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NUMERICAL COMPARISONS OF SEVERAL ALGORITHMS FOR TREATING INCONSISTENT DATA IN A LEAST-SQUARES ADJUSTMENT OF THE FUNDAMENTAL CONSTANTS

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U.S. DEPARTMENT OF COMMERCE, **Malcolm Baldrige**, ***Secretary***

NATIONAL BUREAU OF STANDARDS, **Ernest Ambler**, ***Director***

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ABSTRACT

A number of recently proposed algorithms for treating inconsistent or discrepant data in a least-squares adjustment of the fundamental physical constants, along with several new but related algorithms, are compared in detail. The comparisons are first made by means of the numerical results the algorithms yield when applied to the same data considered by Cohen and Taylor in their 1973 adjustment which led to the recommended set of constants adopted by CODATA and in current use. A selected number of the algorithms are then further compared through the numerical results they yield when applied to the data considered by Taylor, Parker, and Langenberg in their 1969 adjustment and by Cohen and DuMond in their 1963 adjustment. The principal conclusion of this paper is that the actual algorithm used to carry out an adjustment is much less important than the data finally selected for inclusion in the adjustment.

Key Words: Data analysis; discrepant data; fundamental constants; inconsistent data; least-squares adjustments; physical constants.

1. INTRODUCTION

In the three most recent least-squares adjustments of the fundamental physical constants, those carried out by Cohen and DuMond in 1963 [1],* Taylor, Parker, and Langenberg in 1969 [2], and Cohen and Taylor in 1973 [3], the input data were considerably censored because of large discrepancies among the data. For example, in the 1963 adjustment the value of the fine-structure constant α, determined from the hydrogen ground-state hyperfine splitting, was discarded; in the 1969 adjustment two "high values" of the magnetic moment of the proton in H_2O in units of the nuclear magneton μ_p'/μ_N, were eliminated; and in the 1973 adjustment two values of the Faraday constant F, were censored. While in each case there was some additional supporting evidence for discarding the suspect data, the principal justification was their significant disagreement with the remaining data.[1]

Not surprisingly, "Murphy's Law" was apparently operating in each of these three adjustments -- subsequent measurements indicated that it was highly likely that the wrong data had been discarded. For example, the value of α derived from Parker, Taylor, and Langenberg's ac Josephson effect determination of 2e/h in 1967 showed that the hydrogen hyperfine splitting value of α discarded by Cohen and DuMond in their 1963 adjustment was more nearly correct than the value of α they retained which had been derived from the deuterium fine-structure measurements of Lamb and coworkers. The two values of μ_p'/μ_N discarded by Taylor and colleagues in their 1969 adjustment were subsequently shown to be more nearly correct than the three values they retained by the highly consistent, sub-part-per-million (ppm) determinations of Petley and Morris and of Mamyrin and coworkers. And the two values of F eliminated by Cohen and Taylor in their 1973 adjustment have received signif-

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*Footnotes begin on p. 43 and literature references begin on p. 46.

icant support from the recent high precision, coulometric redetermination of F by Bower and Davis; and from Kibble and Hunt's measurement of the proton gyromagnetic ratio in H_2O by the so-called high field method. (See Refs. [1-6] for further details.)

Censoring the wrong data in the 1963 and 1969 adjustments has led to large changes (in comparison with their assigned uncertainties) in the recommended values of several constants from one adjustment to another. For example, the 1963 recommended value of α^{-1} was 137.0388(6) (4.6 ppm) while the 1969 recommended value was 137.03602(21) (1.5 ppm), a decrease of 20 ppm or over four combined standard deviations;[2] the 1969 recommended value of μ_p'/μ_N was 2.792709(17) (6.2 ppm) while the 1973 (and present) recommended value is 2.7927740(11) (0.38 ppm), an increase of 23 ppm or nearly four combined standard deviations; and because of the situation with the Faraday constant discussed above, similar significant changes in the 1973 recommended values of several constants are likely to result with the completion of the 1982 adjustment which is currently being carried out by Cohen and Taylor under the auspices of the CODATA Task Group on Fundamental Constants.

The author has argued [7] that such changes are really not as serious as they might first appear and that the insight gained during the course of the critical review which necessarily accompanies a least-squares adjustment of the constants is much more important than the recommended values themselves. However, others, in particular a group at the Mendeleyev Institute of Metrology (VNIIM) U.S.S.R., are very much disturbed by such large changes in the recommended values. Moreover, this group feels that such large variations can be significantly reduced and values closer to the true (but unknown) values obtained by including all of the data available at a given epoch in the adjustment, even if some are discrepant. To this end, the VNIIM group [8-12] has developed a specific algorithm for including discrepant data in a least squares adjustment and has applied it to modified versions of the 1963, 1969, and 1973 data as well as to more recent data. They conclude that in general, the recommended values of the constants from one adjustment to another are in better agreement if all of the data available at the time of each adjustment are included using the new algorithm, and that the values are closer to the true values.

The theoretical basis for the VNIIM algorithm is not well founded. In an attempt to place the handling of discrepant data in a least-squares adjustment on firmer theoretical ground, E.R. Cohen [13-17] has examined the proposed algorithm, has developed a number of alternatives, and has compared several of them with the VNIIM algorithm by applying them to a modification of the 1973 data.

Although the papers of both the VNIIM group and Cohen have included numerical examples, these have not been given in any great detail nor have they been particularly extensive. Since it is likely that the 1982 adjustment will be carried out using one of the proposed algorithms or an appropriate modification thereof, the author feels it is important that they become more widely known and that the detailed results of their application to the uncensored and unmodified data of past adjustments be made available for critical review. Thus, it is the purpose of this paper first to compare these various algorithms, as well as several new but closely related algorithms developed by the author, through the detailed numerical results they yield when applied to the same data considered by Cohen and Taylor in their 1973 adjustment; and then to compare further a selected number of the algorithms through the numerical results they yield when applied to the data considered by Taylor, Parker, and Langenberg in 1969 and by Cohen and DuMond in 1963.

2. SUMMARY OF ALGORITHMS

We summarize here the various algorithms for treating discrepant data which will be compared in detail in the present paper.[3] However, it should be borne in mind throughout this paper that the underlying reason for trying to devise such algorithms is the belief that if discrepant data are included in an adjustment, and it turns out at a later date that these data are in fact reliable, then the changes in the recommended values of the constants from one adjustment to the next will be less than if the discrepant data had been discarded. More importantly, the recommended values should be closer to the true values. Although the discussion in Section 1 implies that this might possibly have been the case with the last three adjustments, it would definitely not have been the case if the unreliable nature of the discarded data had in fact been confirmed by subsequent experiments. If this had occurred, then including the discrepant data would also have led to disturbing variations in the constants from one adjustment to the next and shifts from their true values. In other words, including discrepant data will reduce such variations and shifts if it later turns out that the originally discrepant data are reliable, but will contribute to such variations and shifts if it later turns out that the discrepant data are in fact unreliable. (Guess right and you win, guess wrong and you loose!)

2.1. Traditional Approaches

2.1.1. <u>Birge Ratio Algorithm</u>. One of the simplest ways of objectively treating discrepant data in a least-squares adjustment, and one which sees considerable use in data analysis, involves the Birge ratio $R_B \equiv (\chi^2/F)^{1/2}$, where χ^2 is the familiar statistic chi-squared for the adjustment and F is the number of degrees of freedom: $F = N - M$ where N is the number of items of stochastic input data and M the number of unknowns or adjustable constants. In this approach, one first carries out an adjustment using all of the available data with their a priori assigned uncertainties σ_i (weights $w_i = 1/\sigma_i^2$), computes R_B, and then carries out a second adjustment with new uncertainties $\sigma_i' = R_B\sigma_i$ (weights $w_i' = 1/R_B^2\sigma_i^2$). Since this leads to a value of chi-squared for the second adjustment equal to its statistically expected value $\chi'^2 = F$, it may be viewed as being statistically consistent.[4,5]

While the adjusted values of the unknowns are the same in the two adjustments because each a priori uncertainty in the second adjustment is changed by the same multiplicative factor R_B, the uncertainties of the adjusted values of the unknowns in the two adjustments differ by this multiplicative factor. For the discrepant data case ($R_B > 1$), the increase in uncertainty of the adjusted values is a direct result of the inconsistencies among the data with the larger uncertainties better reflecting the true reliability of the adjusted values of the unknowns. (This approach may also be described as calculating the uncertainties on the basis of external consistency; see Refs. [1-3].)

Although the Birge ratio method of handling discrepant data is simple and objective, it inappropriately treats the uncertainties assigned to all of the stochastic input data as equally reliable and all values of $(\sigma_i'/\sigma_i)^2$ within some range as equally probable. This leads to the undesirable result of having the a priori uncertainty of a highly consistent input datum (residual much less than unity -- see Footnote 1) with possibly a well defined uncertainty increased by the same factor as a highly discrepant datum (residual much greater than unity) with perhaps a poorly defined uncertainty.

2.1.2. Two-Stage Birge Ratio Algorithm. There is an obvious generalization of the above approach which was used in part in the 1973 adjustment and which partially responds to this last difficulty. It involves using the Birge ratio in two distinct stages. First, the weighted average of each subgroup of stochastic input data of the same kind (e.g., all the values of α) is computed and the uncertainty of each weighted average expanded by its Birge ratio if greater than unity. Then, these individual weighted averages with their uncertainties (expanded or not as the case may be) are used as input data in an adjustment to obtain the adjusted values of the unknowns. If R_B for this second adjustment is greater than unity, then it is repeated with the uncertainties of the individual weighted averages expanded by this Birge ratio. Thus, each data subset is first made internally consistent, and then the weighted averages of the different subsets are made consistent with each other.

While this approach is also simple and objective, and to some extent reduces the influence of a discrepant input datum on the bulk of the more reliable data, it does have the drawback of possibly giving greater weight in the adjustment to the indirect values of a particular quantity than to the directly measured values. That is, the first step of the procedure decreases the weight that the directly measured values would have had in determining the adjusted values of the unknowns (assuming R_B for the weighted average is greater than unity), thus allowing the indirect values to have a greater influence in determining the unknowns. (See Refs. [1-3] for a further discussion of direct and indirect values.)

2.2. The VNIIM Algorithm

The VNIIM group [8-12] suggests altering the usual least-squares procedure by changing the *a priori* uncertainties σ_i assigned each input datum to new values σ_i' such that the sum

$$S = \sum_{i=1}^{N} (R_i^2 - 1)^2 \tag{1}$$

is minimized, where $R_i^2 \equiv (\sigma_i'/\sigma_i)^2$. This minimization is to be carried out subject to the constraint that the value of the statistic χ'^2 calculated for the adjustment employing the new σ_i' is equal to its expected value $F = N - M$, the degrees of freedom of the adjustment. The adjusted values of the unknowns $\overline{X}_\alpha'$ ($\alpha = 1...M$), are those which minimize the usual least-squares expression

$$Q' = \sum_{i=1}^{N} [A_i - \overline{A}_i(X_\alpha)]^2/\sigma_i'^2 \equiv \sum_{i=1}^{N} [r_i'(X_\alpha)]^2 \quad , \tag{2}$$

where A_i is the numerical value of the i^{th} input datum and $\overline{A}_i(X_\alpha)$ is A_i expressed in terms of the unknowns or adjustable constants X_α. The sum of the squared normalized residuals r_i' evaluated with $X_\alpha = \overline{X}_\alpha'$ is the statistic chi-squared for the adjustment and thus we have

$$\chi'^2 = F = \sum_{i=1}^{N} [r_i'(\overline{X}_\alpha')]^2 \quad . \tag{3}$$

The values of $\overline{X}_\alpha'$ and σ_i' are found, therefore, by simultaneously minimizing Eqs. (1) and (2) subject to $Q'(\min) = F$, which is Eq. (3).

The simultaneous minimization of Eqs. (1) and (2) may be carried out by combining them into a single relation through a Lagrange multiplier μ, and minimizing

$$Q'^* = \sum_{i=1}^{N} [r_i'^2 - \mu(R_i^2 - 1)^2] \quad . \tag{4}$$

Since the term multiplying μ in Eq. (4) is independent of the X_α, the least-squares solution $\overline{X}_\alpha'$ of Eq. (4) obtained by setting $\partial Q'^*/\partial X_\alpha = 0$ is the same as the usual least-squares solution but carried out using the revised uncertainties σ_i'. The latter may be obtained by setting $\partial Q'^*/\partial\sigma_i' = 0$ and using Eq. (3) to eliminate μ. The result is

$$R_i^4(R_i^2 - 1) = R_i^2(r_i'^2/F) \sum_{j=1}^{N} R_j^2(R_j^2 - 1) \quad . \qquad (5)$$

In practice, Eq. (5) may be solved iteratively by, for example, first carrying out an adjustment using the original σ_i and taking as the first iterate $R_i = R_B$, where R_B is the Birge ratio. These values of R_i are then substituted into the right-hand side of Eq. (5) with r_i' replaced by r_i, and the new values of R_i to be used in the next iteration determined by solving Eq. (5) with the R_i on the left-hand side taken as unknowns.[6] The procedure is then continued until $|\chi'^2 - F|$ is sufficiently small, say 10^{-5}, and/or the R_i and $\overline{X}_\alpha'$ change by negligible amounts from one iteration to the next.

As noted earlier, a firm theoretical basis for Eq. (1) is yet to be given. However, the VNIIM group has justified it on the following grounds: One may introduce a probability density function $\phi_i(R_i^2)$ on the assumption that the R_i^2 are random quantities each with average value equal to unity and with the same variance σ^2. This means there is a certain probability that the _a priori_ assigned uncertainty σ_i of each stochastic input datum will differ from its true value and that all of the _a priori_ assigned uncertainties are equally reliable. The values of the R_i^2 which give a consistent adjustment can then be found from the equation

$$\prod_{i=1}^{N} \phi_i(R_i^2) = \text{maximum} \quad , \qquad (6)$$

with the requirement that $\chi'^2 = F$. The choice of the functions ϕ_i in the form of a uniform distribution leads to the solution $R_i = R_B$ for all i (the Birge ratio algorithm of Section 2.1.1), while the choice of the ϕ_i in the form of a normal distribution leads to Eqs. (1) and (5). The uniform distribution corresponds to assuming that all values of R_i^2 (within a specified range) are equally probable, while the normal distribution corresponds to assuming that the more R_i^2 differs from unity, the less probable is that value of R_i^2.[7]

In summary, the algorithm proposed by the VNIIM group replaces the somewhat subjective approach of completely discarding discrepant data with an objective technique which retains all of the data but alters each of the *a priori* assigned uncertainties in accordance with what Cohen [13] terms a "cost function", that is, Eq. (1). This cost function, in combination with the requirement $\chi'^2 = F$, leads to larger values of σ_i'/σ_i for the discrepant data than for the consistent data, thereby reducing but not entirely eliminating their influence on the adjusted values of the unknowns. However, the VNIIM algorithm still suffers from the improbable assumption that all of the *a priori* assigned uncertainties are equally reliable. One well knows that the uncertainties in some experiments are much easier to evaluate than in others, that some workers carry out their experiments with greater care than others, etc.

2.3. Generalizations of the VNIIM Algorithm

In order to better understand Eq. (1) Cohen [13] has generalized it to

$$S = \sum_i W_i \; f_i(w_i, w_i') = \text{minimum} \quad , \tag{7}$$

where the *a priori* weights w_i and new weights w_i' are related to the uncertainties σ_i and σ_i' in the usual way, that is, $w_i = 1/\sigma_i^2$ and $w_i' = 1/\sigma_i'^2$;

and W_i is a weighting factor which may be different for each cost function component f_i, but which we presently assume to be unity because a completely objective way of arriving at such weights is not at all clear.[8] The only restrictions needed on the f_i are that they be a positive definite function of w_i' such that $f_i = 0$ for $w_i' = w_i$ and $f_i > 0$ for $w_i' \neq w_i$.

In terms of weights, the VNIIM group's proposed cost function, Eq. (1), may be written as

$$S = \sum_{i=1}^{N} (\Delta w_i/w_i')^2 = \sum_{i=1}^{N} (R_i^2 - 1)^2 \quad , \tag{8}$$

where $\Delta w_i \equiv w_i - w_i'$ and as before $R_i^2 \equiv (\sigma_i'/\sigma_i)^2$.

2.3.1. <u>Inverse Algorithm</u>. Cohen notes [13] that an expression similar to Eq. (8) and perhaps equally valid is

$$S = \sum_{i=1}^{N} (\Delta w_i/w_i)^2 = \sum_{i=1}^{N} (1/R_i^2 - 1)^2 \quad , \tag{9}$$

which we term the "inverse" of the VNIIM algorithm because of the way R_i^2 enters. Equations (8) and (9) are approximately equal if w_i' and w_i are approximately equal, but differ significantly if w_i' and w_i differ significantly. Proceeding as before with the constraint $\chi'^2 = F$, one can show that the expression corresponding to Eq. (5) for this cost function is

$$1 - 1/R_i^2 = R_i^2(r_i'^2/F) \sum_{j=1}^{N} (1/R_j^2 - 1/R_j^4) \quad . \tag{10}$$

A major difficulty with Eq. (9) is that it leads to meaningless negative values of w_i' for highly discrepant items of stochastic input data because it is finite for $w_i' = 0$ ($R_i^2 = \infty$). For such data, one must impose the additional constraint $w_i' \geq 0$ $(0 < R_i^2 \leq \infty)$. The end result is that this algorithm essentially discards highly discrepant items of input data by diminishing their weights to negligible values ($R_i^2 >> 1$). Since these items are not available to influence the adjusted values of the unknowns, some of the latter can differ markedly from the values which result from the VNIIM algorithm and other algorithms which do not require the additional constraint $w_i' \geq 0$. (The numerical examples in Section 3 will clearly demonstrate this.) Thus, the inverse algorithm would not seem to completely satisfy the underlying reason for devising such algorithms as was discussed at the start of Section 2.

2.3.2. Natural Log Algorithm. Cohen also suggests a cost function intermediate between Eqs. (8) and (9):

$$S = \sum_{i=1}^{N} \ell n^2 (w_i/w_i') = \sum_{i=1}^{N} \ell n^2 R_i^2 \quad , \tag{11}$$

which leads to

$$R_i^2 \ \ell n \ R_i = R_i^2 (r_i'^2/F) \sum_{j=1}^{N} \ell n \ R_j \quad . \tag{12}$$

He points out that this expression has several advantages. First, it is invariant as to whether S is considered to be a function of the weights w_i or uncertainties σ_i. Second, the penalty is exactly the same whether $R_i = p$ or $R_i = 1/p$, for example, whether σ_i is doubled or cut in half. This desirable property is approximately true for Eqs. (7) and (8) only when R_i is very near unity.

2.3.3. Geometrical Mean Algorithm. It occurred to us that the geometrical mean $(w_i w_i')^{1/2}$ could have been taken as the "normalization" of Δw_i in Eqs.

(8) and (9) (i.e., the denominators in the expression for S) in place of either w_i' or w_i which lead to the VNIIM and inverse algorithms, respectively. The resulting cost function would be

$$S = \sum_{i=1}^{N} (\Delta w_i)^2/w_i w_i' = \sum_{i=1}^{N} (R_i^2 - 1)^2/R_i^2 \quad , \tag{13}$$

with

$$R_i^4 - 1 = R_i^2(r_i'^2/F) \sum_{j=1}^{N} (R_j^4 - 1)/R_j^2 \quad . \tag{14}$$

Comparing Eqs. (13) and (14) with Eqs. (8) and (5) leads to the conclusion that the geometrical mean and VNIIM algorithms should yield similar results.

2.3.4. Simple Mean Algorithm. If the normalization of Δw_i is taken to be the simple mean $(w_i + w_i')/2$, instead of the geometrical mean, then one obtains the cost function

$$S = 4\sum_{i=1}^{N} (\Delta w_i)^2/(w_i + w_i')^2 = 4\sum_{i=1}^{N} (R_i^2 - 1)^2/(R_i^2 + 1)^2 \ , \tag{15}$$

with

$$R_i^4(R_i^2 - 1)/(R_i^2 + 1)^3 = R_i^2(r_i'^2/F) \sum_{j=1}^{N} R_j^2(R_j^2 - 1)/(R_j^2 + 1)^3 . \tag{16}$$

Comparing Eqs. (15) and (16) with Eqs. (9) and (10) leads to the conclusion that the simple mean and inverse algorithms should also yield similar results. In particular, the same additional constraint imposed for the inverse algorithm, namely $w_i' \geq 0$, must be imposed in this case as well since Eq. (15) is finite for $w_i' = 0$ ($R_i^2 = \infty$).

2.4. Algorithms in Which Random and Systematic Uncertainties Are Separated

It is common practice to report the total uncertainty σ of an experimental result as the square root of the sum of the squares of the random uncertainty component σ_R and the systematic uncertainty component σ_S:

$$\sigma = (\sigma_R^2 + \sigma_S^2)^{1/2} \quad . \tag{17}$$

In Ref. [13] Cohen suggests that in many fundamental constant determinations the random uncertainty is much better defined than the assigned systematic uncertainty. The reason is that σ_R is usually obtained in a well prescribed, statistically sound way from repeated observations. In contrast, σ_S is often based on rough estimates using approaches which differ widely from one experimenter to another. Moreover, in some cases important sources of systematic error may be entirely overlooked. He therefore suggests that in treating discrepant data, it may not be unreasonable to require that σ_R remain fixed and to vary σ solely by varying σ_S.

On the other hand, one could argue that Eq. (17) is not statistically well justified and that in complex experiments requiring several ancillary measurements, each with its own random and systematic undertainty components, the random and systematic uncertainty components of the final result cannot really be separated in a meaningful way. Furthermore, the uncertainty (variance) in σ_R is given approximately by [18]

$$\mathrm{var}(\sigma_R) \doteq \sigma_R^2/2n \ , \tag{18}$$

where n is the number of repeated measurements. In many experiments n is so small that σ_R is quite uncertain (e.g., for n = 10 the one-standard-deviation uncertainty in σ_R is 0.22 σ_R). As we shall see in Section 3, even for a set of input data with some highly discrepant items the majority of the values of σ_i'/σ_i required to achieve $\chi'^2 = F$ using one of the more reasonable of the proposed algorithms is 1.5 or less, with only a few as large as 2.0. Since such changes in σ_i are not at all inconsistent with the typical uncertainty in σ_R, there would seem to be no compelling reason to allow only σ_S to vary. Nevertheless, for the purpose of better understanding the various proposed algorithms as well as for completeness, we shall examine this possibility.

The idea of only allowing σ_S to vary can in principle be applied to all of the algorithms so far presented. However, in light of the above discussion, but also because it would lead to a large increase in the number of algorithms we would have to consider, only two representative examples will be examined. Since the natural log and geometrical mean algorithms yield results similar to the VNIIM algorithm, we shall not explore them here. Similarly, because the inverse and simple mean algorithms are alike and because they lead to values of σ_i'/σ_i for highly discrepant data which are so large that these data are essentially completely discarded from the adjustment, we shall disregard them as well. We shall also disregard the two-stage Birge ratio algorithm since it is not meaningful to separate the uncertainties of the weighted averages obtained in the first stage into random and systematic components. Thus, only the Birge ratio and VNIIM algorithms will be considered.

