRR. Furthermore, trial-and-error solutions may find only one answer and fail to indicate the existence of other mathematically correct solution values. The AIRR, as calculated in equations 7.4 and 7.5, avoids this problem.[2]

A second limitation arises with both the IRR and the AIRR measures when the MARR is expected to change over the lifetime of the investment. For example, if the MARR were expected to be 17% for the first four years of a 10 year project and 9% for the last six years, it is not clear how efficient that project would be if it had an IRR or AIRR of 15%.

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[1]Another way of describing such a cash flow is that over the project life some periods are revenue earning (positive cash flow) periods and others are investment or cash disbursement (negative cash flow) periods.

[2]The modified rate of return (MROR), a more complicated method than the IRR, provides a percentage measure of economic worth while avoiding some of the problems inherent in the IRR. For a discussion of the method and flow chart of the procedures for calculating it, see American Telephone and Telegraph Company, Engineering Economy: A Manager's Guide to Economic Decision Making (New York, New York: McGraw Hill, 1977), pp. 317-321.

8.2 NB or IRR Method

Another controversial issue is which of the two methods, NB or IRR, gives the proper answer for specific investment questions. Some authors suggest the NB method for some investment decisions and the IRR method for others.[1] A few authors recommend the NB method exclusively over the IRR method.[2]

The position taken in this report is summarized by table 8.1, which lists typical economic questions asked by the building community and indicates if the NB method, the IRR method, or both are appropriate for answering those questions.[3]

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[1]See J. Fred Weston and Brigham, Managerial Finance, 7th ed. (Hinsdale, Illinois: The Dryden Press, 1981), p. 413 and p. 423, and Marshall and Ruegg, Simplified Energy Design Economics, p. 13.

[2]Kerr, "A Final Word on FMRR," p. 102, asserts that "... it is correct to use the NB method in all investment decisions." Lee G. Anderson and Russel F. Settle, Benefit-Cost Analysis: A Practical Guide (Lexington, Massachusetts: D.C. Heath and Company, 1977), pp. 96-98, and Peter G. Sassone and William A. Shaffer, Cost-Benefit Analysis: A Handbook (New York, New York: Academic Press, 1978), p. 28, agree with the exception that they recommend the benefit-to-cost ratio method for choosing among independent projects with a budget constraint.

[3]The NB and IRR methods are not the only economic methods for solving these questions. The savings-to-investment ratio and benefit-cost ratio methods (see Harold E. Marshall and Rosalie T. Ruegg, Recommended Practice for Measuring Benefit/Cost and Savings-to-Investment Ratios), for example, can be used to answer most of the questions for which the IRR is identified in table 8.1. Also, the life-cycle cost method (see Rosalie T. Ruegg, Stephen R. Petersen, and Harold E. Marshall, Recommended Practice for Measuring Life-Cycle Costs) is substitutable in most cases for the NB method.

Table 8.1 Economic Questions and Recommended Methods[a]

	NB	IRR
1. HOW can benefits be compared to costs?	●	●
2. HOW large an investment to make?	●	▲
3. HOW can projects directly competing for the same purpose be compared?	●	
4. HOW can different-purpose projects competing for the same budget be compared?		●
5. HOW to find the rate of return on investment?	·	●

● Based on total, present value cost and benefit figures.
▲ Based on incremental, present value cost and benefit figures.

[a]The table is adapted from Rosalie T. Ruegg and Harold E. Marshall, "Economics of Building Design," <u>Solar Age</u> (July 1981), p. 27.

Table 8.1 shows in the first row that both the NB and IRR methods can be used to indicate whether a given project is cost effective (i.e., if benefits exceed costs). If the PVNB is greater than zero, the investor's minimum acceptable rate of return (MARR), as reflected in the discount rate, has been achieved, and the project is cost effective.[1] If the PVNB is less than zero, then the project is not cost effective. If the AIRR is greater than the MARR for a given project, then the project is cost effective (i.e., benefits exceed costs in present value terms). If the IRR measure is greater than the MARR, the project may also be cost effective, but the calculated IRR value will be correct (i.e., equal to the AIRR value) only when r = IRR.

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[1]The MARR and the discount rate can be assumed to be equal when each is based on the anticipated return on the next best investment.

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The second question, how large an investment to make, is generally answered with NB analysis, but can be answered with the IRR method if the IRR or AIRR is calculated for marginal investment changes. Unless rate of return data are available for other reasons, however, increasing scale until the PVNB is maximized (i.e., using the NB method) is probably a more efficient approach from the standpoint of making the calculations.[1]

Typical size or scale examples from the building industry are as follows:

- o How large a building should be constructed?

- o How large a dam should be constructed?

- o How much insulation should be put in a house?

- o How many square feet of collector area should be installed in a solar energy system?

Taking the example of how large a building to construct, if the objective of an owner/builder is to maximize profits (net benefits) from a building, the builder can be expected to increase the square feet, number of floors, and height variables as long as the increase in scale yields more new benefits (income) than costs. The size at which the PVNB is maximized will be the most economically efficient building scale.

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[1]If computer assistance is available, the choice for computational efficiency is less important.

Figure 8.1 illustrates graphically how the NB method is used to choose the economically efficient level of energy conservation. Conservation costs are shown to increase at an increasing rate as the physical quantity of inputs to conserve energy (Q_i) is increased (e.g., increased insulation). Conservation benefits, as measured by energy savings, also increase with additional inputs to energy conservation, but at a decreasing rate. The difference between dollar conservation benefits and costs at any given level of conservation inputs is the PVNB. The level of energy conservation where the PVNB is maximized is Q_e. Any smaller (Q_1) or larger investments (Q_2 or Q_3) than Q_e would be economically inefficient, because the potential PVNB (profits) is greatest at Q_e.[1] To find the economically efficient level of insulation for a residence, therefore, the PVNB is computed for increasing applications of insulation until the level is found where the PVNB is maximized.

Table 8.2 and figure 8.2 together illustrate how project size can be selected on the basis of incremental AIRR analysis. Table 8.2 presents five size alternatives (A, B, C, D, and E) with corresponding AIRR's for all possible size changes. Reading table 8.2 from left to right, the top row gives the AIRR's on total investment at each of those size levels (size A is assumed to be a project size of zero). Although size B has the largest AIRR (35%), it is not necessarily the one that yields maximum net benefits, which is the objective in selecting the most efficient size.

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[1]Note that the efficient size could be smaller if the investment budget were limited and if other projects were available with incremental benefit-to-cost ratios greater than one. For a discussion of these conditions, see Harold E. Marshall and Rosalie T. Ruegg, <u>Recommended Practice for Measuring Benefit/Cost and Savings-to-Investment Ratios</u>, pp. 28-29.

Figure 8.1 Level of Energy Conservation That Maximizes the PVNB
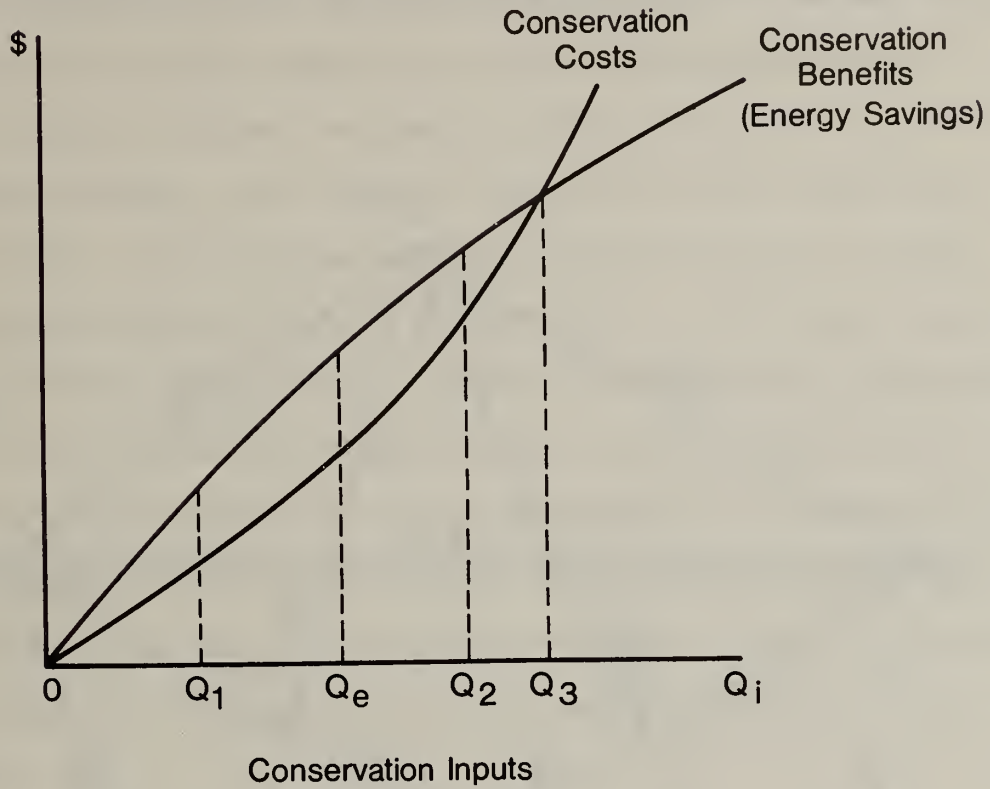
Table 8.2 AIRR's For Project Size Changes
(%)

From Size \ To Size	B	C	D	E
A	35	27	19	14
B		13	11	8
C			7	4
D				3

Subsequent rows of table 8.2 give the incremental AIRR's calculated on the differences between project sizes. Thus the incremental AIRR from expanding size from B to C is 13%; from B to D, 11%; and from C to E, 4%.

