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TAGGED PHOTONS
An Analysis of the Bremsstranlung Differential Cross Section in the Range of Interest for A Tagged Photon System

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ABSTRACT

We consider in detail the differential cross section for bremsstrahlung for angles and energies in the range of interest for a tagging system. We derive a high energy, small angle approximation for the differential cross section for bremsstrahlung, eq (I.1). We use this approximation to determine the maxima and minimum of the cross section and to evaluate it at these extrema. It is shown that the differential cross section has a very sharp dip in the region of small momentum transfers. Coulomb corrections to the Born approximation are considered, and do not fill in this dip.

Key Words: Bethe-Heitler cross section; bremsstrahlung differential cross section; bremsstrahlung monochromator; photon beams; photonuclear research; tagged photon method

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I. INTRODUCTION

During the last twenty years there has been a strong trend toward the utilization of "monochromatic" photon beams rather than bremsstrahlung photon beams for the measurement of photonuclear reaction cross sections. This trend clearly represents progress toward the "ideal" photon beam which would have the characteristics outlined in the recent survey of Beil and Bergère [1]:[1]

1) An energy resolution as small as possible,
2) A flux within the desired energy resolution as high as possible,
3) Low counting rates associated with unwanted photons, both within and outside the desired energy range, and
4) A well defined, and easily and continuously adjustable photon energy.

At present there are two main experimental techniques which have had the greatest success in achieving good compromises between these sometimes conflicting requirements; the positron annihilation-in-flight technique, and the bremsstrahlung monochromator or photon "tagging" technique. The tagged photon technique detects the electron after bremsstrahlung to determine the emitted photon energy. This detected electron is put in time coincidence with a nuclear decay product (i.e., n, p, or γ') following the photon induced reaction. The laboratory at Saclay has specialized in the production and utilization of photon beams obtained using the positron annihilation-in-flight

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[1] Figures in brackets indicate literature references at the end of this paper.

technique, first at the AL60 accelerator [2], and now using the positron beams obtainable at the ALS (Accélérateur Linéaire de Saclay) accelerator [3]. The photon tagging technique has become increasingly popular as electron beams of duty cycle approaching 100% have become available. At present there are photon monochromators installed both at the University of Illinois [4] and at the University of Mainz [5]. Both of the installations are at present restricted to modest photon energies due to the maximum electron beam energies available (67 MeV at Illinois and 14 MeV at Mainz). However, the first results obtained at these installations provide a clear indication of the advantages of the photon tagging technique with high duty cycle electron beams. A monochromator capable of tagging photons of energies up to 390 MeV has been installed [6] at the University of Bonn, utilizing the internal beam of their 5% duty cycle, 500 MeV electron synchrotron. The excellent characteristics of the tagged photon beam obtained at Bonn despite the restriction of a 5% duty cycle has led the Saclay group to consider the possibility of developing a photon tagging apparatus using the 2% duty cycle beams available at the ALS.

In order to investigate the experimental possibilities of tagged photon beams at intermediate energies without disrupting the ongoing program of photonuclear experiments using the positron annihilation-in-flight photon beams, the Saclay group has adapted the magnet normally used to dump the positron beam so that it can also function as a photon monochromator. To further reduce conflicts with the ongoing program, the group has chosen to test the monochromator using the "parasite" electron beam which can be obtained in the low energy (BE) experimental hall whenever the ALS is producing a high current beam by collecting electrons scattered through a few degrees from a small tungsten wire inserted into the main beam. The development of a photon tagging system which operates within the constraints imposed by the use of this parasite beam, which has high emittance and low current, and by the use of an existing magnet, which was not designed for use as a tagging spectrometer, necessitates a careful study of the theory of the tagging process. Specifically, one must understand whether the available magnet can provide a practical tagging system with good efficiency and energy resolution when a large emittance electron beam is used. It is also of interest to understand how the characteristics of the available system would improve if the low emittance electron beam obtainable directly from the accelerator were used in place of the "parasite" electron beam. Finally, a more complete understanding of the tagging process can be expected to provide important information for the design and optimization of future photon monochromators.

These considerations lead naturally to a study of the bremsstrahlung process itself. In the remaining sections of this report we consider in detail the differential cross section for bremsstrahlung for angles and energies in the range of interest for a tagging system. For a given photon energy and direction the angular distribution of the tagging electrons is precisely this differential cross section. In this report we consider the cross section summed over both electron spins and photon polarization. In a later report we will consider in detail the bremsstrahlung cross section for polarized photons, still, however, summing over electron spins. One of the more interesting aspects of the problem is the pronounced structure, a sharp dip, which exists in the differential cross section but is absent from both the total cross section and the cross section integrated over the angles of either the photon or the final electron. This dip is shown in figs. 4-10 where one may see that, for the energies and angles considered there, the region of the dip extends over an angular region of approximately $0.2°$ in the polar angle, θ_2, and $0.5°$ in the azimuthal angle $(0 \leqq \phi \lesssim 0.5°)$ of the final electron. This marked difference between the differential and integrated cross section may be observed in a number of elementary electrodynamic processes, and is of importance both from a theoretical point of view and for the analysis of experiments using electrons, positrons, and photons as probes. A detailed knowledge of the differential cross section in the region of the dip is of particular importance for the analysis of experiments using finite size detectors. An evaluation of the cross section for a particular set of angles, especially for $\phi = 0°$, could then give totally incorrect values for the cross section integrated over the finite detection solid angles. We therefore devote the next section to this point, with the aim of situating the present specific problem in a larger context. In the following sections we present the details leading to our high energy, small angle, approximation to the Bethe-Heitler differential cross section for bremsstrahlung. This is given by eqs (IV.1) and (VI.49):