2.4.1. Internal Birge Ratio Algorithm. Assuming the random uncertainty to be fixed, Cohen [13] has shown that the Birge ratio algorithm of Section 2.1.1 defined by $\sigma_i' = R_B\sigma_i$ with $\chi'^2 = F$ becomes

$$\sigma_i'^2 = \sigma_{Ri}^2 + \sigma_{Si}'^2 = \sigma_{Ri}^2 + R_{IB}^2\,\sigma_{Si}^2 \quad , \tag{19a}$$

or equivalently,

$$\sigma_i' = [(\sigma_{Ri}^2 + \sigma_{Si}'^2)/(\sigma_{Ri}^2 + \sigma_{Si}^2)]^{1/2}\,\sigma_i \tag{19b}$$

$$= [(T_i^2 + R_{IB}^2)/(T_i^2 + 1)]^{1/2}\,\sigma_i \quad , \tag{19c}$$

with once again $\chi'^2 = F$.[9] Here as before σ_i and σ_i' are, respectively, the a priori and altered total uncertainties of the i^{th} input datum; σ_{Ri} is the datum's a priori and assumed fixed random uncertainty; σ_{Si}' its altered systematic uncertainty; σ_{Si} the datum's a priori assigned systematic uncertainty; $T_i^2 \equiv (\sigma_{Ri}/\sigma_{Si})^2$; and R_{IB} an "internal Birge ratio" such that when all of the σ_{Si} are increased by the same factor R_{IB} (i.e., $\sigma_{Si}' = R_{IB}\,\sigma_{Si}$), the resulting σ_i' are such that the adjustment is consistent, that is, $\chi'^2 = F$. Equation (19a) is, of course, what one might intuitively expect. An iterative procedure may be used to calculate R_{IB} with R_B as the first iterate. Since the σ_{Ri} are in general finite, R_{IB} will in general exceed R_B. Unfortunately, the disadvantages of the Birge ratio algorithm discussed in Section 2.1.1 are still prevalent in this modification.

2.4.2. VNIIM Algorithm for Systematic Uncertainties. Application of the VNIIM algorithm, Eq. (1), only to the systematic uncertainty components yields as a cost function

$$S = \sum_{i=1}^{N} [(\sigma_{Si}'/\sigma_{Si})^2 - 1]^2 = \sum_{i=1}^{N} (R_{Si}^2 - 1)^2 \quad , \tag{20}$$

where $R_{Si}^2 \equiv (\sigma_{Si}'/\sigma_{Si})^2$. As above, Eq. (17) is used to relate σ_i to σ_{Ri} and σ_{Si}; and σ_i' to σ_{Ri} and σ_{Si}'. With the usual constraint $\chi'^2 = F$, the expression analogous to Eq. (5), obtained by differentiating Eq. (4) with respect to σ_{Si}' rather than σ_i' (and with R_i^2 replaced by R_{Si}^2), is

$$(T_i^2 + R_{Si}^2)^2(R_{Si}^2 - 1) = (T_i^2 + R_{Si}^2)(r_i'^2/F)\sum_{j=1}^{N}(T_j^2 + R_{Sj}^2)(R_{Sj}^2 - 1), \tag{21}$$

where, as before, $T_i^2 \equiv (\sigma_{Ri}/\sigma_{Si})^2$. Equation (21) may be treated in a manner similar to Eq. (5).

Since it is assumed that the random uncertainties remain unchanged, the cost of changing σ_i to σ_i' is borne solely by the systematic uncertainties. However, the form of Eq. (20) is such that the cost of this change is the same whether σ_{Si} is a major or minor contributor to the total a priori assigned uncertainty σ_i -- in other words, whether the change in σ_{Si} leads to a large or small change in σ_i. Intuitively, one would expect that the penalty should be less for changes in σ_{Si} which do not lead to large changes in σ_i.

At first glance, one might consider addressing this last criticism by altering Eq. (20) to the following:

$$S = \sum_{i=1}^{N} [(\sigma_{Si}'^2 - \sigma_{Si}^2)/\sigma_i^2]^2 = \sum_{i=1}^{N} (R_{Si}^2 - 1)^2/(T_i^2 + 1)^2, \tag{22}$$

where, as before, $T_i^2 \equiv (\sigma_{Ri}/\sigma_{Si})^2$ and $R_{Si}^2 \equiv (\sigma_{Si}'/\sigma_{Si})^2$. The desirable feature of this expression is that the cost of changing a particular a priori systematic uncertainty depends on the size of the corresponding random uncertainty. That is, for a given change in σ_{Si}, the cost is greater for σ_{Si} larger than σ_{Ri} ($T_i < 1$) than for σ_{Si} smaller than σ_{Ri} ($T_i > 1$). If $T_i >> 1$, then large changes in σ_{Si} still cost relatively little. This is as one might expect since it is difficult to investigate potential systematic errors in an experiment dominated by a large random uncertainty. Thus, changing the a priori assigned systematic uncertainty by comparatively large amounts for such experiments is not particularly disturbing and should not result in a large penalty. Surprisingly, however, it is readily shown that Eq. (22) is identical to the original VNIIM algorithm and therefore offers nothing new.

2.4.3. VNIIM Algorithm with Weights. As pointed out in Footnote 7 in connection with the discussion of Eq. (7), the weights W_i may be used to take into account the relative reliability of the *a priori* assigned uncertainties σ_i. One possible choice for such weights is

$$W_i^* \equiv \sigma_{Si}^2/(\sigma_{Ri}^2 + \sigma_{Si}^2) = \sigma_{Si}^2/\sigma_i^2 = 1/(T_i^2 + 1) \quad , \tag{23}$$

where, as before, $T_i^2 \equiv (\sigma_{Ri}/\sigma_{Si})^2$. This weighting factor behaves as seems appropriate: W_i^* is near unity for experiments dominated by systematic uncertainty, thus leading to a large contribution to the cost function S from the cost function component f_i; and is significantly diminished for experiments dominated by random uncertainty, leading to a small contribution to S from f_i. For example, $W_i^* = 0.9$ for $T_i = 1/3$ while $W_i^* = 0.1$ for $T_i = 3$. Although this again assumes that experiments in which σ_R is dominant are inherently more reliable than experiments in which σ_S is dominant, the difficulties discussed earlier (p. 13, second paragraph) regarding the validity of allowing only σ_{Si} to vary are not prevalent in the use of Eq. (23) for the weights W_i since there is no special assumption that σ_{Ri} is fixed.

From Eq. (7) the generalization of the VNIIM algorithm to the case where the weights are constants (i.e., independent of σ_i' and X_α) such as are given by Eq. (23) yields the cost function

$$S = \sum_{i=1}^{N} W_i^*(R_i^2 - 1)^2 \quad . \tag{24}$$

The expression analogous to Eq. (5) is then

$$R_i^4(R_i^2 - 1) = R_i^2(r_i'^2/W_i^*F) \sum_{j=1}^{N} W_j^*R_j^2(R_j^2 - 1) \; . \tag{25}$$

While we could explore the effect of including weights for the other algorithms we have discussed, as well as a weighting factor other than Eq. (23), we choose not to for essentially the same reasons given at the end of Section 2.4. The VNIIM algorithm should be sufficiently representative of the method. We do note that all of the previously given equations analogous to Eqs. (1) and (5) can be "converted" to their proper equivalents of Eqs. (24) and (25) by following the exact form of the latter with regard to the position of the factor W_i*.[10]

2.5. The Extended Least-Squares Algorithm

In Refs. [14,15,17], Cohen has attempted to develop an algorithm for treating discrepant data which is more firmly based on sound statistical principles than is the VNIIM algorithm and its various derivatives, and which also does not have the limitations of the Birge ratio algorithms. The basic idea behind his approach is to recognize that the a priori uncertainty assigned each stochastic input datum is itself uncertain, and to assume that this uncertainty may be characterized by a "confidence parameter" v_i. He defines ε_i as the true (but unknown) error in the i^{th} input datum, σ_i^2 as the true (but unknown) variance of ε_i, and s_i^2 as the a priori estimate of this true variance, that is, s_i is the a priori assigned uncertainty of the i^{th} input datum. (Note that this notation differs from what we have been using up to this point.) Cohen further assumes that the probability distribution for the errors ε_i is approximately Gaussian, and makes other reasonable postulates about the quantities ε_i, σ_i, and s_i. He also introduces a factor of 2 into his definition

$$< s_i^2 \; s_j^2 > = \sigma_i^2 \; \sigma_j^2 (1 + 2\delta_{ij}/v_i) \; , \qquad (26)$$

where δ_{ij} is the Kronecker delta, so that v_i can be identified with the number of independent observations or degrees of freedom characteristic of the experiment which determined the i^{th} datum. That is, Eq. (26) implies that the uncertainty of the a priori assigned uncertainty of the i^{th} datum (variance of its variance) is

$$var(s_i^2) = 2\sigma_i^4/v_i \quad , \tag{27a}$$

or

$$var(s_i) = \sigma_i^2/2v_i \quad . \tag{27b}$$

The identification of v_i in Eq. (27b) with n in Eq. (18) is obvious. Cohen emphasizes, however, that it is unnecessary to give v_i such a strict interpretation. One may identify it with the "effective" number of independent observations which defines the variance of the a priori assigned variance, or as stated above, as simply a parameter which measures the reliability of the a priori uncertainty assigned the i^{th} input datum.

Cohen goes on to define $\hat{\sigma}_i^2$ and $\hat{X}_\alpha$ as the minimum variance, unbiased estimators of σ_i^2 and the adjustable constants or unknowns, respectively. For $\hat{X}_\alpha$ he takes a linear combination of the stochastic input data, and for $\hat{\sigma}_i^2$ a linear combination of s_i^2 and the differences between all of the input data and their adjusted values. His approach to altering the a priori assigned uncertainties s_i is thus embodied in his expression for $\hat{\sigma}_i^2$. The requirement that the estimators be both unbiased and minimum variance allows him to solve for both $\hat{\sigma}_i^2$ and $\hat{X}_\alpha$ with the result

$$\hat{\sigma}_i^2 = (v_i s_i^2 + \sigma_i^2 \chi^2)/(v_i + F) \ , \tag{28}$$

where as before F = N - M is the number of degrees of freedom for the adjustment and χ^2 is the statistic chi-squared. The latter depends implicitly on the $\hat{\sigma}_i^2$ since the $\hat{X}_\alpha$ are obtained in the usual way using $\hat{\sigma}_i$ as the uncertainty to be associated with the i^{th} input datum (i.e., the weight of the i^{th} input datum is $w_i = 1/\hat{\sigma}_i^2$). But as Cohen points out, Eq. (28) is hardly a solution since it expresses $\hat{\sigma}_i^2$, the estimate for the unknown variance σ_i^2, in terms of the unknown variance itself. If it were known, then there would be no need to estimate it!

Cohen notes though that it is possible to obtain a meaningful solution by assuming σ_i^2 and var(s_i) to be independently estimated, resulting in the replacement of σ_i^2 in Eq. (28) by its estimate $\hat{\sigma}_i^2$ and of v_i by its estimate $\hat{\sigma}_i^2/2\mathrm{var}(s_i)$ as obtained from Eq. (27b). If then var(s_i) is estimated by $s_i^2/2v_i$ [see Eq. (18)], Eq. (28) finally becomes

$$\hat{\sigma}^2 = [1 + (\chi^2 - F)/v_i]\, s_i^2 \quad , \tag{29}$$

or in terms of the notation used with the other algorithms we have previously discussed,

$$\sigma_i' = [1 + (\chi'^2 - F)/v_i]^{1/2}\, \sigma_i \quad . \tag{30}$$

Equation (30) gives the new, expanded uncertainty σ_i' for the i^{th} stochastic input datum in terms of its known <u>a priori</u> assigned uncertainty σ_i; the known confidence parameter for σ_i, v_i; the known degrees of freedom F; and the value of χ'^2 calculated for the adjustment using the σ_i'. Of course, the relation is not as straightforward as it appears since χ'^2 depends implicitly on all of the σ_i'. Consequently, as for several of the other algorithms previously discussed, some sort of iterative procedure must be used until a sufficiently self consistent result is obtained. Cohen gives one such approach in Ref. [17].

There are four especially interesting features of this algorithm as it is represented in Eqs. (29) or (30). First, items of input data with the same confidence parameter v_i will have their <u>a priori</u> uncertainties σ_i expanded by the same factor. This means that if ν_i is the same for all of the data, the adjusted values of the unknowns will be the same as for the Birge ratio algorithm but with smaller uncertainties. Second, the more reliable the input datum (i.e., the larger its confidence parameter), the less its a priori uncertainty will be altered (as $\nu_i \to \infty$, $\sigma_i'/\sigma_i \to 1$). Third, since the a priori uncertainties σ_i are positive and since the σ_i' must also be real and positive, $\chi'^2 > F - \nu_i(\min)$. That is, χ'^2 for the adjustment must be greater than the number of degrees of freedom for the adjustment minus the smallest confidence parameter.[11] Fourth, the algorithm will yield the expected value $\chi'^2 = F$, in only two cases: if $\nu_i = 0$ for each datum (it then reduces to the Birge ratio algorithm); and if σ_i for each datum just happens to be equal to σ_i'. In general, however, the algorithm will yield a value of χ'^2 between F and the value of χ^2 obtained using the a priori uncertainties σ_i. If most of the ν_i are large, implying that the σ_i are reliable (small uncertainty), then χ'^2 will be closer to χ^2 than to F. On the other hand, if the ν_i are generally small (e.g., near unity), implying that the σ_i are rather uncertain, then χ'^2 will be closer to F.

The principal difficulty with this algorithm is arriving at meaningful confidence parameters ν_i for each datum. In Ref. [17] Cohen applies the algorithm to the spectroscopic data bearing on α^{-1} considered in the 1973 adjustment of Cohen and Taylor [3] and attempts to derive a value of ν_i in an objective way for each datum by separating its total uncertainty into two components: Component A, characterized by variance σ_{Ai}^2 and degrees of freedom ν_{Ai}, is that portion of the total uncertainty or variance which is based on a statistical analysis of repeated measurements (essentially, the random component σ_R of Section 2.4). Component B, characterized by variance σ_{Bi}^2 and degrees of freedom ν_{Bi}, is then the remaining variance (essentially the systematic component σ_S of Section 2.4). These quantities are related by

$$\frac{\sigma_i^4}{v_i} = \frac{\sigma_{Ai}^4}{v_{Ai}} + \frac{\sigma_{Bi}^4}{v_{Bi}} \quad , \tag{31a}$$

with

$$\sigma_i^2 = \sigma_{Ai}^2 + \sigma_{Bi}^2 \quad . \tag{31b}$$

In calculating v_i for the data in question, Cohen assumed $v_{Bi} = 1$. He also decided to restrict v_{Ai} to 10 or less because the actual values of v_{Ai} were so large for a number of experiments (due to a large number of separate line-center measurements) that they implied a reliability for these experiments which could not be justified by past experience. Thus, a considerable degree of subjectivity had to be injected into the calculation.

Because of the difficulty in obtaining meaningful values of v_i, we shall only investigate the extended least-squares algorithm in this paper by applying it to the 1973 data. To obtain values of v_i for these data, we use the relation

$$v_i = 1/2x_i^2 \quad , \tag{32}$$

where $x_i\sigma_i$ is the assumed one-standard-deviation uncertainty of the a priori assigned uncertainty σ_i. Simply stated, if σ_i is assumed to be uncertain by 10%, $x_i = 0.1$ and $v_i = 50$; if the assumed uncertainty in σ_i is 50%, $x_i = 0.5$ and $v_i = 2$; etc. [Eq. (32) follows from Eqs. (18) and (27b)]. A brief discussion of how values for the x_i were arrived at for the 1973 data will be given in Section 3.

3. NUMERICAL RESULTS AND DISCUSSION

3.1. Summary of Input Data

Here we present and discuss the numerical results which the algorithms of Section 2 yield when applied to the data considered for use in the 1973, 1969, and 1963 least-squares adjustments. These algorithms are succinctly summarized in Table 1 for easy reference,* while the relevant data are summarized in Tables 2, 2a, 3, 3a, 4, and 4a. The following comments apply to these data.

3.1.1. 1973 Adjustment Data. Table 2 is essentially Cohen and Taylor's Table 27.1 [3] and includes all of the items of input data they considered for use in their 1973 adjustment. No data discussed by these authors in Ref. [3] were excluded from their Table 27.1 solely because they were discrepant. The several items they discussed but did not include in the table were discarded on the basis of a very large uncertainty compared with the uncertainties of other quantities of the same type, or because the experiment was incomplete or preliminary. However, there was no discussion in Ref. [3] of the deuterium fine-structure measurement of α^{-1} by Lamb and coworkers or the high field determination of γ_p' by Thomas and colleagues. The accuracy of both of these experiments was still competitive in 1973 and there were no compelling experimental or theoretical reasons for disbelieving them. Cohen and Taylor's principal basis for excluding these experiments was their historical gross disagreement with a large amount of other data thought to be reliable. Thus, if one were strictly to adhere to the philosophy discussed at the start of Section 2, one could argue that these data should be included in Table 2. We choose not to, however, because the evidence against the reliability of these experiments seems incontrovertible. The point is that one should not blindly include a datum so inconsistent with all of the remaining data that it is clearly wrong. While there obviously has to be some subjectivity associated with such decisions, and each situation has

*The tables begin on p. 49.

to be handled separately, as a rule of thumb one might arbitrarily take something like a five standard deviation discrepancy, or more precisely a normalized residual of five, as an indication that a particular datum should at least be considered for exclusion.

In this regard, we note that item 10.4 of Table 2, the value of α^{-1} obtained from the Kaufman, Lamb et al. determination of the $\Delta E - \mathcal{L}$ splitting in hydrogen, is discrepant by over seven standard deviations (normalized residual of 7.9 -- see Table 6, algorithm 2). One might, therefore, consider discarding this item. Indeed, the VNIIM group did just that in applying their algorithm to the 1973 data [8]. However, we shall initially retain it in order to investigate how the various algorithms handle highly discrepant items.

Our decomposition of the a priori assigned uncertainties σ_i into their random and systematic components is similar to that given by Cohen in Ref. [13] and is based on the detailed discussion contained in Refs. [2] and [3] and the original papers. However, this decomposition was done with only limited accuracy because we did not feel that the time and effort required to improve it significantly was warranted in light of the historical nature of much of the data and the limited purpose of the present work -- our aim here is not to derive a complete set of constants of unquestioned reliability but rather to examine the various algorithms and their limitations. Furthermore, for many experiments it is extremely difficult and often meaningless to try and separate unequivocally the random and systematic components of uncertainty.

Our estimate of the value of each x_i, and thus the confidence parameter v_i through the relation $v_i = 1/2x_i^2$, is based on personal knowledge of the experiments in question, their historical difficulty, the degree of conservatism exhibited by the experimenters in assigning their final uncertainties, and other similar factors. Although they are a rather subjective set of numbers which could differ significantly from a set derived by another author on the same basis or by other means, the extended least-squares algorithm is relatively insensitive to the values of v_i assumed and our set of x_i should suffice to demonstrate the basic features of the algorithm. It is clear from Table 2 that we feel only a few a priori assigned uncertainties are highly reliable.

The auxiliary constants employed with the 1973 data are those used by Cohen and Taylor (Table 11.1 of Ref. [3]) rather than the present best values. Similarly, the auxiliary constants employed with the 1969 and 1963 data are those used at the time (Table XI of Ref. [2] and Table III of Ref. [1], respectively). We choose this approach in order to minimize any effect of hindsight. That is, one of our goals is to answer the following question: If these algorithms had been available at the time of the 1963, 1969 and 1973 least-squares adjustments, what would the result have been, all other things being equal?

Table 2a (and its counterparts for the 1969 and 1963 data, Tables 3a and 4a) gives the input data to be used in the application of the two-stage Birge ratio algorithm. These data follow directly from Table 2 (Tables 3 and 4 for the 1969 and 1963 data, respectively) and are simply the weighted averages of the various groups of like data using their assigned a priori uncertainties σ_i. The uncertainties given in Table 2a (Tables 3a and 4a for the 1969 and 1963 data, respectively) are those resulting from the weighted averages but increased by the Birge ratios for the weighted averages when greater than unity.

The unknowns or adjustable constants used in the application of the various algorithms to the 1973 data are those used in the 1973 adjustment: α^{-1}, A_{BI69}/A, N_A, Ω_{BI69}/Ω, Λ, and μ_μ/μ_p.

3.1.2. <u>1969 Adjustment Data</u>. Table 3 is essentially Taylor, Parker, and Langenberg's Table XVI [2] and includes all of the data they considered for use in their 1969 adjustment. As for the 1973 data just discussed, the only items initially discarded by Taylor et al. on the basis of inconsistency were the Thomas et al. value of γ_p'(high) and the Lamb et al. value of α^{-1}. The other data they discussed but did not include for consideration were discarded on the basis of a large uncertainty compared with the uncertainties of similar quantities, an incomplete or preliminary experiment, or insufficient information. We use item 22a rather than 22b of Table XVI, Ref. [2], because even in 1969 the theoretical uncertainties in the expression for ΔE_H were negligible. This places the full burden of any inconsistencies on the experimental measurements of $(\Delta E - \mathcal{L})_H$ and $\mathcal{L}_H$. As noted earlier, we choose not to apply to either the 1969 or 1963 data the algorithms requiring separation of random and systematic uncertainties because of the difficulty in separating the uncertainties in this way for historical data and because of the limited justification for the approach. The application of these algorithms to the 1973 data should suffice to demonstrate their properties. Similarly, because of the difficulty in reliably estimating the confidence parameter v_i, the extended least-squares algorithm is applied only to the 1973 data.

The adjustable constants used to analyze the 1969 data are α^{-1}, e, A_{NBS}/A, N_A, and Λ.

3.1.3. <u>1963 Adjustment Data</u>. Table 4 is taken from Cohen and DuMond's Table X [1] with two exceptions. First, so that the results using the 1963 data may be readily compared with the results using the 1969 and 1973 data, the proton moment in nuclear magnetons and the proton gyromagnetic ratio do not include the 26.0 ppm correcton for diamagnetism. Second, the <u>a priori</u> assigned uncertainties of the two values of $N_A\Lambda^3$ are taken from Table V of Ref. [1]

rather than from Table X because in the latter these uncertainties were arbitrarily expanded by a factor of three to compensate for the inconsistency of the two values. All of the other data listed in Table X of Ref. [1] were excluded for consideration by Cohen and DuMond on the basis of a comparatively large uncertainty, an incomplete or preliminary experiment or theoretical calculation, non-availability of the result at the time the 1963 adjustment was actually carried out, lack of sufficient information to evaluate the reliability of the experiment, insurmountable difficulties in assuring that the results of certain X-ray experiments were expressed on the same kxu scale, etc. It should be especially noted that in contrast to the 1969 and 1973 adjustments, the Lamb et al. value of α^{-1} (item number 6.2) and the Thomas et al. value of γ_p' (item number 2.3) had been included for consideration. Also, because Cohen and DuMond took the value of the ratio of the as-maintained to absolute ampere as an exactly known auxiliary constant, it does not appear as an item of stochastic input data. This also has the effect of eliminating the distinction between low and high field values of the proton gyromagnetic ratio -- all three values of γ_p' in Table 4 are expressed in SI units and should in principle be identical to each other.

The adjustable constants used to analyze the 1963 data are α^{-1}, e, N_A and Λ.

3.2. Results and Discussion

The results of our calculations are given in Tables 5 through 13 and in Figs. 1 through 11. In performing the numerical work, a sufficient number of iterations were carried out where appropriate to ensure that $\left|\chi^2 - F\right| \leq 10^{-5}$, and that σ_i'/σ_i showed essentially no change in the third decimal place and the adjusted values in the tenth. We have attempted to make each table and figure as self contained and understandable as possible so as not to require detailed explanation. However, a number of clarifying remarks, comments, and observations are in order.

3.2.1. Table 5. The data used to obtain the 1973 recommended values were strongly censored (items 3.1, 3.2, 9.1, and 10.4 of Table 2 were deleted -- see Alg. 1, Table 6). Hence the adjusted values for a number of constants resulting from the application of most of the algorithms of Table 1 to the uncensored 1973 data differ significantly from the corresponding recommended values in comparison with the uncertainties of the latter (compare for example Alg. 2 with Alg. 1). This was alluded to in Section 1.

Algorithm 3 differs from Alg. 2 only in that the uncertainties of the output values of Alg. 3 are 2.18 times the uncertainties of the corresponding output values of Alg. 2. The factor 2.18 is, of course, the Birge ratio of Alg. 2.

The 1973 data exhibit not only inconsistencies among quantities of the same kind (see Table 2a), but major inconsistencies between the different data subgroups. This is evidenced by a value of 2.21 for the second Birge ratio of Alg. 4, that is, the Birge ratio characterizing the adjustment involving the 12 weighted averages of Table 2a.