Figure 8.2 Network Diagram Using Incremental AIRR Analysis for Project Sizing
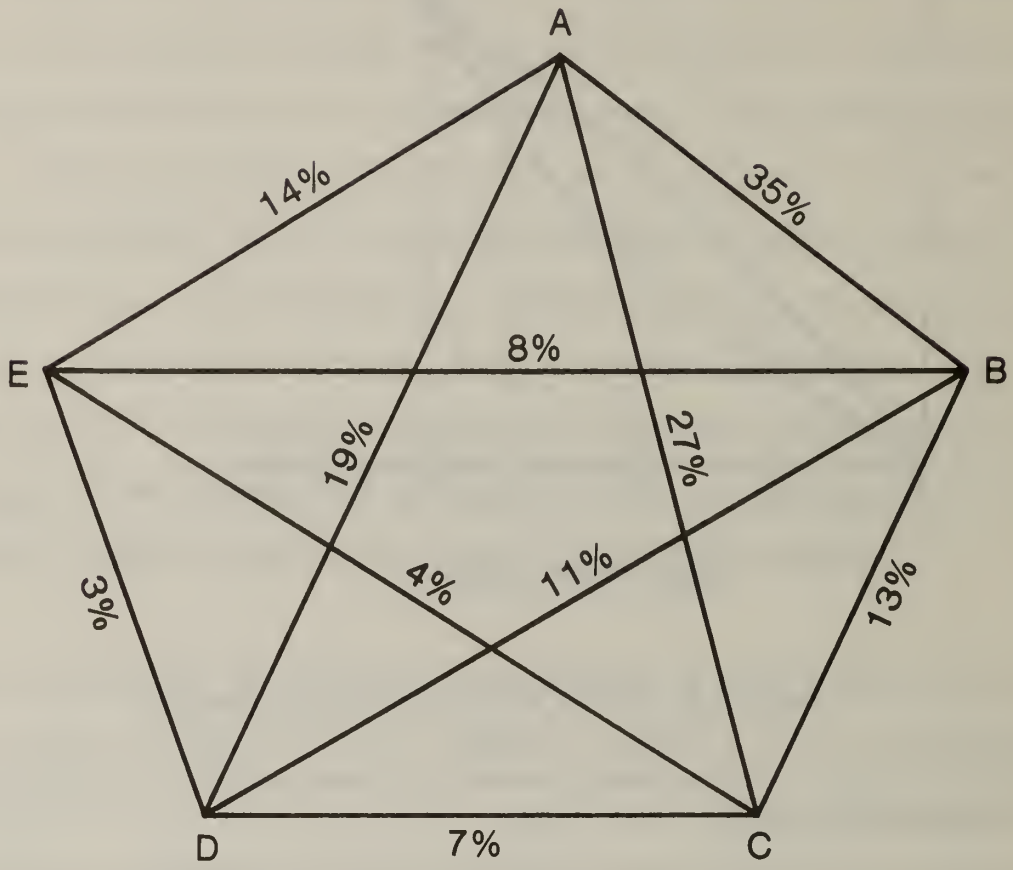
Figure 8.2 shows all of the potential sizing options in a network diagram.
Each line in the network connects two lettered points and represents a
potential sizing decision. The number by each connecting line is the AIRR for
that sizing increment. For example, the line from A to D shows size D with a
return of 19%. Looking at the successive increments up to D, however, shows A
to B earning an AIRR of 35%; B to C 13%, and C to D 7%. Only if the MARR is
less than 7% will an investment in size D be the economically efficient
decision. This is the case because the last increment returns only 7%, even
though the overall AIRR is 19%. If the MARR were 12%, the efficient size
would be C, because the increment B to C is the last one to earn more than the
12% MARR. Thus the rule for selecting the efficient size is to increase size
by each separable increment until the incremental AIRR equals or just exceeds
the MARR.

Note that failing to expand size whenever the incremental AIRR exceeds the
MARR or failing to reduce size whenever the incremental AIRR is less than the
MARR will result in potential PVNB foregone. This is equivalent to choosing a
Q_i in figure 8.1 that is respectively less than or higher than Qe. Note
further that choosing a scale where the AIRR on the total investment is just
equal to the MARR (say choosing D when the MARR is 19%) would result in PVNB
being equal to zero (Q_3 in figure 8.1). Clearly this is not the most economic
choice.

As long as net benefits are calculated with a discount rate equal to the MARR, the efficient sizes indicated by the NB and incremental AIRR measures will be the same.[1]

The NB method is recommended for answering the third question in table 8.1, how to compare projects or designs competing for the same purpose to see which is most efficient (i.e., cost effective).[2] Typical examples from the building industry include:

   o How to select between single, double, or triple glazing?

   o How to choose between a solar energy system and a conventional
      energy system?

   o How to choose between a large dam and a small dam with levees to
      provide flood control?

The most efficient project in each case would be the one with the greatest PVNB. Applying equation 6.1, for example, to the selection of a flood control project, if PVNB is greater for the small dam and levees than for the large dam, then the small dam and leeves are the economically preferred system.

In comparing projects competing for the same purpose, the analyst must some-times normalize the PVNB with respect to time in order to have a valid economic comparison. The PVNB of projects with identical expected lives can be compared directly. If the expected lives are different, however, adjustments

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[1]This assumes the same reinvestment rates for net cash flows are used in both methods.

[2]Under certain circumstances the incremental IRR method can also be used to compare projects or designs competing for the same purpose. For example, one case of comparing mutually exclusive designs is comparing scale or size as was discussed under question 2 in table 8.1.

are required. A common adjustment is to convert each project's life to the least common multiple of the lives of all projects under consideration. By making assumptions about reinvestment costs and earnings, a time-normalized PVNB can then be calculated for each project for comparison over the common study period.

A second approach in comparing projects of unequal lives is to convert the PVNB calculated on the basis of each project's life to an annual value of net benefits (AVNB). (See equation 6.2.) The AVNB figures will yield a valid economic comparison if replacement assets in the short-lived projects repeat the costs and earnings of the original assets.

A third approach is to assume a perpetual or infinite period of earnings through continual reinvestment.[1] The PVNB of each project is converted to an AVNB. Dividing the AVNB by the discount rate for each competing project yields PVNB measures for projects with infinite lives. This normalization with respect to lives results in PVNB figures that provide valid comparisons among projects.

Note that an implicit assumption typically made in all of these approaches is that the replacement investment is identical to the original investment and will repeat the returns of the original investment.[2]

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[1] For further descriptions of these three approaches, see Grant and Ireson, Principles of Engineering Economy, pp. 77-81 and 92-98; Gerald W. Smith, Engineering Economy: Analysis of Capital Expenditures, 2nd ed. (Ames, Iowa; Iowa State University Press, 1973), pp. 107-108; and Weston and Brigham, Managerial Finance, pp. 436-438.

[2] This same assumption is implicit in the IRR method. However, the assumption may not reflect actual reinvestment opportunities.

A mistake analysts sometimes make is to use the IRR measure, based on total present value cost and benefit figures, for selecting among projects competing for the same purpose. Table 8.3 helps illustrate how using the unadjusted IRR measure instead of the NB method to select between projects competing for the same purpose will lead to inefficient choices under certain circumstances.[1] Cash flows for projects H and I are given in table 8.3. The two example projects are designed to reflect high payoffs early in one project and late in the other so as to highlight the inconsistent answers that occur when the cash flows differ among projects. Figure 8.3 illustrates the potential inconsistency of the two methods by comparing plots or profiles of the PVNB for the two projects, H and I, over a range of discount rates. The profiles are derived the same way as the graph shown in figure 7.1.

Table 8.3   Project Data for Comparing Projects by NB and IRR Methods

Year (t)	Project H B_t	Project H C_t	Project H $(B_t - C_t)$	Project I B_t	Project I C_t	Project I $(B_t - C_t)$
0	$0	$1,000	$-1,000	$0	$1,000	$-1,000
1	100	0	100	1,100	0	1,100
2	125	0	125	200	0	200
3	1,300	0	1,300	50	0	50
			$525			$350

[1]The circumstances that lead to different answers are typically that the costs and benefits of one project far exceed those of the other project or that the cash flows have a different timing sequence.