$$\frac{d^3\sigma}{d\Omega_{p_2} d\Omega_k dk} = Z^2 \frac{e^2}{\hbar c}\left(\frac{e^2}{mc^2}\right)^2 \frac{[1 - F(q)]^2}{q^4} \frac{1}{k} \frac{p_2}{p_1} \frac{1}{(2\pi)^2}$$

$$\times \left\{f(v)(v-u+\sigma)^2 + g(v)\sin^2\tfrac{1}{2}\phi + \rho\right\} \quad , \qquad (I.1)$$

where $f(y)$, $g(y)$, ρ, and σ are given in eqs (VI.44), (VI.45), (VI.47), and (VI.48). All other variables are defined in section III. This expression serves as the basis for our detailed description of the cross section in the region of interest for the present tagged photon facility, viz., the region of very small momentum transfers, of the order of the kinematically allowed minimum momentum transfer. When we speak of eq (I.1) as being a high energy, small angle approximation to the differential cross section, we mean that the terms neglected in arriving at eq (I.1) are of relative order θ^2, i.e., relative to those terms retained in eq (I.1), for all angles and energies, including the region of the dip. The expression (I.1) has in addition the virtue that each of the three terms in $\{\ \}$ is positive. It is thus free of the very large cancellations which occur in the original expression, (IV.1). Specifically, in section III we present the kinematics to be used throughout this report. We also present there the relations between the angles in the system with z-axis along the incident beam and the angles in the system with z-axis along the emitted photon direction. It is this latter system that will be used in all the expressions presented in this report. In section IV we make some general remarks concerning the subject of high energy, small angle approximations to the cross section. In section V we develop in detail our high energy small angle approximation to the expression for q^2, the square of the momentum transfer. This section also serves as introduction to the procedures used in section VI, where we develop in detail the high energy, small angle approximation to the Born approximation (Bethe-Heitler) differential cross section. In section VII we use the approximation developed in section VI in order to obtain analytic expressions for the maxima and minimum of this cross section in the region of very small momentum transfers. In section VIII we present a summary of the essential formulas derived in sections V, VI, and VII for the reader who may be more concerned with applying these formulas than with their derivation. In section IX we discuss the Coulomb corrections to the differential cross section.

We wish to emphasize that this report is intended to be a working paper. The inclusion of mathematical details in the text is intentional.

II. CONSIDERATIONS OF A GENERAL NATURE CONCERNING DIFFERENTIAL AND INTEGRATED CROSS SECTIONS

In the analysis of photonuclear experiments, the fundamental electrodynamic processes of bremsstrahlung and pair production intervene inevitably. Indeed, in some of these analyses it is the lack in our knowledge of the theoretical cross sections for the electrodynamic processes that constitutes the major uncertainty in the determination of the desired nuclear information. These processes may, moreover, enter in the analysis in various guises. The field in which the bremsstrahlung or pair production takes place may be provided either by the nucleus or by the atomic electrons. In the case of bremsstrahlung, the incident beam of interest may be either electrons or positrons. And in each case, the differences in the nature of the target or of the beam are reflected in significant differences in the details of the cross section. Furthermore, in the analysis of a given experiment, one may be interested either in the differential cross section for the particular process or in an integrated cross section (integrated over either angles alone or both angles and energy). Thus, for example, in the use of a positron beam to provide "monoenergetic" photons by annihilation-in-flight ([1], [3]), the relative importance of the background bremsstrahlung photons may be diminished by utilizing photons emitted at an appropriate (relatively large) angle with respect to the incident beam of positrons. For the determination of the relative intensity of the background bremsstrahlung, the differential cross section for the process is required. Similarly, in connection with the present experimental arrangement in which monoenergetic photons are produced by tagging the associated scattered electrons, it is the differential cross section for bremsstrahlung that is of concern. On the other hand, in the use of the entire forward cone of the bremsstrahlung spectrum as a source of photons, we are concerned with cross sections integrated over angles. And in the case of total photoabsorption measurements, the principal background to the photonuclear absorption is due to pair production. In this case we require the cross section integrated over both angles and energies.