The internal Birge ratio R_{IB} required to make $\chi'^2 = F$ for Alg. 10 is 2.54, which may be compared with the value $R_B = 2.18$ for Alg. 2. As pointed out in Section 2.4.1, since the σ_{Ri} are in general finite, R_{IB} will always be larger than R_B.

For the extended least-squares algorithm, Alg. 13, $\chi'^2 = 29.5$. It may be compared with $\chi^2 = 119.1$ for Alg. 2 and with $F = 25$. As noted in Section 2.5, this algorithm will always yield $F \leq \chi'^2 \leq \chi^2$, the actual value depending on the v_i. For example, Alg. 13 was repeated with each x_i of Table 2 reduced by a factor of 2 and thus each v_i increased by a factor of 4. The adjusted values of the various constants changed by only 0 to 0.3 ppm compared with the values given in Alg. 13, Table 5, but $\chi'^2 = 36.7$ ($R_B' = 1.21$). The uncertainties of the adjustable constants were reduced by roughly 10%. (Note that for larger values of v_i, the σ_i' are generally smaller and thus the uncertainties of the output values are smaller). The fact that this algorithm is more dependent on the relative values of the v_i than on the actual values themselves should simplify the assignment of a meaningful set of confidence parameters to a given set of data.

The most interesting point to note in Table 5 is that with the exception of Alg. 1 for which the data were highly censored, the changes which occur in the adjusted values of the constants (in comparison with their uncertainties) from one algorithm to another are surprisingly small. This feature, quite astonishing when one considers that the input data include the extremely discrepant item 10.4 and Alg. 6 and Alg. 9 actually discard it (see Table 6), becomes quite clear when the results in Table 5 for a particular constant are graphically compared* as in Figs. 1 through 8.[12] With the possible exception of the α^{-1} plot, which can be explained by the direct dependence of α^{-1} on item 10.4, a major portion of the points in each figure are contained within each other's error bars. Table 5 and its counterparts, Table 7 (1973 data with item 10.4 deleted), Table 9 (1969 data), and Table 11 (1963 data), lead us to the principle conclusion of the present work: The actual algorithm used to carry out a least-squares adjustment of the constants is much less important than the particular items of stochastic input data selected for inclusion in the adjustment. Indeed, Alg. 2 of Table 5, for which all of the 1973 data were included without any alterations in their a priori uncertainties, does not differ in a major way from most of the other algorithms, even with regard to the

*The figure captions and figures begin on p. 69.

uncertainties of the resulting adjusted values. This emphasizes the need in any adjustment to evaluate the reliability of the available experimental data as carefully as possible.

3.2.2. Table 6. The two-stage Birge ratio algorithm has not been included because it is the 12 weighted averages of the 12 different data subgroups rather than the 31 separate items of input data which enter the final adjustment.

The censored and altered nature of the data used to obtain the 1973 recommended values is evident in Alg. 1.

As first noted in Section 3.1.1, the highly discrepant nature of item 10.4 is apparent from its normalized residual $r_{10.4} = 7.91$ in Alg. 2. By comparison, all of the other items are significantly less discrepant. In particular, one might wonder why items 3.1 and 3.2 were deleted from Alg. 1.

In Alg. 3, all of the *a priori* uncertainties have been expanded by the factor $R_B = 2.18$ of Alg. 2 and therefore each r_i in Alg. 3 is reduced by this factor in comparison with its corresponding value in Alg. 2.

The functioning of the VNIIM algorithm is clearly seen in Alg. 5. The large increase in the *a priori* uncertainties of the two most discrepant items, $(\sigma'/\sigma)_{10.4} = 2.87$ and $(\sigma'/\sigma)_{9.1} = 2.10$, is apparent. These expansions of σ_i are in marked contrast to the negligible expansions of σ_i for the highly consistent items 1.1, 4.2, and 6.1. Because item 10.4 is significantly smaller than the other values of α^{-1} in Table 2, and its weight is significantly reduced in Alg. 5, the adjusted value of α^{-1} resulting from Alg. 5 is rather larger than the value resulting from Alg. 2 (see Table 5 and Fig. 1).

As pointed out in Section 2.3.1, the inverse algorithm tends to discard highly discrepant items and this is apparent in Alg. 6 where we see that $(\sigma'/\sigma)_{10.4} = (\sigma'/\sigma)_{9.1} = \infty$. The absence of these items severely reduces the overall "strain" [1] in the adjustment and leads to much smaller increases in the remaining σ_i than for the corresponding σ_i in the VNIIM algorithm (compare a particular σ_i'/σ_i for Alg. 6 with the same σ_i'/σ_i for Alg. 5). The fact that Alg. 6 completely discards item 10.4 explains why its resulting value of α^{-1} differs so markedly from the value of Alg. 2 (see Table 5 and Fig. 1).

The natural log algorithm, Alg. 7, is similar to the VNIIM algorithm but tends to expand the σ_i of highly discrepant items by a larger factor. For example, Alg. 7 gives $(\sigma'/\sigma)_{10.4} = 4.22$ compared with 2.87 for Alg. 5. As a result, the values of σ_i'/σ_i for the remaining items of input data are generally rather less than the corresponding values for Alg. 5.

The geometrical mean algorithm yields results between the VNIIM and natural log algorithms as can be seen by comparing Alg. 8 with Algs. 7 and 5.

As pointed out in Section 2.3.4, the simple mean algorithm is similar to the inverse algorithm. However, it is less severe in its treatment of highly discrepant items and as can be seen from Alg. 9, censors only item 10.4 while Alg. 6 censors both 10.4 and 9.1. As a result, the remaining values of σ_i'/σ_i for Alg. 9 are somewhat larger than the corresponding values for Alg. 6 because the overall strain in the adjustment is not reduced to the same extent.

For the internal Birge ratio algorithm, $R_{IB} = 2.54$ is the expansion factor by which each <u>a priori</u> systematic uncertainty σ_{Si} is increased to achieve $\chi'^2 = F = 25$. Since $\sigma'^2 = \sigma_{Ri}^2 + R_{IB}^2\sigma_{Si}^2$, the expansion factor for each σ_i depends on the ratio $T_i = \sigma_{Ri}/\sigma_{Si}$ -- the smaller T_i, the more nearly σ_i'/σ_i will equal σ_{Si}'/σ_{Si}; for large values of T_i, σ_i'/σ_i will be closer to unity. This is apparent in Alg. 10 where we see that $(\sigma'/\sigma)_{1.1} = 2.54$ and $(\sigma'/\sigma)_{3.1} = 1.19$. This feature of the algorithm can have an important effect if a particular datum is somewhat discrepant but has a relatively large value of T_i. For example, $(\sigma'/\sigma)_{3.1} = 1.19$ means that this particular input value for the Faraday constant F carries a relatively large weight in the adjustment and determines the final output value of F to a greater extent than it does in some of the other algorithms where $(\sigma'/\sigma)_{3.1}$ is as high as 1.75. This explains the relatively large difference between the value of F resulting from Alg. 10 and from some of the other algorithms (see Table 5 and Fig. 8). A similar but opposite effect also explains why α^{-1} is so different for Alg. 10 than for the other algorithms. Item 4.2, a measurement of the proton gyromagnetic ratio, determines a highly accurate, indirect value of α^{-1} which is numerically larger by about 13 ppm than item 10.4. Because $T_{4.2}$ is relatively small, the uncertainty of item 4.2 is expanded by a relatively large factor: $(\sigma'/\sigma)_{4.3} = 2.43$. The uncertainty of the indirect value of α^{-1} is similarly increased with the net result that it carries much less weight in Alg. 10 relative to item 10.4 than it does for the other algorithms.

The general comments made regarding Alg. 10 apply to Alg. 11 although the corresponding values of σ_{Si}'/σ_{Si} and σ_i'/σ_i are radically different between the two.

The VNIIM with weights algorithm, Alg. 12, yields normalized residuals and values of σ_i'/σ_i which are quite similar to those of the VNIIM algorithm as one might expect since the weights $W_i{}^* = (T_i{}^2 + 1)^{-1}$ are rather similar for most of the data. (If $W_i{}^*$ was the same for each datum, the two algorithms would be identical.) However, for those items with unusually large values of T_i (small $W_i{}^*$) such as items 3.1 and 10.6, the differences in σ_i'/σ_i for the two algorithms are rather large as can be seen by comparing Alg. 12 with Alg. 5 for these two items. As discussed in Section 2.4.3, the reason is that for small $W_i{}^*$, the "cost" of expanding σ_i by a large factor is significantly reduced.

As pointed out in Section 2.5, for the extended least-squares algorithm, Alg. 13, those items of input data with identical confidence parameters v_i end up with identical values of σ_i'/σ_i. It is also clear from Alg. 13 that for those items of input data with reliable uncertainties or large v_i, σ_i'/σ_i is near unity. For example, $v_{6.2} = 50$ and $(\sigma'/\sigma)_{6.2} = 1.04$. It is of interest to note that if all of the v_i were identical and equal to 2 ($x_i = 0.5$), then $\sigma_i'/\sigma_i \equiv \sigma'/\sigma = 1.97$ and $\chi'^2 = 30.8$; if all the $v_i = 10$ ($x_i = 0.22$), then $\sigma'/\sigma = 1.67$ and $\chi'^2 = 42.9$. These numbers should be compared with $R_B = 2.18$ and $\chi^2 = 119.1$ for Alg. 2.[13]

3.2.3. Table 7. Table 7 is identical to Table 5 except that the highly discrepant item 10.4 has been deleted from the 1973 data. Many of the more general comments and observations made with regard to Table 5 are therefore applicable to Table 7.

Deleting item 10.4 reduces the Birge ratio from 2.18 to 1.39 for Alg. 2 and thus the uncertainties of the output values of Alg. 3, Table 7, are 1.39 times larger than the uncertainties of the corresponding output values of Alg. 2, same table.

The inconsistencies among the various values of α^{-1} are significantly reduced with the deletion of item 10.4 (compare $R_B = 2.90$ for the weighted average 10a in Table 2 with $R_B = 0.95$ for the weighted average 10b). The inconsistencies between the different data subgroups is also reduced: The value for the second Birge ratio of Alg. 4, Table 7, is 2.13 compared with 2.21 for Alg. 4, Table 5.

The value of the internal Birge ratio R_{IB} for Alg. 10, Table 7, is 1.63 compared with 2.54 for Alg. 10, Table 5.

For Alg. 13, Table 7, the extended least squares algorithm, $\chi'^2 = 25.3$ with $F = 24$ compared to $\chi'^2 = 29.5$ with $F = 25$ for Alg. 13, Table 5. Thus, eliminating the discrepant item 10.4 leads to a value rather closer to F. For the case where each v_i is increased by a factor of 4 (see Section 3.2.1),

$\chi'^2 = 27.9$ with $F = 24$ for Alg. 13, Table 7, compared to $\chi'^2 = 36.7$ with $F = 25$ for Alg. 13, Table 5.

With the removal of item 10.4 the changes which occur in the adjusted values of the constants from one algorithm to another are even smaller than those in Table 5. This is clear from Fig. 9 where we graphically compare the values of α^{-1} resulting from the 13 algorithms of Table 7. In comparing Fig. 9 to its counterpart from Table 5, Fig. 1, we see that the variations are signficantly reduced. This is as expected since α^{-1} is directly dependent on item 10.4. The reduction in the variations of the other constants is not so dramatic because they depend less critically on item 10.4, but of course, their variations were less in the first place. Table 7 reinforces the principal conclusion of this paper: The algorithm used to carry out an adjustment is much less important than the data selected for inclusion in the adjustment.

It is of interest to consider how "robust" the various algorithms are with respect to the deletion of a highly discrepant datum. In comparing the adjusted values of like constants resulting from the same algorithms in Tables 7 and 5, we find surprisingly small changes. For example, the value of N_A resulting from Alg. 5, Table 7, exceeds the value of N_A resulting from Alg. 5, Table 5, by only 0.7 ppm. (The uncertainties of these respective values of N_A are 5.0 and 6.0 ppm.) Figures 10 and 11 graphically compare in this way Table 7 with Table 5 for selected constants. As can be seen from the figures, Alg. 2 (and the similar Alg. 3) is the algorithm most sensitive to the deletion of the discrepant item 10.4, while Alg. 9 is the least sensitive. This is not surprising since Alg. 2 does not expand the <u>a priori</u> uncertainties of discrepant data in any way and thus they carry full weight. In contrast, Alg. 9 expands the uncertainties of highly discrepant items to such an extent that item 10.4 was mathematically discarded from Alg. 9, Table 5 (see Table 6 and Section 3.2.2). The actual data were therefore essentially the same for Alg. 9 in both Tables 5 and 7. (Although the inverse algorithm treats discrepant data in a similar way, it shows greater variations in Fig. 10 because of the way it treats item 3.1 -- compare the change in $(\sigma'/\sigma)_{3.1}$ between Tables 6 and 8 for Alg. 6 with the change for Alg. 9.)

3.2.4. Table 8. Table 8 is again identical to its counterpart, Table 6, with the exception that the highly discrepant item 10.4 has been deleted. Thus, many of the general remarks made regarding Table 6 apply to it as well.

With the deletion of item 10.4, the most discrepant datum is item 9.1 as can be seen in Alg. 2 where $r_{9.1} = -3.26$. While this large a residual is not necessarily unacceptable, the residuals for the remaining data are significantly less and quite acceptable. Thus Alg. 3, in which each σ_i has been expanded by the factor $R_B = 1.39$ of Alg. 2, might have been quite a reasonable treatment of the 1973 data in the context of the viewpoint of the VNIIM group. Indeed, if one strictly adheres to their viewpoint, there is no justification for discarding any input datum except item 10.4 from the 1973 data (see also the discussion of Section 3.1.1).

With the removal of the highly discrepant item 10.4, the overall strain in each adjustment is significantly reduced and the individual values of σ_i'/σ_i for 11 of the 13 algorithms in Table 8 are generally much smaller than the corresponding values in Table 6. The two exceptions are Alg. 6 and Alg. 9. Since these algorithms had already mathematically discarded item 10.4 in Table 6, the changes in σ_i'/σ_i between Tables 6 and 8 for the two algorithms are much smaller.

3.2.5. Tables 9 and 10. Since the general features of the application of the various algorithms to the 1969 (and 1963) data are the same as for the 1973 data, we shall severely limit our comments concerning Tables 9 and 10 (and Tables 11 and 12).

The two main points to note concerning Table 9 are first that because the data used to obtain the 1969 recommended values were highly censored (items 6.3, 6.4, 7.1, 8.1, 10.1, 10.2, 11.2 and 11.3 of Table 3 were discarded -- see Alg. 1, Table 10), the adjusted values for several constants resulting from Algs. 2 through 9 differ significantly from the corresponding recommended values. These differences are mainly due to the deletion of items 6.3 and 6.4 from the 1969 data, the so-called "high values" of μ_p'/μ_N (see Section 1).

Second, as we now expect, the changes which occur in the adjusted values of the constants from one algorithm to another are relatively small with perhaps the exception of the inverse algorithm, Alg. 6. As can be seen from Table 10, this algorithm mathematically discards the discrepant item 6.4 and thus this datum has no influence whatsoever on the adjusted values resulting from Alg. 6. In contrast, all of the remaining algorithms, including the somewhat similar Alg. 9, retain it but with an expanded uncertainty. This again points up the fact that Alg. 9 is less severe in its treatment of discrepant data than Alg. 6. However, a close examination of Table 10 shows that it is more severe in its treatment of such data than the VNIIM or other VNIIM related algorithms and thus might provide a good compromise between discarding discrepant items entirely, and giving them excessive weight. Indeed, we recall that Fig. 10 showed Alg. 9 to be the least sensitive of the algorithms to the deletion of the highly discrepant item 10.4 from the 1973 data.

With reference to Table 10, it is evident from Alg. 2 that in contrast to the 1973 data, the 1969 data contain no single item so discrepant that one could justify discarding it if one were strictly adhering to the viewpoint of the VNIIM group.

3.2.6. Tables 11 and 12. The 1963 recommended values given in Table 11 are those actually calculated from the input data of Table 4 and differ slightly from the values given in Ref. [1]. That is, we were not able to duplicate exactly the recommended values given in Ref. [1] from the given input data. However, the differences are relatively minor and have little bearing on the conclusions to be drawn. It should also be noted that a recommended value for Λ was never actually given in Ref. [1] so we have chosen a plausible value resulting from one of the exploratory adjustments carried out in Ref. [1]. Since it serves primarily as a fiducial point for comparing the results of the various algorithms, its value is not critical.

We make three observations regarding Tables 11 and 12. First, in contrast to the situation which prevails with the 1973 and 1969 data, there are only small differences (in comparison with their uncertainties) between the 1963 recommended values of the constants and the values which result when all of the 1963 data are included (compare Alg. 2 with Alg. 1, Table 11). This is somewhat surprising in view of the degree to which the 1963 data were censored in order to obtain the 1963 recommended values -- items 2.3, 3.1, 4.1, 4.2, 4.3, 5.1, 5.2, and 6.2 of Table 4 were deleted (see Alg. 1, Table 12). Although this might lead one to conclude that the 1963 data were in relatively good agreement, the value R_B = 1.92 for Alg. 2 and the several large residuals for Alg. 2 in Table 12 show that the 1963 data were in fact highly inconsistent. But it does point up a peculiarity which can occur in an adjustment -- including discrepant data does not always have as large an impact as one might expect.

Second, there are interesting variations in the way Algs. 5 through 9 treat the discrepant items 2.9 and 5.1. For example, referring to Table 12, Alg. 6 mathematically discards item 5.1 and retains item 2.3 even though item 2.3 is the more discrepant datum (i.e., in Alg. 2, $r_{2.3}$ = -3.40 while $r_{5.1}$ = -2.51).

Algorithms 7, 8, and 9 also expand the uncertainty of item 5.1 by a larger factor than for item 2.3, although for Alg. 8 the difference is entirely negligible. On the other hand, for Alg. 5, the VNIIM algorithm, item 2.3 is expanded by a larger factor than for item 5.1. Nevertheless, in spite of these differences, the variations in the adjusted values of the constants (in comparison with their uncertainties) from one algorithm to another are surprisingly small as can be seen from Table 11.

Third, referring to Table 12, Alg. 2, one sees that as for the 1969 data, there is no single input datum so discrepant among the 1963 data that one could justify discarding it if one were strictly adhering to the viewpoint of the VNIIM group.

3.2.7. Table 13. The aim of Table 13 is to give some indication of which if any of several selected algorithms of Table 1 is "best", that is, would have led to the smallest changes in the recommended values of the constants from one adjustment to the next (i.e., from 1963 to 1969 to 1973), and would have given values closest to the true values. (Note however, that these are not necessarily compatible requirements!)

Table 13 follows directly from Tables 7, 9, and 11 and in general is to be read horizontally; reading the three "data" columns vertically and comparing like constants is equivalent to reading Tables 7, 9, and 11 horizontally.

As a further aid to reading the table, we point out the following: the 1969 recommended value for N_A has an uncertainty of 6.6 ppm and exceeds the 1973 recommended value by 20.6 ppm; the 1963 recommended value for N_A has an uncertainty of 15 ppm and exceeds the 1973 recommended value by 82 ppm. The 1963 value thus exceeds the 1969 value by 61 ppm.

Continuing in the same vein, for the application of the VNIIM algorithm, Alg. 5, to the 1973 data (but with item 10.4 deleted because of its highly discrepant nature), one finds that the value of N_A has an uncertainty of 5.0 ppm and exceeds the 1973 recommended value by 11.0 ppm; that the value of N_A resulting from the application of Alg. 5 to the 1969 data has an uncertainty of 7.0 ppm and exceeds the value of N_A resulting from the application of Alg. 5 to the 1973 data by 0.1 ppm; and that the value of N_A resulting from the application of Alg. 5 to the 1963 data has an uncertainty of 15 ppm and exceeds the value of N_A resulting from the application of Alg. 5 to the 1973 data by 69 ppm. Thus, the 1963 Alg. 5 value exceeds the 1969 Alg. 5 value by 69 ppm.

We see then that the results from the application of a particular algorithm to the 1969 and 1963 data are given relative to the results for the application of the same algorithm to the 1973 data (with item 10.4 deleted), while the results for the application of the algorithm to the 1973 data are given relative to the 1973 recommended values. This means that the algorithm for which the 1969 and 1963 results for a particular constant as given in the table are closest in value to each other, and for which the magnitudes of the values are closest to zero, is the "best" in the sense discussed above. This assumes that the application of the particular algorithm in question to the 1973 data (with item 10.4 deleted) results in values of the constants which are "correct," that is, the true values. With the completion of the 1982 and subsequent future adjustments, each of which will presumably be closer to the truth than its predecessor, we will be able to extend Table 13 and test more critically for the best algorithms. For the moment, however, we must content ourselves with looking only at the 1963 and 1969 results for a particular algorithm relative to each other and to the result of applying that algorithm to the 1973 data -- until the gathering and analysis of the 1982 data is completed, there will be nothing to which the latter may be compared.

A detailed examination of Table 13 leads us to conclude that there is no clearly best algorithm. The relative changes in the recommended values between 1963 and 1969 and 1969 and 1973, as well as the magnitude of the changes, are not all that different for any of the algorithms. For example, in comparing the VNIIM algorithm, Alg. 5, to the recommended values algorithm, Alg. 1, we see that while the numbers in the table for the 1963 data are generally smaller for Alg. 5 than for Alg. 1, the reverse is true for the 1969 data. The changes in the constants from 1963 to 1969 for the two algorithms are also similar. Although we leave it to the reader to draw his or her own conclusions from Table 13, we believe that it does not in any way demonstrate the supposed benefit of routinely including discrepant data in a least-squares adjustment of the constants.

4. CONCLUSIONS

It is the author's opinion that the development of a purely objective means based on sound statistical principles for optimally incorporating discrepant data in a least-squares adjustment of the constants is a goal yet to be achieved. The various Birge ratio approaches, the VNIIM algorithm and its variations, as well as the extended least-squares algorithm of E. R. Cohen, all have their limitations as discussed in this paper.

It is the author's view that a computer program cannot and should not replace the sound judgement of the conscientious, thoughtful reviewer who, through his personal knowledge of the experiments and experimenters, is best able to judge the reliability of a measurement. In retrospect, perhaps more faith should have been placed in this judgement during the course of past adjustments than in statistical measures of inconsistency. That is, there was perhaps too great a tendency to discard only mildly discrepant items of input data rather than to retain them and adjust for the inconsistencies via, for example, the Birge ratio. For as we believe we have clearly demonstrated in this paper, the actual algorithm one uses to carry out an adjustment is not nearly as critical as the actual items of input data one selects for inclusion in the adjustment.

Of course, there is still the basic question which in the author's opinion is yet to be clearly resolved: Does including what appears to be discrepant data in an adjustment really lead both to smaller changes in the recommended values of various constants from one adjustment to the next and to values which are closer to the true values? Although discarding some apparently discrepant items in the last three adjustments has led to undesirable consequences, if subsequent experiments had in fact substantiated the unreliable nature of the discarded data, then the decisions to discard would have gone unnoticed. Indeed, the reader should not forget that the decision to discard the highly discrepant value of α^{-1} of Kaufman, Lamb et al. (item 10.4 of Table 2) from both the 1969 and 1973 adjustments has been well supported by recent experiments [5,6] and that if this datum had been included, it would have led to highly erroneous recommended values for the fine-structure constant. Thus, in contrast to the VNIIM Group's point of view, the author feels that we must wait for the completion of several more adjustments before we can say with any degree of confidence that indiscriminantly including all but the most highly discrepant data is really advantageous.

Finally, although it is premature to say just what algorithm will be used to carry out the 1982 adjustment presently underway, it is certainly our intention to explore the implications of at least several of the algorithms discussed in this paper. This should lead to a good appreciation of the inherent reliability of the final recommended values resulting from that adjustment and mark yet another milestone on a long journey begun in 1929 by R. T. Birge.

5. ACKNOWLEDGEMENTS

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6. FOOTNOTES

[1]The inconsistency of the i^{th} stochastic input datum in an adjustment is manifested by the excessive contribution of its normalized residual r_i to the statistic χ^2 for the adjustment, where $\chi^2 = \Sigma r_i{}^2$. The normalized residual of an input datum is its deviation from its least-squares adjusted value divided by (or normalized to) its a priori uncertainty. In a typical adjustment, r_i should be unity or less for a consistent datum. A value of r_i significantly greater than unity implies that the datum is discrepant. (See Refs. [1,2,3,16] for a detailed discussion of the least-squares technique as applied to the fundamental constants.)