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Figure 8.3 Graphical Illustration of Inconsistent Project Rankings by NB and IRR Methods
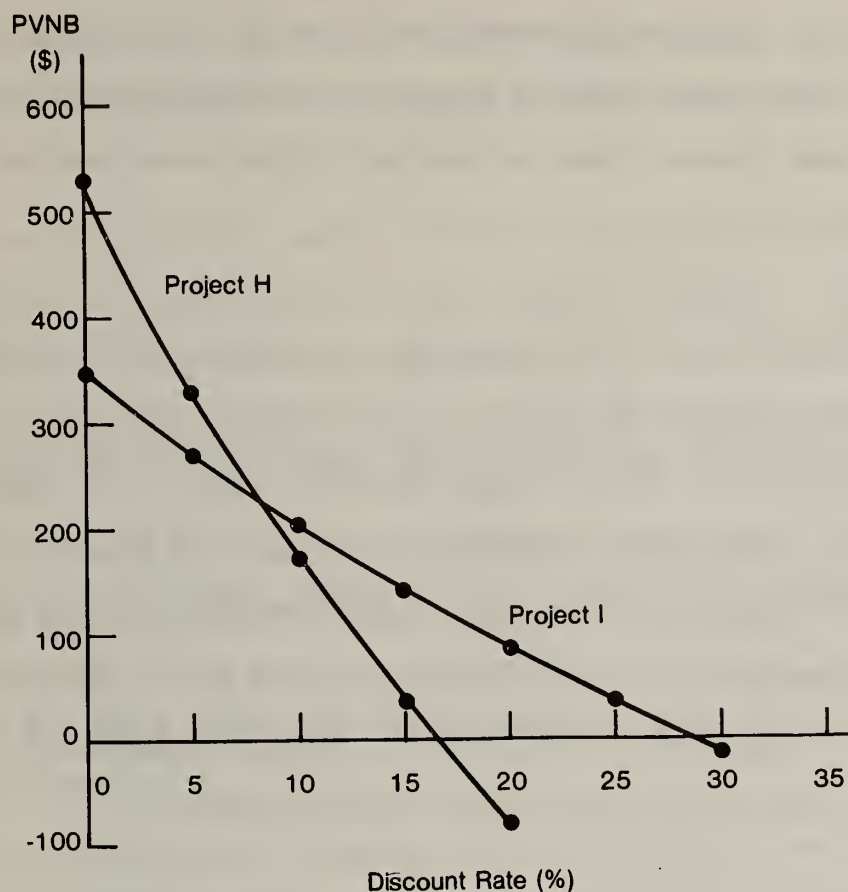
Given a discount rate of zero, the PVNB would be the arithmetic sum over time of the net cash flows--$525 for project H, and $350 for project I. Thus the function for H intersects the vertical axis at $525 and for I at $350. Discount rates in 5% increments are substituted into equation 6.1 to find the PVNB at those rates. Connecting the PVNB values for each project for the assumed discount rates gives a function approximating the PVNB for a wide range of discount rates. The function for project H crosses the horizontal axis at an IRR of approximately 16.5%. For project I, the unadjusted IRR appears to be approximately 28.5%. Thus project I appears to be the more attractive investment. This is consistent with the results from applying the NB method for all discount rates greater than the crossover rate (in this case about 8%) where the two functions cross. That is, for all discount rates

greater than 8%, the PVNB is greater for I than H. This is what would be expected in that I earns benefits early during the project life and is not affected by high discount rates as much as H which earns benefits late in the project life.

The inconsistency between the two methods occurs whenever the discount rate used to compute the PVNB is less than the crossover rate. Thus in figure 8.3, for example, project H yields higher PVNB than project I for every discount rate below 8%. Again this is expected because project H's major benefits accrue late in the project life. But the IRR measure still favors project I while the NB method favors H. Therefore, as long as the objective is maximization of the PVNB, the NB method is preferred, since the unadjusted IRR will not consistently lead to the efficient project.

The AIRR, if computed with a reinvestment rate equal to the discount rate, would yield similar results to the NB method for the problem described in table 8.3. Under certain circumstances, however, such as one project having benefits and costs that far exceed those of the other project, the AIRR will indicate the incorrect alternative (i.e., it will be higher for the project with a lower PVNB). Thus the NB method is preferred to both the IRR and AIRR measures.

Figure 8.3 also shows that different conclusions result within the NB method itself as the discount rate changes. For example, at rates less than 8%, project H is preferred, whereas at rates greater than 8%, project I is preferred. This does not indicate any limitation in the method. It

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shows only that, as the opportunity cost of funds changes, so does the

relative worth of projects with different payouts over time.

The AIRR measure is used to answer the fourth question, how can different-

purpose projects competing for the same budget be compared. Typical examples

from the building industry are as follows:

o Which conservation retrofits are most cost effective for the
  homeowner when the budget permits only some of the economically
  feasible projects to be undertaken?

o Which packages of energy conservation projects for its buildings
  should a government agency select, given a limited budget, among
  the many opportunities identified by field offices?

o Which water resource construction projects should an agency select
  (e.g., among flood control, irrigation, and recreation projects) when
  the construction budget will not cover all the economically feasible
  alternatives?

Each of the examples above describes a set of non-mutually-exclusive,

independent projects competing for a limited budget.[1] Given that the budget

cannot fund all of the projects for which the AIRR exceeds the MARR, the

aggregate PVNB from the limited budget will be maximized generally by under-

taking projects in descending order of their AIRR's until the budget is

exhausted. If the AIRR's fall below the MARR before the available budget is

exhausted, then project acceptance should terminate with the last project

whose AIRR exceeds the MARR. It is economically efficient to hold back part

[1]For non-mutually-exclusive, independent projects with no budget constraint,
the appropriate selection criterion is simply to take every project where the
PVNB exceeds zero. If one or more non-mutually-exclusive projects were
dependent (i.e., the PVNB of any one project depends on whether or not another
is built), then the set of projects that would maximize the PVNB in
combination would be economically efficient. If a budget constraint were pre-
sent, then there would be fewer sets to choose from.
    For a description of how to choose combinations of interdependent projects,
see Sassone and Schaffer, Cost-Benefit Analysis, pp. 25-29.

of the budget if the remaining available projects have AIRR's less than the MARR. If projects with high AIRR's cost more than the available budget, lower cost projects with lower AIRR's (but AIRR's greater than the MARR) may be substituted until the budget is exhausted.

The AIRR rankings as described above will not indicate the optimal project mix in every case. For that reason, the set of projects selected by the AIRR rankings should be used only as a guideline and checked against other project mixes to make sure that no other combination will yield a higher aggregate PVNB for the limited budget. An example of a readjustment of the project mix that would maximize the total PVNB for a limited budget is illustrated in table 8.4.

Table 8.4   Optimal Ranking Using AIRR as a Guideline[a]

| | | | | | Allocation of $10K Budget | |
| | Initial Costs (C_o) | PVNB | AIRR (%) | AIRR Ranking | AIRR Project Set | Optimal Project Set |
Projects						
M	$4,000	$5,222	30	1	M	M
N	1,000	895	25	2	N	
O	6,000	4,488	23	3		O
P	2,000	391	14	4	P	
Q	3,000	283	12	5	Q	
				Aggregate PVNB =	$6,791	$9,710

[a]The five hypothetical projects used to illustrate optimal ranking are evaluated over a 5-year study period at a 10% reinvestment rate.

Five projects, each yielding earnings for five years and having expected rein-vestment rates of 10%, are ranked according to their AIRR's. Selecting projects in descending order of the AIRR's yields a project mix of M, N, P, and Q, given a $10,000 budget. Project O is passed by since inadequate funds remain after choosing M and N. An aggregate PVNB of $6,791 results from the four projects chosen. Alternative project mixes including project O should be examined, however, to be certain that the aggregate PVNB is maximized at $6,791. Inspection of table 8.4 shows that spending the $10,000 budget on M and O would yield an aggregate PVNB of $9,710, almost $3,000 more than the project set indicated by the AIRR ranking.[1] Thus to assure selection of the optimal set of projects, alternative mixes should be compared against the initial AIRR ranking whenever the AIRR ranking passes over cost-effective projects more costly than the remaining budget.

The description of table 8.4 above explains in part the confusion arising from some textbooks recommending IRR methods while others recommend NB methods for choosing among non-mutually-exclusive, independent projects. Both recommendations are correct under certain circumstances. Project selection according to the AIRR rankings, for example, will generally lead to the mix that maximizes the aggregate PVNB, with the exception illustrated in table 8.4. The NB method, on the other hand, is correct if the aggregate PVNB of a group of projects is maximized for a limited budget. Selection on the basis

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[1]It is assumed for this problem that all or most of the budget must be spent. That is, remaining budget funds are not retained by the investor. However, if the remaining budget could be retained, and the objective were to maximize the sum of the aggregate PVNB and the remaining budget, then projects M and N would be preferred to M and O, because the aggregate PVNB plus remaining budget for M and N ($11,117) exceed that of M and O ($9,710).

of individial project net benefits, however, will __not__ necessarily lead to the
optimal mix.[1] Yet project analysts sometimes rank projects according to the
PVNB of individual projects. The following case example is provided to show
the economic implications of such rankings.

