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NORMAL FORM AND REPRESENTATION THEORY

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Normal form and representation theory

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Representation theory of Lie algebras is called upon to develop a procedure for normalizing a dynamical system with two degrees of freedom in the neighbourhood of an equilibrium when the Hamiltonian $H(x, y, X, Y)$ in the coordinates (x, y) and their conjugate momenta (X, Y) is of the type $H = (X^2 + Y^2)/2 + V(x, y, X, Y)$, the potential energy V being a sum of homogeneous polynomials in the phase variables of degree strictly greater than two. The fact that the resulting potential V' is a polynomial in the new coordinates (x', y') and the angular momentum $G' = x' Y' - y' X'$ implies that the normalization is a rotation in the configuration space from a fixed frame to an ideal frame. The technique is intended for normalizing an Hamiltonian in equilibrium at the origin when the Lie derivative associated with the quadratic part is not semi-simple, e.g. the planar Restricted Problem of Three Bodies at the equilateral equilibrium L_4 when the basic frequencies are equal (Routh's singular case).

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1. INTRODUCTION

The literature about normalization deals mainly with semi-simple systems in equilibrium at the origin. The Hamiltonian being a formal series

$$H \equiv H(x, y, X, Y) = \sum_{n\geq 0} \frac{1}{n!} H_n \qquad (1)$$

whose terms H_n are homogeneous polynomials of degree n + 2 in the coordinates (x, y) and their conjugate momenta (X, Y), it is generally assumed that the dominant term H_0 is a quadratic form reducible to the type

$$J = \frac{1}{2} (X^2 + \varepsilon_1 \omega_1^2 x^2) + \frac{1}{2} \varepsilon (Y^2 + \varepsilon_2 \omega_2^2 y^2) , \qquad (2)$$

in which the frequencies ω_1 and ω_2 are real and > 0, the factors ε, ε_1 and ε_2 being either +1 or -1. Such systems are called <u>semi-simple</u> because their dominant term leads to a linear Hamiltonian vector field that is semi-simple. Let

$$\underset{\sim}{L}_J : F \to \underset{\sim}{L}_J(F) = (F; J)$$

be the Lie derivative associated with the Hamiltonian linear vector field derived from J. For any n ≥ 2, the restriction of the differential operator

$$\underset{\sim}{L}_J = (X \frac{\partial}{\partial x} + \varepsilon_1 \omega_1^2 x \frac{\partial}{\partial X}) + \varepsilon (Y \frac{\partial}{\partial y} + \varepsilon_2 \omega_2^2 y \frac{\partial}{\partial Y})$$

to the vector space $\underset{\sim}{P}_n$ of homogeneous polynomials of degree n in (x, y, X, Y) is an endomorphism of $\underset{\sim}{P}_n$ that is semi-simple; hence, with respect to $\underset{\sim}{L}_J$, $\underset{\sim}{P}_n$ may be decomposed into the direct sum

$$\underset{\sim}{P}_n = \mathrm{Im}\ \underset{\sim}{L}_J + \mathrm{Ker}\ \underset{\sim}{L}_J \,. \tag{3}$$

The concept of normalization for semi-simple systems in equilibrium at the origin must be credited to Whittaker[1] who created it first by adapting a method proposed by Delaunay[2] for eliminating periodic terms from the main problem of Lunar Theory. Later, Whittaker[3] carried out the normalization as a canonical transformation (x, y, X, Y) → (x', y', X', Y') defined through the implicit equations

$$X = \frac{\partial S}{\partial x}, \qquad Y = \frac{\partial S}{\partial y}, \qquad x' = \frac{\partial S}{\partial X'}, \qquad y' = \frac{\partial S}{\partial Y'}$$

derived from a generating function

$$S \equiv S(x, y, X', Y') = \sum_{n \geq 0} S_n$$

where $S_0 = x X' + y Y'$ and, for any $n \geq 1$, the term S_n is a homogeneous polynomial of degree $n + 2$ in (x, y, X', Y'). Although devised by Poincaré[4] as one of his " méthodes nouvelles ", the procedure is referred to in some quarters of celestial mechanics as von Zeipel's method. Nowadays[5] the normalization is executed as a Lie transformation[6] in the sense of to convert the formal power series (1) into a formal power series

$$H' \equiv H'(x', y', X', Y') = \sum_{n \geq 0} \frac{1}{n!} H'_n \tag{4}$$

such that (i) $H'_0 = J(x', y', X', Y')$, and (ii) for each $n > 0$, H'_n belongs to the kernel of $\underset{\sim}{L}_J$. In action- and angle- variables (ϕ, ψ, Φ, Ψ), a polynomial in (x, y, X, Y) becomes a trigonometric sum in (ϕ, ψ), its component in Im $\underset{\sim}{L}_J$ consists of short-period terms whereas its component in Ker $\underset{\sim}{L}_J$ groups the terms which are either secular or of long period. The normalization is justified in this framework as a technique for removing short-period effects from the perturbation. The requirement that $(H'; J) = (H'_0; H') = 0$ implies at once that the dominant term H'_0 as a function of the normalizing variables is a (formal) integral of the system, hence calls for a reduction of the Hamiltonian system from two to one degree of freedom.

Whittaker[7] in his theory of Integration by Series, and most textbooks following him, considers exclusively semi-simple Hamiltonians in equilibrium. It must be realized[8] though that a non-degenerate quadratic form in (u, v, U, V) is reducible by a real symplectic linear transformation $(u, v, U, V) \to (x, y, X, Y)$ either to type (2) or to the type

$$J = \omega\,(xY - yX) - \frac{1}{2}\,\varepsilon\,\omega^2\,(x^2 + y^2). \tag{5}$$

In the second case, the Lie derivative

$$\underset{\sim}{L}_J = \omega\,\left(x \frac{\partial}{\partial X} - y \frac{\partial}{\partial Y}\right) + \omega\,\left[(Y - \varepsilon\omega x)\frac{\partial}{\partial X} - (X + \varepsilon\omega y)\frac{\partial}{\partial Y}\right] \tag{6}$$

will be decomposed into the sum

$$\underset{\sim}{L}_J = \omega \, \underset{\sim}{L}_G + \varepsilon \, \omega^2 \, \underset{\sim}{L}_D \tag{7}$$

where the differential operators

$$\underset{\sim}{L}_G = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} + X \frac{\partial}{\partial Y} - Y \frac{\partial}{\partial X} , \tag{8}$$

$$\underset{\sim}{L}_D = x \frac{\partial}{\partial X} + y \frac{\partial}{\partial Y} \tag{9}$$

are the Lie derivatives corresponding to the Hamiltonian linear vector fields derived from the functions

$$G = xY - yX \qquad \text{and} \qquad D = - \frac{1}{2} (x^2 + y^2). \tag{10}$$

The sum (7) realizes a Jordan decomposition of the endomorphism $\underset{\sim}{L}_J : \underset{\sim}{P}_n \to \underset{\sim}{P}_n$, that is, $\underset{\sim}{L}_G$ is semi-simple, $\underset{\sim}{L}_D$ is nilpotent, and these operators commute since

$$\underset{\sim}{L}_D \, \underset{\sim}{L}_G = \underset{\sim}{L}_G \, \underset{\sim}{L}_D = x \frac{\partial}{\partial Y} - y \frac{\partial}{\partial X} . \tag{11}$$

For any $n \geqslant 0$, the vector space $\underset{\sim}{P}_n$ turns out to be the direct sum

$$\underset{\sim}{P}_n = \text{Im} \, \underset{\sim}{L}_G \; + \; \text{Ker} \, \underset{\sim}{L}_G ,$$

...

and, on account of the commutativity relation (11), the restriction of $\mathbf{L}_D$ to the kernel of $\mathbf{L}_G$ is an endomorphism of Ker $\mathbf{L}_G$. Hence the normalization of a non semi-simple system in equilibrium at the origin could proceed in two steps. First a Lie transformation $\phi : (x, y, X, Y) \rightarrow (x', y', X', Y')$ normalizes the system with respect to the semi-simple component $\mathbf{L}_G$, thereby changing (1) into a power series (4) in the kernel of $\mathbf{L}_G$. In the new phase variables, the angular momentum $G' = x' Y' - y' X'$ is a (formal) integral; hence the term $\omega G'$ may be omitted from the dominant part in the transformed Hamiltonian, then reduced to $J' = \varepsilon \omega^2 D'$. Rather than analyzing the partially normalized problem as a system reduced to one degree of freedom by means of the integral of angular momentum, van der Meer[9] proposes that the normalization be continued with a Lie transformation $\psi : (x', y', X', Y') \rightarrow (x'', y'', X'', Y'')$ to convert (4) into a formal power series confined to a remarkable vector subspace of Ker $\mathbf{L}_G$. Indeed, relative to the Lie derivative

$$\mathbf{L}_K = X \frac{\partial}{\partial x} + Y \frac{\partial}{\partial y} \tag{12}$$

associated with the Hamiltonian vector field derived from

$$K = \frac{1}{2} (X^2 + Y^2) , \tag{13}$$

the kernel of $\mathbf{L}_G$ may be decomposed into the direct sum

$$\text{Ker } \mathbf{L}_G = (\text{Im } \mathbf{L}_D \quad \text{Ker } \mathbf{L}_G) \quad + \quad (\text{Ker } \mathbf{L}_K \qquad \text{Ker } \mathbf{L}_G) .$$

This makes it possible for the transformation ψ to convert (4) into a (formal) power series

$$H'' \equiv H''(x'', y'', X'', Y'') = \sum_{n \geqslant 0} \frac{1}{n!} H''_n$$

that belongs to Ker L_G Ker L_K. As a combined effect of the transformations ϕ and ψ, the kinetic energy $H''_0 = \varepsilon\,\omega^2\,D''$ and the angular momentum G'' in the third set of variables come out formally as integrals. Because they admit two independent integrals in involution, *non semi-simple systems with two degrees of freedom in equilibrium at the origin are (formally) integrable*.

van der Meer did not concern himself with developing an algorithm for generating the second normalization. As for ourselves, while engaged in designing such a procedure, we noticed that our techniques apply to a class of systems wider than the nilpotent part of a non semi-simple system in equilibrium at the origin. In fact exchanging the coordinates and the momenta transposes the Hamiltonian of the latter system into one whose dominant term H'_0 is K' instead of D'. Such Hamiltonians represent motions of a particle subject to weak perturbations in the neighbourhood of the origin. In that general context, the main contribution of this article is summed up in the following

Theorem .- Given a Hamiltonian system with two degrees of freedom represented by the formal series

$$H \equiv H(x, y, X, Y) = \frac{1}{2} (X^2 + Y^2) + \sum_{n \geq 1} \frac{1}{n!} H_n(x, y, X, Y) \tag{14}$$

where, for $n \geq 1$, the perturbation H_n is a homogeneous polynomial of degree $n + 2$, one may build formally a Lie transformation $(x, y, X, Y) \to (x', y', X', Y')$ to convert H into a formal power series

$$H' \equiv H'(x', y', X', Y') = \frac{1}{2} (X'^2 + Y'^2) + \sum_{n \geq 1} \frac{1}{n!} H'_n(x', y', X', Y')$$

where, for each $n \geq 1$, H'_n is in the kernel of L_D. More precisely,

$$H'_n = \sum_{\alpha+\beta+2\gamma=n+2} h_{\alpha,\beta,\gamma} \; x'^{\alpha} y'^{\beta} G'^{\gamma} \; .$$

The physical meaning of the normalization is exposed below in the corollary to the theorem. For the resulting potential

$$V' \equiv V'(x', y', G') = \sum_{n \geq 1} \frac{1}{n!} H'_n(x', y', G')$$

of the forces acting on the particle in the neighbourhood of the origin, let the differential be written as the 1-form

$$dV' = \partial_1 V' \, dx' + \partial_2 V' \, dy' + \partial_3 V' \, dG' \ .$$

Also, let (C) be the original Cartesian frame of reference in the configuration plane (x, y) with $\underset{\sim}{i}$ and $\underset{\sim}{j}$ standing for the orthonormal unit vectors in the directions of the reference axes. Finally let (I) designate the moving frame obtained by rotating (C) about the origin at the adjusted angular velocity $-\partial_3 V'$, $\underset{\sim}{i}'$ and $\underset{\sim}{j}'$ denoting the orthonormal unit vectors in the directions of the coordinate axes. In these notations

Corollary .- The particle's position $\underset{\sim}{x}$, its velocity $\underset{\sim}{v}$ and its acceleration $\underset{\sim}{a}$ relative to the frame (C) are such that

$$\underset{\sim}{x} = x \, \underset{\sim}{i} + y \, \underset{\sim}{j} = x' \, \underset{\sim}{i}' + y' \, \underset{\sim}{j}' \ , \tag{15}$$

$$\underset{\sim}{v} = X \, \underset{\sim}{i} + Y \, \underset{\sim}{j} = X' \, \underset{\sim}{i}' + Y' \, \underset{\sim}{j}' \ , \tag{16}$$

$$\underset{\sim}{a} = - \partial_1 V' \, \underset{\sim}{i}' - \partial_2 V' \, \underset{\sim}{j}' \ . \tag{17}$$

Thanks to the normalization, the motion in the frame (C) is decomposed into a rotation about the normal to the plane at the variable rate $-\partial_3 V'(x', y', G')$, and a motion with respect to the moving frame (I). In the latter frame, whereas one would have expected Coriolis forces and centrifugal repulsion to compensate for the slow rotation, one finds that the forces are reduced to the gradient of the force function $U' = -V$. Thus the moving frame (I) constitutes what Hansen calls an ideal frame, and the normalization may be regarded as a procedure to extract from Hamiltonian (14) the instantaneous rate at which, along each particular orbit, the frame (C) should be set in rotation so that it becomes the ideal frame proper to that particular orbit.