It may appear on first consideration that this distinction between the differential and integrated cross section is an essentially trivial one, implying nothing more profound than an integration over the unobserved quantities. We would like to stress that this is decidedly not the case. The differential cross section on the one hand, and the integrated cross section on the other have each very distinctive properties that are not shared by the other, as we will illustrate by a number of examples. The integrated cross section in general will reflect the completeness of the functions over which one integrates. It may show invariance properties and is associated with sum rules. On the other hand the integrated cross section only reflects that part of the differential cross section which contributes significantly to the integral. The characteristics of the differential cross section in a region of angles and energies which does not contribute significantly are not manifest in the integral cross section. Likewise, any fine structure in the differential cross section, even if it appears in an otherwise important region, is lost in the integration. It is precisely such structure, or other particular properties occurring for special angles and energies, that may be utilized in an experiment, which are specific to the differential cross section. Let us illustrate these statements with a few examples, the last of which will be the present case of tagged bremsstrahlung photons.

For our first example we consider jointly the processes of bremsstrahlung and pair production. As is well known [7], in first Born approximation (the Bethe-Heitler cross section) the differential cross section for either process can be obtained from that of the other by a simple reversal of sign of the appropriate momenta and energy (and, of course, an appropriate modification of the density of final states). However, once the Coulomb corrections (required for high Z target nuclei) are included, this is no longer true [8]. The expression for the differential cross section for bremsstrahlung is then totally different from that for pair production. This results from the asymptotic condition which must be imposed on the wave function for each of the particles [8]. In pair production, both the electron and the positron go out from the reaction and hence both are characterized asymptotically by a plane wave plus incoming spherical waves. In the case of bremsstrahlung the final electron goes out, but the initial electron is incident upon the reaction. The final electron is therefore characterized by a plane wave plus incoming spherical waves, whereas the initial electron is characterized by a plane wave plus outgoing spherical waves. This difference in the asymptotic character of the two wave functions results in a matrix element quite different from that for pair production, where both particles have the same asymptotic character [8]. However, if we consider the cross section for each of these processes integrated over the angles of the final electron, then the two cross sections are indeed related as in the Born approximation [9]. This result is a reflection of the fact that the set of wave functions with either ingoing or outgoing spherical waves separately form a complete set. By integrating over the direction of the final electron we have summed over the functions of the complete set, with the consequence that the particular radiation condition -- ingoing waves for the final particle --

is no longer reflected. In the integrated cross sections for the two processes we thus have only one wave function with a specified radiation condition, rather than two. The corresponding matrix elements are then simply related by the changes of sign of the appropriate momenta and energies. Thus to study or make use of characteristics which distinguish bremsstrahlung from pair production one must measure the completely differential cross section. For example, at high energies the Coulomb corrections for these two processes arise in different regions: for relatively large momentum transfers, $q \sim mc$, in the case of bremsstrahlung, and for very small momentum transfers, $q \sim mc/E$, for pair production. This distinction can be observed only in a measurement of the completely differential cross section. If one of the final particles is not observed, then one effectively integrates over the momentum transfer. Since the Coulomb corrections to the integrated cross sections are identical, there is then no significant difference between the two processes.

For our second example we consider the bremsstrahlung spectrum produced by a beam of electrons (or positrons), of energy ε, (in units of mc^2) incident upon a target, of atomic number Z. If the entire forward cone of photons is utilized, then at high energies, in first Born approximation, the cross section consists of two parts: $Z^2\sigma_{BH}$ from bremsstrahlung produced in the nuclear field, $Z\sigma_{BH}$ from bremsstrahlung produced in the field of the atomic electrons, the total being the familiar expression $Z(Z + 1)\sigma_{BH}$, where σ_{BH} is the Bethe-Heitler cross section for a point change, e. The spectrum has the familiar shape shown in figure 1, and the tip of the spectrum is given by

$$k_{max} = \varepsilon - 1 \quad .$$

Now, however, let us consider the experimental setup described in the introduction, where one utilizes the quasi-monoenergetic photons produced by annihilation-in-flight of a beam of positrons. There the background radiation, the bremsstrahlung produced by the beam of positrons, is diminished relative to the annihilation photons by using photons emitted at an angle θ (relative to the incident beam) which, although fairly small, is nonetheless large compared to $1/\varepsilon$. In this case we require, for a description of the background, the differential cross section for the bremsstrahlung emitted at an angle θ, rather than the integrated cross section. The tip of the spectrum for photons produced in the nuclear field is

$$k^N_{max} = \varepsilon - 1 \quad ,$$

as before. However, the tip of the spectrum for photons produced in the field of the atomic electrons is [1], [10],

$$k^e_{max} = \frac{\dfrac{\varepsilon - 1}{\varepsilon + 1}}{1 - \sqrt{\dfrac{\varepsilon-1}{\varepsilon+1}}\cos\theta}$$

$$= \frac{\varepsilon - 1}{1 + \dfrac{1}{\varepsilon + p} + 2p \sin^2 \dfrac{\theta}{2}} \quad , \qquad (p^2 = \varepsilon^2 - 1) \quad .$$

For high energies and small angles ($\theta \ll 1$, but not necessarily $\theta \sim 1/\varepsilon$), this can be written

$$k^e_{max} \cong \frac{\varepsilon - 1}{1 + \frac{1 + u^2}{2\varepsilon}} \quad , \qquad u = p\theta \quad .$$