Table 8.5 lists seven non-mutually-exclusive, independent projects (column 1).
They are ranked by the AVNB (column 7) and by the AIRR (column 9). (The AVNB
measure is used here instead of the PVNB measure to account for the unequal
project lives.) AIRR's (column 8) are calculated on the basis of uniform net
cash flows per year (column 3) and the initial project cost (column 4). The
study period, determined by the number of earning years for each project,
ranges from 3 to 6 years. The AVNB (column 6) for each project is calculated
by subtracting the annual value of initial costs (column 5) from the uniform
annual net cash flows (column 3). A 12% discount rate was used to find annual
values of cash flows, and 12% was also assumed to be the MARR and the rein-
vestment rate for net cash flows.

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[1]A computer program that calculates aggregate PVNB for every possible project
mix for a given budget would be the most efficient approach to project
selection.

Table 8.5 NB Versus IRR Method for Ranking Projects[a]

Projects	Earning Years	Annual Net Cash Flows $(B_t - C_t)$	C_o	Annual Value of C_o	AVNB	Ranking by AVNB	AIRR	Ranking by AIRR
(1)	(2)	(3)	(4)	(5)	(6)=(3)-(5)	(7)	(8)	(9)
A	3	395	1,000	$416	$-21	7	10.1%	6
B	4	1,094	3,000	987	107	3	14.9%	3
C	6	162	500	122	40	6	17.5%	1
D	4	1,401	4,000	1,316	85	4	13.7%	5
E	3	4,007	9,000	3,744	263	1	14.5%	4
F	5	350	1,000	277	73	5	17.3%	2
G	4	1,641	4,500	1,481	160	2	14.9%	3

[a] A 12% discount rate is assumed for calculating the PVNB and the AVNB. A 12% rate is also assumed to be the MARR and the reinvestment rate for net cash flows.

Note that project A is clearly inefficient by both methods because the AVNB is negative and the AIRR is less than the MARR. However, rankings for the rest of the projects are inconsistent between the two methods and may lead to different packages of selected projects, depending on the budget constraint. For example, if the budget were $9,000, the AIRR ranking would indicate projects C, F, and G and B, in that order, for an aggregate AVNB of $380.[1] For the same budget, the AVNB ranking, on the other hand, would indicate project E alone, with an AVNB of $263. Ranking with the AVNB measure in this case

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[1] Projects G and B have the same AIRR and are therefore equally ranked.

example results in annual-value, economic efficiency losses of $117 ($380 - $263).[1] Thus the table shows how ranking with the AVNB measure may bias project selection to one or a few large projects with a high AVNB per project but a relatively low return on the limited budget.

The last question of table 8.1, finding the rate of return on an investment, can be answered only with the IRR or AIRR measures. Neither the NB method nor any of the other economic evaluation tools for capital investment expresses the "value" of an investment in percentage terms.

The guidelines for applying the NB and IRR methods described above cover many of the questions that the private and public building communities must answer in making economically efficient decisions in the design, operation, maintenance, and retrofit of buildings. While other methods could be used successfully in answering most of these questions, the focus of this report is on the NB and IRR methods. For an alternative to the NB method, see the National Bureau of Standards (NBS) report Recommended Practice for Measuring Life-Cycle Costs of Buildings and Building Systems. For an alternative to the IRR method, see the NBS report Recommended Practice for Measuring Benefit/Cost and Savings-to-Investment Ratios for Buildings and Building Systems.

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[1]Ranking these seven projects with the unadjusted IRR would give exactly the same priority as with the AIRR, but the spread of returns would be much wider (9% to 23%). However, under different cash flow assumptions the AIRR and IRR measures might yield different rankings.

APPENDIX A

DISCOUNT FACTOR TABLES*

*Tables needed for problems illustrated in this report were taken from American Society for Testing and Materials, <u>Discount Factor Tables</u>, to be published as an Adjunct to "Recommended Practice for Measuring Life-Cycle Costs of Buildings and Building Systems" (Philadelphia, Pennsylvania: ASTM, 1983).

Table 5 DISCRETE RATE-OF-RETURN FACTORS*

	Single Compound Amount Factor SCA	Single Present Value Factor SPV	Uniform Capital Recovery Factor UCR	Uniform Present Value Factor UPV	Uniform Sinking Fund Factor USF	Uniform Compound Amount Factor UCA
For-mula	$(1+i)^n$	$\dfrac{1}{(1+i)^n}$	$\dfrac{i(1+i)^n}{(1+i)^n-1}$	$\dfrac{(1+i)^n-1}{i(1+i)^n}$	$\dfrac{i}{(1+i)^n-1}$	$\dfrac{(1+i)^n-1}{i}$
Given To Find	P F	F P	P A	A P	F A	A F
Period n	$i = 5\%$					

n	SCA	SPV	UCR	UPV	USF	UCA
1	1.050	.9524	1.050	.9524	1.000	1.000
2	1.103	.9070	.5378	1.859	.4878	2.050
3	1.158	.8638	.3672	2.723	.3172	3.153
4	1.216	.8227	.2820	3.546	.2320	4.310
5	1.276	.7835	.2310	4.329	.1810	5.526
6	1.340	.7462	.1970	5.076	.1470	6.802
7	1.407	.7107	.1728	5.786	.1228	8.142
8	1.478	.6768	.1547	6.463	.1047	9.549
9	1.551	.6446	.1407	7.108	.0907	11.03
10	1.629	.6139	.1295	7.722	.0795	12.58
11	1.710	.5847	.1204	8.306	.0704	14.21
12	1.796	.5568	.1128	8.863	.0628	15.92
13	1.886	.5303	.1065	9.394	.0565	17.71
14	1.980	.5051	.1010	9.899	.0510	19.60
15	2.079	.4810	.0963	10.38	.0463	21.58
16	2.183	.4581	.0923	10.84	.0423	23.66
17	2.292	.4363	.0887	11.27	.0387	25.84
18	2.407	.4155	.0855	11.69	.0355	28.13
19	2.527	.3957	.0827	12.09	.0327	30.54
20	2.653	.3769	.0802	12.46	.0302	33.07
21	2.786	.3589	.0780	12.82	.0280	35.72
22	2.925	.3419	.0760	13.16	.0260	38.51
23	3.072	.3256	.0741	13.49	.0241	41.43
24	3.225	.3101	.0725	13.80	.0225	44.50
25	3.386	.2953	.0710	14.09	.0209	47.73
26	3.556	.2812	.0696	14.38	.0196	51.11
27	3.733	.2678	.0683	14.64	.0183	54.67
28	3.920	.2551	.0671	14.90	.0171	58.40
29	4.116	.2429	.0660	15.14	.0160	62.32
30	4.322	.2314	.0651	15.37	.0151	66.44
31	4.538	.2204	.0641	15.59	.0141	70.76
32	4.765	.2099	.0633	15.80	.0133	75.30
33	5.003	.1999	.0625	16.00	.0125	80.06
34	5.253	.1904	.0618	16.19	.0117	85.07
35	5.516	.1813	.0611	16.37	.0111	90.32
36	5.792	.1727	.0604	16.55	.0104	95.84
37	6.081	.1644	.0598	16.71	.0098	101.6
38	6.385	.1566	.0593	16.87	.0093	107.7
39	6.705	.1491	.0588	17.02	.0088	114.1
40	7.040	.1420	.0583	17.16	.0083	120.8

* All formulae assume end-of-period payments.

P = a present sum of money; F = a future sum of money, equivalent to P at the end of n periods of time at discount rate of i; A = an end-of-period payment (or receipts) in a uniform series of payments (or receipts) over n periods at i interest rates.