Symmetry Lie algebras provide a natural framework in which to consider the normalization of Hamiltonian systems. A case in point is the class of semi-simple systems in equilibrium which admit a 1:1 resonance. Credit for having discovered there the relationship between symmetry Lie algebras and normalization goes to Kummer[10]. As matter of fact, an algorithm can be set up to produce immediately the reduced Hamiltonian as a function over the Lie algebra su(2) spanned by the symmetry generators[11]. In that light, normalization of semi-simple systems in 1:1 resonance is but an application of the Reduction Theorem.[12] A similar situation occurs for perturbed two-dimensional Keplerian problems where the Delaunay normalization[13] builds the reduced Hamiltonian as a function over the Lie algebra so(3). The present article shows that, for systems of type (14), the connection between symmetry Lie algebras and normalization is equally decisive, but of a different nature. For the symmetry Lie algebra is here solvable -- and not semi-simple. However the Lie derivative $\underset{\sim}{L}_K$ may be embedded in a simple Lie algebra, namely $sl(2, \underset{\sim}{R})$, whose representation specifies the requirements imposed by the normalization.

2. SYMPLECTIC SYMMETRIES

The task of developing a normalization algorithm for nilpotent systems of type (14) when K is given by (13) begins with studying the linear infinitesimally symplectic symmetries for the Hamiltonian K. The Lie algebra $sp(4, \underset{\sim}{R})$ of all infinitesimally symplectic linear maps $(x, y, X, Y) \to (x', y', X', Y')$ is isomorphic to the Lie algebra of quadratic Hamiltonians under the Poisson bracket[14]: to the infinitesimally symplectic matrix w in $sp(4, \underset{\sim}{R})$ corresponds the quadratic form

$$W(x, y, X, Y) = -(Jw\, \underset{\sim}{v}, \underset{\sim}{v})$$

where

$$J = \begin{matrix} 0 & I \\ -I & 0 \end{matrix} \qquad \text{and} \qquad \underset{\sim}{v} = (x, y, X, Y),$$

in which case W is said to generate the infinitesimally symplectic linear map w. The latter is called a (linear infinitesimally symplectic) symmetry of K if there is a scalar ν such that $[w, k] = \nu k$ if k is the infinitesimally symplectic linear matrix generated by K, or equivalently

$$(W; K) = \nu K .$$

The definition adopted here for a symmetry is borrowed from Cartan[15]; it allows for a reparametrization of the integral curves of the vector field derived from K by the symmetries generated from W.

A quick evaluation shows that the infinitesimally linear symplectic symmetries form a five-dimensional solvable Lie algebra, a basis of which is generated by the sum

$$W = \varepsilon_1 W_1 + \varepsilon_2 W_2 + \varepsilon_3 W_3 + \varepsilon_4 W_4 + \varepsilon_5 W_5$$

whose terms are $W_1 = K$, $W_3 = G$,

$$W_2 = S = xX + yY, \qquad W_4 = \frac{1}{2}\,(X^2 - Y^2)\ , \qquad W_5 = XY\ .$$

In G, one recognizes the generator of the rotations about the origin. From the infinitesimal symmetry s defined by the equations

$$x' = x\ , \qquad y' = y\ ,$$
$$X' = -X\ , \qquad Y' = -Y$$

derived from S, the exponential mapping $e^{\varepsilon s}$ produces the symplectic symmetry

$$x' = e^{\varepsilon} X\ , \qquad y' = e^{\varepsilon} Y\ , \qquad X' = e^{-\varepsilon} X\ , \qquad Y' = e^{-\varepsilon} Y$$

in finite form. It corresponds to the similarity due to the homogeneity in dimensions : multiplying the coordinates by a constant λ (= e^{ε}) and dividing the velocities by the same constant requires, for preserving the dimensional homogeneity, that the time itself be multiplied by λ^2. This rule of similarity can also be verified by checking that the symplectic symmetry changes Cartan's form

$$\omega = X\, dx + Y\, dy - K\, dt$$

into the 1-form

$$\omega = X'\, dx' + Y'\, dy' - \frac{1}{2} \lambda^2 (X'^2 + Y'^2)\, dt ,$$

thus suggesting that the time t be replaced by the independent variable $t' = \lambda^2 t$.

The symmetry generators K, G, W_4 and W_5 are evidently integrals for the free particle. The function K and G are respectively the particle's energy and its angular momentum; the integrals W_4 and W_5 restate the principle of conservation of linear momentum since they combine to yield that $X + Y$ and $X - Y$, hence the components X and Y of the velocity, are integrals. The generator S itself is not an integral of the free particle. However the symmetry condition $(S; K) = 2K$ gives rise to the differential relation $\dot{S} = (S; K) = 2\, K$, and thus to the integral $S - 2\, K\, t = c_1$ from which, by yet another quadrature, is derived the so-called Jacobi integral

$$D + S\, t - K\, t^2 = c_0 .$$

The infinitesimal symmetries w_i derived from the generators W_i $(1 \leqslant i \leqslant 5)$ span a subalgebra in the linear Lie algebra $sp(4, \underset{\sim}{R})$ of 4×4 symplectic matrices. The family is completed into a basis w_i $(1 \leqslant i \leqslant 10)$

TABLE I of $sp(4, \underset{\sim}{R})$. Table I lists the generator and the non zero elements $w_{i,j,k}$ (in row j and column k) for each symplectic matrix w_i in the basis.

For the basis of $sp(4, \underset{\sim}{R})$ established in Table I, one computes easily (at least on a home computer) the commutators $[w_i, w_j] = w_i w_j - w_j w_i$ ($1 \leq i, j \leq 10$), the results of which have been entered in the respective i-th row and j-th column of Table II.

TABLE II

Basic to the normalization are the following facts which can be read immediately from Table II of the commutators.

a) Each element a of a Lie algebra A over a field F determines an endomorphism ad a : $b \to [a, b] : A \to A$ of the F-vector space. The element a is said to be ad-nilpotent if $(\text{ad } a)^m = 0$ for some integer $m > 0$. A quick calculation using Table II shows that $(\text{ad } w_1)^3 = 0$, or that the symmetry w_1 is ad-nilpotent in the linear algebra $sp(4, \underset{\sim}{R})$.

b) In a finite-dimensional semi-simple Lie algebra A over the real or the complex field, every ad-nilpotent element may be embedded in a subalgebra B of A isomorphic to the <u>special</u> linear Lie algebra $sl(2, \underset{\sim}{R})$ of 2×2 matrices with zero trace[16]. More precisely, there exists a pair (b, c) of elements in A such that

$$[a, b] = 2b \, , \qquad [a, c] = -2c \, , \qquad [b, c] = a \, .$$

In application of the embedding theorem to the ad-nilpotent matrix w_1 in $sp(4, \underset{\sim}{R})$, one reads from Table II that

$$[w_2, w_1] = 2w_1 \, , \qquad [w_2, w_6] = -2w_6 \, , \qquad [w_1, w_6] = w_2 \, . \tag{16}$$

c) Each matrix w_i in Table I is, by construction, the Hamiltonian linear vector field derived from the corresponding generator W_i. But, given two vector fields α and β, the Lie derivative of their commutator $[\alpha, \beta]$ is usually[17] defined as the differential operator

$$\underset{\sim}{L}_{[\alpha\beta]} = [\underset{\sim}{L}_\alpha, \underset{\sim}{L}_\beta] = \underset{\sim}{L}_\beta \underset{\sim}{L}_\alpha - \underset{\sim}{L}_\alpha \underset{\sim}{L}_\beta \, .$$

Hence the embedding relations (16) transposed in terms of Lie derivatives yield

Lemma 1.- $[\underset{\sim}{L}_S, \underset{\sim}{L}_K] = 2\underset{\sim}{L}_K$, $[\underset{\sim}{L}_S, \underset{\sim}{L}_D] = -2\underset{\sim}{L}_D$, $[\underset{\sim}{L}_K, \underset{\sim}{L}_D] = \underset{\sim}{L}_S$.

The lemma could have been obtained by evaluating long-hand the commutators of these Lie derivatives from their expressions as differential operators, or by evaluating the Poisson brackets $(W_i; W_j)$ since

$$[\underset{\sim}{L}_{W_i}, \underset{\sim}{L}_{W_j}] = \underset{\sim}{L}_{(W_i;\, W_j)} = \underset{\sim}{L}_{[w_j,\, w_i]} \, .$$

While an algebraist interprets the results in Lemma 1 as saying that $\underset{\sim}{L}_K$ and $\underset{\sim}{L}_D$ are eigenvectors of ad $\underset{\sim}{L}_S$ with eigenvalues respectively equal to 2 and -2, a physicist reads them as meaning that $\underset{\sim}{L}_S$ is a symmetry of the vector field derived from K that <u>shortens</u> the time along the integral curves of K while, as a symmetry of the vector field corresponding to D, it <u>slows down</u> the time along the integral curves. This interpretation brings forth a close analogy between the normalization proposed in this article for perturbed free particles and the conventional averaging procedures applied to conditionally periodic systems: in both cases, the algorithm removes the short term effects caused by the perturbations.

From Tables I and II, the reader may collect the following triples:

$$(xX;\ \frac{1}{2}X^2) = X^2\ , \qquad (xX;\ -\frac{1}{2}x^2) = x^2\ , \qquad (\frac{1}{2}X^2;\ -\frac{1}{2}x^2) = xX\ ;$$

$$(yY;\ \frac{1}{2}Y^2) = Y^2\ , \qquad (yY;\ -\frac{1}{2}Y^2) = y^2\ , \qquad (\frac{1}{2}Y^2;\ -\frac{1}{2}y^2) = yY\ ;$$

$$(xX+yY;\ XY) = 2XY\ , \qquad (xX+yY;\ -xy) = 2xy\ , \qquad (XY;\ -xy) = xX+yY\ ;$$

$$(xX-yY;\ Xy) = 2Xy\ , \qquad (xX-yY;\ xY) = -2xY\ , \qquad (Xy;\ xY) = xX-yY\ .$$

As will be seen in the next sections, to each of them corresponds a representation of the simple algebra $sl(2, \underset{\sim}{R})$ in the general linear algebra $gl(\underset{\sim}{P}_n)$ relative to the vector space $\underset{\sim}{P}_n$, hence a normalization scheme for a certain class of Hamiltonians. For example, the first triple concerns Hamiltonians of the type

$$H \equiv H(x,\ y,\ X,\ Y) = \frac{1}{2}X^2 + \sum_{n\geq 1} \frac{1}{n!} H_n(x,\ y,\ X,\ Y)$$

which may be normalized into Hamiltonians

$$H' \equiv H'(x',\ y',\ X',\ Y') = \frac{1}{2}X'^2 + \sum_{n\geq 1} \frac{1}{n!} H'_n(x',\ y',\ Y')$$

with the momentum X' eliminated from the perturbations.

3. DECOMPOSITION OF THE PERTURBATIONS

The objective of this section is to prove that any homogeneous polynomial p may be written in a unique way as the sum $p = p_K + p_D$ of two homogeneous polynomials of the same degree such that $\underset{\sim}{L}_D\, p_D = 0$ and $p_K = \underset{\sim}{L}_K\, q$ for some homogeneous polynomial q. It is to be noted that, in this section, $\underset{\sim}{L}_K$ and $\underset{\sim}{L}_D$ stand respectively for the restrictions of $\underset{\sim}{L}_K$ and $\underset{\sim}{L}_D$ to the vector space $\underset{\sim}{P}_n$. The decomposition leads to a procedure for constructing a Lie transformation which will strip any perturbation term H_n in (14) of its component in Im $\underset{\sim}{L}_K$.

Actually the decomposition of $\underset{\sim}{P}_n$ as the direct sum of Ker $\underset{\sim}{L}_D$ and Im $\underset{\sim}{L}_K$ is but a particular case of a more general result about the representations of the special linear Lie algebra sl(2, $\underset{\sim}{R}$). To get at the essence of the problem, we need however to fix notations and terminology, and to recall a few basics from Representation Theory.

Let V be a real vector space of finite dimension; in the algebra of endomorphisms of V, [a, b] denotes the commutator ab - ba. For readability, the image $\phi(x)$ of a vector x in V by an endomorphism ϕ will be written simply as ϕx. One says that V is an sl(2, $\underset{\sim}{R}$) - module or, equivalently that sl(2, $\underset{\sim}{R}$) is represented on V, if there exist three non-zero endomorphisms x, y, h of V satisfying the relations

$$[h, x] = 2x, \qquad [h, y] = -2y, \qquad [x, y] = h. \tag{17}$$

By virtue of Weyl's theorem, an sl(2, $\underset{\sim}{R}$) - module V may be decomposed into a direct sum of real vector subspaces

$$V = V_1 + V_2 + \ldots + V_s \tag{18}$$

whose summands are invariant and irreducible under the set of endomorphisms (x, y, h). There may be more than one way of accomplishing the decomposition, but the number s of summands and the equivalence classes of the irreducible representations are uniquely determined.

Proposition 1 .- *Relative to a real vector space* V *of finite dimension that is an* $sl(2, \mathbb{R})$ - *module for the three endomorphisms* x, y, h *satisfying the commutator relations* (17),

(i) *the endomorphisms* x *and* y *are nilpotent*;

(ii) *the endomorphisms* xy *and* yx *are semi-simple*;

(iii) $\mathrm{Ker}\, xy = \mathrm{Ker}\, y$ *and* $\mathrm{Im}\, xy = \mathrm{Im}\, x$; *likewise*, $\mathrm{Ker}\, yx = \mathrm{Ker}\, x$ *and* $\mathrm{Im}\, yx = \mathrm{Im}\, y$;

(iv) $V = \mathrm{Ker}\, x + \mathrm{Im}\, y$; *likewise*, $V = \mathrm{Ker}\, y + \mathrm{Im}\, x$;

(v) $\dim \mathrm{Ker}\, x = \dim \mathrm{Ker}\, y = s$ *where* s *is the number of summands in any decomposition of* V *into a direct sum of irreducible vector subspaces*.