This formula serves to illustrate the point made earlier. For those angles which contribute significantly to the integrated cross sections (viz, $\theta \sim 1/\varepsilon$), k^e_{max} is very close to k^N_{max}, differing only by terms of relative order $1/\varepsilon$. Thus if the experimental setup is such that one observes essentially the cross section integrated over angles, then $k^e_{max} \approx k^N_{max}$, the spectra are similar in shape and the cross section is, approximately, $Z(Z + 1)\sigma_{BH}$. However, for angles $\theta \gg 1/\varepsilon$, which do not contribute significantly to the total cross section, k^e_{max} may be significantly smaller than k^N_{max}. Thus, for example, for an incident beam of 100 MeV ($\varepsilon \cong 200$) and an angle $\theta = 4°$, we have $k^e_{max} \cong 67$ MeV. The two spectra are then as shown in figure 2. The cross section is then approximately $Z(Z + 1)\, d\sigma_{BH}/d\Omega$ for $k < k^e_{max}$ and $Z^2\, d\sigma_{BH}/d\Omega$ for $k^e_{max} < k < k^N_{max}$.

Thus, the characteristics of the bremsstrahlung spectra that are distinctive to the mass of the target (nucleus or electron), are manifest in the differential cross section, but not in the integrated cross section. This observation can be inverted; by utilizing photons emitted at angles $\theta \gg 1/\varepsilon$ (more precisely, for angles $\theta \gtrsim \sqrt{(1/\varepsilon)}$), we have at our disposal a reasonably wide range of energies, $k^e_{max} < k < k^N_{max}$, in which the bremsstrahlung photons are emitted solely in the nuclear field. This part of the spectrum can then serve either to study the properties of nuclear bremsstrahlung by itself, or as a source of photons whose cross section is accurately known: [A fairly simple analysis shows that for this portion of the spectrum, the associated momentum transfers, q, are such that $q \geqq 1$ (in units of mc). There are as well, therefore, essentially no effects of screening in this region of the spectrum.]

Our final example is provided by the analysis of the differential cross section for bremsstrahlung in the range of angles of interest for the tagged photon system which constitutes the primary concern of this report. This system is characterized by high energies $(\varepsilon >> 1)$ of the scattered electron and emitted photon, and small angles $(\theta << 1)$ between the directions of the scattered electron, emitted photon, and incident beam. With these restrictions, the momentum transfer, q, to the nucleus, lies in the range

$$q_{min} = \delta(1 + u^2) \leqslant q \lesssim O(u) \quad ,$$

where $u = p_1\theta_1$, and $\delta = k/(2\varepsilon_1\varepsilon_2)$, in which $\varepsilon_1, \varepsilon_2$, and k are the energies (in units of mc^2) of the initial electron, final electron and emitted photon, $\underline{p}_1$, $\underline{p}_2$, and $\underline{k}$ are the corresponding momenta (in units of mc), and θ_1 is the angle of emission of the photon with respect to the incident beam (in radians).

Now in the region $q_{min} \leqslant q \lesssim O(u)$ all momentum transfers contribute significantly to the cross section integrated over the angles of the final electron, the momentum distribution being essentially of the form dq/q, [11]. Thus we have equal contribution from the region of small momentum transfers, $q \sim q_{min}$, and from the region of "large" momentum transfers, $q \sim u$. However, the solid angle of the final electron corresponding to $q \sim q_{min}$ is much smaller, of order q^2_{min}/u^2

relative to the solid angle corresponding to $q \sim u$. Consequently the cross section in the region $q \sim q_{min}$ is (for a given solid angle of the final electron) larger by u^2/q_{min}^2 than that from the region $q \sim u$. Thus, from the experimental standpoint, it is the region of small momentum transfers, $q \sim q_{min}$, that is of interest, the cross section being so much larger there. The primary concern of this report is a detailed examination of the differential cross section for bremsstrahlung in the region of momentum transfers $q \sim q_{min}$. When we examine the differential cross section in detail in the region of small momentum transfers, $q \sim q_{min}$, what we find is that the cross section, considered as a function of the angle θ_2 between the photon and the final electron, has a very sharp dip for sufficiently small values of the azimuthal angle ϕ between $\underline{p}_1$ and $\underline{p}_2$ (see fig. 3a). The ratio of the cross section in the dip to that at the neighboring maxima, is (as will be shown in the analysis that follows) given (for the case in which $\phi = 0$ and $u = \varepsilon_1\theta_1 >> 1$) approximately by

$$\left(\frac{2k\varepsilon_1}{\varepsilon_1{}^2 + \varepsilon_1{}^2}\right)^2 \theta_1{}^2 \quad .$$

If we consider this expression for the values used in most of the figures presented here, viz., $\varepsilon_1 = 140$ MeV, $k = 95$ MeV, $\theta = 1°$, then we have

$$\left(\frac{2k\varepsilon_1}{\varepsilon_1{}^2 + \varepsilon_2{}^2}\right)^2 \theta_1{}^2 \approx 5 \times 10^{-4} \quad .$$

This sharp dip for $\phi = 0$ may be seen in figure 4.