Table 10 DISCRETE RATE-OF-RETURN FACTORS*

Period n	Single Compound Amount Factor SCAF	Single Present Value Factor SPVF	Uniform Capital Recovery Factor UCRF	Uniform Present Value Factor UPVF	Uniform Sinking Fund Factor USFF	Uniform Compound Amount Factor UCAF
Formula	$(1+i)^n$	$\dfrac{1}{(1+i)^n}$	$\dfrac{i(1+i)^n}{(1+i)^n-1}$	$\dfrac{(1+i)^n-1}{i(1+i)^n}$	$\dfrac{i}{(1+i)^n-1}$	$\dfrac{(1+i)^n-1}{i}$
Given	P	F	P	A	F	A
To Find	F	P	A	P	A	F
				$i=10\%$		
1	1.100	.9091	1.100	.9091	1.000	1.000
2	1.210	.8264	.5762	1.736	.4762	2.100
3	1.331	.7513	.4021	2.487	.3021	3.310
4	1.464	.6820	.3155	3.170	.2155	4.641
5	1.611	.6209	.2638	3.791	.1638	6.105
6	1.772	.5645	.2296	4.355	.1296	7.716
7	1.949	.5132	.2054	4.868	.1054	9.487
8	2.144	.4665	.1874	5.335	.0874	11.44
9	2.358	.4241	.1736	5.759	.0736	13.58
10	2.594	.3855	.1627	6.145	.0627	15.94
11	2.853	.3505	.1540	6.495	.0540	18.53
12	3.138	.3186	.1468	6.814	.0468	21.38
13	3.452	.2897	.1408	7.103	.0408	24.52
14	3.798	.2633	.1357	7.367	.0357	27.98
15	4.177	.2394	.1315	7.606	.0315	31.77
16	4.595	.2176	.1278	7.824	.0278	35.95
17	5.054	.1978	.1247	8.022	.0247	40.54
18	5.560	.1799	.1219	8.201	.0219	45.60
19	6.116	.1635	.1195	8.365	.0195	51.16
20	6.728	.1486	.1175	8.514	.0175	57.28
21	7.400	.1351	.1156	8.650	.0156	64.00
22	8.140	.1228	.1140	8.772	.0140	71.40
23	8.954	.1117	.1126	8.883	.0126	79.54
24	9.850	.1015	.1113	8.984	.0113	88.50
25	10.83	.0923	.1102	9.077	.0102	98.35
26	11.92	.0839	.1092	9.161	.0092	109.2
27	13.11	.0763	.1083	9.237	.0083	121.1
28	14.42	.0693	.1075	9.307	.0074	134.2
29	15.86	.0630	.1067	9.370	.0067	148.6
30	17.45	.0573	.1016	8.427	.0061	164.5
31	19.19	.0521	.1055	9.479	.0055	181.9
32	21.11	.0474	.1050	9.526	.0050	201.1
33	23.23	.0431	.1045	9.569	.0044	222.3
34	25.55	.0391	.1041	9.609	.0001	245.5
35	28.10	.0356	.1037	9.644	.0037	271.0
36	30.91	.0323	.1033	9.676	.0033	299.1
37	34.00	.0294	.1030	9.706	.0030	330.0
38	37.40	.0267	.1027	9.733	.0027	364.0
39	41.14	.0243	.1025	9.757	.0025	401.4
40	45.26	.0221	.1023	9.779	.0023	442.6

* All formulae assume end-of-period payments.

P = a present sum of money; F = a future sum of money, equivalent to P at the end of n periods of time at discount rate of i; A = an end-of-period payment (or receipts) in a uniform series of payments (or receipts) over n periods at i interest rates.

Table 14 DISCRETE RATE-OF-RETURN FACTORS*

	Single Compound Amount Factor SCAF	Single Present Value Factor SPVF	Uniform Capital Recovery Factor UCRF	Uniform Present Value Factor UPVF	Uniform Sinking Fund Factor USFF	Uniform Compound Amount Factor UCAF
Formula	$(1+i)^n$	$\dfrac{1}{(1+i)^n}$	$\dfrac{i(1+i)^n}{(1+i)^n-1}$	$\dfrac{(1+i)^n-1}{i(1+i)^n}$	$\dfrac{i}{(1+i)^n-1}$	$\dfrac{(1+i)^n-1}{i}$
Given	P	F	P	A	F	A
To Find	F	P	A	P	A	F
Period n	$i=14\%$					

n						
1	1.140	.8772	1.140	.8772	1.000	1.000
2	1.300	.7695	.6073	1.647	.4673	2.140
3	1.482	.6750	.4307	2.322	.2907	3.440
4	1.689	.5921	.3432	2.914	.2032	4.921
5	1.925	.5194	.2913	3.433	.1513	6.610
6	2.195	.4556	.2572	3.889	.1172	8.536
7	2.502	.3996	.2332	4.288	.0932	10.73
8	2.853	.3506	.2156	4.639	.0756	13.23
9	3.252	.3075	.2022	4.946	.0622	16.09
10	3.707	.2697	.1917	5.216	.0517	19.34
11	4.226	.2366	.1834	5.453	.0434	23.04
12	4.818	.2076	.1767	5.660	.0367	27.27
13	5.492	.1821	.1712	5.842	.0312	32.09
14	6.261	.1597	.1666	6.002	.0266	37.58
15	7.138	.1401	.1628	6.142	.0228	43.84
16	8.137	.1229	.1596	6.265	.0196	50.98
17	9.276	.1078	.1569	6.373	.0169	59.12
18	10.58	.0949	.1546	6.467	.0146	68.39
19	12.06	.0829	.1527	6.550	.0127	78.97
20	13.74	.0728	.1510	6.623	.0110	91.02
21	15.67	.0638	.1495	6.687	.0095	104.8
22	17.86	.0560	.1483	6.743	.0083	120.4
23	20.36	.0491	.1472	6.792	.0072	138.3
24	23.21	.0431	.1463	6.835	.0063	158.7
25	26.46	.0378	.1455	6.873	.0055	181.9
26	30.17	.0331	.1448	6.906	.0048	208.3
27	34.39	.0291	.1442	6.935	.0042	238.5
28	39.20	.0255	.1437	6.961	.0037	272.9
29	44.69	.0224	.1432	6.983	.0032	312.1
30	50.95	.0196	.1428	7.003	.0028	356.8
31	58.08	.0172	.1425	7.020	.0025	407.8
32	66.21	.0151	.1421	7.035	.0021	465.8
33	75.48	.0132	.1419	7.048	.0020	532.0
34	86.05	.0116	.1416	7.060	.0016	607.5
35	98.10	.0102	.1414	7.070	.0014	693.6
36	111.8	.0089	.1413	7.079	.0013	791.7
37	127.5	.0078	.1411	7.097	.0011	903.5
38	145.3	.0069	.1410	7.094	.0010	1031
39	165.7	.0060	.1409	7.100	.0009	1176
40	188.9	.0053	.1407	7.105	.0007	1342

* All formulae assume end-of-period payments.
P = a present sum of money; F = a future sum of money, equivalent to P at the end of n periods of time at discount rate of i; A = an end-of-period payment (or receipts) in a uniform series of payments (or receipts) over n periods at i interest rates.

Table 15 DISCRETE RATE-OF-RETURN FACTORS*

	Single Compound Amount Factor SCAF	Single Present Value Factor SPVF	Uniform Capital Recovery Factor UCRF	Uniform Present Value Factor UPVF	Uniform Sinking Fund Factor USFF	Uniform Compound Amount Factor UCAF
For-mula	$(1+i)^n$	$\dfrac{1}{(1+i)^n}$	$\dfrac{i(1+i)^n}{(1+i)^n-1}$	$\dfrac{(1+i)^n-1}{i(1+i)^n}$	$\dfrac{i}{(1+i)^n-1}$	$\dfrac{(1+i)^n-1}{i}$
Given	P	F	P	A	F	A
To Find	F	P	A	P	A	F
Period n	$i=15\%$					

n	SCAF	SPVF	UCRF	UPVF	USFF	UCAF
1	1.150	.8696	1.150	.8696	1.000	1.000
2	1.323	.7561	.6151	1.626	.4651	2.150
3	1.521	.6575	.4380	2.283	.2880	3.473
4	1.749	.5718	.3503	2.855	.2003	4.993
5	2.011	.4972	.2983	3.352	.1483	6.742
6	2.313	.4323	.2642	3.784	.1142	8.754
7	2.660	.3760	.240	4.160	.0904	11.07
8	3.059	.3269	.2230	4.487	.0729	13.73
9	3.518	.2843	.2096	4.772	.0596	16.79
10	4.046	.2472	.1993	5.019	.0493	20.30
11	4.652	.2149	.1911	5.234	.0411	24.35
12	5.350	.1869	.1845	5.421	.0345	29.00
13	6.153	.1625	.1791	5.583	.0291	34.35
14	7.076	.1413	.1747	5.724	.0297	40.50
15	8.137	.1229	.1710	5.847	.0210	47.58
16	9.358	.1069	.1679	5.954	.0180	55.72
17	10.76	.0929	.1654	6.047	.0154	65.08
18	12.38	.0808	.1632	6.128	.0132	75.84
19	14.23	.0703	.1613	6.198	.0113	88.21
20	16.37	.0611	.1598	6.259	.0098	102.4
21	18.82	.0531	.1584	6.312	.0084	118.8
22	21.64	.0462	.1573	6.359	.0073	137.6
23	24.89	.0402	.1563	6.399	.0063	159.3
24	28.63	.0349	.1554	6.434	.0054	184.2
25	32.92	.0304	.1547	6.464	.0047	212.8
26	37.86	.0264	.1541	6.491	.0041	245.7
27	43.54	.0230	.1535	6.514	.0035	283.6
28	50.07	.0200	.1531	6.534	.0031	327.1
29	57.58	.0174	.1527	6.551	.0027	377.2
30	66.21	.0151	.1523	6.566	.0023	434.7
31	76.14	.0131	.1520	6.579	.0020	501.0
32	87.57	.0114	.1517	6.591	.0017	577.1
33	100.7	.0099	.1515	6.600	.0015	664.7
34	115.8	.0086	.1513	6.609	.0013	765.4
35	133.2	.0075	.1511	6.617	.0011	881.2
36	153.2	.0065	.1510	6.623	.0010	1014
37	176.1	.0057	.1509	6.629	.0009	1168
38	202.5	.0049	.1507	6.634	.0007	1344
39	232.9	.0043	.1506	6.638	.0006	1546
40	267.9	.0037	.1506	6.642	.0006	1779

* All formulae assume end-of-period payments.