The proposition is proved in two stages. First is considered the particular case when $s = 1$, that is, V itself is irreducible; then the results are extended to the general case where $s > 1$.

When it is irreducible, V admits, according to Humphreys[18], a basis $(v_0, v_1, \ldots, v_m)$ such that, for $0 \leq i \leq m$,

$$hv_i = (m-2i)\; v_i\,, \tag{19}$$

$$yv_i = (i+1)\; v_{i+1}\,, \tag{20}$$

$$xv_i = (m-i+1)\; v_{i-1}\,, \tag{21}$$

with the convention that $v_{-1} = v_{m+1} = 0$. As a consequence, $x^{m-1}v_i = y^{m-1} v_i = 0$, which proves that the endomorphisms x and y are nilpotent. There follows also that $(xy)v_i = (i+1)(m-i)v_i$; hence the vectors (v_i) $(0 \leq i \leq m)$ are a basis of eigenvectors for the endomorphism xy (and likewise for the endomorphism yx), which means that xy and yx are semi-simple. For that reason, $V = \mathrm{Ker}\, xy + \mathrm{Im}\, xy$, and also $V = \mathrm{Ker}\, yx + \mathrm{Im}\, yx$. More precisely, since, on the one hand, $(xy)v_i = 0$ if and only $i = m$, and, on the other hand, $v_i = (xy)[(i+1)^{-1}(m-i)^{-1}v_i]$ for $0 \leq i \leq m-1$, it turns out that Ker xy is the one-dimensional vector subspace generated by v_m, while Im xy is the m-dimensional vector subspace generated by the vectors v_i $(0 \leq i \leq m-1)$. In view of this decomposition of V, statements (iv) and (v) in the proposition are immediate corollaries of statement (iii). There remains thus to prove point (iii). To this end, observe that, on account of relation (21), v_i is in the image of x if and only if $0 \leq i < m$, while $xv_m = 0$; therefore, being the vector subspace generated by the vectors v_i $(0 \leq i \leq m-1)$, Im x is identical to Im xy. Similarly, by virtue of (20), a linear combination $a_0 v_0 + a_1 v_1 + \ldots + a_m v_m$ is in the kernel of y if and only if $a_i = 0$ for $0 \leq i \leq m-1$; hence, being the one-dimensional subspace generated by v_m, Ker y is identical to Ker xy.

The proof of Proposition 1 in the general case where V is completely reducible although not irreducible rests on decomposition (18) of V into a direct sum of irreducible submodules. It has just been proved that the restrictions of x and y to each of the summands are nilpotent, and that the restrictions of xy and yx are semi-simple; hence the endomorphisms x and y themselves are nilpotent in V, whereas the endomorphisms xy and yx are semi-simple in V. Demonstration of assertions (iii) - (v) involves the following

Lemma 2.- *Let* V *be a vector space that is a direct sum of the subspaces* (V_i) $(1 \leqslant i \leqslant n)$. *Then, for any vector subspace* W *of* V, *the following statements are equivalent:*

(i) W *is the direct sum of the vector subspaces* $(W \quad V_i)$ $(1 \leqslant i \leqslant n)$;

(ii) *For any element* w *in* W, *the decomposition* $w = w_1 + w_2 + \ldots + w_n$ *such that* w_i *belongs to* V_i *for* $1 \leqslant i \leqslant n$ *implies that each component* w_i *belongs to* W.

Intuitively speaking, the lemma says that W is the direct sum of its intersections with the V_i's if and only if the components of any vector in W along every "direction" V_i lie in W. Elementary as it may be, this lemma is not mentioned in the major textbooks in Linear Algebra; it is therefore in order to skecth its proof. Considering that V is the direct sum of the subspaces V_i, any element w in W may be decomposed into the sum $w = w_1 + w_2 + \ldots + w_n$ with w_i in V_i for each i. But, assuming that (i) holds, the same element w may be decomposed into the sum $w = u_1 + u_2 + \ldots + u_n$ with u_i in $W \quad V_i$ for each i. The decomposition of w in V being unique, there follows that $w_i = u_i$, hence that w_i belongs to W for each i, which shows that (i) implies (ii). Conversely, if (ii) holds, then w_i belongs to $W \quad V_i$ for each i. Such a decomposition being unique, there results that W is the direct sum of the subspaces $W \quad V_i$, hence that (ii) implies (i).

Lemma 2 is used to prove that

$$\mathrm{Im}\ z = (V_1 \quad \mathrm{Im}\ z) + (V_2 \quad \mathrm{Im}\ z) + \ldots + (V_s \quad \mathrm{Im}\ z), \qquad (22)$$

$$\mathrm{Ker}\ z = (V_1 \quad \mathrm{Ker}\ z) + (V_2 \quad \mathrm{Ker}\ z) + \ldots + (V_s \quad \mathrm{Ker}\ z) \qquad (23)$$

when the endomorphism z is either x, y, xy or yx. Indeed, take w in Im z; in view of Weyl's theorem, it may be decomposed into a sum $w = w_1 + w_2 + \ldots + w_s$ where w_i belongs to V_i for $1 \leqslant i \leqslant s$. But $w = zv$ for some v in V which in turn may be decomposed into a sum $v = v_1 + v_2 + \ldots + v_s$ with v_i in V_i for $1 \leqslant i \leqslant s$; therefore $zv = zv_1 + zv_2 + \ldots + zv_s$. Now, since z leaves each vector subspace V_i invariant and since the decomposition of w into its components in the V_i's is unique, there follows that $w_i = zv_i$ for $1 \leqslant i \leqslant s$, or that each component lies in Im z, hence formula (22) on account of Lemma 2. Similarly, for w in Ker z, there results that $0 = zw = zw_1 + zw_2 + \ldots + zw_s$, hence that $zw_i = 0$ for $1 \leqslant i \leqslant n$, since $0 = 0 + \ldots + 0$ is the unique way of decomposing the null vector in the direct sum (18). Because each component w_i belongs to Ker z, formula (23) results also from Lemma 2.

Proposition 1 having been proved when V is irreducible, there follows that Im $xy|V_i$ = Im $x|V_i$ and Ker $xy|V_i$ = Ker $y|V_i$ for $1 \leqslant i \leqslant s$. Then, by reason of the relations (22)-(23), one concludes that Im xy = Im x and Ker xy = Ker y, which thus proves statement (iii) in Proposition 1. The next statement is then a consequence of the fact that the endomorphism xy is semi-simple. Finally, since Ker $x|V_i$ and Ker $y|V_i$ are 1-dimensional for $1 \leqslant i \leqslant s$, one concludes from formula (23) that Ker x and Ker y are of dimension s. This completes the demonstration of Proposition 1.

Corollary - The vector space $\underset{\sim}{P}_n$ is the direct sum Im $\underset{\sim}{L}_K$ + Ker $\underset{\sim}{L}_D$. ●●●

If V is an $sl(2, \underset{\sim}{R})$ - module, the real number λ is called a weight of the representation when the vector subspace V_λ of elements x such that $hx = \lambda x$ is not the null space. In Representation Theory, it is proved that the number s of summands in the direct decomposition (18) is equal to dim V_0 + dim V_1. In the present case, the semi-simple endomorphism involved in the representation of $sl(2, \mathbb{R})$ is the differential operator

$$L_S = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} - X \frac{\partial}{\partial X} - Y \frac{\partial}{\partial Y} .$$

From now on, in order to disencumber the notations, monomials like $x^\alpha y^\beta X^\gamma Y^\delta$ will be denoted $e(\alpha, \beta, \gamma, \delta)$. In those terms,

$$L_S \; e(\alpha, \beta, \gamma, \delta) = (\alpha + \beta - \gamma - \delta) \; e(\alpha, \beta, \gamma, \delta) \; ;$$

hence the monomials $e(\alpha, \beta, \gamma, \delta)$ constitute a basis of the vector subspace V_k of P_n if and only if

$$\alpha + \beta + \gamma + \delta = n \qquad \text{and} \qquad \alpha + \beta - \gamma - \delta = k.$$

Therefore, when $n = 2m$, V_1 reduces to the null space whereas $\dim V_0 = (m+1)^2$; otherwise, when $n = 2m+1$, it is V_0 that is reduced to the null space while $\dim V_1 = (m+1)(m+2)$. Hence

Proposition 2.- The dimension of Ker L_D *in* P_n *is equal to* $(m+1)^2$ *when* $n = 2m$, *and to* $(m+1)(m+2)$ *when* $n = 2m + 1$.

By consulting Table III, the reader will gain a measure of appreciation for the extent to which the normalization simplifies the perturbed system (14). An arbitrary polynomial that is homogeneous of degree n in (x, y, X, Y) is the sum of $\binom{n+3}{3}$ monomials. Thus, from degree n to degree (n+1), the perturbation term H_n grows in complexity by $(n+2)(n+3)/2$ terms; by contrast, Ker $\underset{\sim}{L}_D$ increases only by $1 + (n+1)/2$ terms.

As the next section will indicate, there is however more to the normalization than a drastic reduction in algebraic complexity.

4. THE ANGULAR MOMENTUM INSIDE THE PERTURBATION

The decomposition vouchsafed by the corollary to Proposition 1 owes its physical interest to the algebraic nature of the kernel of $\underset{\sim}{L}_D$ as one can see from

Proposition 3.- *With* $\alpha+\beta+2\gamma = n$, *the polynomials* $g(\alpha, \beta, \gamma) = x^\alpha y^\beta G^\gamma$ *form a basis of* Ker $\underset{\sim}{L}_D$ *in* $\underset{\sim}{P}_n$, *and the polynomials* $G(\alpha, \beta, \gamma) = X^\alpha Y^\beta G^\gamma$, *a basis of* Ker $\underset{\sim}{L}_K$ *in* $\underset{\sim}{P}_n$.

The proof of Proposition 3 rests on yet another decomposition of $\underset{\sim}{P}_n$ into a direct sum of vector subspaces which will be detailed first. In what follows, most of the time, a monomial $e(\alpha, \beta, \gamma, \delta)$ will be identified with the quadruple $(\alpha, \beta, \gamma, \delta)$ of its exponents. Clearly, an arbitrary quadruple $(\alpha, \beta, \gamma, \delta)$ of integers represents a monomial if and only if its elements are non-negative, in which case the monomial belongs to $\underset{\sim}{P}_n$ if and only if $\alpha + \beta + \gamma + \delta = n$; let $\underset{\sim}{E}_n$ be the set of all quadruples satisfying these two conditions. The relation $\underset{\sim}{R}$ defined by

$$(\alpha', \beta', \gamma', \delta')\ \underset{\sim}{R}\ (\alpha, \beta, \gamma, \delta)$$
$$\text{\underline{if there exists an integer} } k \text{ \underline{such that}}$$
$$(\alpha', \beta', \gamma', \delta') = (\alpha, \beta, \gamma, \delta) + (-k, k, k, -k)$$

is an equivalence among quadruples. For any quadruple $(\alpha, \beta, \gamma, \delta)$, let $E(\alpha, \beta, \gamma, \delta)$ designate the intersection with $\underset{\sim}{E}_n$ of the class of quadruples equivalent to $(\alpha, \beta, \gamma, \delta)$ *modulo* $\underset{\sim}{R}$; given the integer k, the quadruple $(\alpha-k, \beta+k, \gamma+k, \delta-k)$ is an element of $E(\alpha, \beta, \gamma, \delta)$ if and only if

$$\alpha-k \geqslant 0, \qquad \beta+k \geqslant 0, \qquad \gamma+k \geqslant 0, \qquad \delta-k \geqslant 0,$$

or, equivalently, if and only if

$$-\min(\beta, \gamma) \leqslant k \leqslant \min(\alpha, \delta);$$

this shows in particular that the number of quadruples in the equivalence class $E(\alpha, \beta, \gamma, \delta)$ is equal to $1 + \min(\alpha, \delta) + \min(\beta, \gamma)$. The natural ordering on the integers k provides an ordering on $E(\alpha, \beta, \gamma, \delta)$. In that order, the lowest element in $E(\alpha, \beta, \gamma, \delta)$ is of the form $(\alpha_0, \beta_0, \gamma_0, \delta_0)$ with $\beta_0\gamma_0 = 0$. Repeated addition of $(-1, 1, 1, -1)$ to the lowest element produces all the quadruples in the equivalence class $E(\alpha_0, \beta_0, \gamma_0, \delta_0)$ up to the highest element which is of the form $(\alpha_1, \beta_1, \gamma_1, \delta_1)$ with $\alpha_1\delta_1 = 0$.

Now let $P(\alpha, \beta, \gamma, \delta)$ denote the vector space generated by those monomials whose exponents belong to the class $E(\alpha, \beta, \gamma, \delta)$. The monomials in $\underset{\sim}{E}_n$ being a basis of $\underset{\sim}{P}_n$, the preceding discussion establishes that $\underset{\sim}{P}_n$ is the direct sum of the vector subspaces $P(\alpha, \beta, \gamma, \delta)$, and that $\dim P(\alpha, \beta, \gamma, \delta) = \min(\alpha, \delta) + \min(\beta, \gamma) + 1$. However the vector subspaces $P(\alpha, \beta, \gamma, \delta)$ are not in general invariant under the operators $\underset{\sim}{L}_K$ and $\underset{\sim}{L}_D$; as a matter of fact, the way in which these operators act on $P(\alpha, \beta, \gamma, \delta)$ is given in the next statement.