[2]Uncertainties are given as one standard deviation estimates throughout this paper.

[3]Much of the discussion in Section 2 follows Refs. [13-17].

[4]The standard deviation of χ^2 is $(2F)^{1/2}$ [1] and thus taking σ_i'/σ_i in the interval $R_B/\sqrt{1+(2/F)^{1/2}} \leq \sigma_i'/\sigma_i \leq R_B/\sqrt{1-(2/F)^{1/2}}$ so that χ'^2 lies in the interval $F - (2F)^{1/2} \leq \chi'^2 \leq F + (2F)^{1/2}$ would not be unreasonable. Taking $\sigma_i'/\sigma_i = R_B$ so that $\chi'^2 = F$, its expected value, may be viewed as a good compromise between excessive pessimism and excessive optimism concerning the overall reliability of the input data.

[5]Throughout, the prime means that the so indicated quantities are based on the adjustment carried out with the altered uncertainties σ_i'. Unprimed quantities are based on the adjustment carried out with the a priori uncertainties σ_i.

[6]Equation (5) and its counterparts throughout this paper are written so as to minimize the explicit dependence of the right-hand side on σ_i'; the quantity $R_i^2 r_i'^2$ is only implicitly dependent on σ_i' through the values of the adjustable constants or unknowns.

[7]For the uniform distribution, $\phi_i(R_i^2) = 0$ for $R_i^2 < 1 - \Delta$ and $R_i^2 > 1 + \Delta$, $\phi_i(R_i^2) = 1/2\Delta$ for $1 - \Delta \leq R_i^2 \leq 1 + \Delta$, with $\Delta = \sigma\sqrt{3}$; for the normal distribution, $\phi_i(R_i^2) = (1/\sigma\sqrt{2\pi}) \exp[-(R_i^2 - 1)^2/2\sigma^2]$. However, see Ref. [16], pp. 591-593, for a detailed discussion of the problem of justifying the VNIIM algorithm on the basis of a normalizable probability distribution.

[8]Taking $W_i \neq 1$ is one approach to overcoming the assumption that all <u>a priori</u> uncertainties are equally reliable. The question is how to arrive at such weights with a minimum of subjectivity. One particular approach will be discussed in Section 2.4.3.

[9]It is pointed out in Ref. [13] that the condition $\chi'^2 = F - 1$ is perhaps more reasonable since the additional variable R_{IB} has been introduced into the problem and the number of degrees of freedom has been reduced by one. However, we shall use F since it is consistent with what was done for the previous algorithms. The difference between the two choices is not significant for the typical adjustment with a large number of items of input data (i.e., large N). See also Footnote 4.

[10]In this regard, it is interesting to note that the equivalent of Eq. (21) for the VNIIM algorithm in the form represented by Eq. (22) may be derived from Eq. (21) by introducing the weighting factor $W_i^{*2} = [1/(T_i^2 + 1)]^2$ in the latter.

[11]If σ_i^2 and v_i had been considered to be independently estimated instead of σ_i^2 and var(s_i), σ_i^2 in Eq. (28) would have been replaced by $\hat{\sigma}_i^2$ to yield, in analogy with Eq. (30), $\sigma_i' = [1 + (F - \chi'^2)/v_i]^{-1/2}\,\sigma_i$. However, this expression requires the physically unreal constraint that $\chi'^2 < F + v_i(\min)$. That is, χ'^2 would be determined by the input datum with the least reliable uncertainty or smallest confidence parameter, independent of the size of the uncertainty itself and the datum's level of agreement with the other experiments. The constraint imposed by Eq. (30) is much more physically real since it implies that there is a minimum overall level of consistency as characterized by the value of χ'^2 which can be achieved for the adjustment and that this level is dependent upon the input datum with the least reliable uncertainty.

[12]Plots are not given for Ω_{BI69}/Ω or μ_p'/μ_N because of their obvious negligible variations (mainly due to their small a priori uncertainties), nor for γ_p' or m_e because their variations are nearly identical to those of F and N_A^{-1}, respectively.

[13]It can be shown that if all of the v_i are identical and equal to v, then $\sigma_i'/\sigma_i \equiv \sigma'/\sigma = (\sqrt{2}/2)\bullet[\sqrt{(F/v - 1)^2 + 2FR_B^2/v} - (F/v - 1)]^{1/2}$ and $\chi'^2 = \chi^2(\sigma'/\sigma)^2$, where R_B and χ^2 are, respectively, the Birge ratio and chi-squared for the adjustment carried out using the σ_i (i.e., Alg. 2). It also follows that $R_B' = R_B/(\sigma'/\sigma)$, where R_B' is the Birge ratio which would result from the adjustment carried out using the σ_i'.

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8. TABLES

Table 1. Summary of algorithms to be considered in the present work.

Algorithm[a]	Identifying characteristic[b]	Section where discussed
1. 1973, 1969, or 1963 recommended values	Varies from one adjustment to another	Refs. [3,2,1]
2. _A priori_ assigned uncertainties and all available data used	$\sigma_i' = \sigma_i$	-
3. Birge ratio	$\sigma_i' = R_B\sigma_i$	2.1.1
4. Two-stage Birge ratio	$\sigma_i' = R_B\sigma_i$ (twice)	2.1.2
5. VNIIM	$S = \sum_{i=1}^{N} (R_i^2 - 1)^2$, $\chi'^2 = F$	2.2, 2.3
6. Inverse	$S = \sum_{i=1}^{N} (R_i^{-2} - 1)^2$, $\chi'^2 = F$	2.3.1
7. Natural log	$S = \sum_{i=1}^{N} \ell n^2 R_i^2$, $\chi'^2 = F$	2.3.2
8. Geometrical mean	$S = \sum_{i=1}^{N} (R_i^2 - 1)^2/R_i^2$, $\chi'^2 = F$	2.3.3
9. Simple mean	$S = 4\sum_{i=1}^{N} (R_i^2 - 1)^2/(R_i^2 + 1)^2$, $\chi'^2 = F$	2.3.4
10. Internal Birge ratio	$\sigma_i'^2 = \sigma_{Ri}^2 + R_{IB}^2\sigma_{Si}^2$, $\chi'^2 = F$	2.4.1
11. VNIIM-systematic uncertainties	$S = \sum_{i=1}^{N} (R_{Si}^2 - 1)^2$, $\chi'^2 = F$	2.4.2
12. VNIIM with weights	$S = \sum_{i=1}^{N} W_i^*(R_i^2 - 1)^2$, $\chi'^2 = F$	2.4.3
13. Extended least-squares	$\sigma_i' = [1 + (\chi'^2 - F)/\nu_i]^{1/2}\,\sigma_i$	2.5

[a] The algorithm number will be referenced in the tables, figures, and text.

[b] $R_i^2 \equiv (\sigma_i'/\sigma_i)^2$; $R_{Si}^2 \equiv (\sigma_{Si}'/\sigma_{Si})^2$; $T_i^2 \equiv (\sigma_{Ri}/\sigma_{Si})^2$;
$W_i^* = \sigma_{Si}^2/\sigma_i^2 = 1/(T_i^2 + 1)$

Table 2. Summary of data considered for use in the 1973 adjustment of Cohen and Taylor [3].

Item No.[a]	Quantity	Value[b]	Uncertainty[c] (ppm) σ_i	σ_{Ri}	σ_{Si}	T_i	x_i	v_i	Description	Orig. Eq. No.[d]
1.1	Ω_{BI69}/Ω	$1 - 0.54 \times 10^{-6}$	0.19	0.01	0.19	0.02	0.50	2.00	Calculable capacitor	4.4
2.1	A_{BI69}/A	$1 + 1.8 \times 10^{-6}$	9.7	8.5	4.8	1.76	0.40	3.13	NBS Pellat balance	12.1
2.2	A_{BI69}/A	$1 - 1.2 \times 10^{-6}$	7.7	6.1	4.8	1.29	0.40	3.13	NBS current balance	12.2
2.3	A_{BI69}/A	$1 + 0.0 \times 10^{-6}$	5.5	4.4	3.2	1.37	0.30	5.56	NPL current balance	12.3
3.1	F	9.648672	6.8	6.5	1.9	3.48	0.30	5.56	Silver coulometer	13.1
3.2	F	9.648695	9.6	4.3	8.7	0.49	0.70	1.02	Benzoic and oxalic acids	13.2
4.1	γ_p'(low)	2.6751156	4.0	2.0	3.5	0.58	0.50	2.00	ETL	14.1
4.2	γ_p'(low)	2.6751370	2.0	0.6	1.9	0.34	0.15	22.22	NBS	14.2
4.3	γ_p'(low)	2.6751187	4.0	1.0	3.9	0.26	0.30	5.56	NPL	14.3
4.4	γ_p'(low)	2.6751100	6.0	1.2	5.9	0.20	0.60	1.39	VNIIM	14.4
5.1	γ_p'(high)	2.675130	7.4	2.0	7.1	0.28	0.50	2.00	KhGNIIM (Kharkov)	14.5
5.2	γ_p'(high)	2.675075	16	11	12	0.96	0.50	2.00	NPL	14.6
6.1	μ_p'/μ_N	2.7927738	0.43	0.20	0.38	0.53	0.50	2.00	Mass spectrometer	15.7
6.2	μ_p'/μ_N	2.7927748	0.82	0.14	0.81	0.17	0.10	50.00	NPL Omegatron	15.8
7.1	Λ	1.002027	33	21	26	0.79	0.70	1.02	Ruled grating	16.3
7.2	Λ	1.0020655	9.8	7.5	6.3	1.19	0.70	1.02	Ruled grating	16.5
7.3	Λ	1.002041	33	10	32	0.31	0.70	1.02	Short wavelength limit	16.7

8.1	$N_A\Lambda^3$	6.059768	16	2.8	15	0.18	0.70	1.02	XRCD - silicon	17.1
8.2	$N_A\Lambda^3$	6.05961	28	21	19	1.08	0.70	1.02	XRCD - calcite	17.2
9.1	λ_C	24.21416	15	10	11	0.88	0.70	1.02	Annihilation - tantalum	18.2
9.2	λ_C	24.21315	33	22	24	0.92	0.70	1.02	Annihilation - tantalum	18.3
10.1	α^{-1}	137.03563	3.1	2.0	2.3	0.85	0.40	3.13	Electron anomalous moment	19.8
10.2	α^{-1}	137.03597	1.6	0.0	1.6	0.00	0.70	1.02	Hydrogen hyperfine splitting	22.6
10.3	α^{-1}	137.03544	3.9	3.4	2.0	1.69	0.70	1.02	Hydrogen fine structure	23.4
10.4	α^{-1}	137.03416	1.5	0.8	1.2	0.64	0.70	1.02	Hydrogen fine structure	23.5
10.5	α^{-1}	137.03508	3.3	2.7	1.9	1.41	0.70	1.02	Hydrogen fine structure	23.6
10.6	α^{-1}	137.03563	2.3	2.1	0.8	2.55	0.70	1.02	Hydrogen fine structure	23.7
11.1	μ_μ/μ_p	3.1833467	2.6	1.2	2.3	0.52	0.40	3.13	Muon precession	21.1
11.2	μ_μ/μ_p	3.183356	9.6	6.9	6.7	1.04	0.60	1.39	Muon precession	21.2
11.3	μ_μ/μ_p	3.183350	4.7	2.7	3.8	0.71	0.70	1.02	Muonium Zeeman transitions	21.3
12.1	ν(Mhfs)	4463303.8	2.0	0.4	2.0	0.20	0.70	1.02	Muonium hyperfine splitting	22.2

[a] This is the item identification number to be used in the tables and discussion of the present work.

[b] The units for F are $10^4\ A_{BI69}\cdot s\cdot mol^{-1}$; for γ_p'(low), $10^8\ s^{-1}\cdot T_{BI69}^{-1}$; for γ_p'(high), $10^8\ A_{BI69}\cdot s\cdot kg^{-1}$; for $N_A\Lambda^3$, $10^{23}\ mol^{-1}$; for λ_C, 10^{-3} kxu; and for ν(Mhfs), kHz.

[c] $\sigma_i^2 = \sigma_{Ri}^2 + \sigma_{Si}^2$, $T_i^2 \equiv (\sigma_{Ri}/\sigma_{Si})^2$ -- see Section 2.4; $v_i = 1/2x_i^2$, $x_i^2 = \mathrm{var}(\sigma_i)/\sigma_i^2$ -- see Section 2.5.

[d] These are the identification numbers for the indicated items used in Ref. [3].

Table 2a. Input data derived from Table 2 to be used in the application of the two-stage Birge ratio algorithm to the 1973 data; see text.

Item No.[a]	Quantity	Value[b] (weighted average)	Birge ratio of weighted average	Uncertainty[c] (ppm)	Degrees of freedom[d]
1	Ω_{BI69}/Ω	$1 - 0.54 \times 10^{-6}$	-	0.19	-
2	A_{BI69}/A	$1 + 0.0 \times 10^{-6}$	0.17	4.1	2
3	F	9.648679	0.21	5.6	1
4	γ_p'(low)	2.6751290	1.43	2.3	3
5	γ_p'(high)	2.675121	1.16	7.8	1
6	μ_p'/μ_N	2.7927740	0.39	0.38	1
7	Λ	1.0020609	0.89	9.1	2
8	$N_A\Lambda^3$	6.059730	0.82	14	1
9	λ_C	24.21398	1.15	16	1
10a	α^{-1}	137.03516	2.90	2.5	5
10b	α^{-1}	137.03571	0.95	1.1	4
11	μ_μ/μ_p	3.1833479	0.24	2.2	2
12	ν(Mhfs)	4463303.8	-	2.0	-

[a]These identification numbers correspond to those given in Table 2. Item 10a is the weighted average of all of items 10 in Table 2; 10b is the weighted average of items 10 with 10.4 deleted.

[b]Since there is only one value of type 1 and 12 data in Table 2, a weighted average cannot be carried out; the values remain unchanged. The units for F are $10^4\ A_{BI69}\cdot s\cdot mol^{-1}$; for γ_p'(low), $10^8\ s^{-1}\cdot T_{BI69}^{-1}$; for γ_p'(high), $10^8\ A_{BI69}\cdot s\cdot kg^{-1}$; for $N_A\Lambda^3$, $10^{23}\ mol^{-1}$; for λ_C, 10^{-3} kxu; and for ν(Mhfs), kHz.

[c]This column gives the total ppm uncertainty of the weighted average, multiplied by the corresponding Birge ratio if the latter is greater than unity.

[d]The degrees of freedom for each weighted average is the number of items of like data minus one.

Table 3. Summary of data considered for use in the 1969 adjustment of Taylor, Parker and Langenberg [2].

Item No.[a]	Quantity	Value[b]	Uncertainty (ppm)	Description	Original No.[c]
1.1	2e/h	4.835976	2.4	Josephson effect	1
2.1	A_{NBS}/A	$1 + 10.2 \times 10^{-6}$	9.7	NBS Pellat balance	2
2.2	A_{NBS}/A	$1 + 9.2 \times 10^{-6}$	7.7	NBS current balance	3
2.3	A_{NBS}/A	$1 + 8.0 \times 10^{-6}$	6.0	NPL current balance	4
3.1	F	9.648570	6.8	Silver coulometer	5
4.1	γ_p'(low)	2.6751525	3.7	NBS	6
4.2	γ_p'(low)	2.675144	5.8	NPL	7
5.1	γ_p'(high)	2.675105	7.4	KhGIMIP (Kharkov)	8
6.1	μ_p'/μ_N	2.792690	11	NBS Omegatron	9
6.2	μ_p'/μ_N	2.792701	26	Inverse cyclotron	10
6.3	μ_p'/μ_N	2.792832	20	Cyclotron	11
6.4	μ_p'/μ_N	2.792794	6.2	Mass spectrometer	12
6.5	μ_p'/μ_N	2.792746	19	NPL Omegatron	13
7.1	hc/e	12373.15	33	Short-wavelength limit	14
8.1	Λ	1.002030	38	Ruled gratings	17
9.1	$N_A\Lambda^3$	6.05972	37	XRCD-calcite	15
9.2	$N_A\Lambda^3$	6.059768	16	XRCD-silicon	16
10.1	λ_C	24.21263	38	Annihilation-H_2O	18
10.2	λ_C	24.21421	15	Annihilation-tantalum	19
11.1	α^{-1}	137.03591	2.6	Hydrogen hyperfine splitting	20
11.2	α^{-1}	137.03545	4.3	Hydrogen fine-structure	21
11.3	α^{-1}	137.03505	2.4	Hydrogen fine-structure	22a

[a]This is the item identification number to be used in the tables and discussion of the present work.

[b]The units for 2e/h are 10^{14} Hz/V_{NBS}; for F, 10^4 $A_{NBS}\cdot s\cdot mol^{-1}$; for γ_p'(low), 10^8 $s^{-1}\cdot T_{NBS}^{-1}$; for γ_p'(high), 10^8 $A_{NBS}\cdot s\cdot kg^{-1}$; for hc/e, $V_{NBS}\cdot$kxu; for $N_A\Lambda^3$, 10^{23} mol^{-1}; and for λ_C, 10^{-3} kxu.

[c]These are the identification numbers for the indicated items used in Ref. [2].

Table 3a. Input data derived from Table 3 to be used in the application of the two-stage Birge ratio algorithm to the 1969 data; see text.

Item No.[a]	Quantity	Value[b] (weighted average)	Birge ratio of weighted average	Uncertainty[c] (ppm)	Degrees of freedom[d]
1	$2e/h$	4.835976	-	2.4	-
2	A_{NBS}/A	$1 + 8.8 \times 10^{-6}$	0.14	4.3	2
3	F	9.648570	-	6.8	-
4	γ_p'(low)	2.6751501	0.46	3.1	1
5	γ_p'(high)	2.675105	-	7.4	-
6	μ_p'/μ_N	2.792768	1.68	8.2	4
7	hc/e	12373.15	-	33	-
8	Λ	1.002030	-	38	2
9	$N_A\Lambda^3$	6.059761	0.20	14	1
10	λ_C	24.21399	1.59	22	1
11	α^{-1}	137.0354	1.28	2.1	2

[a]These identification numbers correspond to those given in Table 3.

[b]Since there is only one value of type 1, 3, 5, 7, and 8 data in Table 3, a weighted average cannot be carried out; the values remain unchanged. The units for $2e/h$ are 10^{14} Hz/V_{NBS}; for F, 10^4 $A_{NBS}\cdot s\cdot mol^{-1}$; for γ_p'(low), 10^8 $s^{-1}\cdot T_{NBS}^{-1}$; for γ_p'(high), 10^8 $A_{NBS}\cdot s\cdot kg^{-1}$; for hc/e, $V_{NBS}\cdot$kxu; for $N_A\Lambda^3$, 10^{23} mol^{-1}; and for λ_C, 10^{-3} kxu.

[c]This column gives the ppm uncertainty of the weighted average, multiplied by the corresponding Birge ratio if the latter is greater than unity.

[d]The degrees of freedom for each weighted average is the number of items of like data minus one.

Table 4. Summary of data considered for use in the 1963 adjustment of Cohen and DuMond [1].

Item No.[a]	Quantity	Value[b]	Uncertainty (ppm)	Description	Original No.[c]
1.1	F	9.648682	6.8	NBS Silver coulometer	6
2.1	γ_p'	2.6751224	3.0	NBS low field	7
2.2	γ_p'	2.6751184	3.0	NPL low field	8
2.3	γ_p'	2.6752104	9.3	NBS high field	9
3.1	μ_p'/μ_N	2.792833	20	Cyclotron	3
3.2	μ_p'/μ_N	2.792684	9.0	NBS Omegatron	4
3.3	μ_p'/μ_N	2.792697	25	Inverse cyclotron	5
4.1	Λ	1.002020	35	Ruled grating, $CuK\alpha_1$	10
4.2	Λ	1.002110	75	Ruled grating, $CuK\alpha_1$	11
4.3	Λ	1.002011	33	Ruled grating, $AlK\alpha_1\alpha_2$	12
5.1	$N_A\Lambda^3$	6.06018	18	XRCD, $CuK\alpha_1$	13
5.2	$N_A\Lambda^3$	6.05972	17	XRCD, $MoK\alpha_1$	14
6.1	α^{-1}	137.0352	12	Hydrogen hyperfine splitting	1
6.2	α^{-1}	137.0388	4.6	Hydrogen fine-structure	2

[a]This is the item identification number to be used in the tables and discussion of the present work.

[b]The units for F are $10^4\ \mathrm{C \cdot mol^{-1}}$; for γ_p', $10^8\ \mathrm{s^{-1} \cdot T^{-1}}$; and for $N_A\Lambda^3$, $10^{23}\ \mathrm{mol^{-1}}$.

[c]These are the identification numbers for the indicated items used in Ref. [1].

Table 4a. Input data derived from Table 4 to be used in the application of the two-stage Birge ratio algorithm to the 1963 data; see text.

Item No.[a]	Quantity	Value[b] (weighted average)	Birge ratio of weighted average	Uncertainty[c] (ppm)	Degrees of freedom[d]
1	F	9.648682	-	6.8	-
2	γ_p'	2.6751248	2.50	5.1	2
3	μ_p'/μ_N	2.792708	1.72	13	2
4	Λ	1.002024	0.86	23	2
5	$N_A\Lambda^3$	6.05993	3.09	38	1
6	α^{-1}	137.0384	2.00	8.5	1

[a]These identification numbers correspond to those given in Table 4.

[b]Since there is only one value of type 1 data in Table 4, a weighted average cannot be carried out; the value remains unchanged. The units for F are 10^4 $C \cdot mol^{-1}$; for γ_p', 10^8 $s^{-1} \cdot T^{-1}$; and for $N_A\Lambda^3$, 10^{23} mol^{-1}.

[c]This column gives the ppm uncertainty of the weighted average, multiplied by the corresponding Birge ratio if the latter is greater than unity.

[d]The degrees of freedom for each weighted average is the number of items of like data minus one.

Table 5. Comparison of the adjusted values of selected constants resulting from the application of the algorithms of Table 1 to all of the data of the 1973 adjustment as given in Table 2.

Quantity[a]	1. 1973 recommended value and ppm uncertainty[b]		New adjusted value (expressed as a ppm change relative to the corresponding 1973 recommended value) and its ppm uncertainty for the indicated algorithm: 2. *A priori* uncertainty[c]		3. Birge ratio		4. Two-stage Birge ratio		5. VNIIM		6. Inverse		7. Natural log		8. Geometrical mean		9. Simple mean	
α^{-1}	137.03604(11)	0.82	-2.1	0.54	-2.1	1.2	-0.11	1.8	-0.98	0.72	-0.12	0.63	-0.62	0.68	-0.78	0.68	-0.25	0.65
Ω_{BI69}/Ω	$1 - 0.53 \times 10^{-6}$	0.19	0.06	0.19	0.06	0.42	-0.01	0.42	0.00	0.19	0.00	0.19	0.00	0.19	0.00	0.19	0.00	0.19
N_A	6.022045(31)	5.0	10.8	3.8	10.8	8.2	13.2	8.8	10.3	6.0	9.3	4.5	9.5	5.1	9.4	5.3	10.3	4.7
A_{BI69}/A	$1 + 0.7 \times 10^{-6}$	2.6	-4.4	1.9	-4.4	4.2	-6.5	4.6	-4.7	3.0	-4.6	2.3	-4.4	2.6	-4.3	2.7	-5.0	2.4
Λ	1.0020772(54)	5.3	-6.1	4.0	-6.1	8.8	-6.1	9.0	-6.2	5.8	-2.6	4.5	-5.7	5.2	-6.0	5.4	-4.7	4.8
μ_μ/μ_p	3.1833402(72)	2.3	-2.2	1.6	-2.2	3.5	-0.1	3.9	-0.4	2.5	0.0	1.7	0.2	2.0	0.3	2.1	0.0	1.8
e	1.6021892(46)	2.9	-2.2	2.1	-2.2	4.6	-6.4	5.4	-3.7	3.2	-4.5	2.5	-3.8	2.8	-3.5	2.9	-4.7	2.6
h	6.626176(36)	5.4	-6.6	4.0	-6.6	8.8	-12.9	9.9	-8.3	6.2	-9.0	4.7	-8.2	5.3	-7.8	5.5	-9.7	4.9
m_e	9.109534(47)	5.1	-10.8	3.8	-10.8	8.2	-13.1	8.8	-10.3	6.0	-9.3	4.5	-9.4	5.1	-9.3	5.3	-10.2	4.6
F	9.648456(27)	2.8	8.6	2.0	8.6	4.4	6.8	4.9	6.6	3.1	4.9	2.4	5.7	2.7	5.9	2.8	5.5	2.5
γ_p'	2.6751301(75)	2.8	8.5	2.0	8.5	4.3	6.7	4.9	6.6	3.1	4.8	2.4	5.6	2.7	5.8	2.8	5.5	2.5
μ_p'/μ_N	2.7927740(11)	0.38	-0.06	0.38	-0.06	0.83	-0.06	0.84	-0.04	0.40	-0.04	0.38	-0.03	0.39	-0.03	0.39	-0.04	0.38

[a] The units for N_A are 10^{23} mol^{-1}; for e, 10^{-19} C; for h, 10^{-34} J·s; for m_e, 10^{-31} kg; for F, 10^4 C·mol^{-1}; and for γ_p', 10^8 $s^{-1}\cdot T^{-1}$.