P = a present sum of money; F = a future sum of money, equivalent to P at the end of n periods of time at discount rate of i; A = an end-of-period payment (or receipts) in a uniform series of payments (or receipts) over n periods at i interest rates.

Table 18 DISCRETE RATE-OF-RETURN FACTORS*

	Single Compound Amount Factor SCAF	Single Present Value Factor SPVF	Uniform Capital Recovery Factor UCRF	Uniform Present Value Factor UPVF	Uniform Sinking Fund Factor USFF	Uniform Compound Amount Factor UCAF
Formula	$(1+i)^n$	$\dfrac{1}{(1+i)^n}$	$\dfrac{i(1+i)^n}{(1+i)^n-1}$	$\dfrac{(1+i)^n-1}{i(1+i)^n}$	$\dfrac{i}{(1+i)^n-1}$	$\dfrac{(1+i)^n-1}{i}$
Given	P	F	P	A	F	A
To Find	F	P	A	P	A	F
Period n				$i=18\%$		

n	SCAF	SPVF	UCRF	UPVF	USFF	UCAF
1	1.180	.8475	1.180	.8475	1.000	1.000
2	1.392	.7182	.6387	1.566	.4587	2.180
3	1.643	.6086	.4600	2.174	.2799	3.572
4	1.939	.5158	.3717	2.690	.1917	5.215
5	2.288	.4371	.3198	3.127	.1398	7.154
6	2.700	.3704	.2859	3.498	.1059	9.442
7	3.185	.3139	.2624	3.812	.0824	12.14
8	3.759	.2660	.2452	4.078	.0652	15.33
9	4.435	.2255	.2324	4.303	.0524	19.09
10	5.234	.1911	.2225	4.494	.0425	23.52
11	6.176	.1619	.2148	4.656	.0348	28.76
12	7.288	.1372	.2086	4.793	.0286	34.93
13	8.600	.1163	.2037	4.909	.0237	42.22
14	10.15	.0985	.1997	5.008	.0197	50.82
15	11.97	.0835	.1964	5.091	.0164	60.97
16	14.13	.0708	.1937	5.162	.0137	72.94
17	16.67	.0600	.1915	5.222	.0115	87.07
18	19.67	.0508	.1896	5.273	.0096	103.7
19	23.21	.0431	.1881	5.316	.0081	123.4
20	27.39	.0365	.1868	5.352	.0068	146.6
21	32.32	.0309	.1857	5.384	.0057	174.0
22	38.142	.0262	.1848	5.410	.0048	206.3
23	45.01	.0222	.1841	5.432	.0041	244.5
24	53.11	.0188	.1835	5.451	.0035	289.5
25	62.67	.01595	.1829	5.467	.0029	342.6
26	73.95	.0135	.1825	5.480	.0025	405.3
27	87.26	.0115	.1821	5.492	.0021	479.2
28	103.0	.0097	.1818	5.502	.0018	566.5
29	121.5	.0082	.1815	5.510	.0015	669.4
30	143.4	.0070	.1813	5.517	.0013	790.9
31	169.2	.0060	.1811	5.523	.0011	934.3
32	199.6	.0050	.1809	5.528	.0009	1103
33	235.6	.0042	.1808	5.532	.0008	1303
34	278.0	.0036	.1807	5.536	.0006	1538
35	328.0	.0030	.1806	5.539	.0006	1816
36	387.0	.0026	.1805	5.541	.0005	2144
37	456.7	.0022	.1804	5.543	.0004	2531
38	538.9	.0019	.1803	5.545	.0003	2988
39	635.9	.0016	.1803	5.547	.0003	3527
40	750.4	.0013	.1802	5.548	.0002	4163

* All formulae assume end-of-period payments.

P = a present sum of money; F = a future sum of money, equivalent to P at the end of n periods of time at discount rate of i; A = an end-of-period payment (or receipts) in a uniform series of payments (or receipts) over n periods at i interest rates.

Table 19 DISCRETE RATE-OF-RETURN FACTORS*

	Single Compound Amount Factor SCAF	Single Present Value Factor SPVF	Uniform Capital Recovery Factor UCRF	Uniform Present Value Factor UPVF	Uniform Sinking Fund Factor USFF	Uniform Compound Amount Factor UCAF
Formula	$(1+i)^n$	$\dfrac{1}{(1+i)^n}$	$\dfrac{i(1+i)^n}{(1+i)^n-1}$	$\dfrac{(1+i)^n-1}{i(1+i)^n}$	$\dfrac{i}{(1+i)^n-1}$	$\dfrac{(1+i)^n-1}{i}$
Given	P	F	P	A	F	A
To Find	F	P	A	P	A	F
Period n	$i=19\%$					

n						
1	1.190	.8403	1.190	.8403	1.000	1.000
2	1.416	.7062	.6467	1.547	.4567	2.190
3	1.685	.5934	.4673	2.140	.2773	3.606
4	2.005	.4987	.3790	2.639	.1890	5.291
5	2.386	.4190	.3271	3.058	.1371	7.297
6	2.840	.3521	.2933	3.409	.1033	9.683
7	3.379	.2959	.2699	3.706	.0799	12.52
8	4.021	.2487	.2529	3.954	.0629	15.90
9	4.785	.2190	.2402	4.163	.0502	19.92
10	5.695	.1756	.2305	4.339	.0405	24.71
11	6.777	.1476	.2229	4.487	.0329	30.40
12	8.064	.1240	.2169	4.611	.0269	37.18
13	9.596	.1042	.2121	4.715	.0221	45.24
14	11.42	.0876	.2082	4.802	.0182	54.84
15	13.59	.0736	.2051	4.876	.0151	66.26
16	16.17	.0618	.2025	4.938	.0125	79.85
17	19.24	.0520	.2004	4.990	.0104	96.02
18	22.90	.0437	.1987	5.033	.0087	115.27
19	27.25	.0367	.1972	5.070	.0072	138.2
20	32.43	.0308	.1960	5.101	.0060	165.4
21	38.59	.0259	.1951	5.127	.0051	197.8
22	45.92	.0218	.1942	5.149	.0042	236.4
23	54.65	.0183	.1935	5.167	.0035	282.4
24	65.03	.0154	.1930	5.182	.0030	337.0
25	77.39	.0129	.1925	5.195	.0025	402.0
26	92.09	.0109	.1921	5.206	.0021	479.4
27	109.6	.0091	.1918	5.215	.0017	571.5
28	130.4	.0077	.1915	5.223	.0015	681.1
29	155.2	.0064	.1912	5.229	.0012	811.5
30	184.7	.0054	.1910	5.235	.0010	966.7
31	219.8	.0046	.1909	5.239	.0009	1151
32	261.5	.0038	.1907	5.243	.0007	1371
33	311.2	.0032	.1906	5.246	.0006	1632
34	370.3	.0027	.1905	5.249	.0005	1943
35	440.7	.0023	.1904	5.251	.0004	2314
36	524.4	.0019	.1904	5.253	.0004	2754
37	624.1	.0016	.1903	5.255	.0003	3279
38	742.7	.0013	.1903	5.256	.0003	3903
39	893.8	.0011	.1902	5.257	.0002	4646
40	1051	.0009	.1902	5.258	.0002	5529

* All formulae assume end-of-period payments.

P = a present sum of money; F = a future sum of money, equivalent to P at the end of n periods of time at discount rate of i; A = an end-of-period payment (or receipts) in a uniform series of payments (or receipts) over n periods at i interest rates.

Table 20 DISCRETE RATE-OF-RETURN FACTORS*

	Single Compound Amount Factor SCAF	Single Present Value Factor SPVF	Uniform Capital Recovery Factor UCRF	Uniform Present Value Factor UPVF	Uniform Sinking Fund Factor USFF	Uniform Compound Amount Factor UCAF
For-mula	$(1+i)^n$	$\dfrac{1}{(1+i)^n}$	$\dfrac{i(1+i)^n}{(1+i)^n-1}$	$\dfrac{(1+i)^n-1}{i(1+i)^n}$	$\dfrac{i}{(1+i)^n-1}$	$\dfrac{(1+i)^n-1}{i}$
Given	P	F	P	A	F	A
To Find	F	P	A	P	A	F
Period n	$i=20$					