Lemma 3.- <u>For any</u> $(\alpha, \beta, \gamma, \delta)$ <u>in</u> $\underset{\sim}{E}_n$ <u>and with the convention that</u> $P(\alpha, \beta, \gamma, \delta)$ <u>designates the null vector space when</u> $E(\alpha, \beta, \gamma, \delta)$ <u>is empty</u>,

(i) $\underset{\sim}{L}_K$ <u>maps</u> $P(\alpha, \beta, \gamma, \delta)$ <u>into</u> $P(\alpha-1, \beta, \gamma+1, \delta)$;

(ii) $\underset{\sim}{L}_D$ <u>maps</u> $P(\alpha, \beta, \gamma, \delta)$ <u>into</u> $P(\alpha+1, \beta, \gamma-1, \delta)$;

(iii) $\underset{\sim}{L}_D\underset{\sim}{L}_K$ <u>maps</u> $P(\alpha, \beta, \gamma, \delta)$ <u>into itself</u>.

Indeed, for any monomial $e(\alpha, \beta, \gamma, \delta)$,

$$\underset{\sim}{L}_K\, e(\alpha,\ \beta,\ \gamma,\ \delta) = \alpha\ e(\alpha-1,\ \beta,\ \gamma+1,\ \delta) + \beta\ e(\alpha,\ \beta-1,\ \gamma,\ \delta+1) \qquad (24)$$

by virtue of definition (12). But $(\alpha-1,\ \beta,\ \gamma+1,\ \delta) = (\alpha,\ \beta-1,\ \gamma,\ \delta+1) + (-1,\ 1,\ 1,\ -1)$, hence both monomials in the right-hand member of (24) are equivalent modulo $\underset{\sim}{R}$, and (i) is thus proved. By an analogous argument, (ii) is a consequence of the identity

$$\underset{\sim}{L}_D\, e(\alpha,\ \beta,\ \gamma,\ \delta) = \gamma\ e(\alpha+1,\ \beta,\ \gamma-1,\ \delta) + \delta\ e(\alpha,\ \beta+1,\ \gamma,\ \delta-1) \qquad (25)$$

resulting from definition (9). Then (iii) follows by composing (i) and (ii).

Against this background information, Propsition 3 will now be proved, but for $\underset{\sim}{L}_K$ only, since the result for $\underset{\sim}{L}_D$ follows from swapping the coordinates $(x,\ y)$ and the momenta $(X,\ Y)$. Because $\underset{\sim}{L}_K$ is a derivation, the relations $\underset{\sim}{L}_K\ X = \underset{\sim}{L}_K\ Y = \underset{\sim}{L}_K\ (xY-yX) = 0$ imply that $\underset{\sim}{L}_K\ G(\alpha,\ \beta,\ \gamma) = 0$; hence there remains to show that the polynomials $G(\alpha,\ \beta,\ \gamma)$ are linearly independent and that they span Ker $\underset{\sim}{L}_K$. On the one hand, the binomial expansion

$$X^\alpha Y^\beta (xY - yX)^\gamma = X^\alpha Y^\beta \sum_{0\leqslant k\leqslant\gamma} (-1)^k \binom{\gamma}{k} x^{\gamma-k} y^k X^k Y^{\gamma-k}$$

$$= \sum_{0\leqslant k\leqslant\gamma} (-1)^k \binom{\gamma}{k} e(\gamma-k,\ k,\ \alpha+k,\ \beta+\gamma-k)$$

shows that $G(\alpha,\ \beta,\ \gamma)$ belongs to the vector subspace $P(\gamma,\ 0,\ \alpha,\ \beta+\gamma)$. The equivalence classes $E(\gamma,\ 0,\ \alpha,\ \beta+\gamma)$ and $E(\gamma',\ 0,\ \alpha',\ \beta'+\gamma')$ being disjoint when $(\alpha, \beta, \gamma, \delta) \neq (\alpha', \beta', \gamma', \delta')$, distinct polynomials $G(\alpha, \beta, \gamma)$ belong to distinct vector subspaces $P(\gamma, 0, \alpha, \beta+\gamma)$, which means that the polynomials $G(\alpha, \beta, \gamma)$ are linearly independent.

On the other hand, take a polynomial p in P_n, and let $p = \sum_\iota p_\iota$ be its decomposition relative to the subspaces $P(\alpha, \beta, \gamma, \delta)$. Then $L_K\, p = \sum_\iota L_K\, p_\iota$; Since, according to Lemma 3, for $\iota \neq \kappa$, the images $L_K\, p_\iota$ and $L_K\, p_\kappa$ belong to distinct subspaces $P(\alpha, \beta, \gamma, \delta)$, the relation $L_K\, p = 0$ implies that $L_K\, p_\iota = 0$ for each index ι. There results by reason of Lemma 2 that Ker L_K is the direct sum of the vector subspaces Ker L_K $P(\alpha, \beta, \gamma, \delta)$. As will be seen, all of these summands are identical to the null subspace save the intersections Ker L_K $P(\gamma, 0, \alpha, \beta+\gamma)$ which are of dimension one and are in fact generated by a polynomial $G(\alpha, \beta, \gamma)$ (see Lemma 4 below). This will prove that such special polynomials span the kernel of L_K, and therefore constitute a basis of Ker L_K as is announced in Proposition 3. There remains thus to examine the trace of Ker L_K on each vector subspace $P(\alpha, \beta, \gamma, \delta)$, which will be done by studying in detail the action of L_K on each of them. Such an analysis will prove useful also in Section 5 where the normalization algorithm will be developed; it will show in particular that the decomposition of a polynomial into its components in Im L_K and Ker L_D reduces to inverting a few matrices of very low dimension.

TABLE IV

Depending on the type presented by the lowest element $(\alpha_0, \beta_0, \gamma_0, \delta_0)$ in $P(\alpha, \beta, \gamma, \delta)$, the five cases mentioned in Table IV have to be considered. It is a question of setting proper bases for representing the linear map L_K : $P(\alpha, \beta, \gamma, \delta) \rightarrow P(\alpha-1, \beta, \gamma+1, \delta)$ as a matrix. To this end, in each case, the monomials ordered from lowest to highest are adopted as a basis, the coefficients of a polynomial are regarded as a column vector, and the operator

$\underset{\sim}{L}_K$ is represented as a matrix acting by multiplication to the left.

Class I can be disposed of at once. For it is clear that $P(0, 0, \gamma_0, \delta_0)$ is of dimension 1, being generated as it is by the monomial $X^{\gamma_0}Y^{\delta_0}$, and that it is mapped by $\underset{\sim}{L}_K$ onto the null space.

In class II, the monomials

$$e(\alpha_0, 0, \gamma_0, \delta_0), \ldots, e(0, \alpha_0, \gamma_0+\alpha_0, \delta_0-\alpha_0) \tag{26}$$

form a basis of $P(\alpha_0, 0, \gamma_0, \delta_0)$, and the monomials

$$e(\alpha_0-1, 0, \gamma_0+1, \delta_0), \ldots, e(0, \alpha_0-1, \gamma_0+\alpha_0, \delta_0-\alpha_0+1), \tag{27}$$

a basis of $P(\alpha_0-1, 0, \gamma_0+1, \delta_0)$. Restricted to $P(\alpha_0, 0, \gamma_0, \delta_0)$, $\underset{\sim}{L}_K$ is, by virtue of (24), represented by the rectangular band matrix

$$\begin{bmatrix} \alpha_0 & 1 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & \alpha_0-1 & 2 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 0 & \alpha_0-2 & \cdots & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \cdots & \alpha_0-2 & 0 & 0 \\ 0 & 0 & 0 & \cdots & \cdots & 2 & \alpha_0-1 & 0 \\ 0 & 0 & 0 & \cdots & \cdots & 0 & 1 & \alpha_0 \end{bmatrix} \tag{28}$$

The first α_0 columns of (28) constitute a square matrix whose determinant (= $\alpha_0!$) is not zero. Hence (28) is of rank α_0 and of nullity 1, which means that

Ker $\underset{\sim}{L}_K$ $P(\alpha_0, 0, \gamma_0, \delta_0)$ is generated by the special polynomial $G(\gamma_0, \delta_0-\alpha_0,$ ••• $\alpha_0)$.

In class III, the restriction of $\underset{\sim}{L}_K$ corresponds to the band matrix

$$\begin{bmatrix} \alpha_0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \alpha_0-1 & 2 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \alpha_0-2 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \alpha_0-\delta_0+2 & \delta_0-1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \alpha_0-\delta_0+1 & \delta_0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \alpha_0-\delta_0 \end{bmatrix} \tag{29}$$

when the vectors

$$e(\alpha_0, 0, \gamma_0, \delta_0), \ldots, e(\alpha_0-\delta_0, \delta_0, \gamma_0+\delta_0, 0) \tag{30}$$

are choosen as a basis in $P(\alpha_0, 0, \gamma_0, \delta_0)$ while the vectors

$$e(\alpha_0-1, 0, \gamma_0+1, \delta_0), \ldots, e(\alpha_0-\delta_0-1, \delta_0, \gamma_0+\delta_0+1, 0) \tag{31}$$

are taken for the basis of $P(\alpha_0-1, 0, \gamma_0+1, \delta_0)$. Matrix (29) is square, and it is clearly non-singular. Therefore $\underset{\sim}{L}_K$ is an isomorphism of $P(\alpha_0, 0, \gamma_0, \delta_0)$ onto $P(\alpha_0-1, 0, \gamma_0+1, \delta_0)$.

Now, in class IV, the bases in $P(\alpha_0, \beta_0, 0, \delta_0)$ and $P(\alpha_0, \beta_0-1, 0, \delta_0+1)$ are chosen to be respectively

$$e(\alpha_0,\ \beta_0,\ 0,\ \delta_0),\ \ldots,\ e(0,\ \beta_0+\alpha_0,\ \alpha_0,\ \delta_0-\alpha_0) \tag{32}$$

and

$$e(\alpha_0,\ \beta_0-1,\ 0,\ \delta_0+1),\ \ldots,\ e(0,\ \beta_0+\alpha_0-1,\ \alpha_0,\ \delta_0-\alpha_0+1), \tag{33}$$

so that the square matrix representing the restriction of $\underset{\sim}{L}_K$ is

$$\begin{bmatrix} \beta_0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \alpha_0 & \beta_0+1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \alpha_0-1 & \beta_0+2 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \beta_0+\alpha_0-2 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 2 & \beta_0+\alpha_0-1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & \beta_0+\alpha_0 \end{bmatrix} \tag{34}$$

which is obviously non-singular.

Finally, in class V, the monomials

$$e(\alpha_0,\ \beta_0,\ 0,\ \delta_0),\ \ldots,\ e(\alpha_0-\delta_0,\ \beta_0+\delta_0,\ \delta_0,\ 0) \tag{35}$$

are chosen as the basis in $P(\alpha_0,\ \beta_0,\ 0,\ \delta_0)$, while the monomials

$$e(\alpha_0,\ \beta_0-1,\ 0,\ \delta_0+1),\ \ldots,\ e(\alpha_0-\delta_0-1,\ \beta_0+\delta_0,\ \delta_0+1,\ 0) \tag{36}$$

form the basis in $P(\alpha_0,\ \beta_0-1,\ 0,\ \delta_0+1)$. In this way, the operator $\underset{\sim}{L}_K$ is given by the band matrix

$$\begin{bmatrix} \beta_0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \alpha_0 & \beta_0+1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \alpha_0-1 & \beta_0+2 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \alpha_0-\delta_0+2 & \beta_0+\delta_0-1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \alpha_0-\delta_0+1 & \beta_0+\delta_0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \alpha_0-\delta_0 \end{bmatrix} \tag{37}$$

The first $(\delta_0 + 1)$ rows of (37) have the non-zero product $\beta_0\ (\beta+1)\ \ldots\ (\beta_0+\delta_0)$ for determinant; therefore matrix (37) has rank $\delta_0 + 1$ and nullity 0.

That portion of the results just obtained which is needed in the proof of Proposition 3 is summed up in the following alternative:

Lemma 4 .- Let $(\alpha_0, \beta_0, \gamma_0, \delta_0)$ be the lowest quadruple in an equivalence class $E(\alpha, \beta, \gamma, \delta)$. If $\beta_0 = 0$ and $\alpha_0 < \delta_0$, then Ker $\underset{\sim}{L}_K\ \ P(\alpha, \beta, \gamma, \delta) = 0$ is of dimension one, and is generated by the special polynomial $G(\gamma_0, \delta_0-\alpha_0, \alpha_0)$. Otherwise Ker $\underset{\sim}{L}_K\ \ P(\alpha, \beta, \gamma, \delta) = 0$.

Proposition 3 affords an easy way of decomposing $\underset{\sim}{P}_n$ into a direct sum of irreducible weight spaces : each polynomial $v_0 = g(\alpha, \beta, \gamma)$ generates a basis formed of the chain of polynomials v_k $(k \geq 0)$ such that $kv_k = \underset{\sim}{L}_K\ v_{k-1}$ for $k \geq 1$. Such polynomials span a weight space V_λ of weight $\lambda = \alpha + \beta - 2\gamma$. Yet attempts at using the complete reduction (18) to decompose any polynomial into its constituents in Im $\underset{\sim}{L}_K$ and Ker $\underset{\sim}{L}_D$ have not resulted in clear and elegant software procedures. All the same, an algorithm based on the classes enumerated in Table V proved to be both expedient and easy to code.