As in the two previous examples, this structure is particular to the totally differential cross section. The region over which it occurs is sufficiently small that it gives a negligible contribution to the cross section integrated over the angles of either the final electron or the emitted photon. The familiar bremsstrahlung spectrum (in which the final electron is not observed) thus shows none of the structure particular to the spectrum of tagged photons.

III. KINEMATICS

In this section we present the details pertinent to the kinematics and define the various quantities which are used throughout this report.

Unless specified otherwise, we take energies to be in units of the electron rest energy, mc^2, and momenta in units of mc. We have

$\varepsilon_1, \underline{p}_1$: Energy and momentum of the incident electron (or positron) $(\varepsilon_1{}^2 - \underline{p}_1{}^2 = 1)$.

$\varepsilon_2, \underline{p}_2$: Energy and momentum of the final electron (or positron) $(\varepsilon_2{}^2 - \underline{p}_2{}^2 = 1)$.

$k, \underline{k}$: Energy and momentum of the emitted photon.

$\underline{q} = \underline{p}_1 - \underline{p}_2 - \underline{k}$: Momentum transferred to the target nucleus.

The energy transferred to the nucleus, which is in general $q_0 = \varepsilon_1 - \varepsilon_2 - k$, is taken throughout this report to be zero, as in the Bethe-Heitler cross section. This is equivalent to the assumption of an infinitely heavy target nucleus. We thereby neglect the effects of the recoil of the target nucleus, both kinematic and dynamic (this latter due to photon emission by the nucleus). For high energies and small angles (in which case $q \lesssim O(u)$), these effects are completely negligible, since they give contributions of relative order $qZ(m/M)$ in the region of the dip, and less elsewhere, M being the mass of the target nucleus.

In the system with z-axis in the direction of $\underline{k}$, the angles of $\underline{p}_1$ are (θ_1, ϕ_1), the angles of $\underline{p}_2$ are (θ_2, ϕ_2). These vectors are shown in fig. 3a. The components of $\underline{p}_1$ and $\underline{p}_2$ perpendicular to $\underline{k}$ are then

$$\underline{u} = \underline{p}_{1\perp}$$

$$\underline{v} = \underline{p}_{2\perp}$$

The angle between the vectors $\underline{u}$ and $\underline{v}$, which lie in the plane perpendicular to $\underline{k}$, is then

$$\phi = \phi_2 - \phi_1 \quad .$$

The component of $\underline{q}$ perpendicular to $\underline{k}$ is

$$\underline{q}_\perp = \underline{u} - \underline{v} \quad .$$

We have then

$$q_\perp{}^2 = u^2 - 2uv \cos\phi + v^2 \quad .$$

The component of $\underline{q}$ in the direction of $\underline{k}$ is

$$q_z = p_1\cos\theta_1 - p_2\cos\theta_2 - k \quad .$$

The magnitude of the vectors $\underline{u}$ and $\underline{v}$ is given by

$$u = |\underline{u}| = p_1\sin\theta_1$$

$$v = |\underline{v}| = p_2\sin\theta_2 \quad .$$

Throughout the analysis given in this report we use the coordinate system with z-axis in the direction of $\underline{k}$. However from the experimental standpoint, it may be convenient to use the coordinate system with z-axis in the direction of the incident beam ($\underline{p}_1$). We therefore present here some details of the relation between these two systems.

In the system with z-axis in the direction of $\underline{p}_1$ we denote the polar angle of $\underline{p}_2$ by ψ. The polar angle of $\underline{k}$ is again θ_1. The azimuthal angle between the vectors $\underline{p}_2$ and $\underline{k}$ in this system is denoted by ω. These vectors are shown in fig. 3b. It should be noted that if ϕ, the azimuthal angle of $\underline{p}_2$ relative to $\underline{p}_1$ in the $\underline{k}$-oriented system, is small $(0 \leqslant \phi \ll 1)$ then, ω, the azimuthal angle of $\underline{p}_2$ relative to $\underline{k}$ in the p_1-oriented system, will be near π. Since we will be concerned with small ϕ, it is useful to write

$$\omega = \pi + \omega' \quad ,$$

in which case $0 \leqslant \phi \ll 1$ implies $0 \leqslant \omega' \ll 1$.

The angles just defined are related by

$$\cos\psi = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos\phi \tag{III.1}$$

$$\begin{aligned}\cos\theta_2 &= \cos\theta_1\cos\psi + \sin\theta_1\sin\psi\cos\omega \\ &= \cos\theta_1\cos\psi - \sin\theta_1\sin\psi\cos\omega' \quad .\end{aligned} \tag{III.2}$$