n						
1	1.200	.8333	1.200	.8333	1.000	1.000
2	1.440	.6944	.6545	1.528	.4545	2.200
3	1.728	.5787	.4747	2.106	.2747	3.640
4	2.074	.4823	.3863	2.589	.1863	5.368
5	2.488	.4019	.3344	2.991	.1344	7.442
6	2.986	.3349	.3007	3.326	.1007	9.930
7	3.583	.2791	.2774	3.605	.0774	12.92
8	4.300	.2326	.2606	3.837	.0606	16.50
9	5.160	.1938	.2481	4.031	.0481	20.80
10	6.192	.1615	.2385	4.192	.0385	25.96
11	7.430	.1346	.2311	4.327	.0311	32.15
12	8.916	.1122	.2253	4.439	.0253	39.58
13	10.70	.0935	.2206	4.533	.0206	48.50
14	12.84	.0779	.2169	4.611	.0169	59.20
15	15.41	.0649	.2139	4.675	.0139	72.04
16	18.49	.0541	.2114	4.730	.0114	87.44
17	22.19	.0451	.2094	4.775	.0094	105.9
18	26.62	.0376	.2078	4.812	.0078	128.1
19	31.95	.0313	.2065	4.844	.0065	154.7
20	38.34	.0261	.2054	4.870	.0057	186.7
21	46.01	.0217	.2044	4.891	.0044	225.0
22	55.21	.0181	.2037	4.909	.0037	271.0
23	66.25	.0151	.2031	4.925	.0031	326.2
24	79.50	.0126	.2025	4.937	.0025	392.5
25	95.40	.0104	.2021	4.948	.0021	472.0
26	114.4	.0087	.2018	4.956	.0018	567.4
27	137.4	.0073	.2015	4.964	.0015	681.9
28	164.8	.0061	.2012	4.970	.0012	819.2
29	197.8	.0051	.2010	4.975	.0010	984.1
30	237.4	.0042	.2008	4.979	.0008	1181
31	284.9	.0035	.2007	4.982	.0007	1419
32	341.8	.0029	.2006	4.985	.0006	1704
33	410.2	.0024	.2005	4.988	.0005	2046
34	492.2	.0020	.2004	4.990	.0004	2456
35	590.7	.0017	.2003	4.992	.0003	2948
36	708.8	.0014	.2003	4.993	.0003	3539
37	350.6	.0012	.2002	4.994	.0002	4248
38	1021	.0010	.2002	4.995	.0002	5098
39	1225	.0008	.2002	4.996	.0002	6119
40	1470	.0007	.2001	4.997	.0001	7344

* All formulae assume end-of-period payments.
P = a present sum of money; F = a future sum of money, equivalent to P at the end of n periods of time at discount rate of i; A = an end-of-period payment (or receipts) in a uniform series of payments (or receipts) over n periods at i interest rates.

Table 22   DISCRETE RATE-OF-RETURN FACTORS*

	Single Compound Amount Factor SCAF	Single Present Value Factor SPVF	Uniform Capital Recovery Factor UCRF	Uniform Present Value Factor UPVF	Uniform Sinking Fund Factor USFF	Uniform Compound Amount Factor UCAF
For-mula	$(1+i)^n$	$\dfrac{1}{(1+i)^n}$	$\dfrac{i(1+i)^n}{(1+i)^n-1}$	$\dfrac{(1+i)^n-1}{i(1+i)^n}$	$\dfrac{i}{(1+i)^n-1}$	$\dfrac{(1+i)^n-1}{i}$
Given	P	F	P	A	F	A
To Find	F	P	A	P	A	F
Period n	I=22%					

Period n	SCAF	SPVF	UCRF	UPVF	USFF	UCAF
1	1.220	.8197	1.220	.8197	1.000	1.000
2	1.488	.6719	.6705	1.492	.4505	2.220
3	1.816	.5507	.4897	2.042	.2697	3.708
4	2.215	.4514	.4010	2.494	.1810	5.524
5	2.703	.3700	.3492	2.864	.1292	7.740
6	3.297	.3033	.3158	3.167	.0958	10.44
7	4.023	.2486	.2928	3.416	.0729	13.74
8	4.908	.2038	.2763	3.619	.0563	17.76
9	5.987	.1670	.2641	3.786	.0441	22.67
10	7.306	.1369	.2549	3.923	.0349	28.66
11	8.912	.1122	.2478	4.035	.0278	35.96
12	10.87	.0920	.2423	4.127	.0223	44.87
13	13.26	.0754	.2379	4.203	.0180	55.75
14	16.18	.0618	.2345	4.265	.0145	69.01
15	19.74	.0507	.2317	4.315	.0117	85.19
16	24.09	.0415	.2295	4.357	.0095	104.9
17	29.38	.0340	.2278	4.391	.0078	129.0
18	35.85	.0279	.2263	4.419	.0063	158.4
19	43.74	.0229	.2251	4.442	.0051	194.3
20	53.36	.0187	.2242	4.460	.0042	238.0
21	65.10	.0154	.2234	4.476	.0034	291.3
22	79.42	.0126	.2228	4.488	.0028	356.4
23	96.89	.0103	.2223	4.499	.0023	435.9
24	118.2	.0085	.2219	4.507	.0019	532.6
25	144.2	.0069	.2215	4.514	.0015	651.0
26	175.9	.0057	.2213	4.520	.0013	795.2
27	214.6	.0047	.2210	4.524	.0010	971.0
28	261.9	.0038	.2208	4.528	.0008	1186
29	319.5	.0031	.2207	4.531	.0007	1448
30	399.8	.0026	.2206	4.534	.0006	1767
31	475.5	.0021	.2205	4.536	.0005	2157
32	580.1	.0017	.2204	4.538	.0004	2632
33	707.7	.0014	.2203	4.539	.0003	3212
34	863.4	.0012	.2203	4.540	.0003	3920
35	1053	.0009	.2202	4.541	.0002	4784
36	1285	.0008	.2202	4.542	.0002	5837
37	1568	.0006	.2201	4.543	.0001	7122
38	1913	.0005	.2201	4.543	.0001	8690
39	2334	.0004	.2201	4.544	.0001	10603
40	2847	.0004	.2201	4.544	.0001	12938

* All formulae assume end-of-period payments.

P = a present sum of money; F = a future sum of money, equivalent to P at the end of n periods of time at discount rate of i; A = an end-of-period payment (or receipts) in a uniform series of payments (or receipts) over n periods at i interest rates.

Table 23 DISCRETE RATE-OF-RETURN FACTORS*

	Single Compound Amount Factor SCAF	Single Present Value Factor SPVF	Uniform Capital Recovery Factor UCRF	Uniform Present Value Factor UPVF	Uniform Sinking Fund Factor USFF	Uniform Compound Amount Factor UCAF
Formula	$(1+i)^n$	$\dfrac{1}{(1+i)^n}$	$\dfrac{i(1+i)^n}{(1+i)^n-1}$	$\dfrac{(1+i)^n-1}{i(1+i)^n}$	$\dfrac{i}{(1+i)^n-1}$	$\dfrac{(1+i)^n-1}{i}$
Given	P	F	P	A	F	A
To Find	F	P	A	P	A	F
Period n	$i=23\%$					

n	SCAF	SPVF	UCRF	UPVF	USFF	UCAF
1	1.230	.8130	1.230	.8130	1.000	1.000
2	1.513	.6610	.6784	1.474	.4484	2.230
3	1.861	.5374	.4972	2.011	.2672	3.743
4	2.289	.4369	.4085	2.448	.1785	5.604
5	2.815	.3552	.3567	2.803	.1267	7.893
6	3.463	.2888	.3234	3.092	.0934	10.71
7	4.259	.2348	.3006	3.327	.0706	14.17
8	5.239	.1909	.2843	3.518	.05426	18.43
9	6.444	.1552	.2722	3.673	.0422	23.67
10	7.926	.1262	.2632	3.799	.0332	30.11
11	9.749	.1026	.2563	3.902	.026	38.04
12	11.99	.0834	.2509	3.985	.0209	47.79
13	14.75	.0678	.2467	4.053	.0167	59.78
14	18.14	.0551	.2434	4.108	.0134	74.53
15	22.31	.0448	.2408	4.153	.010	92.67
16	27.45	.0364	.2387	4.189	.0087	115.0
17	33.76	.0296	.2370	4.219	.0070	142.4
18	41.52	.0241	.2357	4.243	.0057	176.2
19	51.07	.0196	.2346	4.263	.0046	217.7
20	62.82	.0160	.2337	4.279	.0037	268.8
21	77.27	.0130	.2330	4.292	.0030	331.6
22	95.04	.0105	.2324	4.302	.0024	408.9
23	116.9	.0086	.2320	4.311	.0020	503.9
24	143.8	.0070	.2316	4.318	.0016	620.8
25	176.9	.0057	.2313	4.323	.0013	764.6
26	217.6	.0046	.2311	4.328	.0011	941.5
27	267.6	.0037	.2309	4.332	.0009	1159
28	329.1	.0030	.2307	4.335	.0007	1427
29	404.8	.0025	.2306	4.337	.0006	1756
30	497.9	.0020	.2305	4.339	.0005	2160
31	612.4	.0016	.2304	4.341	.0004	2658
32	753.3	.0013	.2303	4.342	.0003	3271
33	926.6	.0011	.2302	4.343	.0002	4024
34	1140	.0010	.2302	4.344	.0002	4951
35	1402	.0007	.2302	4.345	.0002	6090
36	1724	.0006	.2301	4.345	.0001	7492
37	2121	.0005	.2301	4.346	.0001	9216
38	2609	.0004	.2301	4.346	.0001	11337
39	3208	.0003	.2301	4.346	.0001	13946
40	3946	.0003	.2301	4.347	.0001	17154

* All formulae assume end-of-period payments.
P = a present sum of money; F = a future sum of money, equivalent to P at the end of n periods of time at discount rate of i; A = an end-of-period payment (or receipts) in a uniform series of payments (or receipts) over n periods at i interest rates.