5. THE DECOMPOSITION ALGORITHM

For most dynamical systems, normalization to degree 3 is sufficient; fortunately, at that minimal degree, the decomposition is readily executed by hand. Once Table V has been established, it becomes clear that the polynomial

$$p = \sum_{(\alpha,\ \beta,\ \gamma,\ \delta)} C_{\alpha,\beta,\gamma,\delta}\ x^{\alpha}\ y^{\beta}\ X^{\gamma}\ Y^{\delta}$$

may be written as the sum $p = p_D + \underset{\sim}{L}_K\ q$ where

$$\begin{aligned} q = {} & \frac{1}{3} C_{2,0,1,0}\ x^3 + \frac{1}{3} (C_{1,1,1,0} + C_{2,0,0,1})\ x^2y \\ & + \frac{1}{3} (C_{0,2,1,0} + C_{1,1,0,1})\ xy^2 + \frac{1}{3} C_{0,2,0,1}\ y^3 \\ & + \frac{1}{2} C_{1,0,2,0}\ x^2X + \frac{1}{2} C_{0,1,2,0}\ xyX + \frac{1}{2} C_{0,1,1,1}\ y^2X \\ & + \frac{1}{2} C_{1,0,1,1}\ x^2Y + \frac{1}{2} C_{1,0,0,2}\ xyY + \frac{1}{2} C_{0,1,0,2}\ y^2Y \\ & + C_{0,0,3,0}\ xX^2 + C_{0,0,2,1}\ yX^2 + C_{0,0,1,2}\ xY^2 + C_{0,0,0,3}\ yY^2 \\ & - \frac{1}{2} C_{0,1,2,0}\ xG + \frac{1}{2} C_{1,0,0,2}\ yG \end{aligned} \tag{38}$$

and

$$p_D = C_{3,0,0,0}\, x^3 + C_{2,1,0,0}\, x^2y + C_{1,2,0,0}\, xy^2 + C_{0,3,0,0}\, y^3$$
$$+ (\frac{2}{3} C_{2,0,0,1} - \frac{1}{3} C_{1,1,1,0})\, xG - (\frac{2}{3} C_{0,2,1,0} - \frac{1}{3} C_{1,1,0,1})\, yG\ ,$$

which, by virtue of Proposition 3, is an element of Ker $\underset{\sim}{L}_D$.

However trivial the task appears to be at degree 3, Table III leaves one to gather that the calculations become rapidly voluminous past degree 4, even for a computer program prepared to handle sparse matrices. The next proposition is the foundation of a decomposition algorithm complete to the point of having been coded eventually in APL[19] and run on a DEC-20 at the National Institutes of Health in Bethesda, MD. (N. B. The program is available upon request from the third author).

Let V be a vector space; let also v and w be two endomorphisms of V, and set u = vw. For any x in V, a vector y in V satisfies the relation v(x - wy) = 0 if and only if uy = vx. Thus it is true that, _given any polynomial_ p _in_ $\underset{\sim}{P}_n$, _a polynomial_ q _in_ $\underset{\sim}{P}_n$ _satisfies the relation_ $\underset{\sim}{L}_D\,(p - \underset{\sim}{L}_K\, q) = 0$ _if and only if it is a solution for_ q _of the equation_ $Tq = \underset{\sim}{L}_D\, p$ _where_ T _is the operator_ $\underset{\sim}{L}_D\, \underset{\sim}{L}_K$.

Given an endomorphism u of the vector space V, one can always find an endomorphism u^- such that $u\, u^-\, u = u$. We call u^- a _generalized inverse_ of u, but the reader should note that authors interested in classifying various species of generalized inverses associated with u would name u^- a {1}-_inverse_[20] of u or a _g-inverse_.[21] For any vector x in Im u, $u\, u^- x = x$. In particular, assume that u is the product u = vw of two endomorphisms, and that Im v = Im u; then any vector y of the form $y = (u^- v)x$ is a solution of the equation $uy = vx$. For there exists by hypothesis a vector z such that $vx = uz$, hence $uy = (uu^- u)z = uz = vx$. Applied to the operator $T = L_D L_K : P_n \to P_n$ for which $\mathrm{Im}\, L_D = \mathrm{Im}\, T$ (Proposition 1), the above considerations prove the following

Lemma 5.- <u>Let</u> T^- <u>be a generalized inverse of</u> T <u>and</u> p <u>a polynomial in</u> P_n. <u>Then the polynomial</u> $q = T^- L_D\, p$ <u>is a solution of the equation</u> $Tq = L_D\, p$.

On account of Lemma 5, after a generalized inverse T^- has been produced, the problem of decomposing a polynomial into the sum of its constituents in $\mathrm{Im}\, L_K$ and $\mathrm{Ker}\, L_D$ will be solved by setting $p_K = L_K\, q$ and $p_D = p - p_K$.

Among the many varieties of generalized inverses associated with T, preference should be given here to those which preserve the basic symmetry consisting in exchanging the coordinate x and its conjugate momentum X respectively with the coordinate y and its conjugate momentum Y. There is indeed no reason why the normalization should favour one coordinate more than the other. The symmetry requirement is best expressed by introducing the operator $Z : p \to Zp = p(x, y, X, Y) - p(y, x, Y, X) : P_n \to P_n$. A polynomial p is symmetric in the pairs (x, X) and (y, Y) if and only if $Zp = 0$. Furthermore the differential operator

$$T = L_D\, L_K = \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}\right) + \left(xX \frac{\partial^2}{\partial x \partial X} + yY \frac{\partial^2}{\partial y \partial Y}\right) + \left(xY \frac{\partial^2}{\partial y \partial X} + yX \frac{\partial^2}{\partial x \partial Y}\right)$$

being invariant for the symmetry, the operators T and Z commute over P_n. Our purpose thus is to find a generalized inverse of T with corresponding symetry properties. It should map symmetric polynomials onto symmetric polynomials, that is $ZT^- p$ should be $= 0$ whenever $Zp = 0$; also, by exchanging the pairs

(x, X) and (y, Y) in the image $q(x, y, X, Y) = T^{-} p(x, y, X, Y)$, one should have that $q(y, x, Y, X) = T^{-} p(y, x, Y, X)$, that is, $ZT^{-} = T^{-} Z$. It will be shown that a generalized inverse $T^{\#}$ called the <u>group-inverse</u> of T satisfies these symmetry requirements.

For an endomorphism u of a vector space V, Erdelyi[22] calls group-inverse of u an endomorphism $u^{\#}$ such that

$$u\, u^{\#}\, u = u\ , \qquad u^{\#}\, u\, u^{\#} = u^{\#}\ , \qquad u\, u^{\#} = u^{\#}\, u\ .$$

If it exists, the group-inverse of u is unique. If u is an isomorphism of V onto itself, then $u^{\#} = u^{-1}$; more generally, u admits a group-inverse if and only if V may be decomposed[23] into the direct sum of Ker u and Im u. This is the case when u is semi-simple; then the restriction $u_{\#}$ of u to Im u is bijective, and, in principle at least, the group-inverse $u^{\#}$ may be built as follows:

$$u^{\#} p = \begin{cases} 0 & \text{for } p \text{ in Ker } u, \\ u_{\#}^{-1} p & \text{for } p \text{ in Im } u\ . \end{cases}$$

Lemma 6.- <u>Let</u> V <u>be a vector space, and</u> u <u>an endomorphism of</u> V. <u>If</u> u <u>admits a group-inverse</u> $u^{\#}$, <u>then, for any endomorphism</u> v <u>of</u> V, <u>the relation</u> $vu = uv$ <u>implies the relation</u> $vu^{\#} = u^{\#} v$. <u>In particular, for any vector</u> x <u>of</u> V, <u>the relation</u> $vx = 0$ <u>implies the relation</u> $(vu^{\#})x = 0$.

The lemma is proved when it is shown that $(vu^{\#})x = (v^{\#} u)x$ first for x in Ker u and then for x in Im u. In each case, the demonstration rests on the fact that, because u and v commute, v(Ker u) is contained in Ker u and v(Im u) in Im u.

If $ux = 0$, then, on the one hand, $u^{\#} x = 0$ because $u^{\#}$ admits Ker u as its null space[22], hence $(vu^{\#})x = 0$; on the other hand, $u(vx) = (uv)x = 0$, which implies that $(u^{\#} v)x = u^{\#}(vx) = 0$. Therefore, when restricted to Ker u, $vu^{\#}$ and $u^{\#} v$ are identical, since they are both equal to the null endomorphism.

Now take x in Im u. There is a unique element y in Im u such that $uy = x$ and $u^{\#}x = y$. With x and y both in Im u, the elements vx and vy are also both in Im u. But $u(vy) = (uv)y = (vu)y = v(uy) = vx$, and vy is therefore the unique element of Im u mapped by u onto vx. Hence $vy = u^{\#}(vx)$, and the latter relation implies that $(vu^{\#})x = v(u^{\#} x) = vy = u^{\#}(vx) = (u^{\#} v)x$. This completes the proof of Lemma 5.

The operator $T = \underset{\sim}{L}_D \underset{\sim}{L}_K$ is semi-simple (Proposition 1), hence it admits a group-inverse $T^{\#}$. By virtue of Lemma 6, the group-inverse $T^{\#}$ commutes with the symmetry operator Z, and it maps symmetric polynomials onto symmetric polynomials.

The construction proposed here for the group-inverse $T^{\#}$ of T makes use of the decomposition of $\underset{\sim}{P}_n$ into a direct sum of vector subspaces $P_\iota = P(\alpha, \beta, \gamma, \delta)$ specified in Section 4. Assume that, for each ι, the restriction T_ι of T to P_ι admits a group-inverse; then, for each polynomial p of $\underset{\sim}{P}_n$ decomposed into the sum $p = \sum_\iota p_\iota$ of its components in the summands P_ι, define the image $T^{\#} p = \sum_\iota T_\iota^{\#} p_\iota$. Manifestly $T^{\#}$ is a linear map $\underset{\sim}{P}_n \to \underset{\sim}{P}_n$, and it satisfies the three conditions $T T^{\#} T = T$, $T^{\#} T T^{\#} = T^{\#}$, $T T^{\#} = T^{\#} T$, which means that $T^{\#}$ is the group-inverse of T. The decomposition of a general polynomial p into its components in Ker $\underset{\sim}{L}_D$ and Im $\underset{\sim}{L}_K$ is thereby reduced to the problem of building the group-inverse for the restrictions of T on each of the vector subspaces in the classes enumerated in Table V. Statistics collected in Table VI for a homogeneous polynomial of degree 6 will convince the reader that, however tedious it may be, a careful discussion of each particular situation breaks up the general problem of producing the 84×84 matrix for the group-inverse $T^{\#}$ into the solution of 42 linear systems in at most 3 unknowns, 22 of them being utterly trivial.

TABLE VI

Class I is dealt with at once : T_ι being the null endomorphism, its group-inverse $T_\iota{}^{\#}$ is also the null endomorphism.

In classes III - V, the factor $\underset{\sim}{L}_K$ is injective; since Ker T_ι is equal to the kernel of $\underset{\sim}{L}_K$ restricted to P_ι (Proposition 1), there follows that T_ι is an isomorphism of P_ι onto itself, hence that $T_\iota{}^{-1}$ is the group-inverse of T_ι. Now a closer examination of classes III and IV will bring forth a straightforward procedure for inverting T_ι.

As was shown in Section 4 for class III, the map $\underset{\sim}{L}_K$ is an isomorphism of $P(\alpha_0, 0, \gamma_0, \delta_0)$ onto $P(\alpha_0-1, 0, \gamma_0+1, \delta_0)$ represented by matrix (29) for the bases (30) in $P(\alpha_0, 0, \gamma_0, \delta_0)$ and (31) in $P(\alpha_0-1, 0, \gamma_0+1, \delta_0)$. But, for the same bases, $\underset{\sim}{L}_D : P(\alpha_0-1, 0, \gamma_0+1, \delta_0) \to P(\alpha_0, 0, \gamma_0, \delta_0)$ is represented by the $(\delta_0 + 1) \times (\delta_0 + 1)$ matrix

$$\begin{bmatrix} \gamma_0+1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \delta_0 & \gamma_0+2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \delta_0-1 & \gamma_0+3 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \gamma_0+\delta_0-1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 2 & \gamma_0+\delta_0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & \gamma_0+\delta_0+1 \end{bmatrix} \tag{39}$$

which is evidently non-singular; hence $\underset{\sim}{L}_D$ is an isomorphism of $P(\alpha_0-1, 0, \gamma_0+1, \delta_0)$ onto $P(\alpha_0, 0, \gamma_0, \delta_0)$. Because matrix (29) is upper triangular, and matrix (39) lower triangular, the equation $T_1 q = p_1$ is solved readily first by forward substitution to obtain a polynomial r such that $\underset{\sim}{L}_D\, r = p_1$, and then by backward substitution to find the polynomial q such that $\underset{\sim}{L}_K\, q = r$.

One meets a similar situation in class IV. The factor $\underset{\sim}{L}_K$ is an isomorphism of $P(\alpha_0, \beta_0, 0, \delta_0)$ onto $P(\alpha_0, \beta_0-1, 0, \delta_0+1)$ represented by matrix (34) when bases (32) and (33) are selected in the subspaces $P(\alpha_0, \beta_0, 0, \delta_0)$ and $P(\alpha_0, \beta_0-1, 0, \delta_0+1)$ respectively. Further, for the same bases, $\underset{\sim}{L}_D : P(\alpha_0, \beta_0-1, 0, \delta_0+1) \to P(\alpha_0, \beta_0, 0, \delta_0)$ is represented by the $(\alpha_0 + 1) \times (\alpha_0 + 1)$ matrix

$$\begin{bmatrix} \delta_0+1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \delta_0 & 2 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \delta_0-1 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \delta_0-a_0+3 & \alpha_0-1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \delta_0-\alpha_0+2 & \alpha_0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \delta_0-\alpha_0+1 \end{bmatrix} \tag{40}$$

which is non-singular; thus $\underset{\sim}{L}_D$ is an isomorphism of $P(\alpha_0, \beta_0-1, 0, \delta_0+1)$ onto $P(\alpha_0, \beta_0, 0, \delta_0)$. Matrix (34) being lower triangular, and matrix (40) upper triangular, the equation $T_1 q = p_1$ is solved first by backward substitution to obtain a solution of the equation $\underset{\sim}{L}_D\, r = p_1$, and then by forward substitution to solve the equation $\underset{\sim}{L}_K\, q = r$.