[b] For this adjustment $\chi^2 = 14.5$, $F = 21$, and $R_B = 0.83$.

[c] For this adjustment $\chi^2 = 119.1$, $F = 25$, and $R_B = 2.18$. For adjustment 4, $\chi'^2 = F = 6$, $R_B' = 1$, and the second Birge ratio = 2.21; for adjustments 5 through 12, $\chi'^2 = F = 25$ and $R_B' = 1$; and for adjustment 13, $\chi'^2 = 29.1$, $F = 25$, and $R_B' = 1.08$.

Table 5 (cont'd). Comparison of the adjusted values of selected constants resulting from the application of the algorithms of Table 1 to all of the data of the 1973 adjustment as given in Table 2.

Quantity	1. 1973 recommended value and ppm uncertainty		New adjusted value (expressed as a ppm change relative to the corresponding 1973 recommended value) and its ppm uncertainty for the indicated algorithm							
			10. Internal Birge ratio[d]		11. VNIIM systematic uncertainties		12. VNIIM with weights		13. Extended least-squares	
α^{-1}	137.03604(11)	0.82	-2.72	1.18	-1.59	0.87	-0.80	0.72	-1.20	0.82
Ω_{BI69}/Ω	$1 - 0.53 \times 10^{-6}$	0.19	0.08	0.48	0.01	0.20	0.00	0.19	0.05	0.33
N_A	6.022045(31)	5.0	13.8	6.2	14.2	5.0	7.8	6.2	11.3	5.8
A_{BI69}/A	$1 + 0.7 \times 10^{-6}$	2.6	-5.6	3.2	-6.3	2.6	-3.5	3.1	-5.1	3.0
Λ	1.0020772(54)	5.3	-8.7	8.6	-8.5	5.8	-4.7	5.8	-6.3	8.8
μ_μ/μ_p	3.1833402(72)	2.3	-2.3	3.7	-0.1	2.7	-0.5	2.4	-0.3	2.9
e	1.6021892(46)	2.9	-2.8	3.7	-4.7	2.9	-2.7	3.3	-3.9	3.2
h	6.626176(36)	5.4	-8.3	6.9	-10.9	5.5	-6.2	6.4	-8.9	6.2
m_e	9.109534(47)	5.1	-13.7	6.2	-14.1	5.0	-7.8	6.2	-11.3	5.8
F	9.648456(27)	2.8	11.0	3.4	9.5	2.7	5.1	3.2	7.4	3.1
γ_p'	2.6751301(75)	2.8	10.9	3.4	9.4	2.7	5.1	3.2	7.4	3.1
μ_p'/μ_N	2.7927740(11)	0.38	-0.11	0.89	-0.06	0.40	-0.03	0.4	0.03	0.56

[d] $R_{IB} = 2.54$.

Table 6. A complement to Table 5 giving the individual normalized residuals and altered uncertainties resulting from the application of algorithms of Table 1 to all of the data of the 1973 adjustment as given in Table 2.

Item No.	A priori uncertainty σ_i (ppm)	Normalized residuals and σ_i'/σ_i for indicated algorithm: 1. 1973 recommended		2. A priori uncertainty		3. Birge ratio		5. VNIIM		6. Inverse		7. Natural log		8. Geometrical mean		9. Simple mean	
1.1	0.19	0.04	1.00	0.36	1.00	0.16	2.18	0.06	1.01	0.03	1.00	0.04	1.00	0.04	1.00	0.03	1.00
2.1	9.7	-0.11	1.00	-0.56	1.00	-0.26	2.18	-0.45	1.32	-0.57	1.02	-0.52	1.09	-0.49	1.12	-0.59	1.06
2.2	7.7	0.24	1.00	-0.32	1.00	-0.15	2.18	-0.30	1.19	-0.34	1.01	-0.32	1.03	-0.30	1.04	-0.39	1.02
2.3	5.5	0.12	1.00	-0.69	1.00	-0.31	2.18	-0.53	1.39	-0.70	1.03	-0.61	1.13	-0.57	1.17	-0.73	1.09
3.1	6.8	deleted		-1.48	1.00	-0.68	2.18	-0.99	1.75	-1.41	1.42	-1.20	1.59	-1.13	1.67	-1.34	1.38
3.2	9.6	deleted		-1.30	1.00	-0.60	2.18	-0.88	1.68	-1.35	1.24	-1.09	1.47	-1.03	1.56	-1.21	1.29
4.1	4.0	1.07	1.43	2.56	1.00	1.17	2.18	1.10	1.83	1.31	1.21	1.18	1.56	1.14	1.68	1.26	1.31
4.2	2.0	-0.66	1.43	1.12	1.00	0.51	2.18	0.03	1.00	-0.80	1.05	-0.32	1.03	-0.17	1.01	-0.65	1.07
4.3	4.0	0.87	1.43	2.27	1.00	1.04	2.18	0.98	1.75	1.15	1.13	1.07	1.44	1.04	1.57	1.11	1.23
4.4	6.0	0.95	1.43	2.05	1.00	0.94	2.18	0.97	1.74	1.22	1.15	1.08	1.45	1.04	1.57	1.16	1.25
5.1	7.4	-0.09	1.00	1.65	1.00	0.76	2.18	0.86	1.66	1.07	1.10	0.95	1.33	0.89	1.42	1.09	1.22
5.2	16	1.23	1.00	2.04	1.00	0.93	2.18	1.07	1.81	1.39	1.31	1.18	1.57	1.12	1.66	1.35	1.39
6.1	0.43	0.16	1.00	0.02	1.00	0.01	2.18	0.07	1.01	0.07	1.00	0.08	1.00	0.09	1.00	0.07	1.00
6.2	0.82	-0.35	1.00	-0.42	1.00	-0.19	2.18	-0.33	1.22	-0.39	1.01	-0.37	1.05	-0.37	1.07	-0.39	1.02
7.1	33	1.17	1.28	1.32	1.00	0.60	2.18	0.81	1.62	1.23	1.16	0.98	1.36	0.92	1.44	1.11	1.23
7.2	9.8	0.93	1.28	0.57	1.00	0.26	2.18	0.43	1.31	0.87	1.06	0.55	1.10	0.51	1.13	0.66	1.07
7.3	33	0.84	1.28	0.89	1.00	0.41	2.18	0.61	1.46	0.93	1.07	0.75	1.20	0.71	1.26	0.83	1.12
8.1	16	-0.97	1.28	-1.72	1.00	-0.79	2.18	-1.00	1.77	-1.05	1.10	-1.14	1.52	-1.10	1.63	-1.18	1.27
8.2	28	0.19	1.28	-0.02	1.00	-0.01	2.18	-0.05	1.01	0.29	1.01	-0.02	1.00	-0.06	1.00	0.11	1.00
9.1	15	deleted		-3.05	1.00	-1.40	2.18	-1.52	2.10	0.00	∞	-1.53	2.13	-1.51	2.14	-1.54	2.19
9.2	33	-0.35	1.28	-0.14	1.00	-0.06	2.18	-0.19	1.09	-0.36	1.01	-0.24	1.02	-0.22	1.02	-0.29	1.01
10.1	3.1	0.69	1.40	0.29	1.00	0.13	2.18	0.48	1.36	0.88	1.06	0.67	1.15	0.61	1.19	0.80	1.11
10.2	1.6	0.23	1.40	-0.99	1.00	-0.46	2.18	-0.26	1.15	0.25	1.00	-0.07	1.00	-0.17	1.01	0.16	1.00
10.3	3.9	0.78	1.40	0.57	1.00	0.26	2.18	0.59	1.45	0.99	1.08	0.78	1.21	0.71	1.26	0.91	1.14
10.4	1.5	deleted		7.91	1.00	3.62	2.18	3.02	2.87	0.00	∞	2.11	4.22	2.52	3.49	0.00	∞
10.5	3.3	1.49	1.40	1.46	1.00	0.67	2.18	1.01	1.77	1.41	1.46	1.20	1.59	1.12	1.66	1.40	1.44
10.6	2.3	0.93	1.40	0.38	1.00	0.17	2.18	0.60	1.45	1.12	1.12	0.83	1.25	0.75	1.29	1.01	1.18
11.1	2.6	-0.57	1.40	-1.67	1.00	-0.77	2.18	-0.63	1.48	-0.76	1.04	-0.63	1.14	-0.58	1.17	-0.72	1.09
11.2	9.6	-0.38	1.40	-0.76	1.00	-0.35	2.18	-0.43	1.31	-0.52	1.02	-0.47	1.07	-0.45	1.10	-0.51	1.04
11.3	4.7	-0.46	1.40	-1.12	1.00	-0.51	2.18	-0.52	1.38	-0.62	1.03	-0.54	1.10	-0.51	1.13	-0.60	1.06
12.1	2.0	0.73	1.40	1.98	1.00	0.91	2.18	1.02	1.76	1.04	1.09	1.14	1.52	1.14	1.69	1.06	1.20

Table 6 (cont'd). A complement to Table 5 (cont'd) giving the individual normalized residuals and altered uncertainties resulting from the application of algorithms of Table 1 to all of the data of the 1973 adjustment as given in Table 2.

Item No.	A priori uncertainty (ppm) σ_i	σ_{Si}	T_i	x_i	v_i	Normalized residuals, σ_i'/σ_i, and σ_{Si}'/σ_{Si} for indicated algorithms[a]: 10. Internal Birge ratio			11. VNIIM systematic uncertainties			12. VNIIM with weights		13. Extended least-squares	
1.1	0.19	0.19	0.02	0.50	2.00	0.18	2.54	2.54	0.07	1.03	1.03	0.05	1.00	0.17	1.74
2.1	9.7	4.8	1.76	0.40	3.13	-0.45	1.53	2.54	-0.66	1.15	1.52	-0.33	1.40	-0.42	1.52
2.2	7.7	4.8	1.29	0.40	3.13	-0.27	1.75	2.54	-0.48	1.18	1.43	-0.18	1.13	-0.27	1.52
2.3	5.5	3.2	1.37	0.30	5.56	-0.53	1.70	2.54	-0.79	1.31	1.75	-0.38	1.38	-0.62	1.32
3.1	6.8	1.9	3.48	0.30	5.56	-0.79	1.19	2.54	-1.03	1.05	1.49	-0.83	2.57	-1.18	1.32
3.2	9.6	8.7	0.49	0.70	1.02	-0.40	2.33	2.54	-0.63	1.62	1.73	-1.03	1.70	-0.60	2.23
4.1	4.0	3.5	0.58	0.50	2.00	1.27	2.26	2.54	1.15	2.01	2.25	1.09	1.76	1.21	1.74
4.2	2.0	1.9	0.34	0.15	22.22	0.71	2.43	2.54	0.44	1.47	1.52	-0.14	1.04	0.21	1.09
4.3	4.0	3.9	0.26	0.30	5.56	1.04	2.48	2.54	0.98	2.07	2.12	1.00	1.63	1.38	1.32
4.4	6.0	5.9	0.20	0.60	1.39	0.90	2.50	2.54	0.93	2.04	2.07	1.01	1.62	0.88	1.98
5.1	7.4	7.1	0.28	0.50	2.00	0.87	2.47	2.54	0.99	2.06	2.12	0.73	1.45	0.92	1.74
5.2	16	12	0.96	0.50	2.00	1.16	1.96	2.54	1.24	1.78	2.28	0.98	1.81	1.16	1.74
6.1	0.43	0.38	0.53	0.50	2.00	-0.04	2.30	2.54	0.03	1.00	1.00	0.09	1.02	0.13	1.74
6.2	0.82	0.81	0.17	0.10	50.00	-0.19	2.51	2.54	-0.31	1.35	1.36	-0.33	1.16	-0.30	1.04
7.1	33	26	0.79	0.70	1.02	0.59	2.09	2.54	0.78	1.59	1.86	0.83	1.64	0.59	2.23
7.2	9.8	6.3	1.19	0.70	1.02	0.17	1.81	2.54	0.29	1.10	1.22	0.49	1.46	0.24	2.23
7.3	33	32	0.31	0.70	1.02	0.33	2.45	2.54	0.52	1.57	1.62	0.66	1.41	0.39	2.23
8.1	16	15	0.18	0.70	1.02	-0.81	2.51	2.54	-0.95	2.07	2.10	-1.01	1.62	-0.78	2.23
8.2	28	19	1.08	0.70	1.02	-0.10	1.88	2.54	-0.16	1.04	1.08	0.02	1.00	-0.01	2.23
9.1	15	11	0.88	0.70	1.02	-1.38	2.02	2.54	-1.49	1.98	2.50	-1.53	2.16	-1.41	2.23
9.2	33	24	0.92	0.70	1.02	-0.01	1.99	2.54	-0.09	1.02	1.04	-0.23	1.14	-0.08	2.23
10.1	3.1	2.3	0.85	0.40	3.13	0.04	2.04	2.54	0.37	1.23	1.37	0.51	1.39	0.39	1.52
10.2	1.6	1.6	0.00	0.70	1.02	-0.54	2.54	2.54	-0.44	1.54	1.54	-0.17	1.05	-0.19	2.23
10.3	3.9	2.0	1.69	0.70	1.02	0.26	1.55	2.54	0.61	1.15	1.48	0.54	1.65	0.36	2.23
10.4	1.5	1.2	0.64	0.70	1.02	3.38	2.21	2.54	2.81	2.94	3.43	3.11	2.83	3.82	2.23
10.5	3.3	1.9	1.41	0.70	1.02	0.76	1.68	2.54	1.11	1.45	2.07	0.94	1.96	0.78	2.23
10.6	2.3	0.8	2.55	0.70	1.02	0.08	1.31	2.54	0.58	1.04	1.29	0.52	1.84	0.35	2.23
11.1	2.6	2.3	0.52	0.40	3.13	-0.74	2.30	2.54	-0.54	1.52	1.63	-0.68	1.46	-0.61	1.52
11.2	9.6	6.7	1.04	0.60	1.39	-0.41	1.91	2.54	-0.44	1.22	1.42	-0.43	1.36	-0.28	1.98
11.3	4.7	3.8	0.71	0.70	1.02	-0.53	2.15	2.54	-0.48	1.37	1.52	-0.54	1.39	-0.32	2.23
12.1	2.0	2.0	0.20	0.70	1.02	1.02	2.50	2.54	1.14	2.23	2.27	0.97	1.60	0.91	2.23

[a]For algorithms 12 and 13, σ_{Si}'/σ_{Si} by itself has no meaning. Thus, only the value for σ_i'/σ_i is given.

Table 7. Comparison of the adjusted values of selected constants resulting from the application of algorithms of Table 1 to all of the data of the 1973 adjustment as given in Table 2 but with item No. 10.4 deleted.

Quantity[a]	1. 1973 recommended value and ppm uncertainty[b]		New adjusted value (expressed as a ppm change relative to the corresponding 1973 recommended value) and its ppm uncertainty for the indicated algorithm: 2. A priori uncertainty[c]		3. Birge ratio		4. Two-stage Birge ratio		5. VNIIM		6. Inverse		7. Natural log		8. Geometrical mean		9. Simple mean	
α^{-1}	137.03604(11)	0.82	-0.26	0.58	-0.26	0.81	-0.54	1.46	-0.31	0.69	-0.10	0.63	-0.30	0.67	-0.31	0.67	-0.25	0.66
Ω_{BI69}/Ω	$1 - 0.53 \times 10^{-6}$	0.19	0.01	0.19	0.01	0.27	0.00	0.40	0.00	0.19	0.00	0.19	0.00	0.19	0.00	0.19	0.00	0.19
N_A	6.022045(31)	5.0	12.0	3.8	12.0	5.3	12.9	8.4	11.0	5.0	7.9	4.6	10.2	4.8	10.4	4.8	10.0	4.7
A_{BI69}/A	$1 + 0.7 \times 10^{-6}$	2.6	-5.8	1.9	-5.8	2.7	-6.2	4.3	-5.3	2.5	-3.9	2.3	-5.0	2.4	-5.0	2.5	-4.8	2.4
Λ	1.0020772(54)	5.3	-6.6	4.0	-6.6	5.6	-6.0	8.7	-6.2	5.0	-2.3	4.5	-5.6	5.0	-5.9	5.0	-4.6	4.8
μ_μ/μ_p	3.1833402(72)	2.3	-0.3	1.6	-0.3	2.3	-0.6	3.5	0.0	2.0	0.0	1.7	0.1	1.9	0.1	1.9	0.1	1.8
e	1.6021892(46)	2.9	-5.6	2.2	-5.6	3.0	-5.6	4.9	-5.0	2.7	-3.8	2.5	-4.7	2.7	-4.7	2.7	-4.6	2.6
h	6.626176(36)	5.4	-11.4	4.1	-11.4	5.7	-11.8	9.1	-10.3	5.2	-7.7	4.8	-9.6	5.1	-9.7	5.1	-9.4	5.0
m_e	9.109534(47)	5.1	-11.9	3.8	-11.9	5.3	-12.9	8.4	-11.0	4.9	-7.9	4.6	-10.2	4.8	-10.3	4.8	-9.9	4.7
F	9.648456(27)	2.8	6.4	2.0	6.4	2.8	7.3	4.6	6.0	2.6	4.1	2.5	5.6	2.6	5.7	2.6	5.4	2.5
γ_p'	2.6751301(75)	2.8	6.3	2.0	6.3	2.8	7.3	4.5	6.0	2.6	4.1	2.4	5.6	2.5	5.6	2.6	5.3	2.5
μ_p'/μ_N	2.7927740(11)	0.38	-0.06	0.38	-0.06	0.53	-0.06	0.81	-0.04	0.39	-0.04	0.38	-0.04	0.38	-0.04	0.38	-0.04	0.38

[a] The units for N_A are 10^{23} mol^{-1}; for e, 10^{-19} C; for h, 10^{-34} J·s; for m_e, 10^{-31} kg; for F, 10^4 $C \cdot mol^{-1}$; and for γ_p' 10^8 $s^{-1} \cdot T^{-1}$.

[b] For this adjustment, $\chi^2 = 14.5$, $F = 21$, and $R_B = 0.83$.

[c] For this adjustment $\chi^2 = 46.6$, $F = 24$, and $R_B = 1.39$. For adjustment 4, $\chi'^2 = F = 6$, $R_B' = 1$, and the second Birge ratio = 2.13; for adjustments 5 through 12, $\chi'^2 = F = 24$ and $R_B' = 1$; and for adjustment 13, $\chi'^2 = 25.3$, $F = 24$, and $R_B' = 1.03$.

Table 7 (cont'd). Comparison of the adjusted values of selected constants resulting from the application of algorithms of Table 1 to all of the data of the 1973 adjustment as given in Table 2 but with item No. 10.4 deleted.

Quantity	1. 1973 recommended value and ppm uncertainty		New adjusted value (expressed as a ppm change relative to the corresponding 1973 recommended value) and its ppm uncertainty for the indicated algorithm							
			10. Internal Birge ratio[d]		11. VNIIM systematic uncertainties		12. VNIIM with weights		13. Extended least-squares	
α^{-1}	137.03604(11)	0.82	-0.69	0.87	-0.75	0.68	-0.08	0.69	-0.29	0.71
Ω_{BI69}/Ω	$1 - 0.53 \times 10^{-6}$	0.19	0.02	0.31	0.00	0.19	0.00	0.19	0.00	0.24
N_A	6.022045(31)	5.0	13.4	4.8	13.4	4.5	9.0	5.1	11.5	4.6
A_{BI69}/A	$1 + 0.7 \times 10^{-6}$	2.6	-6.3	2.5	-6.3	2.3	-4.4	2.6	-5.6	2.3
Λ	1.0020772(54)	5.3	-8.4	5.9	-8.2	5.2	-5.0	4.9	-6.5	6.0
μ_μ/μ_p	3.1833402(72)	2.3	-0.5	2.5	0.1	2.2	0.0	1.9	-0.2	2.3
e	1.6021892(46)	2.9	-5.6	2.9	-5.6	2.6	-4.3	2.8	-5.3	2.6
h	6.626176(36)	5.4	-11.9	5.3	-11.9	4.9	-8.8	5.3	-10.9	4.9
m_e	9.109534(47)	5.1	-13.3	4.8	-13.4	4.6	-8.9	5.1	-11.5	4.6
F	9.648456(27)	2.8	7.8	2.7	7.9	2.4	4.6	2.7	6.2	2.5
γ_p'	2.6751301(75)	2.8	7.7	2.6	7.8	2.4	4.6	2.7	6.2	2.4
μ_p'/μ_N	2.7927740(11)	0.38	-0.10	0.58	-0.06	0.39	-0.03	0.39	-0.03	0.46

[d] $R_{IB} = 1.63$.

Table 8. A complement to Table 7 giving the individual normalized residuals and altered uncertainties resulting from the application of algorithms of Table 1 to all of the data of the 1973 adjustment as given in Table 2 but with item 10.4 deleted.