Table 24 DISCRETE RATE-OF-RETURN FACTORS*

	Single Compound Amount Factor SCAF	Single Present Value Factor SPVF	Uniform Capital Recovery Factor UCRF	Uniform Present Value Factor UPVF	Uniform Sinking Fund Factor USFF	Uniform Compound Amount Factor UCAF
Formula	$(1+i)^n$	$\dfrac{1}{(1+i)^n}$	$\dfrac{i(1+i)^n}{(1+i)^n-1}$	$\dfrac{(1+i)^n-1}{i(1+i)^n}$	$\dfrac{i}{(1+i)^n-1}$	$\dfrac{(1+i)^n-1}{i}$
Given	P	F	P	A	F	A
To Find	F	P	A	P	A	F
Period n				$i = 24\%$		

1	1.240	.8065	1.240	.8065	1.000	1.000
2	1.538	.6504	.6864	1.457	.4464	2.240
3	1.907	.5245	.5047	1.981	.2647	3.778
4	2.364	.4230	.4159	2.404	.1759	5.684
5	2.932	.3411	.3642	2.745	.1242	8.048
6	3.635	.2751	.3311	3.020	.0911	10.98
7	4.508	.2218	.3084	3.242	.0684	14.62
8	5.590	.1789	.2923	3.421	.0523	19.12
9	6.931	.1443	.2805	3.566	.0405	24.71
10	8.59	.1164	.2716	3.682	.0316	31.64
11	10.66	.0938	.2649	3.776	.0249	40.24
12	13.21	.0757	.2596	3.851	.0198	50.90
13	16.39	.0610	.2556	3.912	.0156	64.11
14	20.32	.0492	.2524	3.962	.0124	80.50
15	25.20	.0397	.2499	4.001	.0100	100.8
16	31.24	.0320	.2479	4.033	.0080	126.0
17	38.74	.0258	.2464	4.059	.0064	157.3
18	48.04	.0208	.2451	4.080	.0051	196.0
19	59.57	.0168	.2441	4.097	.0041	244.0
20	73.86	.0135	.2433	4.110	.0033	303.6
21	91.59	.0109	.2426	4.121	.0027	377.5
22	113.6	.0088	.2421	4.130	.0021	469.1
23	140.8	.0071	.2417	4.137	.0017	582.6
24	174.6	.0057	.2414	4.143	.0014	723.5
25	216.5	.0046	.2411	4.147	.0011	898.1
26	268.5	.0037	.2409	4.151	.0009	1115
27	333.0	.0030	.2407	4.154	.0007	1383
28	412.9	.0024	.2406	4.157	.0006	1716
29	512.0	.0020	.2405	4.159	.0005	2129
30	634.8	.0016	.2404	4.160	.0004	2641
31	787.2	.0013	.2403	4.161	.0003	3276
32	976.1	.0010	.2402	4.162	.0002	4063
33	1210	.0008	.2402	4.163	.0002	5039
34	1501	.0007	.2402	4.164	.0002	6249
35	1861	.0005	.2401	4.164	.0001	7750
36	2308	.0004	.2401	4.165	.0001	9611
37	2862	.0003	.2401	4.165	.0001	11919
38	3548	.0003	.2401	4.165	.0001	14781
39	4400	.0002	.2401	4.166	.0001	18329
40	5456	.0002	.2400	4.166	.0000	22729

* All formulae assume end-of-period payments.

P = a present sum of money; F = a future sum of money, equivalent to P at the end of n periods of time at discount rate of i; A = an end-of-period payment (or receipts) in a uniform series of payments (or receipts) over n periods at i interest rates.

Table 25 DISCRETE RATE-OF-RETURN FACTORS*

	Single Compound Amount Factor SCAF	Single Present Value Factor SPVF	Uniform Capital Recovery Factor UCRF	Uniform Present Value Factor UPVF	Uniform Sinking Fund Factor USFF	Uniform Compound Amount Factor UCAF
Formula	$(1+i)^n$	$\dfrac{1}{(1+i)^n}$	$\dfrac{i(1+i)^n}{(1+i)^n-1}$	$\dfrac{(1+i)^n-1}{i(1+i)^n}$	$\dfrac{i}{(1+i)^n-1}$	$\dfrac{(1+i)^n-1}{i}$
Given	P	F	P	A	F	A
To Find	F	P	A	P	A	F
Period n	i = 25%					

n	SCAF	SPVF	UCRF	UPVF	USFF	UCAF
1	1.2500	.8000	1.250	.8000	1.000	1.000
2	1.563	.6400	.6944	1.440	.4444	2.250
3	1.953	.5126	.5123	1.952	.2623	3.813
4	2.441	.4096	.4234	2.362	.1734	5.766
5	3.052	.3277	.3718	2.689	.1218	8.207
6	3.815	.2621	.3388	2.951	.0889	11.26
7	4.768	.2097	.3163	3.161	.0663	15.07
8	5.961	.1678	.3004	3.329	.0504	19.84
9	7.451	.1342	.2888	3.463	.0388	25.80
10	9.313	.1074	.2801	3.570	.0301	33.25
11	11.64	.0859	.2735	3.656	.0235	42.57
12	14.55	.0687	.2684	3.725	.0184	54.21
13	18.19	.0550	.2645	3.780	.0145	68.76
14	22.74	.0440	.2615	3.824	.0115	86.95
15	28.42	.0352	.2591	3.859	.0091	109.7
16	35.53	.0281	.2572	3.887	.0072	138.1
17	44.41	.0225	.2558	3.910	.0059	173.6
18	55.51	.0180	.2546	3.928	.0046	218.0
19	69.39	.0144	.2537	3.942	.0037	273.6
20	86.74	.0115	.2529	3.954	.0030	342.9
21	108.4	.0092	.2523	3.963	.0023	429.7
22	135.5	.0074	.2519	3.970	.0019	538.1
23	169.4	.0060	.2515	3.976	.0015	673.6
24	211.8	.0047	.2512	3.981	.0012	843.0
25	264.7	.0038	.2509	3.985	.0010	1055
26	330.9	.0030	.2508	3.988	.0008	1319
27	413.6	.0024	.2506	3.990	.0006	1650
28	517.0	.0019	.2505	3.992	.0005	2064
29	646.2	.0015	.2504	3.994	.0004	3581
30	207.8	.0012	.2503	3.995	.0003	3227
31	1010	.0010	.2502	3.996	.0002	4035
32	1262	.0008	.2502	3.997	.0002	5045
33	1578	.0006	.2502	3.997	.0002	6307
34	1972	.0006	.2501	3.998	.0001	7885
35	2465	.0004	.2501	3.998	.0001	9857
36	3081	.0003	.2501	3.999	.0001	12322
37	3952	.0003	.2501	4.000	.0001	15403
38	4815	.0002	.2501	4.000	.0001	19255
39	6019	.0002	.2500	4.000	.0000	24070
40	7253	.0001	.2500	4.000	.0000	30089

* All formulae assume end-of-period payments.

P = a present sum of money; F = a future sum of money, equivalent to P at the end of n periods of time at discount rate of i; A = an end-of-period payment (or receipts) in a uniform series of payments (or receipts) over n periods at i interest rates.

APPENDIX B
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RECOMMENDED PRACTICE FOR MEASURING NET BENEFITS AND INTERNAL RATES OF RETURN
FOR INVESTMENTS IN BUILDINGS AND BUILDING SYSTEMS

5. AUTHOR(S)
Harold E. Marshall

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10. SUPPLEMENTARY NOTES

☐ Document describes a computer program; SF-185, FIPS Software Summary, is attached.

11. ABSTRACT *(A 200-word or less factual summary of most significant information. If document includes a significant bibliography or literature survey, mention it here)*

This report describes how to calculate net benefits (savings) and internal rates of
return and how to use them in selecting building designs and building systems that
will be cost effective over time. Net benefits are the difference between positive
benefits or cost reductions (savings) and project costs, measured in present value
or annual value dollars. The net benefits method is used to decide if a given
project is cost effective and which size or design for a given purpose is most cost
effective when no budget constraint exists. The internal rate of return on an
investment is the compound rate of interest that, when used to discount the streams
of dollar benefits (savings) and dollar costs over a defined study period, will make
the two equal. The internal rate of return is used to determine if a given project
is cost effective, to compare the relative cost effectiveness of different purpose
projects competing for a limited budget, and, when calculated on incremental changes
in benefits and costs, to evaluate which size or design for a given purpose is most
cost effective. The report describes formulas for calculating net benefits and
the internal rate of return, their applications in selecting cost-effective projects,
and limitations in their use. This recommended practice for measuring net benefits
and internal rates of return will assist the private and public building communities
in making cost-effective decisions in the design, operation, maintenance, and
retrofit of buildings.

12. KEY WORDS *(Six to twelve entries; alphabetical order; capitalize only proper names; and separate key words by semicolons)*
adjusted internal rate of return; benefit-cost analysis; building economics; building
economics standard; constrained optimization; cost effectiveness; discounting; economic
efficiency; internal rate of return; net benefits; net present value

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