The solution is not that simple in Class V. On the one hand, $\underset{\sim}{L}_K$ is injective but not surjective. On the other hand, the linear map $\underset{\sim}{L}_D$: $P(\alpha_0, \beta_0-1, 0, \delta_0+1) \to P(\alpha_0, \beta_0, 0, \delta_0)$ is surjective but not injective. Indeed, with (35) and (36) chosen as the bases in $P(\alpha_0, \beta_0, 0, \delta_0)$ and $P(\alpha_0, \beta_0-1, 0, \delta_0+1)$ respectively, $\underset{\sim}{L}_D$ is represented by the band matrix

$$\begin{bmatrix} \delta_0+1 & 1 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & \delta_0 & 2 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 0 & \delta_0-1 & \cdots & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \cdots & \delta_0-1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & \cdots & 2 & \delta_0 & 0 \\ 0 & 0 & 0 & \cdots & \cdots & 0 & 1 & \delta_0+1 \end{bmatrix} \tag{41}$$

with $(\delta_0 + 1)$ rows and $(\delta_0 + 2)$ columns, and it is easily seen that matrix (41) is of maximum rank $\delta_0 + 1$ and of nullity 1. For the restrictions of T_l in class V, there seems to be no quicker way of inverting T_l than by actually multiplying matrices (41) and (37) row by column, thereafter calculating explicitly the inverse of their product.

In both classes I and II, the operator T_l is not invertible. While, in class I, the group-inverse of T_l is the null endomorphism, in class II is encountered the case where a way of computing the group-inverse $T_l^{\#}$ must be prescribed. But the conditions defining the group-inverse characterize u, in the terminology of Drazin,[24] as a pseudo-invertible element in the associative algebra End V of the endomorphisms of V. Thus the group-inverse $u^{\#}$ is a Drazin pseudo-inverse of u relative to which the index of u is equal to 1. In that context, recall the following statement, which is in fact a particular case of a general theorem proved by Cline.[25] Let U and V be finite-dimensional vector spaces; let also v be an injective linear map $U \to V$, and w a surjective linear map $V \to U$; if the product $u = vw : V \to V$ is pseudo-invertible in the sense of Drazin and if its index is equal to 1, then $wv : U \to U$ is bijective, and

$$u^{\#} = v\ (wv)^{-2}\ w \tag{42}$$

The conditions under which Cline's formula (42) may be applied are satisfied by any endomorphism T_1 in class II. Indeed, from Section 4, it is already known that $\underset{\sim}{L}_K : P(\alpha_0, 0, \gamma_0, \delta_0) \to P(\alpha_0-1, 0, \gamma_0+1, \delta_0)$ is a surjective linear map; there remains to show that $\underset{\sim}{L}_D : P(\alpha_0-1, 0, \gamma_0+1, \delta_0) \to P(\alpha_0, 0, \gamma_0, \delta_0)$ is injective. For the bases (27) and (26) in $P(\alpha_0-1, 0, \gamma_0+1, \delta_0)$ and $P(\alpha_0, 0, \gamma_0, \delta_0)$ respectively, $\underset{\sim}{L}_D$ is represented by the band matrix

$$\begin{bmatrix} \gamma_0+1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \delta_0 & \gamma_0+2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \delta_0-1 & \gamma_0+3 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \delta_0-\alpha_0+3 & \gamma_0+\alpha_0-1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \delta_0-\alpha_0+2 & \gamma_0-\alpha_0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \delta_0-\alpha_0+1 \end{bmatrix} \tag{43}$$

with α_0+1 rows and α_0 columns. Clearly, the determinant made of the first α_0 rows of (43) is not zero; hence $\underset{\sim}{L}_D$ restricted to $P(\alpha_0-1, 0, \gamma_0+1, \delta_0)$ is injective. In application of formula (42), the computer program computes the group-inverse

$$T_1^{\#} = (\underset{\sim}{L}_D \underset{\sim}{L}_K)^{\#} = \underset{\sim}{L}_D (\underset{\sim}{L}_K \underset{\sim}{L}_D)^{-2} \underset{\sim}{L}_K$$

by multiplying matrix (28) by matrix (43), squaring the product, taking its inverse, and multiplying the result to the left by matrix (43) and to the right by matrix (28). Considering that the matrices involved have dimensions of the order of half the degree n, one will admit that the manipulations have been brought down to an elementary level. Furthermore, by taking advantage of the invariance with respect to the symmetry Z, the task of constructing the group-inverse has been cut by almost a half. Alternatively, the invariance may be exploited to check the results coming out of the program.

The fundamental results obtained in the course of discussing the restriction classes mentioned in Table IV are gathered in the following

Proposition 4.- Given a polynomial p in $\underset{\sim}{P}_n$, there exists a unique polynomial q in Im $\underset{\sim}{L}_D$ such that p - $\underset{\sim}{L}_K$ q belongs to Ker $\underset{\sim}{L}_D$. Also Zq belongs to Im $\underset{\sim}{L}_D$, and it is the unique polynomial r in Im $\underset{\sim}{L}_D$ such that Zp - $\underset{\sim}{L}_K$ r belongs to Ker $\underset{\sim}{L}_D$. Whenever Zp = 0, then Zq = 0.

Indeed p may be decomposed in a unique way as the sum $p = p_K + p_D$ with p_k in Im $\underset{\sim}{L}_K$ and p_D in Ker $\underset{\sim}{L}_D$. Hence there exists a polynomial q such that $\underset{\sim}{L}_D$ p = $\underset{\sim}{L}_D$ $\underset{\sim}{L}_K$ q. Taking q = $T^{\#}$ $\underset{\sim}{L}_D$ p guarantees that q belongs to Im T which, by virtue of Proposition 1, is identical to Im $\underset{\sim}{L}_D$. If there is another polynomial q' with the same property, then, on the one hand, q - q' belongs to Im $\underset{\sim}{L}_D$, while, on the other hand, q - q' belongs to Ker T = Ker $\underset{\sim}{L}_K$; but $\underset{\sim}{P}_n$ is the direct sum of Ker $\underset{\sim}{L}_K$ and Im $\underset{\sim}{L}_D$, hence q - q' = 0. In view of the fact that the expressions (9) and (12) for $\underset{\sim}{L}_K$ and $\underset{\sim}{L}_D$ are symmetric in the pairs (x, X) and (y, Y), $\underset{\sim}{L}_D$ Z = Z $\underset{\sim}{L}_D$ and $\underset{\sim}{L}_K$ Z = Z $\underset{\sim}{L}_K$; therefore Zq belongs to Im $\underset{\sim}{L}_D$ and

$\underset{\sim}{L}_D (Zp - \underset{\sim}{L}_K Zq) = \underset{\sim}{L}_D (Z(p - \underset{\sim}{L}_K q)) = Z(\underset{\sim}{L}_D (p - \underset{\sim}{L}_K q)) = 0$. From what has been proved already, there results that Zq is the unique element in Im $\underset{\sim}{L}_D$ to have that property. Finally, the last part of Proposition 4 is an immediate consequence of Lemma 6.

Incidentally the computer program which implements the algorithm produced at degree 3 a polynomial $q = T^{\#} \underset{\sim}{L}_D p$ which differs from the solution in (38) by the quantity

$$\frac{2}{3} G (C_{0,0,2,1} X - C_{0,0,1,2} Y)$$

The discrepancy is admissible since, in agreement with Prposition 1, it is an element of Ker $\underset{\sim}{L}_K$.

6. ELIMINATION OF THE SHORT TERM EFFECTS

A dynamical system described by an Hamiltonian of type (14) is said to be in normal form if, for each $n > 2$, the term H_n belongs to Ker L_D, that is, H_n is a homogeneous polynomial in the coordinates (x, y) and the angular momentum G (Proposition 3). For instance, an Hamiltonian whose potential energy depends only on the coordinates is in normal form; so there is nothing the present normalization can contribute to further its solution. Such is the case for the Monkey Saddle.[26] But, if an Hamiltonian of type (14) is not in normal form, then one can construct a canonical transformation $(x, y, X, Y) \rightarrow (x', y', X', Y')$ which will convert (14) into a series in normal form. It is proposed to construct the normalization as a Lie transformation, that is, as the flow of an Hamiltonian vector field

$$\frac{dx}{d\varepsilon} = \frac{\partial W}{\partial X}, \qquad \frac{dy}{d\varepsilon} = \frac{\partial W}{\partial Y}, \qquad \frac{dX}{d\varepsilon} = -\frac{\partial W}{\partial x}, \qquad \frac{dY}{d\varepsilon} = -\frac{\partial W}{\partial y}$$

derived from a series

$$W \equiv W(x, y, X, Y) = \sum_{n \geq 0} \frac{1}{n!} W_{n+1} .$$

It will be shown that each term W_n in the generator may be obtained as a homogeneous polynomial of degree $n+2$ in the phase variables (x, y, X, Y), so that the normalization may be pursued in a recursive fashion from one degree to the next.

Starting with the infinitesimal contact transformation that is the infinitesimal symplectic transformation tangent to the Lie mapping, the first order terms H'_1 and W_1 in the new Hamiltonian and in the generator respectively are linked by the identity

$$(K;\, W_1) + H_1 = H'_1$$

equivalent, as a matter of fact, to the algebraic relation

$$\underset{\sim}{L}_K\, W_1 + H'_1 = H_1 \ . \tag{44}$$

With the requirement that $\underset{\sim}{L}_D\, H'_1 = 0$, the problem of solving (44) amounts to decomposing the homogeneous polynomial H_1 into its constituents in the direct sum $\underset{\sim}{P}_3 = \text{Ker } \underset{\sim}{L}_D \quad \text{Im } \underset{\sim}{L}_K$.

A change in notation is helpful in following the recursion rules for constructing the normalization past the infinitesimal contact transformation: $H_{n,0}$ will stand for H_n in (14), and $H_{0,n}$ for H'_n. Now assume that all terms W_i $(1 \leq i \leq n-1)$ and $H_{i,j}$ $(0 \leq i,\ j \leq n-1,\ \ i+j \leq n-1)$ have been obtained. Then one is in a position to calculate first

$$\tilde{H}_{n-1,1} = H_{n,0} + \sum_{1 \leq j \leq n-1} \binom{n-1}{j-1} (H_{n-j,0};\, W_j)$$

and thereafter, by decreasing values of i, the terms

$$\tilde{H}_{i,j} = \tilde{H}_{i+1,j-1} + \sum_{0 \leq k \leq i} \binom{i}{k} (H_{i-k,j-1};\, W_{k+1})$$

for $i + j = n$, $0 \leq i, j \leq n$, the last term in the recursive chain being $\tilde{H}_{0,n}$. By prescription of a perturbation algorithm involving a Lie transformation, the terms W_n in the generator and $H_{0,n}$ in the transformed Hamiltonian must satisfy the partial differential identity

$$(H_0; W_n) + \tilde{H}_{0,n} = H_{0,n} \; . \tag{45}$$

But, if p and q are homogeneous polynomials of degree ℓ and m respectively, then their Poisson bracket

$$(p; q) = \frac{\partial p}{\partial x}\frac{\partial q}{\partial X} - \frac{\partial p}{\partial X}\frac{\partial q}{\partial x} + \frac{\partial p}{\partial y}\frac{\partial q}{\partial Y} - \frac{\partial p}{\partial Y}\frac{\partial q}{\partial y}$$

is a homogeneous polynomial of degree $\ell + m - 2$. Hence all terms $\tilde{H}_{i,j}$, for $i + j = n$, are homogeneous polynomials of degree $n + 2$, and the partial differential equation (45) is equivalent to the algebraic equation

$$\tilde{H}_{0,n} = L_K \, W_n + H_{0,n} \; .$$

With the normalization requirement that $\underset{\sim}{L}_D \, H_{0,n} = 0$, the latter is solved by decomposing $\tilde{H}_{0,n}$ into its constituents in the direct sum $\underset{\sim}{P}_{n+2} =$ Im $\underset{\sim}{L}_K$ Ker $\underset{\sim}{L}_D$.