		Normalized residuals and σ_i'/σ_i for indicated algorithm															
Item No.	A priori uncertainty σ_i (ppm)	1. 1973 recommended		2. A priori uncertainty		3. Birge ratio		5. VNIIM		6. Inverse		7. Natural log		8. Geometrical mean		9. Simple mean	
1.1	0.19	0.04	1.00	0.07	1.00	0.05	1.39	0.03	1.00	0.03	1.00	0.02	1.00	0.02	1.00	0.03	1.00
2.1	9.7	-0.11	1.00	-0.71	1.00	-0.51	1.39	-0.57	1.15	-0.50	1.02	-0.58	1.08	-0.58	1.08	-0.57	1.06
2.2	7.7	0.24	1.00	-0.51	1.00	-0.37	1.39	-0.41	1.09	-0.26	1.00	-0.38	1.03	-0.39	1.04	-0.37	1.02
2.3	5.5	0.12	1.00	-0.95	1.00	-0.68	1.39	-0.71	1.20	-0.58	1.02	-0.71	1.12	-0.71	1.13	-0.71	1.09
3.1	6.8	deleted		-1.59	1.00	-1.14	1.39	-1.23	1.40	-1.33	1.66	-1.28	1.43	-1.27	1.43	-1.32	1.43
3.2	9.6	deleted		-1.38	1.00	-0.99	1.39	-1.09	1.35	-1.37	1.32	-1.16	1.34	-1.14	1.35	-1.20	1.32
4.1	4.0	1.07	1.43	1.65	1.00	1.19	1.39	1.21	1.39	1.29	1.22	1.22	1.38	1.21	1.39	1.23	1.34
4.2	2.0	-0.66	1.43	-0.69	1.00	-0.50	1.39	-0.55	1.14	-0.81	1.05	-0.60	1.08	-0.58	1.08	-0.65	1.08
4.3	4.0	0.87	1.43	1.36	1.00	0.98	1.39	1.05	1.33	1.14	1.13	1.08	1.29	1.07	1.30	1.09	1.25
4.4	6.0	0.95	1.43	1.45	1.00	1.04	1.39	1.09	1.35	1.20	1.16	1.12	1.31	1.11	1.32	1.14	1.27
5.1	7.4	-0.09	1.00	1.55	1.00	1.11	1.39	1.07	1.34	0.92	1.17	1.04	1.27	1.05	1.29	1.05	1.22
5.2	16	1.23	1.00	1.99	1.00	1.43	1.39	1.34	1.44	1.35	1.28	1.30	1.45	1.31	1.45	1.32	1.42
6.1	0.43	0.16	1.00	0.01	1.00	0.01	1.39	0.07	1.00	0.08	1.00	0.07	1.00	0.07	1.00	0.07	1.00
6.2	0.82	-0.35	1.00	-0.43	1.00	-0.31	1.39	-0.37	1.07	-0.39	1.01	-0.38	1.03	-0.38	1.04	-0.39	1.03
7.1	33	1.17	1.28	1.30	1.00	0.93	1.39	1.00	1.31	1.23	1.17	1.05	1.27	1.04	1.28	1.09	1.25
7.2	9.8	0.93	1.28	0.51	1.00	0.37	1.39	0.50	1.12	0.90	1.07	0.57	1.07	0.55	1.08	0.67	1.08
7.3	33	0.84	1.28	0.87	1.00	0.62	1.39	0.73	1.21	0.93	1.07	0.79	1.14	0.78	1.15	0.83	1.13
8.1	16	-0.97	1.28	-1.75	1.00	-1.26	1.39	-1.24	1.41	-1.06	1.10	-1.21	1.38	-1.22	1.39	-1.16	1.28
8.2	28	0.19	1.28	-0.04	1.00	-0.03	1.39	-0.03	1.00	0.28	1.01	0.01	1.00	-0.01	1.00	0.11	1.00
9.1	15	deleted		-3.26	1.00	-2.34	1.39	-1.97	1.66	0.00	∞	-1.73	1.92	-1.79	1.84	-1.45	2.34
9.2	33	-0.35	1.28	-0.24	1.00	-0.17	1.39	-0.24	1.03	-0.37	1.01	-0.26	1.01	-0.25	1.02	-0.29	1.02
10.1	3.1	0.69	1.40	0.89	1.00	0.64	1.39	0.72	1.21	0.88	1.06	0.77	1.14	0.76	1.15	0.80	1.12
10.2	1.6	0.23	1.40	0.16	1.00	0.11	1.39	0.12	1.01	0.26	1.00	0.13	1.00	0.12	1.00	0.16	1.00
10.3	3.9	0.78	1.40	1.03	1.00	0.74	1.39	0.82	1.24	0.99	1.09	0.87	1.18	0.86	1.19	0.90	1.15
10.4	1.5	deleted		deleted		deleted		deleted		deleted		deleted		deleted		deleted	
10.5	3.3	1.49	1.40	2.01	1.00	1.44	1.39	1.37	1.46	1.37	1.50	1.35	1.49	1.35	1.48	1.36	1.48
10.6	2.3	0.93	1.40	1.19	1.00	0.85	1.39	0.91	1.28	1.12	1.12	0.96	1.22	0.95	1.23	1.00	1.20
11.1	2.6	-0.57	1.40	-0.90	1.00	-0.65	1.39	-0.68	1.19	-0.75	1.04	-0.69	1.11	-0.69	1.12	-0.71	1.09
11.2	9.6	-0.38	1.40	-0.56	1.00	-0.40	1.39	-0.48	1.11	-0.52	1.02	-0.50	1.06	-0.49	1.06	-0.50	1.04
11.3	4.7	-0.46	1.40	-0.70	1.00	-0.50	1.39	-0.56	1.14	-0.62	1.03	-0.58	1.08	-0.57	1.08	-0.59	1.06
12.1	2.0	0.73	1.40	1.14	1.00	0.82	1.39	1.00	1.32	1.03	1.10	1.06	1.28	1.06	1.29	1.06	1.22

Table 8 (cont'd). A complement to Table 7 (cont'd) giving the individual normalized residuals and altered uncertainties resulting from the application of algorithms of Table 1 to all of the data of the 1973 adjustment as given in Table 2 but with item 10.4 deleted.

						Normalized residuals, σ_i'/σ_i, and σ_{Si}'/σ_{Si} for indicated algorithms[a]									
Item No.	A priori uncertainty (ppm) σ_i	σ_{Si}	T_i	x_i	v_i	10. Internal Birge ratio			11. VNIIM systematic uncertainties			12. VNIIM with weights		13. Extended least-squares	
1.1	0.19	0.19	0.02	0.50	2.00	0.08	1.63	1.63	0.04	1.00	1.00	0.02	1.00	0.05	1.28
2.1	9.7	4.8	1.76	0.40	3.13	-0.64	1.19	1.63	-0.72	1.05	1.20	-0.47	1.19	-0.58	1.19
2.2	7.7	4.8	1.29	0.40	3.13	-0.45	1.27	1.63	-0.54	1.07	1.17	-0.31	1.07	-0.41	1.19
2.3	5.5	3.2	1.37	0.30	5.56	-0.83	1.25	1.63	-0.91	1.14	1.36	-0.58	1.20	-0.82	1.11
3.1	6.8	1.9	3.48	0.30	5.56	-1.24	1.06	1.63	-1.28	1.02	1.21	-1.08	1.92	-1.49	1.11
3.2	9.6	8.7	0.49	0.70	1.02	-0.78	1.53	1.63	-0.85	1.39	1.47	-1.29	1.33	-0.95	1.50
4.1	4.0	3.5	0.58	0.50	2.00	1.25	1.50	1.63	1.23	1.54	1.68	1.19	1.31	1.31	1.28
4.2	2.0	1.9	0.34	0.15	22.22	-0.17	1.58	1.63	-0.19	1.05	1.05	-0.76	1.15	-0.64	1.03
4.3	4.0	3.9	0.26	0.30	5.56	0.99	1.60	1.63	1.03	1.57	1.60	1.04	1.23	1.25	1.11
4.4	6.0	5.9	0.20	0.60	1.39	0.99	1.61	1.63	1.02	1.58	1.60	1.12	1.24	1.06	1.38
5.1	7.4	7.1	0.28	0.50	2.00	1.13	1.59	1.63	1.13	1.61	1.65	0.94	1.20	1.17	1.28
5.2	16	12	0.96	0.50	2.00	1.54	1.36	1.63	1.47	1.44	1.75	1.27	1.42	1.54	1.28
6.1	0.43	0.38	0.53	0.50	2.00	-0.04	1.52	1.63	0.03	1.00	1.00	0.08	1.00	0.08	1.28
6.2	0.82	0.81	0.17	0.10	50.00	-0.29	1.62	1.63	-0.36	1.16	1.16	-0.38	1.04	-0.38	1.01
7.1	33	26	0.79	0.70	1.02	0.88	1.42	1.63	0.95	1.32	1.49	1.04	1.30	0.87	1.50
7.2	9.8	6.3	1.19	0.70	1.02	0.26	1.30	1.63	0.34	1.03	1.08	0.58	1.18	0.35	1.50
7.3	33	32	0.31	0.70	1.02	0.52	1.58	1.63	0.63	1.31	1.34	0.80	1.16	0.58	1.50
8.1	16	15	0.18	0.70	1.02	-1.23	1.61	1.63	-1.18	1.67	1.68	-1.26	1.28	-1.18	1.50
8.2	28	19	1.08	0.70	1.02	-0.13	1.33	1.63	-0.15	1.01	1.02	0.03	1.00	-0.03	1.50
9.1	15	11	0.88	0.70	1.02	-2.22	1.39	1.63	-1.90	1.63	1.98	-2.04	1.66	-2.18	1.50
9.2	33	24	0.92	0.70	1.02	-0.11	1.38	1.63	-0.16	1.01	1.02	-0.28	1.04	-0.16	1.50
10.1	3.1	2.3	0.85	0.40	3.13	0.54	1.40	1.63	0.62	1.17	1.28	0.78	1.21	0.74	1.19
10.2	1.6	1.6	0.00	0.70	1.02	-0.07	1.63	1.63	-0.15	1.04	1.04	0.26	1.02	0.09	1.50
10.3	3.9	2.0	1.69	0.70	1.02	0.77	1.19	1.63	0.85	1.08	1.27	0.79	1.36	0.69	1.50
10.4	1.5	1.2	0.64	0.70	1.02	deleted			deleted			deleted		deleted	
10.5	3.3	1.9	1.41	0.70	1.02	1.51	1.25	1.63	1.48	1.26	1.66	1.33	1.55	1.34	1.50
10.6	2.3	0.8	2.55	0.70	1.02	0.91	1.10	1.63	0.95	1.03	1.20	0.82	1.54	0.79	1.50
11.1	2.6	2.3	0.52	0.40	3.13	-0.65	1.52	1.63	-0.61	1.25	1.31	-0.69	1.14	-0.58	1.50
11.2	9.6	6.7	1.04	0.60	1.39	-0.43	1.34	1.63	-0.48	1.08	1.17	-0.47	1.11	-0.46	1.19
11.3	4.7	3.8	0.71	0.70	1.02	-0.51	1.45	1.63	-0.53	1.16	1.24	-0.56	1.12	-0.49	1.38
12.1	2.0	2.0	0.20	0.70	1.02	0.91	1.61	1.63	1.11	1.63	1.65	0.94	1.19	0.81	1.50

[a]For agorithms 12 and 13, σ_{Si}'/σ_{Si} by itself has no meaning. Thus, only the value for σ_i'/σ_i is given.

Table 9. Comparison of the adjusted values of selected constants resulting from the application of algorithms of Table 1 to all of the data of the 1969 adjustment as given in Table 3.

Quantity[a]	1. 1969 recommended value and ppm uncertainty[b]		New adjusted value (expressed as a ppm change relative to the corresponding 1969 recommended value) and its ppm uncertainty for the indicated algorithm															
			2. *A priori* uncertainty[c]		3. Birge ratio		4. Two-stage Birge ratio		5. VNIIM		6. Inverse		7. Natural log		8. Geometrical mean		9. Simple mean	
α^{-1}	137.03602(21)	1.5	-2.9	1.2	-2.9	1.8	-2.1	2.0	-2.1	1.5	-2.2	1.3	-1.8	1.4	-1.9	1.4	-1.8	1.4
N_A	6.022169(40)	6.6	-11.5	5.8	-11.5	8.5	-8.7	9.0	-9.6	7.0	-3.9	6.4	-8.1	6.5	-8.3	6.6	-7.6	6.4
A_{BI69}/A	$1 + 0.5 \times 10^{-6}$	2.6	-2.5	2.5	-2.5	3.6	-1.8	3.7	-1.9	2.8	0.1	2.6	-1.6	2.7	-1.7	2.7	-1.4	2.6
Λ	1.0020764(53)	5.3	-2.3	4.8	-2.3	7.1	-0.6	7.1	-0.3	5.7	-1.7	5.1	0.0	5.3	0.0	5.3	0.1	5.2
e	1.6021917(70)	4.4	2.0	3.8	2.0	5.5	1.7	5.8	1.4	4.4	3.8	4.0	1.3	4.2	1.3	4.2	1.6	4.1
h	6.626196(50)	7.6	1.1	6.7	1.1	9.9	1.2	10.2	0.7	7.9	5.4	7.1	0.8	7.4	0.7	7.5	1.4	7.3
m_e	9.109558(54)	6.0	-4.6	5.5	-4.6	8.1	-3.0	8.2	-3.6	6.5	1.1	5.9	-2.8	6.0	-3.0	6.1	-2.3	5.9
F	9.648670(54)	5.5	-9.5	4.2	-9.5	6.2	-7.1	7.5	-8.2	5.5	-0.2	5.4	-6.8	5.3	-7.1	5.3	-6.0	5.2
γ_p'	2.6751270(82)	3.1	6.6	2.8	6.6	4.1	4.6	4.2	5.0	3.3	2.7	3.0	4.2	3.1	4.3	3.1	3.9	3.0
μ_p'/μ_N	2.792709(17)	6.2	16.1	4.3	16.1	6.3	11.7	8.4	13.2	5.9	2.9	6.0	10.9	5.7	11.4	5.7	9.8	5.7

[a]The units for N_A are 10^{23} mol^{-1}; for e, 10^{-19} C; for h, 10^{-34} J·s; for m_e, 10^{-31} kg; for F, 10^{4} C·mol^{-1}; and for γ_p', 10^{8} $s^{-1}\cdot T^{-1}$.

[b]For this adjustment, $\chi^2 = 1.36$, $F = 8$, and $R_B = 0.41$.

[c]For this adjustment $\chi^2 = 36.4$, $F = 17$, and $R_B = 1.46$. For adjustment 4, $\chi'^2 = F = 6$, $R_B' = 1$, and the second Birge ratio = 1.40; for all other adjustments in Table 9, $\chi'^2 = F = 17$ and $R_B' = 1$.

Table 10. A complement to Table 9 giving the individual normalized residuals and altered uncertainties resulting from the application of algorithms of Table 1 to all of the data of the 1969 adjustment as given in Table 3.

Item No.	A priori uncertainty σ_i (ppm)	Normalized residuals and σ_i'/σ_i for indicated algorithm															
		1. 1969 recommended		2. A priori uncertainty		3. Birge ratio		5. VNIIM		6. Inverse		7. Natural log		8. Geometrical mean		9. Simple mean	
1.1	2.4	-0.14	1.00	-0.84	1.00	-0.57	1.46	-0.54	1.15	-0.78	1.02	-0.56	1.06	-0.55	1.07	-0.60	1.05
2.1	9.7	-0.11	1.00	-0.37	1.00	-0.25	1.46	-0.28	1.05	-0.10	1.00	-0.27	1.01	-0.27	1.02	-0.25	1.01
2.2	7.7	-0.01	1.00	-0.34	1.00	-0.24	1.46	-0.25	1.04	-0.01	1.00	-0.22	1.01	-0.23	1.01	-0.19	1.00
2.3	6.0	0.19	1.00	-0.23	1.00	-0.16	1.46	-0.12	1.01	0.20	1.00	-0.07	1.00	-0.08	1.00	-0.04	1.00
3.1	6.8	0.17	1.00	-0.85	1.00	-0.58	1.46	-0.64	1.19	0.14	1.00	-0.56	1.06	-0.58	1.07	-0.49	1.03
4.1	3.7	-0.11	1.00	1.00	1.00	0.68	1.46	0.62	1.18	0.63	1.01	0.56	1.06	0.57	1.07	0.55	1.04
4.2	5.8	0.47	1.00	1.18	1.00	0.80	1.46	0.80	1.26	0.93	1.03	0.81	1.13	0.81	1.15	0.83	1.09
5.1	7.4	-0.11	1.00	1.13	1.00	0.77	1.46	0.68	1.20	0.25	1.00	0.62	1.07	0.64	1.09	0.57	1.04
6.1	11	0.63	1.00	2.13	1.00	1.46	1.46	1.28	1.45	0.88	1.02	1.25	1.32	1.24	1.36	1.26	1.23
6.2	26	0.11	1.00	0.72	1.00	0.49	1.46	0.53	1.14	0.22	1.00	0.50	1.05	0.51	1.06	0.47	1.03
6.3	20	deleted		-1.41	1.00	-0.96	1.46	-1.13	1.39	-1.82	1.15	-1.26	1.33	-1.23	1.35	-1.85	1.28
6.4	6.2	deleted		-2.31	1.00	-1.58	1.46	-1.73	1.62	0.00	∞	-1.78	1.77	-1.78	1.74	-1.76	1.89
6.5	19	-0.71	1.00	0.15	1.00	0.10	1.46	-0.01	1.00	-0.56	1.01	-0.13	1.00	-0.10	1.00	-0.18	1.00
7.1	33	deleted		-0.86	1.00	-0.59	1.46	-0.76	1.24	-0.87	1.02	-0.84	1.14	-0.83	1.15	-0.87	1.10
8.1	38	deleted		1.17	1.00	0.80	1.46	0.94	1.31	1.14	1.04	1.02	1.21	1.00	1.23	1.07	1.16
9.1	37	0.18	1.00	-0.31	1.00	-0.21	1.46	-0.10	1.01	-0.06	1.00	-0.03	1.00	-0.04	1.00	-0.02	1.00
9.2	16	-0.08	1.00	-1.25	1.00	-0.86	1.46	-0.62	1.18	-0.65	1.01	-0.56	1.06	-0.57	1.07	-0.53	1.03
10.1	38	deleted		0.42	1.00	0.29	1.46	0.31	1.06	0.36	1.00	0.30	1.02	0.30	1.02	0.30	1.01
10.2	15	deleted		-3.27	1.00	-2.24	1.46	-2.04	1.72	-2.04	1.67	-1.88	1.90	-1.92	1.85	-1.76	2.03
11.1	2.6	0.31	1.00	-0.82	1.00	-0.56	1.46	-0.47	1.12	-0.53	1.01	-0.40	1.03	-0.40	1.04	-0.40	1.02
11.2	4.3	deleted		0.30	1.00	0.20	1.46	0.43	1.10	0.46	1.01	0.51	1.05	0.50	1.06	0.52	1.03
11.3	2.4	deleted		1.77	1.00	1.21	1.46	1.40	1.49	1.81	1.15	1.49	1.49	1.47	1.51	1.56	1.42

Table 11. Comparison of the adjusted values of selected constants resulting from the application of algorithms of Table 1 to all of the data of the 1963 adjustment as given in Table 4.

Quantity[a]	1. 1963 recommended value and ppm uncertainty[b]		New adjusted value (expressed as a ppm change relative to the corresponding 1963 recommended value) and its ppm uncertainty for the indicated algorithm															
			2. *A priori* uncertainty[c]		3. Birge ratio		4. Two-stage Birge ratio		5. VNIIM		6. Inverse		7. Natural log		8. Geometrical mean		9. Simple mean	
α^{-1}	137.03880(62)	4.6	-1.7	4.2	-1.7	8.1	1.1	9.2	-0.5	4.6	-1.3	4.3	-0.5	4.4	-0.5	4.5	-0.7	4.4
N_A	6.022539(90)	15	-4.8	14	-4.8	27	4.5	29	-1.9	15	-5.0	14	-1.7	15	-1.7	15	-2.5	14
Λ	1.002066(14)	14	-0.6	6.0	-0.6	12	-11.6	15	-4.4	8.7	-12.2	7.0	-9.4	7.2	-8.5	7.4	-11.2	7.0
e	1.602098(22)	14	3.1	13	3.1	25	-5.8	28	0.8	14	3.6	13	1.0	13	0.9	14	1.7	13
h	6.62556(15)	23	4.5	21	4.5	41	-10.5	47	1.1	23	5.9	22	1.5	23	1.4	23	2.7	22
m_e	9.10905(13)	14	1.1	13	1.1	25	-8.3	30	0.0	15	3.3	14	0.5	14	0.4	14	1.2	14
F	9.648699(52)	5.4	-1.7	5.2	-1.7	10	-1.3	7.0	-1.1	5.9	-1.3	5.3	-0.7	5.5	-0.7	5.6	-0.8	5.4
γ_p'	2.6751200(56)	2.1	2.0	2.0	2.0	3.9	2.5	5.5	0.8	2.3	0.3	2.1	0.5	2.1	0.5	2.1	0.5	2.1
μ_p'/μ_N	2.792693(15)	5.5	3.7	5.3	3.7	10	3.8	8.2	1.9	6.0	1.7	5.4	1.2	5.6	1.3	5.7	1.2	5.5

[a]The units for N_A are 10^{23} mol^{-1}; for e, 10^{-19} C; for h, 10^{-34} J·s; for m_e, 10^{-31} kg; for F, 10^4 C·mol^{-1}; and for γ_p', 10^8 $s^{-1}\cdot T^{-1}$.

[b]These numbers differ slightly from those given in Ref. [1] -- see text. For this adjustment, $\chi^2 = 0.33$, $F = 3$, and $R_B = 0.33$.

[c]For this adjustment, $\chi^2 = 37.0$, $F = 10$, and $R_B = 1.92$. For adjustment 4, $\chi'^2 = F = 2$, $R_B' = 1$, and the second Birge ratio = 1.14; for all other adjustments in Table 10, $\chi'^2 = F = 10$ and $R_B' = 1$.

Table 12. A complement to Table 11 giving the individual normalized residuals and altered uncertainties resulting from the application of algorithms of Table 1 to all of the data of the 1963 adjustment as given in Table 4.

Item No.	A priori uncertainty σ_i (ppm)	Normalized residuals and σ_i'/σ_i for indicated algorithm															
		1. 1963 recommended		2. A priori uncertainty		3. Birge ratio		5. VNIIM		6. Inverse		7. Natural log		8. Geometrical mean		9. Simple mean	
1.1	6.8	0.26	1.00	0.01	1.00	0.01	1.92	0.10	1.03	0.07	1.00	0.16	1.01	0.15	1.01	0.15	1.00
2.1	3.0	-0.31	1.00	0.37	1.00	0.19	1.92	-0.04	1.01	-0.19	1.00	-0.14	1.01	-0.13	1.01	-0.15	1.00
2.2	3.0	0.19	1.00	0.87	1.00	0.45	1.92	0.36	1.26	0.31	1.00	0.34	1.05	0.35	1.07	0.34	1.02
2.3	9.3	deleted		-3.40	1.00	-1.77	1.92	-1.61	2.19	-0.66	5.44	-1.48	2.41	-1.52	2.34	-1.39	2.57
3.1	20	deleted		-2.31	1.00	-1.20	1.92	-1.23	1.95	-1.81	1.34	-1.27	1.92	-1.25	1.95	-1.45	1.68
3.2	9.0	0.36	1.00	0.78	1.00	0.40	1.92	0.43	1.32	0.54	1.01	0.45	1.09	0.45	1.12	0.48	1.04
3.3	25	-0.01	1.00	0.09	1.00	0.05	1.92	0.02	1.00	0.01	1.00	-0.01	1.00	-0.01	1.00	-0.01	1.00
4.1	35	deleted		1.30	1.00	0.67	1.92	0.75	1.59	0.93	1.04	0.81	1.30	0.78	1.38	0.87	1.14
4.2	75	deleted		-0.59	1.00	-0.31	1.92	-0.47	1.36	-0.73	1.02	-0.61	1.16	-0.58	1.21	-0.68	1.08
4.3	33	deleted		1.65	1.00	0.86	1.92	0.90	1.71	1.21	1.07	0.96	1.44	0.92	1.53	1.07	1.24
5.1	18	deleted		-2.51	1.00	-1.30	1.92	-1.43	2.08	0.00	∞	-1.51	2.50	-1.54	2.36	-1.13	3.63
5.2	17	deleted		1.84	1.00	0.96	1.92	0.80	1.64	-0.28	1.00	0.41	1.07	0.51	1.16	0.07	1.00
6.1	12	deleted		1.99	1.00	1.04	1.92	1.12	1.87	1.68	1.20	1.19	1.76	1.15	1.82	1.38	1.51
6.2	4.6	0.00	1.00	-0.38	1.00	-0.20	1.92	-0.11	1.04	-0.29	1.00	-0.11	1.00	-0.11	1.01	-0.16	1.00

Table 13. Comparison of adjusted values of selected constants resulting from the application of selected algorithms of Table 1 to the 1973 data with item 10.4 deleted, and to the 1969 and 1963 data.[a]

Algorithm	Quantity	Value[b] and ppm uncertainty						1963 - 1969 differences in ppm
		1973 data		1969 data		1963 data		
1. Recommended values	α^{-1}	137.03604(11)	0.82	-0.12	1.5	20	4.6	20
	N_A	6.022045(31)	5.0	20.6	6.6	82	15	61
	e	1.6021892(46)	2.9	1.5	4.4	-57	14	-58
	h	6.626176(36)	5.4	3.1	7.6	-93	23	-96
	m_e	9.109534(47)	5.1	2.6	6.0	-53	14	-56
3. Birge ratio	α^{-1}	-0.26	0.81	-2.7	1.8	19	8.1	21
	N_A	12.0	5.3	-2.8	8.5	65	27	68
	e	-5.6	3.0	9.1	5.5	-48	25	-57
	h	-11.4	5.7	15.6	9.9	-77	41	-93
	m_e	-11.9	5.3	9.9	8.1	-40	25	-50
4. Two-stage Birge ratio	α^{-1}	-0.54	1.46	-1.7	2.0	22	9.2	23
	N_A	12.9	8.4	-1.0	9.0	74	29	75
	e	-5.6	4.9	8.8	5.8	-57	28	-66
	h	-11.8	9.1	16.1	10.2	-92	47	-108
	m_e	-12.9	8.4	12.5	8.2	-49	30	-61
5. VNIIM	α^{-1}	-0.31	0.69	-1.9	1.5	20	4.6	22
	N_A	11.0	5.0	0.1	7.0	69	15	69
	e	-5.0	2.7	7.9	4.4	-51	14	-59
	h	-10.3	5.2	14.1	7.9	-82	23	-96
	m_e	-11.0	4.9	10.0	6.5	-42	15	-52
6. Inverse	α^{-1}	-0.10	0.63	-2.2	1.3	19	4.3	21
	N_A	7.9	4.6	8.8	6.4	69	14	60
	e	-3.8	2.5	9.1	4.0	-49	13	-59
	h	-7.7	4.8	16.2	7.1	-80	22	-96
	m_e	-7.9	4.6	11.6	5.9	-42	14	-54
9. Simple mean	α^{-1}	-0.25	0.66	-1.7	1.4	20	4.4	21
	N_A	10.0	4.7	3.1	6.4	70	14	67
	e	-4.6	2.6	7.7	4.1	-51	13	-58
	h	-9.4	5.0	13.9	7.3	-81	22	-95
	m_e	-9.9	4.7	10.3	5.9	-42	14	-52

[a]Item 10.4 has been deleted from the 1973 data as given in Table 2 because of its extremely discrepant nature as discussed in the text.