It is useful to express the angles of the final electron in the $\underline{k}$-oriented system (θ_2,ϕ) in terms of those in the $\underline{p}_1$-oriented system (ψ,ω), and vice versa, and to do this in a form which lends itself to the small angle approximation. From the above equations we have, first, from eq (III.2),

$$\cos\theta_2 = \cos(\psi + \theta_1) + \sin\theta_1\sin\psi(1 - \cos\omega') \qquad \text{(III.3)}$$

and from eq (III.1),

$$\cos\phi = \frac{\cos\psi - \cos\theta_1\cos\theta_2}{\sin\theta_1\sin\theta_2}$$

$$= \frac{\cos\psi - \cos\theta_1[\cos\theta_1\cos\psi - \sin\theta_1\sin\psi\cos\omega']}{\sin\theta_1\sin\theta_2}$$

$$= \frac{\sin\theta_1\cos\psi + \cos\theta_1\sin\psi\cos\omega'}{\sin\theta_2}$$

$$= \frac{\sin(\psi + \theta_1) - \cos\theta_1\sin\psi(1 - \cos\omega'))]}{\sin\theta_2} \ .$$

We then have

$$\sin^2\theta_2\sin^2\phi = \sin^2\theta_2(1 - \cos^2\phi)$$

$$= \sin^2\theta_2 - (\sin\theta_2\cos\phi)^2$$

$$= \sin^2\theta_2 - [\sin(\psi + \theta_1) - \cos\theta_1\sin\psi(1 - \cos\omega')]^2$$

$$= 1 - \cos^2\theta_2 - [\sin(\psi + \theta_1) - \cos\theta_1\sin\psi(1 - \cos\omega')]^2$$

$$= 1 - [\cos(\psi + \theta_1) + \sin\theta_1\sin\psi(1 - \cos\omega')]^2$$
$$- [\sin(\psi + \theta_1) - \cos\theta_1\sin\psi(1 - \cos\omega')]^2$$

$$= 2\sin\psi(1-\cos\omega')[\sin(\psi+\theta_1)\cos\theta_1-\cos(\psi+\theta_1)\sin\theta_1]-\sin^2\psi(1-\cos\omega')^2$$

$$= \sin^2\psi(1 - \cos\omega')(1 + \cos\omega')$$

$$= \sin^2\psi\sin^2\omega'$$

Thus, finally,

$$\sin\theta_2\sin\phi = \sin\psi\sin\omega' \quad . \qquad \text{(III.4)}$$

Thus if we have, initially, the angles θ_2,ϕ in the $\underline{k}$-oriented system, then the angles ψ,ω' in the $\underline{p}_1$-oriented system are obtained from eq (III.1) (which gives ψ) and eq (III.4) (which then gives ω' in terms of θ_2,ϕ and ψ. If, on the other hand we start with the angles ψ,ω', then the angles θ_2,ϕ are obtained from eq (III.2) (which gives θ_2) and eq (III.4) (which then gives ϕ in terms of ψ,ω' and θ_2).

When all of the angles, θ_1, θ_2, and ϕ, are small, we have, from eqs (III.3) and (III.4),

$$\theta_2 \approx \psi + \theta_1$$

and

$$\theta_2\phi \approx \psi\omega' \quad .$$

IV. HIGH ENERGY, SMALL ANGLE APPROXIMATION OF THE BREMSSTRAHLUNG CROSS SECTION; GENERAL CONSIDERATIONS

We begin our consideration with the first Born approximation differential cross section for bremsstrahlung as given originally by Bethe and Heitler (see references [7], [12], [13]). This expression accounts for the exchange of a single photon between the electron (or positron) and the target in whose field the bremsstrahlung process takes place. The cross section is thus proportional to Z^2, the square of the nuclear charge, and hence does not include Coulomb corrections. A brief discussion of the contribution of Coulomb corrections in the region of energies and angles with which we are concerned here is given at the end of this report, in section IX.

Our starting point is, as we have said, the first Born differential cross section

$$d^3\sigma = Z^2 \frac{e^2}{\hbar c} \left(\frac{e^2}{mc^2}\right)^2 \cdot \frac{[1 - F(q)]^2}{q^4} \frac{dk}{k} \frac{p_2}{p_1} \frac{d\Omega_k d\Omega_{p_2}}{(2\pi)^2} \left\{ \quad \right\} \qquad \text{(IV.1)}$$

where

$$\left\{ \quad \right\} = \frac{p_1^2 \sin^2\theta_1 (4\varepsilon_2^2 - q^2)}{(\varepsilon_1 - p_1\cos\theta_1)^2} + \frac{p_2^2 \sin^2\theta_2 (4\varepsilon_1^2 - q^2)}{(\varepsilon_2 - p_2\cos\theta_2)^2}$$

$$- \frac{2p_1p_2\sin\theta_1\sin\theta_2\cos\phi(4\varepsilon_1\varepsilon_2 - q^2)}{(\varepsilon_1 - p_1\cos\theta_1)(\varepsilon_2 - p_2\cos\theta_2)}$$

$$+ \frac{2k^2(p_1^2\sin^2\theta_1 + p_2^2\sin^2\theta_2 - 2p_1p_2\sin\theta_1\sin\theta_2\cos\phi)}{(\varepsilon_1 - p_1\cos\theta_1)(\varepsilon_2 - p_2\cos\theta_2)}$$