For readers interested either in calculating by hand some of the coefficients in the normalized Hamiltonian, or in checking their automated normalization procedures, listed below are the terms in $H_{0,n}$ $(3 \leq n \leq 6)$ emanating from the right hand members

$$\tilde{H}_{0,n} = \sum_{(\alpha,\, \beta,\, \gamma,\, \delta)} C_{\alpha,\beta,\gamma,\delta} \, x'^{\alpha} \, y'^{\beta} \, X'^{\gamma} \, Y'^{\delta} \; :$$

Degree 4

$$\begin{aligned}
H_{0,4} = {} & C_{4,0,0,0}\, x'^4 && + \frac{1}{4}\,(3\, C_{3,0,0,1} - C_{2,1,1,0})\; x'^2 G' \\
& + C_{3,1,0,0}\, x'^3 y' && + \frac{1}{2}\,(\; C_{2,1,0,1} - C_{1,2,1,0})\; x' y' G' \\
& + C_{2,2,0,0}\, x'^2 y'^2 && - \frac{1}{4}\,(3\, C_{0,3,1,0} - C_{1,2,0,1})\; y'^2 G' \\
& + C_{1,3,0,0}\, x' y'^3 && + \frac{1}{6}\,(2\, C_{2,0,0,2} - C_{1,1,1,1} + 2\, C_{0,2,2,0})\; G'^2 \\
& + C_{0,4,0,0}\, y'^4 \; ;
\end{aligned}$$

Degree 5

$$\begin{aligned}
H_{0,5} = {} & C_{5,0,0,0}\, x'^5 && + \frac{1}{5}\,(4\, C_{4,0,0,1} - C_{3,1,1,0})\; x'^3 G' \\
& + C_{4,1,0,0}\, x'^4 y' && + \frac{1}{5}\,(3\, C_{3,1,0,1} - 2\, C_{2,2,1,0})\; x'^2 y G' \\
& + C_{3,2,0,0}\, x'^3 y'^2 && - \frac{1}{5}\,(3\, C_{1,3,1,0} - 2\, C_{2,2,0,1})\; x' y'^2 G' \\
& + C_{2,3,0,0}\, x'^2 y'^3 && - \frac{1}{5}\,(4\, C_{0,4,1,0} - C_{1,3,0,1})\; y'^3 G' \\
& + C_{1,4,0,0}\, x' y'^4 && + \frac{1}{6}\,(3\, C_{3,0,0,2} - C_{2,1,1,1} + C_{1,2,2,0})\; x' G'^2 \\
& + C_{0,5,0,0}\, y'^5 && + \frac{1}{6}\,(3\, C_{0,3,2,0} - C_{1,2,1,1} + C_{2,1,0,2})\; y' G'^2 \; ;
\end{aligned}$$

<u>degree</u> 6

$$\begin{aligned}
H_{0,6} = {} & C_{6,0,0,0}\, x'^{\ddagger} && + \frac{1}{6}\,(5\, C_{5,0,0,1} - C_{4,1,1,0})\, x'^{9}G' \\
& + C_{5,1,0,0}\, x'\}y' && + \frac{1}{3}\,(2\, C_{4,1,0,1} - 2\, C_{3,2,1,0})\, x'^{8}y'G' \\
& + C_{4,2,0,0}\, x'^{9}y'^{7} && + \frac{1}{2}\,(C_{3,2,0,1} - C_{2,3,1,0})\, x'^{7}y'^{7}G' \\
& + C_{3,3,0,0}\, x'^{8}y'^{8} && - \frac{1}{3}\,(2\, C_{1,4,1,0} - 2\, C_{2,3,0,1})\, x'y'^{8}G' \\
& + C_{2,4,0,0}\, x'^{7}y'^{9} && - \frac{1}{6}\,(5\, C_{0,5,1,0} - C_{1,4,0,1})\, y'^{9}G' \\
& + C_{1,5,0,0}\, x'y'\} && + \frac{1}{20}\,(12\, C_{4,0,0,2} - 3\, C_{3,1,1,1} + 2\, C_{2,2,2,0})\, x'^{7}G'^{7} \\
& + C_{0,6,0,0}\, y'^{\ddagger} && + \frac{1}{10}\,(3\, C_{3,1,0,2} - 2\, C_{2,2,1,1} + 3\, C_{1,3,2,0})\, x'y'G'^{7} \\
& && + \frac{1}{20}\,(12\, C_{0,4,2,0} - 3\, C_{1,3,1,1} + 2\, C_{2,2,0,2})\, y'^{7}G'^{7} \\
& && + \frac{1}{12}\,(3\, C_{3,0,0,3} - C_{2,1,1,2} + C_{1,2,2,1} - 3\, C_{0,3,3,0})\, G'^{8}.
\end{aligned}$$

Let the relativistic corrections for a free particle illustrate the normalization. The kinetic energy

$$E = m_0\, c^7\, (1 - v^7/c^7)^{-1/2}$$

expanded in powers of v^7, after division by the mass at rest m_0 and omission of the constant energy at rest mc^7, gives rise to the Hamiltonian

$$H = \frac{1}{2}\, v^7 - \frac{3}{8}\,\frac{v^9}{c^7} + \frac{5}{16}\,\frac{v^{\ddagger}}{c^9} - \ldots$$

with $v^2 = X^2 + Y^2$. Since

$$\underset{\sim}{L}_K \, (xX + yY)\, v^2 = v^4 \;,$$

there will be no term of degree 4 in the normalized Hamiltonian, and the generator of the infinitesimal contact transformation will be

$$W_1 = -\frac{3}{8}\frac{v^2}{c^2}(xX+yY) \;.$$

Hence, by definition of an infinitesimal contact transformation

$$x = x' + (x';\, W_1), \qquad X = X' + (X';\, W_1),$$

$$y = y' + (y';\, W_1), \qquad Y = Y' + (Y';\, W_1),$$

the relativistic corrections to the first order in v^2/c^2 are:

$$\Delta x' = -\frac{3}{8}\frac{v'^2}{c^2}x' + \frac{3}{4}\frac{G'}{c^2}Y' \;, \qquad \Delta X' = \frac{3}{8}\frac{v'^2}{c^2}X' \;,$$

$$\Delta y' = -\frac{3}{8}\frac{v'^2}{c^2}y' - \frac{3}{4}\frac{G'}{c^2}X' \;, \qquad \Delta Y' = \frac{3}{8}\frac{v'^2}{c^2}Y' \;.$$

A more substantial application of the present scheme for normalization has been made to the Restricted Problem of Three Bodies at the equilateral point L_4 for Routh's critical mass ratio; but this topic requires too much background information in celestial mechanics to be related here.

Unexpected as it comes after a long excursion in Representation Theory of Lie algebras, the physical meaning of the normalization achieved in the present section is in fact very simple. To elucidate this point, the notations will be revised. First the normalized Hamiltonian will be decomposed in the usual way as the sum

$$H' = \frac{1}{2}(X'^2 + Y'^2) - U(x', y', G')$$

of a kinetic energy and of a force function U, with $G' = x' Y' - y' X'$ designating the angular momentum. Next, in order to eliminate ambiguities concerning the partial derivatives, the differential of U will be written as the 1-form

$$dU = \partial_1 U\, dx' + \partial_2 U\, dy' + \partial_3 U\, dG' .$$

In these notations, the equations of normalized motions become

$$\dot{x}' = \frac{\partial H'}{\partial X'} = X' + y'\, \partial_3 U , \qquad \dot{X}' = -\frac{\partial H'}{\partial x'} = \partial_1 U + Y'\, \partial_3 U ,$$

$$\dot{y}' = \frac{\partial H'}{\partial Y'} = Y' - x'\, \partial_3 U , \qquad \dot{Y}' = -\frac{\partial H'}{\partial y'} = \partial_2 U - X'\, \partial_3 U .$$

Consider now a Cartesian frame of reference consisting of an orthonormal basis $(\underset{\sim}{i}', \underset{\sim}{j}', \underset{\sim}{k})$ rotating at the angular velocity $\underset{\sim}{\omega} = \partial_3 U\, \underset{\sim}{k}$ about the normal to the plane of motion; assume also that (x', y') represent the Cartesian coordinates of the particle in the plane $(\underset{\sim}{i}', \underset{\sim}{j}')$, or that $\underset{\sim}{x} = x'\, \underset{\sim}{i}' + y'\, \underset{\sim}{j}'$. Under these conditons, the particle's velocity in a frame (C) fixed in the plane is the vector

$$\dot{\underset{\sim}{x}} = \dot{x}' \underset{\sim}{i}' + \dot{y}' \underset{\sim}{j}' + \underset{\sim}{\omega} \times \underset{\sim}{x} ,$$

equal, by virtue of the normalized equations of motion, to the sum

$$\dot{\underset{\sim}{x}} = X' \underset{\sim}{i}' + Y' \underset{\sim}{j}' .$$

Thus the conjugate momenta X' and Y' are the components in the moving frame of the particle's velocity with respect to the fixed frame. For the same reason,

$$\ddot{\underset{\sim}{x}} = \ddot{X}' \underset{\sim}{i}' + \ddot{Y}' \underset{\sim}{j}' + \underset{\sim}{\omega} \times \dot{\underset{\sim}{x}} = \partial_1 U \underset{\sim}{i}' + \partial_2 U \underset{\sim}{j}' ,$$

which exhibits $\partial_1 U$ and $\partial_2 U$ as the projections of the force on the axes of the rotating frame. The normalization appears now as a procedure to extract the rate at which the frame of reference should be rotated in order to confer the equations of motion the simple form

$$\ddot{\underset{\sim}{x}} = \partial_1 U \underset{\sim}{i}' + \partial_2 U \underset{\sim}{j}'.$$

From that standpoint, the normalization is closely analogous to a method devised by Hansen[27] for handling perturbed Keplerian systems in three dimensions. A slow rate of rotation is imparted to the frame of reference; its axis and its rate are adjusted at each instant so that the rotating frame constitutes what Hansen calls an <u>ideal</u> (i.e. conceptual or virtual) reference system. In the ideal frame, the particle's motion appears to be planar; the forces acting on the mass point are expressed as a two-dimensional gradient, the coupling between the planar motion and the rotation of the ideal frame being accounted for by making the force function dependent explicitly on the angular momentum.

The kinematical interpretation given here to the normalization is further clarified by looking at the equations of motion in the polar variables defined by the canonical extension

$$x' = r'\cos\theta', \qquad X' = R'\cos\theta' - \frac{\Theta'}{r'}\sin\theta',$$

$$y' = r'\sin\theta', \qquad Y' = R'\sin\theta' + \frac{\Theta'}{r'}\cos\theta'.$$

There results at once from the the Cartesian equations of motion that

$$\frac{d}{dt} r' = R', \qquad \frac{d}{dt} R' = \frac{\Theta^2}{r'^3} + \partial_1 U \cos\theta' + \partial_2 U \sin\theta' ,$$

$$\frac{d}{dt} \theta' = \frac{\Theta'}{r'^2} - \partial_3 U , \qquad \frac{d}{dt} \Theta' = r'(\partial_2 U \cos\theta' - \partial_1 U \sin\theta').$$

The angle σ of slow rotation being determined by the 1-form $d\sigma = \partial_3 U\, dt$, and the radial and transversal components of the force being

$$P = \partial_1 U \cos\theta' + \partial_2 U \sin\theta' ,$$

$$Q = \partial_2 U \cos\theta' - \partial_1 U \sin\theta' ,$$

the equations in polar coordinates are given the standard form

$$\frac{d^2}{dt^2} r' = \frac{\Theta'^2}{r'^3} + P , \qquad \frac{d}{dt} (\theta' + \sigma) = \frac{\Theta'}{r'^2} , \qquad \frac{d}{dt} \Theta' = Q .$$

Thanks to the normalization, the particle appears to move under the exclusive action of the gradient of U with respect to the normalizing coordinates.

APPENDIX : CONSTRUCTION OF A GROUP-INVERSE

It has been shown in Section 5 that the decomposition of a polynomial p of $\underset{\sim}{P}_n$ into its components in Ker $\underset{\sim}{L}_D$ and Im $\underset{\sim}{L}_K$ reduces to the construction of the group-inverse $T^{\#}$ of the semisimple operator $\underset{\sim}{L}_D\underset{\sim}{L}_K$. The algorithm given in Section 5 for constructing $T^{\#}$ depends on special properties enjoyed by the action of T on $\underset{\sim}{P}_n$: $\underset{\sim}{P}_n$ can be written as the direct sum of the spaces $P(\alpha,\beta,\gamma,\delta)$, each of which is left invariant by T, and on each subspace $P(\alpha,\beta,\gamma,\delta)$ either T is invertible, or the group-inverse of the restriction of T can be computed via an explicit factorization of its matrix. The decomposition scheme adopted in Section 5 owes its effectiveness to the fact that the subspaces $P(\alpha,\beta,\gamma,\delta)$ are much smaller than $\underset{\sim}{P}_n$: the dimension of $\underset{\sim}{P}_n$ is $\binom{n+3}{3}$, hence of order n^3, while the dimension of $P(\alpha,\beta,\gamma,\delta)$ is at most $n/2 + 1$ (by the discussion preceding Lemna 3). The purpose of this Appendix is to sketch an alternate algorithm for constructing the group-inverse of T. This algorithm hinges on the fact, to be proved next, that T is a <u>diagonalizable</u> operator whose eigenvalues are simple to find; it does not make use of any other properties of T or of $\underset{\sim}{P}_n$.

Lemma. <u>T is a diagonalizable operator whose eigenvalues consist of the products</u> $(n - 2\gamma - i)(i + 1)$, <u>with</u> $0 \leq \gamma \leq [n/2]$ <u>and</u> $0 \leq i \leq n - 2\gamma$. <u>The number of distinct eigenvalues is less than</u> $1 + [(n + 1)/2][(n + 3)/2]/2 < (n + 2)^2/7$.

Proof. $\underset{\sim}{P}n$ may be written, as in (18), as the direct sum of certain subspaces V_k $(1 \leq k \leq s)$, each of which is invariant and irreducible under the set of operators $\{\underset{\sim}{L}_D, \underset{\sim}{L}_K, \underset{\sim}{L}_S\}$. Furthermore, in each V_k, there is a basis $v_0 = v_0^{(k)}, \ldots, v_m = v_{m_k}^{(k)}$ satisfaying relations (19) - (21) (with x, y, h replaced by $\underset{\sim}{L}_D$, $\underset{\sim}{L}_K$, $\underset{\sim}{L}_S$ respectively). It follows from (20) and (21) tha the v_i's $(0 \leq i \leq m_k)$ are a basis of eigenvectors for $T = \underset{\sim}{L}_D\underset{\sim}{L}_K$, with $Tv_i = (m_k - i)(i + 1)\, v_i$. Thus the eigenvalues of T on V_k consist precisely of the products $(m_k - i)(i + 1)$ (for $0 \leq i \leq m_k$). To see which m_k's arise, one may observe, on the one hand, that in each V_k the maximal weight vector $v_0^{(k)}$ is an eigenvector of $\underset{\sim}{L}_S$ of eigenvalue m_k. Furthermore, since the s linearly independent vectors $v_0^{(k)}$ $(1 \leq k \leq s)$ form a basis of Ker $\underset{\sim}{L}_D$, (by relation (21) and Proposition 1), the set $\{m_k : 1 \leq k \leq s\}$ consists of the eigenvalues which $\underset{\sim}{L}_S$ takes on Ker $\underset{\sim}{L}_D$. On the other hand, the nonomials $g(\alpha,\beta,\gamma) = x^\alpha y^\beta G^\gamma$ with $\alpha + \beta + 2\gamma = \dot{n}$ are, by Proposition 3, also a basis of Ker $\underset{\sim}{L}_D$, and are eigenvectors of $\underset{\sim}{L}_S$ as the following formula shows:

$$\underset{\sim}{L}_S\, g(\alpha,\beta,\gamma) = (\alpha + \beta)\, g(\alpha,\beta,\gamma) = (n - 2\gamma)\, g(\alpha,\beta,\gamma).$$

Thus the numbers m_k $(1 \leq k \leq s)$ are exactly the numbers $n - 2\gamma$ $(0 \leq \gamma \leq [n/2])$. Finally, for a fxed γ, there are at most $[(n - 2\gamma + 1)/2] = [(n + 1)/2] - \gamma$ distinct non-zero products among the numbers $(n - 2\gamma - i)(i + 1)$ (for $0 \leq i \leq n - 2\gamma$), so, in all, T has at most $1 + \sum_{0 \leq \gamma \leq (n/2)} [(n + 1)/2] - \gamma = 1 + [(n + 1)/2][(n + 3)/2]/2$ distinct eigenvalues. It is easy to check that this number is always smaller than $(n + 2)^2/7$.