[b]The values for the 1969 and 1963 data are given in ppm relative to the corresponding values for the 1973 data, while the values for the 1973 data are given relative to the 1973 recommended values -- see text. The units for e are 10^{-19} C; for h, 10^{-34} J·s; for m_e, 10^{-31} kg; and for N_A, 10^{23} mol^{-1}.

9. FIGURE CAPTIONS AND FIGURES

Figure 1: Graphical comparison of the values of α^{-1} resulting from the application of the 13 algorithms of Table 1 to the 1973 data (i.e., a plot of the results for α^{-1} given in Table 5). Each value is expressed as a ppm change relative to the 1973 recommended value.

Figure 2: As in Fig. 1 but for N_A.

Figure 3: As in Fig. 1 but for A_{BI69}/A.

Figure 4: As in Fig. 1 but for Λ.

Figure 5: As in Fig. 1 but for μ_μ/μ_p.

Figure 6: As in Fig. 1 but for e.

Figure 7: As in Fig. 1 but for h.

Figure 8: As in Fig. 1 but for F.

Figure 9: As in Fig. 1 but with item 10.4 deleted (i.e., a plot of the results for α^{-1} given in Table 7).

Figure 10: Graphical representation of the changes in the adjusted values of selected constants resulting from the application of the indicated algorithms of Table 1 to the 1973 data, and to the 1973 data with item 10.4 deleted (i.e., a comparison of Tables 7 and 5). The points are the ppm differences between the values given in Table 7 and those given in Table 5. Each division on the vertical scale is one ppm.

Figure 11. As in Fig. 10.

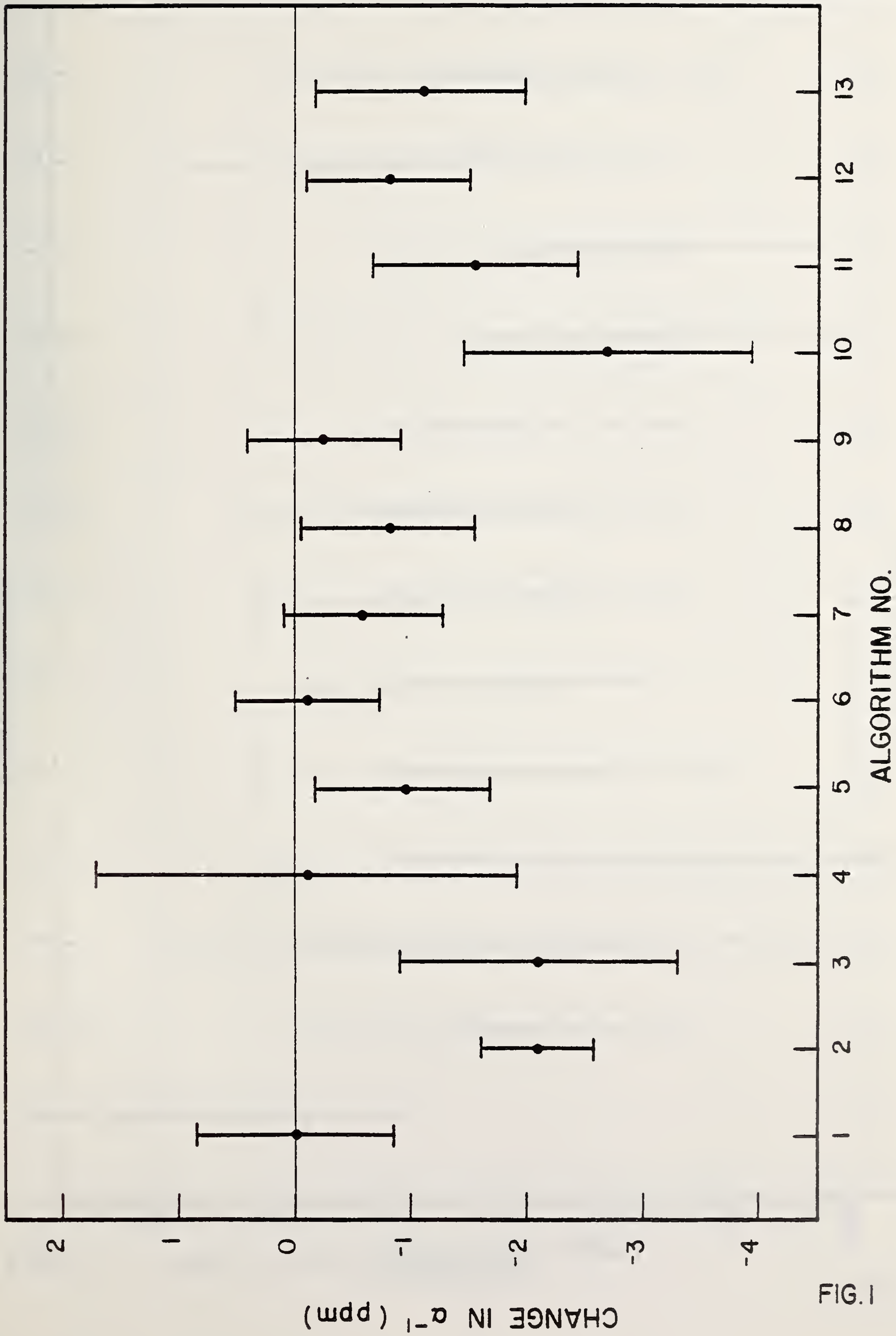

ALGORITHM NO.
1 2 3 4 5 6 7 8 9 10 11 12 13
CHANGE IN α^{-1} (ppm)
2 1 0 -1 -2 -3 -4

FIG. 1

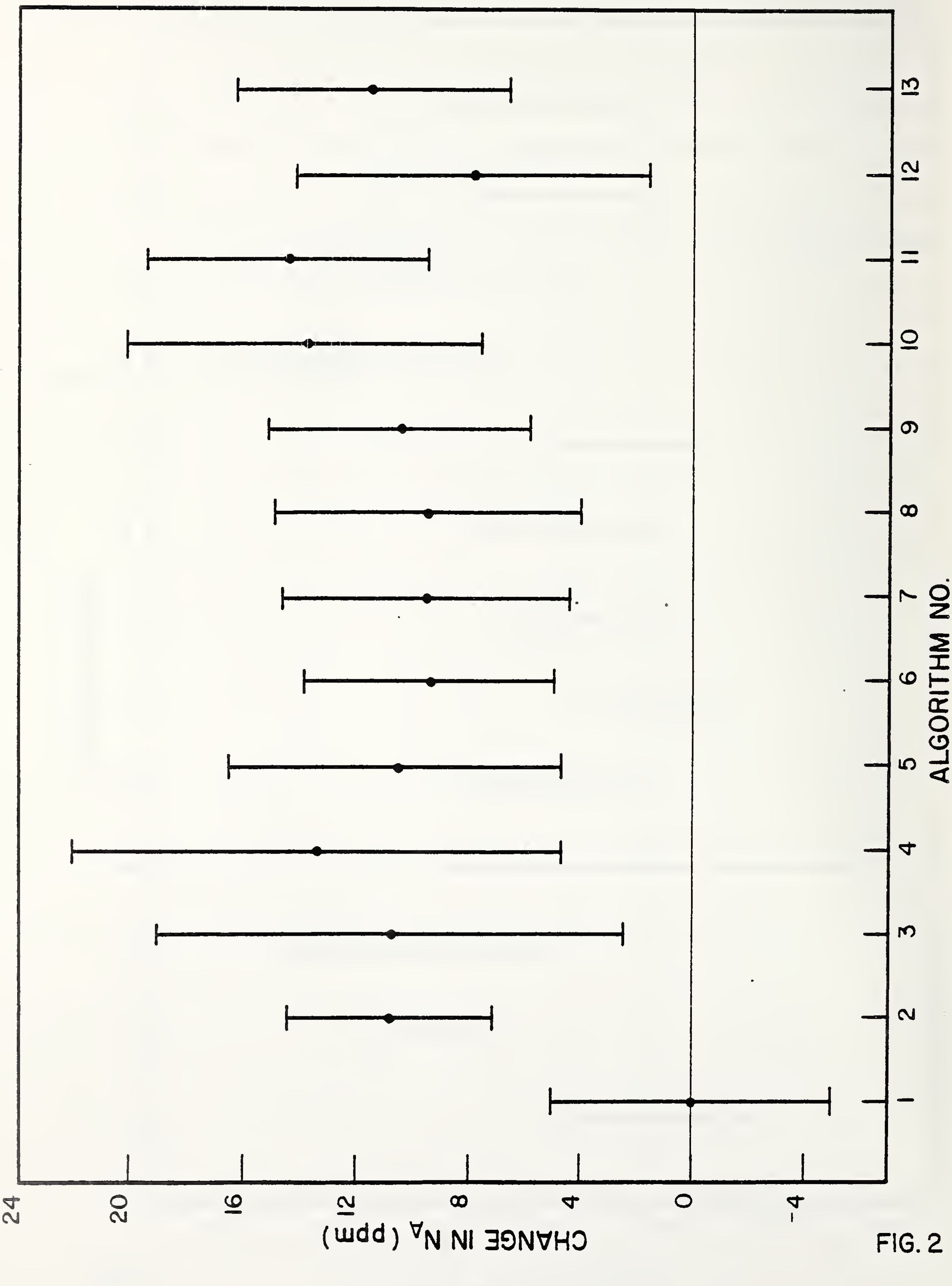

24
20
16
12
8
4
0
-4
CHANGE IN N_A (ppm)
1
2
3
4
5
6
7
8
9
10
11
12
13
ALGORITHM NO.

FIG. 2

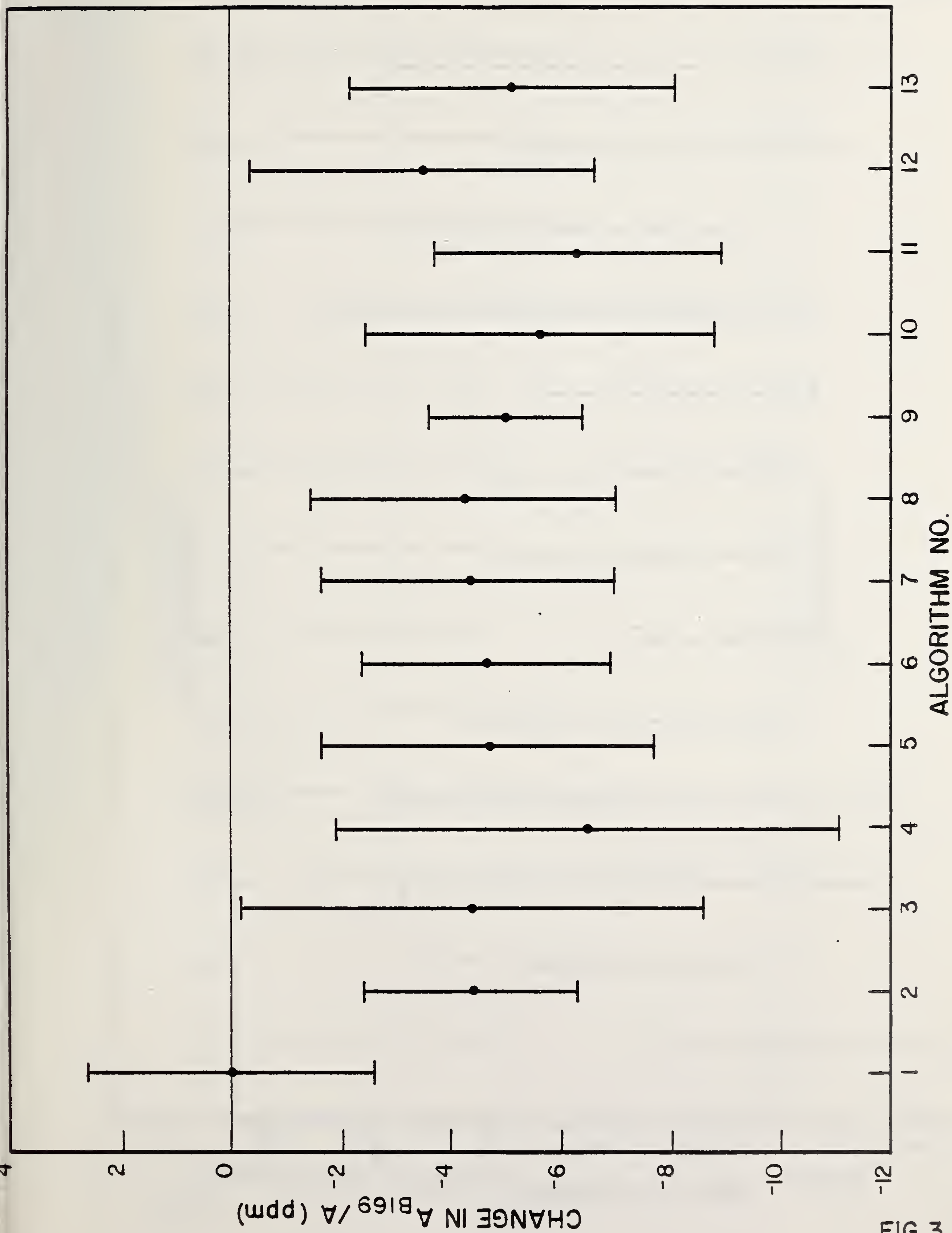

CHANGE IN A B169 /A (ppm)
4
2
0
-2
-4
-6
-8
-10
-12
ALGORITHM NO.
1
2
3
4
5
6
7
8
9
10
11
12
13

FIG. 3

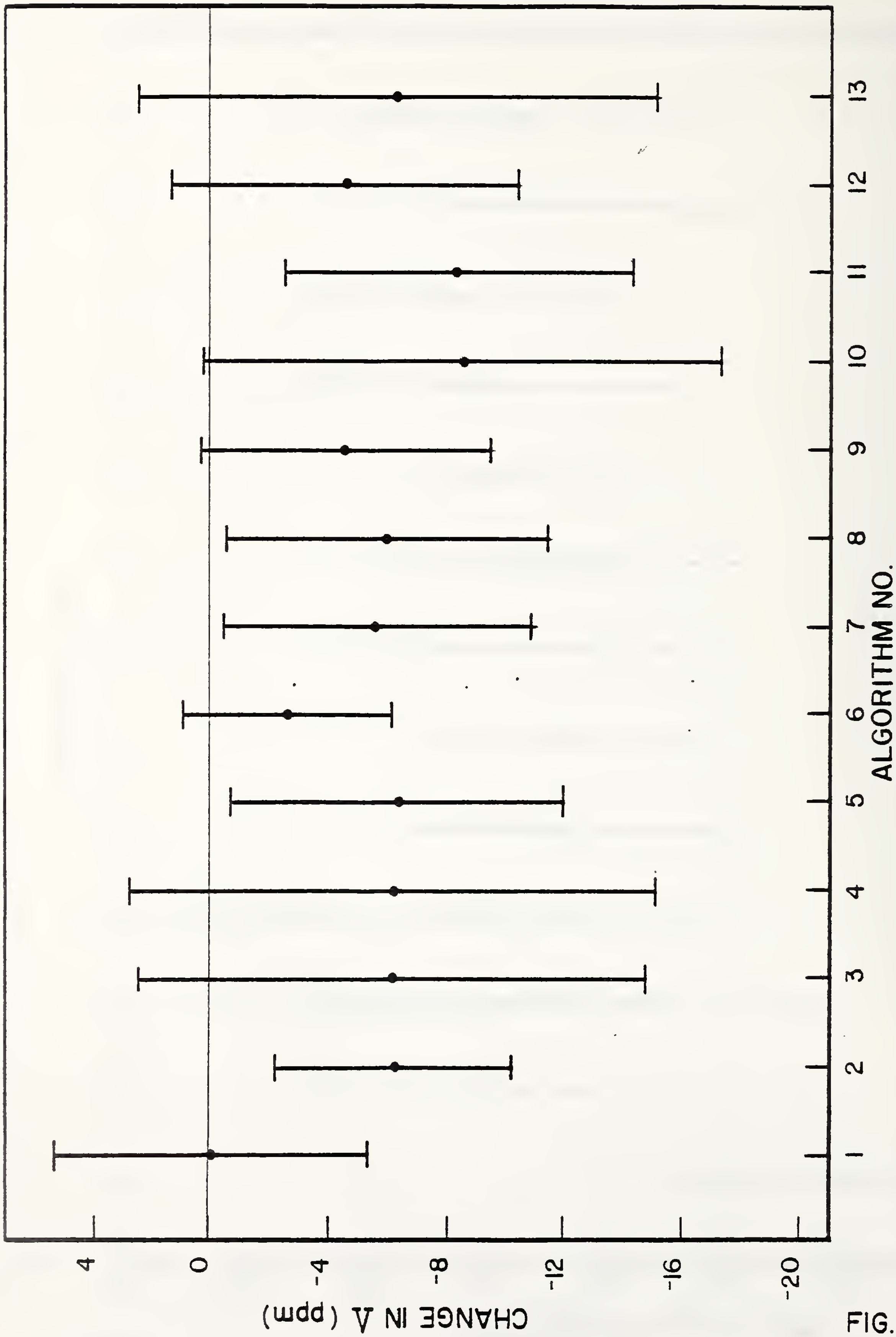

CHANGE IN Δ (ppm)
4
0
-4
-8
-12
-16
-20
ALGORITHM NO.
1
2
3
4
5
6
7
8
9
10
11
12
13

FIG. 4

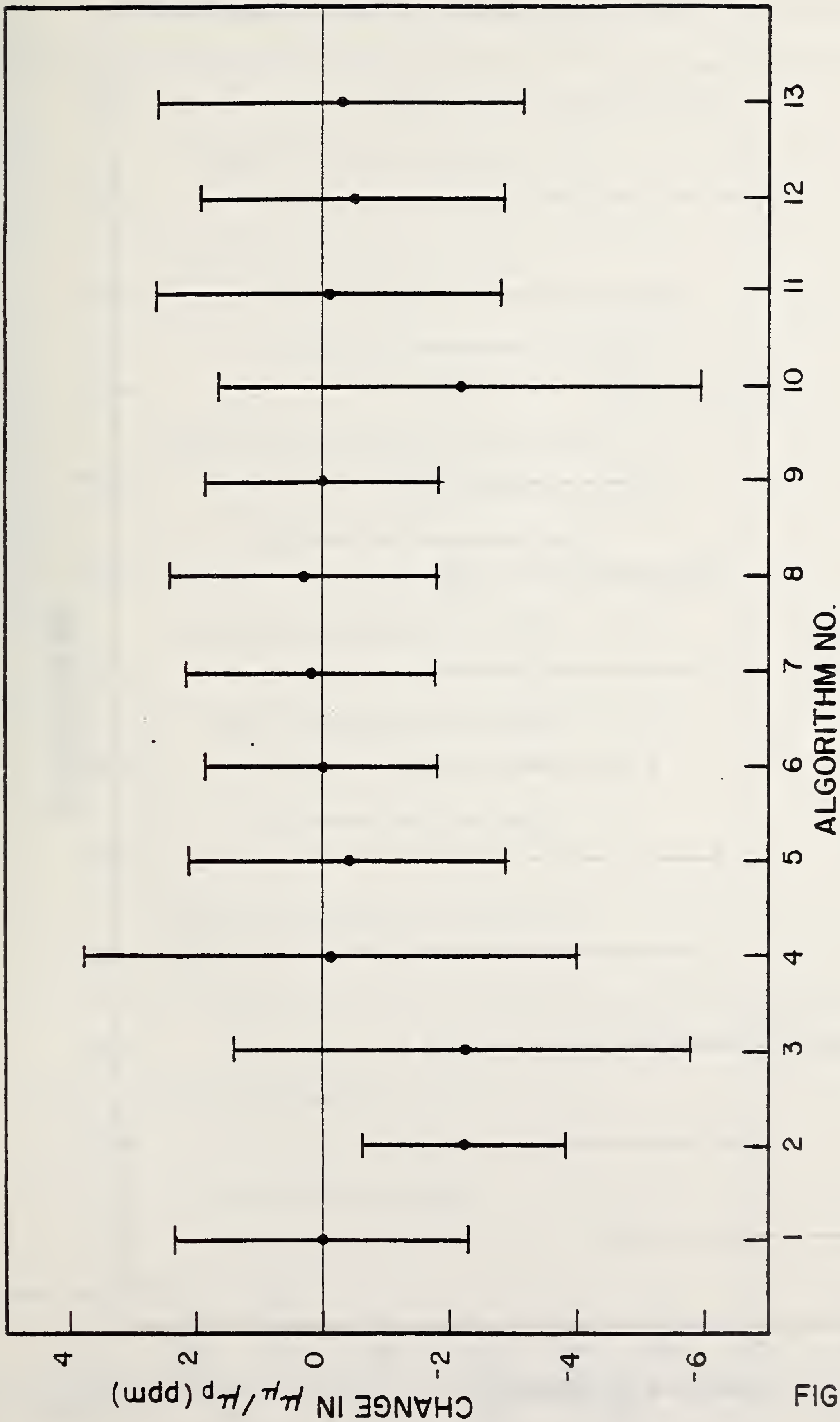

CHANGE IN μ_μ/μ_p (ppm)
4
2
0
-2
-4
-6
ALGORITHM NO.
1
2
3
4
5
6
7
8
9
10
11
12
13