Here $F(q)$ is the atomic form factor, discussed, for example, in Sec. IIE(3) of [13]. As it stands, this expression has no approximations in energy or angles. One could, therefore, deal with it in the form just given. There are several reasons for not doing this, but rather making high energy, small angle approximations. The first is that the cross section without approximations, eq (IV.1), is sufficiently complicated that any of the structure we have mentioned is not at all evident from the expression just given. Second, as we will see in the course of this analysis, there are, for particular energies and angles, extremely large cancellations of different terms in the expression. Thus in a straightforward numerical evaluation of the expression (IV.1) there are cancellations leaving a remainder which is of order $1/\varepsilon^4$ relative to the individual terms. For the energies of concern here ($\varepsilon_1 \approx 140$ MeV $\approx 280\ mc^2$) we note that $1/\varepsilon_1{}^4 \approx 10^{-10}$. One may thereby very easily lose of the order of 10 or more significant digits unless certain precautions are taken to write it in a form which takes account of these cancellations explicitly, as in eq (I.1). While it is possible to do this without making any approximation (in fact we do this, and present the resulting expression in eq (VI.4), as the first step toward our final approximate form), it is nonetheless after making the approximations of high energy and small angles that this explicit cancellation is performed most clearly. Finally, it is only after making these approximations that we have an expression of sufficient simplicity that it permits a detailed, analytical analysis of the structure of the cross section, including a determination of the maxima and minimum of the cross section.

A few preliminary remarks concerning these approximations are in order, however. In most "high energy" analyses, it is assumed that the energies, ε, are large $(\varepsilon >> 1)$ and that the angles are small, of the order of $1/\varepsilon$. This is particularly appropriate if one wants finally to obtain an integrated cross section, since at high energies most of the contribution to the cross section integrated over angles comes from angles $\theta \sim 1/\varepsilon$. However, this is not the case of interest here; for our present purpose we need the totally differential cross section, at angles which are small $(\theta << 1)$, but not necessarily of order $1/\varepsilon$. The first point is therefore that we will consider throughout, for all energies and angles, $\varepsilon >> 1$ and $\theta << 1$, but not require $\theta \sim 1/\varepsilon$. With these assumptions we then have several independent small parameters, $1/\varepsilon_1$, $1/\varepsilon_2$, θ_1, θ_2, in the expression for the cross section. The error in our approximate cross section will be determined by the largest of these, which is θ_2 for the experimental conditions considered here, namely, $\theta_1 > 1/\varepsilon_1$, and $q \approx q_{min}$: as we will see, $q \sim q_{min}$ requires $\varepsilon_1\theta_1 \approx \varepsilon_2\theta_2$, from which $\theta_2 > 1/\varepsilon_2$ and $\theta_2 \approx (\varepsilon_1/\varepsilon_2)\,\theta_1 > \theta_1 > 1/\varepsilon_1$. Our largest small parameter is thus $\theta_2 \approx (\varepsilon_1/\varepsilon_2)\theta_1$. A further point essential to our analysis is that we retain, in the cross section, all terms of order θ_2, neglecting only terms of order θ_2^2 relative to those which are retained. Thus our high energy, small angle approximation is in error by less than 1% for $\theta_2 < 6°$. (In fact the relevant parameter seems to be closer to $\frac{1}{2}\theta_2^2$.) Had we neglected terms of the order of θ_2, the errors in the resulting expression for the cross section would have been of the order of 20% for these same θ_2. In particular, the expression for the cross section in which terms of order θ_2 have been neglected is strictly zero when the component of the momentum transfer perpendicular to the photon direction is zero: $q_\perp = 0$, as may be seen from earlier calculations [14]. It does not, therefore, give a good representation of the differential cross section in the region of the minimum.

The expression (IV.1) for the cross section has two important factors, (1) the expression in $\{\ \}$, in the numerator, and (2) q^4, in the denominator. In the next two sections we examine each of these factors in detail and derive approximate expressions for them, valid for high energies and small angles.

V. APPROXIMATION OF q^2, THE MOMENTUM TRANSFER SQUARED, FOR HIGH ENERGIES AND SMALL ANGLES.

In this section we examine the expansion for q^2 in detail. In particular we derive an approximate expression for q^2, valid for high energies ε_1 and ε_2, and small angles θ_1 and θ_2.

We start with the exact expression for q^2,

$$q^2 = (\underline{p}_1 - \underline{p}_2 - \underline{k})^2 \quad , \tag{V.1}$$

and write

$$q^2 = q_\perp{}^2 + q_z{}^2 \tag{V.2}$$

where

$$\begin{aligned} \underline{q}_\perp &= \underline{p}_{1\perp} - \underline{p}_{2\perp} \\ &= \underline{u} - \underline{v} \end{aligned} \tag{V.3}$$

is the component of $\underline{q}$ perpendicular to $\underline{k}$

and

$$q_z = p_1\cos\theta_1 - p_2\cos\theta_2 - k \tag{V.4}$$

is the component of $\underline{q}$ in the direction of $\underline{k}$.