A glance at Table VI indicates that the bound given in the Lemma is far too generous.

It is perhaps worthwhile to note that a slight extension of the argument just presented yields a description of the decomposition of $\underset{\sim}{P}_n$ into its irreducible components : for each γ satisfying $0 \leq \gamma \leq [n/2]$, the irreducible $sl(2, \underset{\sim}{R})$-module of highest weight $n - 2\gamma$ occurs with the multiplicity $n - 2\gamma + 1$, and no other occur. This can be derived without difficulty from the fact that the polynomials $g(\alpha,\beta,\gamma)$ generate irreducible subspaces of highest weight $n - 2\gamma$.

In order to state the main result of the Appendix, it will be convenient to have some more notation. If λ is a scalar, let $\lambda^{\#}$ denotes λ^{-1} if $\lambda \neq 0$, and 0 otherwise. If $\lambda_0, \ldots, \lambda_t$ is a sequence of distinct scalars, then there is a unique polynomial $f(x)$ of degree $\leq t$ satisfying $f(\lambda_k) = \lambda_k^{\#}$ for $0 \leq k \leq t$. If $f[\lambda_0, \ldots, \lambda_k]$ denotes the k-th divided difference (constructed, for example, from the prescription $f[\lambda_0] = f(\lambda_0)$ and

$$f[\lambda_0, \ldots, \lambda_k] = \frac{f[\lambda_1, \ldots, \lambda_k] - f[\lambda_0, \ldots, \lambda_{k-1}]}{\lambda_k - \lambda_0}$$

for $k \geq 1$), then, as is well-known, $f(x)$ can be written as

$$f(x) = f[\lambda_0] + \sum_{1 \leq k \leq t} f[\lambda_0,\ldots,\lambda_k)\ (x - \lambda_{k-1})\cdots(x - \lambda_0).$$

A number of authors[28] have observed that the group-inverse $T^{\#}$ can be obtained as a polynomial in T. In the proof of the next proposition, it will be seen that, for the case considered here,

$$T^{\#} = f(T) = f[\lambda_0]\ I + \sum_{1 \leq k \leq t} f[\lambda_0,\ldots,\lambda_k]\ (T - \lambda_{k-1},I)\cdots(T - \lambda_0 I).$$

Proposition. *Let* T *be a diagonalizable operator on a finite-dimensional vector space, and suppose the distinct eigenvalues of* T *are* $\lambda_0, \ldots, \lambda_t$. *For any vector* v *in* V, $w = T^{\#} v$ *can be computed by the following algorithm:*

> **Set** $v_0 = v$, $w_0 = f[\lambda_0]\, v_0$;
>
> **For** $k = 1$ **to** t ,
>
>   **set** $v_k = (T - \lambda_{k-1})\, v_{k-1}$, $w_k = w_{k-1} + f[\lambda_0, \ldots, \lambda_k]\, v_k$;
>
> **Then** $w = w_t$.
>
> [N. B. The cycle in k may be stopped as soon as it emcounters a k for which v_k is = 0.]

Proof. It is clear that w_k is just the k-th partial sum of the sum

$$f(T)\, v = f[\lambda_0]\, v + \sum_{1 \leq k \leq t} f[\lambda_0, \ldots, \lambda_k]\, (T - \lambda_{k-1} I) \cdots (T - \lambda_0)\, v,$$

so $w_t = f(T)\, v$. It only remains to see that $f(T)\, v = T^{\#} v$ for all v in V. Since T is diagonalizable, V may be written as the direct sum $V_0 + \cdots + V_t$ of the eigenspaces of T, where $V_k = \{v \quad V : Tv = \lambda_k v\}$ for $0 \leq k \leq t$. But the endomorphism of V which sends $u = u_0 + \cdots + u_t$ ($u_k \quad V_k$ for each k) to $\lambda_0^{\#} u_0 + \cdots + \lambda_t^{\#} u_t$ satisfies the relations for the group-inverse of T, since $Tu = \lambda_0 u_0 + \cdots + \lambda_t u_t$. Thus by uniqueness of the group-inverse, $T^{\#} u = \lambda_0^{\#} u_0 + \cdots + \lambda_t^{\#} u_t$. On the other hand, $f(T)u = \sum f(T)u_t$, and the relation $Tu_k = \lambda_k u_k$ implies easily that $f(T)u_k = \lambda_k^{\#} u_k$ for each k. Hence $f(T\,)u = \sum \lambda_k^{\#} u_k = T^{\#} u$ for all u in V, and the proof is complete.

It is worthwhile to place this algorithm in a somewhat wider context. For, clearly, the method just outlined may be used to compute any polynomial in T, not just the specific polynomial which yields the group-inverse. The range of applicability of this algorithm is thus determined exactly by the following theorem in linear algebra[29]

Theorem. Let S and T be linear operators in a finite-dimensional vector space V. Then S may be written as a polynomial in T if and only if S commutes with every linear operator which commutes with T, that is, if and only if for every linear operator L in V, the relation LT = TL implies LS = SL.

If S and T are diagonalizable, the theorem is quite easy to prove directly. The special case when T is diagonalizable and $S = T^{\#}$ has been confirmed in the Proposition of the Appendix and in Lemma 6.

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TABLE I. A basis in the algebra sp(4, R)

	Generator	Non zero elements	Type n=nilpotent s=semi-simple
w_1	$\frac{1}{2}(X^2 + Y^2)$	$w_{1,1,3} = w_{1,2,4} = 1$	n
w_2	$xX + yY$	$w_{2,1,1} = w_{2,2,2} = -w_{2,3,3} = -w_{2,4,4} = 1$	s
w_3	$xY - yX$	$w_{3,2,1} = w_{3,4,3} = -w_{3,1,2} = -w_{3,3,4} = 1$	s
w_4	$\frac{1}{2}(X^2 - Y^2)$	$w_{4,1,3} = -w_{4,2,4} = 1$	n
w_5	XY	$w_{5,1,4} = w_{5,2,3} = 1$	n
w_6	$-\frac{1}{2}(x^2 + y^2)$	$w_{6,3,1} = w_{6,4,2} = 1$	n
w_7	$xX - yY$	$w_{7,1,1} = -w_{7,2,2} = -w_{7,3,3} = w_{7,4,4} = 1$	s
w_8	$xY + yX$	$w_{8,1,2} = w_{8,2,1} = -w_{8,3,4} = -w_{8,4,3} = 1$	s
w_9	$-xy$	$w_{9,3,2} = w_{9,4,1} = 1$	n
w_{10}	$-\frac{1}{2}(x^2 - y^2)$	$w_{10,3,1} = -w_{10,4,2} = 1$	n

TABLE II. Commutators of the basis matrices in sp(4, R)

	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9	w_{10}
w_1	0	$-2w_1$	0	0	0	w_2	$-2w_4$	$-2w_5$	w_8	w_7
w_2	$2w_1$	0	0	$2w_4$	$2w_5$	$-2w_6$	0	0	$-2w_9$	$-2w_{10}$
w_3	0	0	0	$2w_5$	$-2w_4$	0	$2w_8$	$-2w_7$	$-2w_{10}$	$2w_9$
w_4	0	$-2w_4$	$-2w_5$	0	0	w_7	$-2w_1$	0	$-w_3$	w_2
w_5	0	$-2w_5$	$2w_4$	0	0	w_8	0	$-2w_1$	w_2	w_3
w_6	$-w_2$	$2w_6$	0	$-w_7$	$-w_8$	0	$2w_{10}$	$2w_9$	0	0
w_7	$2w_4$	0	$-2w_8$	$2w_1$	0	$-2w_{10}$	0	$-2w_3$	0	$-2w_6$
w_8	$2w_5$	0	$2w_7$	0	$2w_1$	$-2w_9$	$2w_3$	0	$-2w_6$	0
w_9	$-w_8$	$2w_9$	$2w_{10}$	w_3	$-w_2$	0	0	$2w_6$	0	0
w_{10}	$-w_7$	$2w_{10}$	$-2w_9$	$-w_2$	$-w_3$	0	$2w_6$	0	0	0

TABLE III . Algebraic simplification accomplished by normalization

Degree	3	4	5	6	7	8	...
dim $\underset{\sim}{P}_n$	20	35	56	84	120	165	...
dim Ker $\underset{\sim}{L}_D$	6	9	12	16	20	25	...

TABLE IV . Classes of $\underset{\sim}{L}_K$ restricted to $P(\alpha, \beta, \gamma, \delta)$

Class	$P(\alpha, \beta, \gamma, \delta)$	Matrix $\underset{\sim}{L}_K$			Matrix $\underset{\sim}{L}_D$
	Lowest element	Dimensions	Rank	Nullity	Dimensions
I	$\beta_0 = 0,\quad \alpha_0 = 0$		0	1	
II	$\beta_0 = 0,\quad 1 \le \alpha_0 \le \delta_0$	$\alpha_0 \times (\alpha_0+1)$	α_0	1	$(\alpha_0+1) \times \alpha_0$
III	$\beta_0 = 0,\quad \alpha_0 > \delta_0$	$(\delta_0+1) \times (\delta_0+1)$	δ_0+1	0	$(\delta_0+1) \times (\delta_0+1)$
IV	$\beta_0 \neq 0,\ \gamma_0 = 0,\ \alpha_0 \le \delta_0$	$(\alpha_0+1) \times (\alpha_0+1)$	α_0+1	0	$(\alpha_0+1) \times (\alpha_0+1)$
V	$\beta_0 \neq 0,\ \gamma_0 = 0,\ \alpha_0 > \delta_0$	$(\delta_0+2) \times (\delta_0+1)$	δ_0+1	0	$(\delta_0+1) \times (\delta_0+2)$

TABLE V . Image and co-image of $\underset{\sim}{L}_K$ in the space P_3

$\underset{\sim}{L}_K \frac{1}{3} x^3 = x^2 X$	$\underset{\sim}{L}_K \frac{1}{2} x^2 X = xX^2$	$\underset{\sim}{L}_K xX^2 = X^3$
$\underset{\sim}{L}_K \frac{1}{3} x^2 y = xyX + \frac{1}{3} xG$	$\underset{\sim}{L}_K (\frac{1}{2} xyX - \frac{1}{2} xG) = yX^2$	$\underset{\sim}{L}_K yX^2 = X^2 Y$
$\underset{\sim}{L}_K \frac{1}{3} xy^2 = y^2 X + \frac{2}{3} yG$	$\underset{\sim}{L}_K \frac{1}{2} x^2 Y = xXY$	$\underset{\sim}{L}_K xY^2 = XY^2$
$\underset{\sim}{L}_K \frac{1}{3} x^2 y = x^2 Y - \frac{2}{3} xG$	$\underset{\sim}{L}_K \frac{1}{2} y^2 X = yXY$	$\underset{\sim}{L}_K yY^2 = Y^3$
$\underset{\sim}{L}_K \frac{1}{3} xy^2 = xyY - \frac{1}{3} yG$	$\underset{\sim}{L}_K (\frac{1}{2} xyY + \frac{1}{2} yG) = xY^2$	
$\underset{\sim}{L}_K \frac{1}{3} y^3 = y^2 Y$	$\underset{\sim}{L}_K \frac{1}{2} y^2 Y = yY^2$	

TABLE VI . Count of matrix inversions sufficient to calculate the group-inverse at degree 6.

Classes	Dimensions		
	1 × 1	2 × 2	3 × 3
II or V	5	3	1
III or IV	6	4	2

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11. ABSTRACT *(A 200-word or less factual summary of most significant information. If document includes a significant bibliography or literature survey, mention it here)*

Representation theory of Lie algebras is called upon to develop a procedure for normalizing a dynamical system with two degrees of freedom in the neighbourhood of an equilibrium when the Hamiltonian $H(x,y,X,Y)$ in the coordinates (x,y) and their conjugate momenta (X,Y) is of the type $H = (X^2 + Y^2)/2 + V(x,y,X,Y)$, the potential energy V being a sum of homogeneous polynomials in the phase variables of degree strictly greater than two. The fact that the resulting potential V' is a polynomial in the new coordinates (x',y') and the angular momentum $G' = x'Y' - y'X'$ implies that the normalization is a rotation in the configuration space from a fixed frame to an ideal frame. The technique is intended for normalization is a rotation in the configuration space from a fixed frame to an ideal frame. The technique is intended for normalizing an Hamiltonian in equilibrium at the origin when the Lie derivative associated with the quadratic part is not semi-simple, e.g. the planar Restricted Problem of Three Bodies at the equilateral equilibrium L_4 when the basic frequencies are equal (Routh's singular case).

12. KEY WORDS *(Six to twelve entries; alphabetical order; capitalize only proper names; and separate key words by semicolons)*

Generalized inverses; Hamiltonian Mechanics; Lie algebras; Non-linear oscillations; Normalization; Representation theory

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