FIG. 5

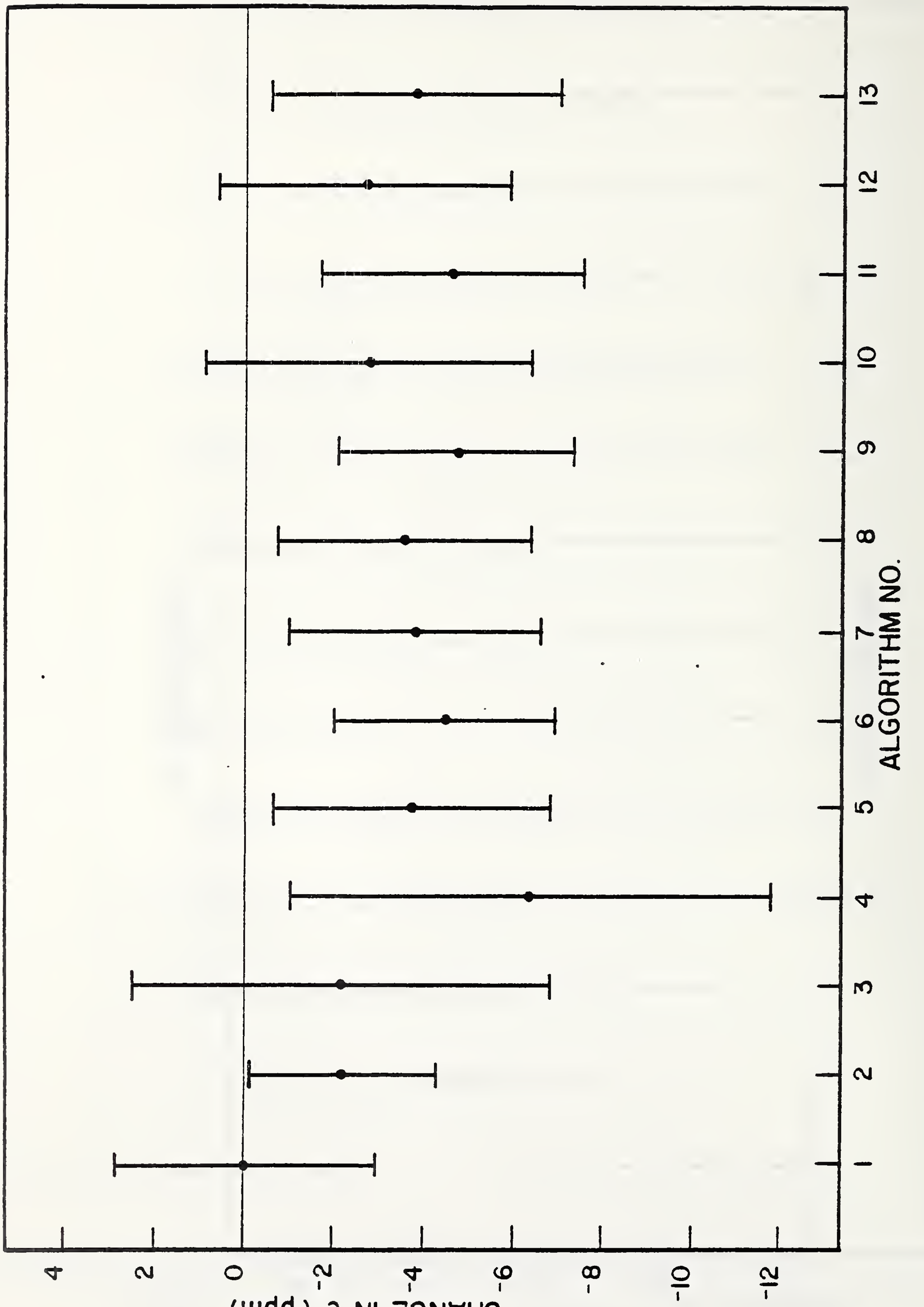

CHANGE IN e (ppm)
4
2
0
-2
-4
-6
-8
-10
-12
ALGORITHM NO.
1
2
3
4
5
6
7
8
9
10
11
12
13

FIG.6

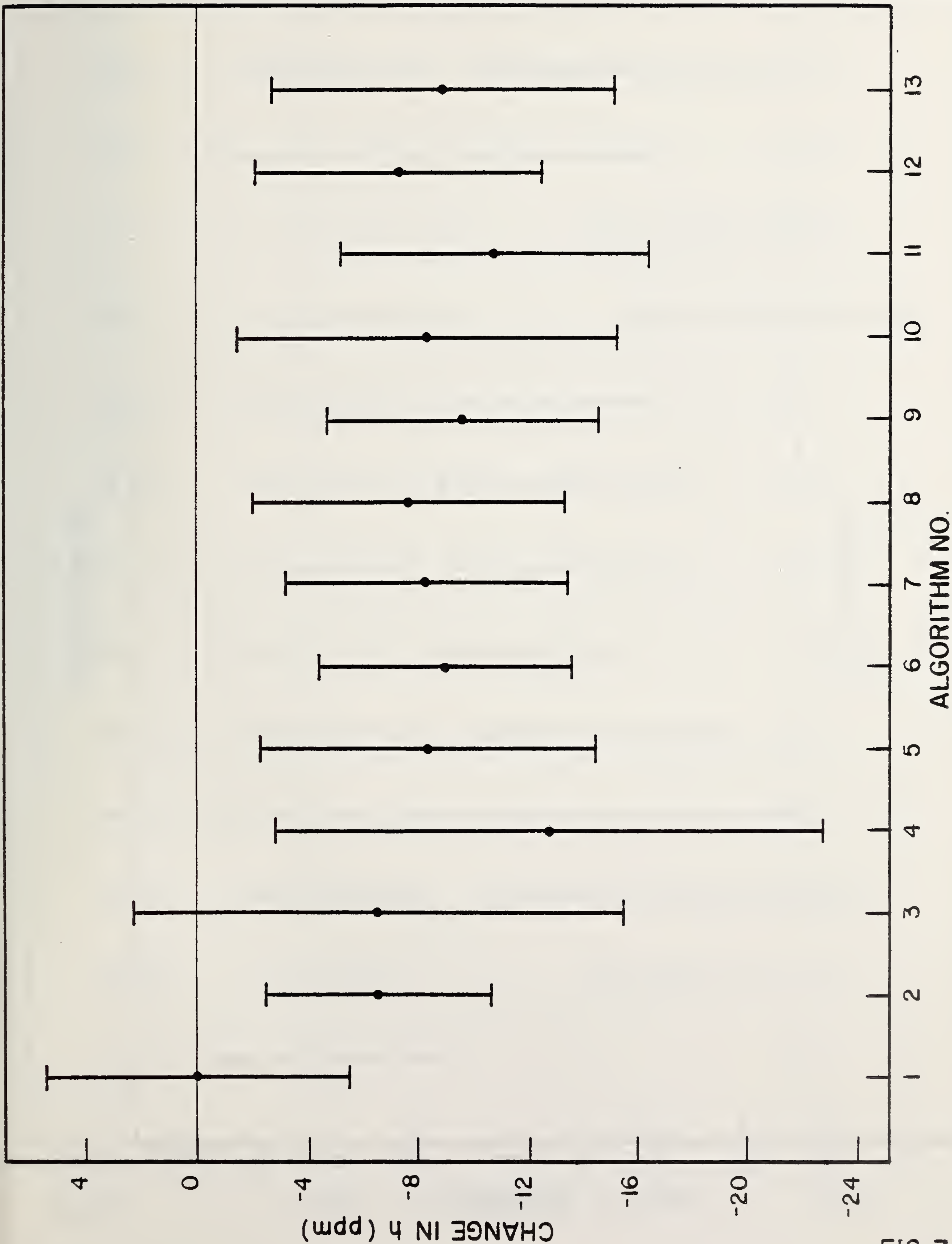

CHANGE IN h (ppm)
4
0
-4
-8
-12
-16
-20
-24
ALGORITHM NO.
1
2
3
4
5
6
7
8
9
10
11
12
13

FIG. 7

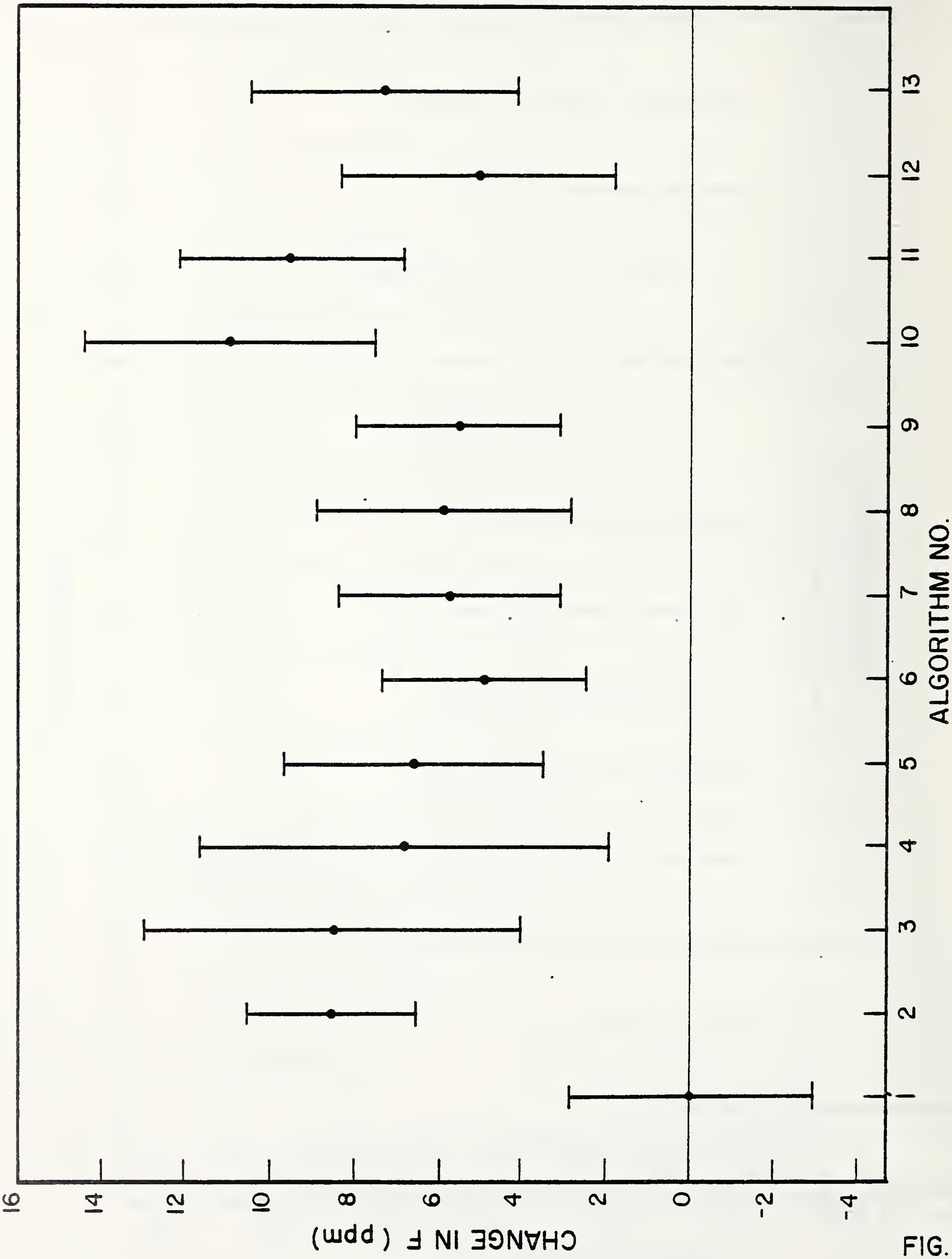
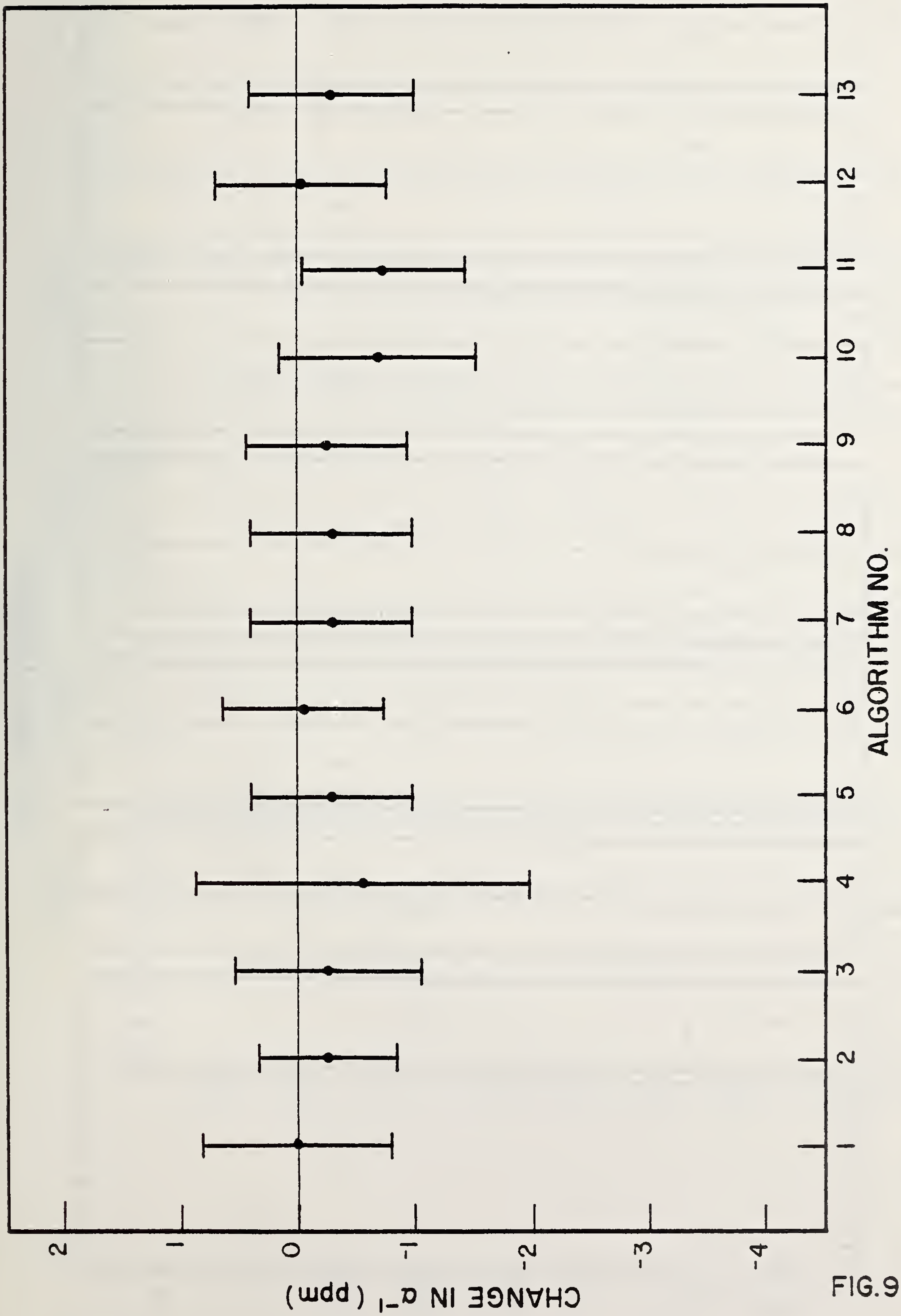

CHANGE IN F (ppm)
16
14
12
10
8
6
4
2
0
-2
-4
ALGORITHM NO.
1
2
3
4
5
6
7
8
9
10
11
12
13

FIG. 8



CHANGE IN α^{-1} (ppm)
2
1
0
-1
-2
-3
-4
ALGORITHM NO.
1
2
3
4
5
6
7
8
9
10
11
12
13

FIG.9

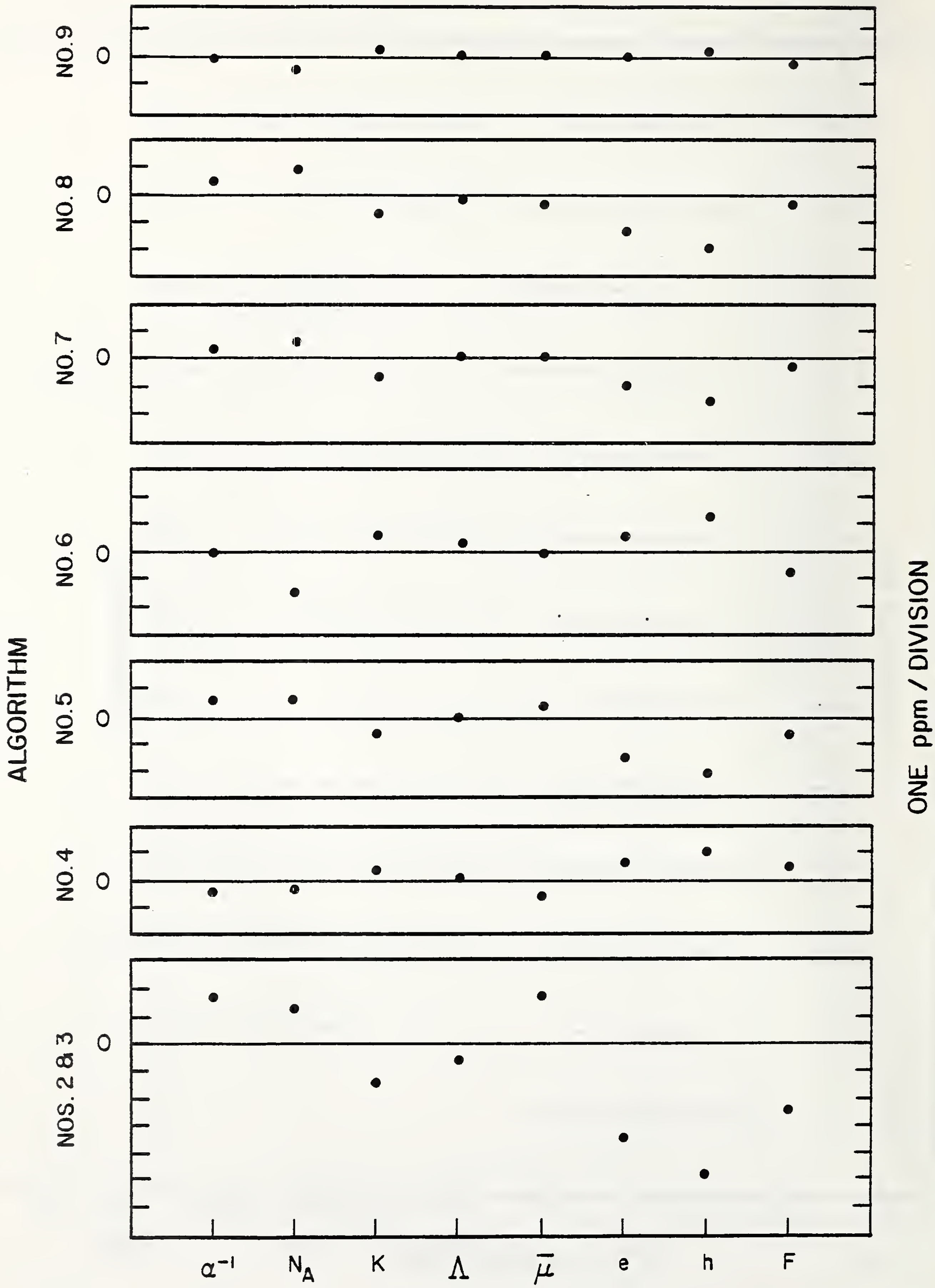

ALGORITHM
NO.9
NO.8
NO.7
NO.6
NO.5
NO.4
NOS. 2 & 3
O
ONE ppm / DIVISION
α^{-1}
N_A
K
Λ
$\bar{\mu}$
e
h
F

FIG.10

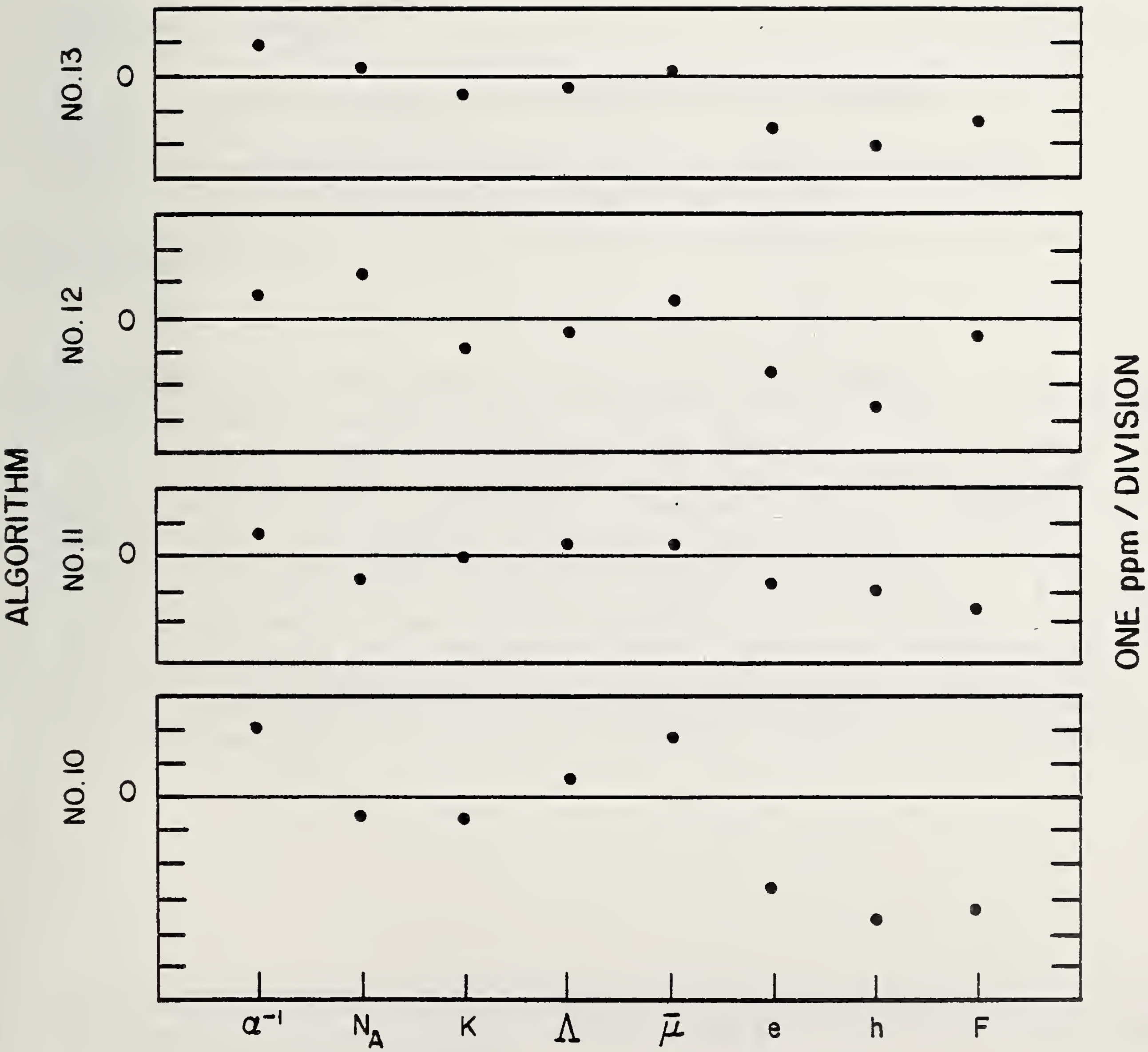

ALGORITHM
NO.13
NO.12
NO.11
NO.10
0
α^{-1}
N_A
K
Λ
$\bar{\mu}$
e
h
F
ONE ppm / DIVISION

FIG.11

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B. N. Taylor

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10. SUPPLEMENTARY NOTES

☐ Document describes a computer program; SF-185, FIPS Software Summary, is attached.

11. ABSTRACT (A 200-word or less factual summary of most significant information. If document includes a significant bibliography or literature survey, mention it here)

A number of recently proposed algorithms for treating inconsistent or discrepant data in a least-squares adjustment of the fundamental physical constants, along with several new but related algorithms, are compared in detail. The comparisons are first made by means of the numerical results the algorithms yield when applied to the same data considered by Cohen and Taylor in their 1973 adjustment which led to the recommended set of constants adopted by CODATA and in current use. A selected number of the algorithms are then further compared through the numerical results they yield when applied to the data considered by Taylor, Parker and Langenberg in their 1969 adjustment and by Cohen and DuMond in their 1963 adjustment. The principal conclusion of this paper is that the actual algorithm used to carry out an adjustment is much less important than the data finally selected for inclusion in the adjustment.

12. KEY WORDS (Six to twelve entries; alphabetical order; capitalize only proper names; and separate key words by semicolons)

Data analysis; discrepant data; fundamental constants; inconsistent data; least-squares adjustments; physical constants.

13. AVAILABILITY

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