Here

$$u = |\underline{u}| = p_1 \sin\theta_1$$

$$v = |\underline{v}| = p_2 \sin\theta_2 \qquad \text{(V.5)}$$

and

$$\begin{aligned} q_z &= p_1\cos\theta_1 - p_2\cos\theta_2 - k \\ &= (\varepsilon_2 - p_2\cos\theta_2) - (\varepsilon_1 - p_1\cos\theta_1) \\ &= d_2 - d_1 \quad , \end{aligned} \qquad \text{(V.6)}$$

where

$$d_1 = \varepsilon_1 - p_1\cos\theta_1$$

$$d_2 = \varepsilon_2 - p_2\cos\theta_2 \quad . \qquad \text{(V.7)}$$

We next obtain expansions for d_1 and d_2 which are useful for high energies and small angles.

$$\begin{aligned} d_1 &= \varepsilon_1 - p_1\cos\theta_1 \\ &= \frac{\varepsilon_1{}^2 - p_1{}^2\cos^2\theta_1}{\varepsilon_1 + p_1\cos\theta_1} \\ &= \frac{\varepsilon_1{}^2 - p_1{}^2 + p_1{}^2(1 - \cos^2\theta_1)}{\varepsilon_1 + p_1\cos\theta_1} \quad . \end{aligned}$$

$$d_1 = \frac{1 + p_1{}^2\sin^2\theta_1}{\varepsilon_1 + p_1\cos\theta_1}$$

$$= \frac{1 + u^2}{\varepsilon_1 + p_1\cos\theta_1}$$

$$= \frac{1 + u^2}{2\varepsilon_1 - (\varepsilon_1 - p_1\cos\theta_1)}$$

$$= \frac{1 + u^2}{2\varepsilon_1 - d_1} \quad .$$

Therefore

$$d_1{}^2 - 2\varepsilon_1 d_1 + 1 + u^2 = 0 \quad , \tag{V.8}$$

and hence

$$d_1 = \varepsilon_1 \pm \sqrt{\varepsilon_1{}^2 - (1 + u^2)} \quad .$$

Since for small angles and large energies $d_1 = 0(1/\varepsilon)$, it is clear that we must take the minus sign. Thus

$$\frac{d_1}{\varepsilon_1} = 1 - \sqrt{1 - \frac{1 + u^2}{\varepsilon_1^2}}$$

$$= 1 - \left[1 - \frac{1}{2}\left(\frac{1+u^2}{\varepsilon_1^2}\right) - \frac{1}{8}\left(\frac{1+u^2}{\varepsilon_1^2}\right)^2 - \frac{1}{16}\left(\frac{1+u^2}{\varepsilon_1^2}\right)^3 - \cdots\right] \tag{V.9}$$

$$= \frac{1}{2}\left(\frac{1+u^2}{\varepsilon_1^2}\right) + \frac{1}{8}\left(\frac{1+u^2}{\varepsilon_1^2}\right)^2 + \frac{1}{16}\left(\frac{1+u^2}{\varepsilon_1^2}\right)^3 + \cdots \ .$$

Since it will be of use later on, we note that

$$\frac{\varepsilon_1}{d_1} = \frac{1}{1 - \sqrt{1 - \frac{(1+u^2)}{\varepsilon_1^2}}} = \frac{1 + \sqrt{1 - \frac{1+u^2}{\varepsilon_1^2}}}{\frac{1+u^2}{\varepsilon_1^2}}$$

$$= \frac{\varepsilon_1^2}{1+u^2}\left[2 - \frac{1}{2}\left(\frac{1+u^2}{\varepsilon_1^2}\right) - \frac{1}{8}\left(\frac{1+u^2}{\varepsilon_1^2}\right)^2 - \frac{1}{16}\left(\frac{1+u^2}{\varepsilon_1^2}\right)^3 - \cdots\right] \tag{V.10}$$

$$= \frac{2\varepsilon_1^2}{1+u^2} - \frac{1}{2} - \frac{1}{8}\left(\frac{1+u^2}{\varepsilon_1^2}\right) - \frac{1}{16}\left(\frac{1+u^2}{\varepsilon_1^2}\right)^2 - \cdots \ .$$

In identical fashion we have

$$\frac{d_2}{\varepsilon_2} = \frac{1}{2}\left(\frac{1+v^2}{\varepsilon_2{}^2}\right) + \frac{1}{8}\left(\frac{1+v^2}{\varepsilon_2{}^2}\right)^2 + \frac{1}{16}\left(\frac{1+v^2}{\varepsilon_2{}^2}\right)^3 + \ldots \qquad \text{(V.11)}$$

and

$$\frac{\varepsilon_2}{d_2} = \frac{2\varepsilon_2{}^2}{1+v^2} - \frac{1}{2} - \frac{1}{8}\left(\frac{1+v^2}{\varepsilon_2{}^2}\right) - \frac{1}{16}\left(\frac{1+v^2}{\varepsilon_2{}^2}\right)^2 - \ldots \qquad \text{(V.12)}$$

We then have

$$\begin{aligned} q_z &= d_2 - d_1 \\ &= \frac{1+v^2}{2\varepsilon_2} - \frac{(1+u^2)}{2\varepsilon_1} \qquad \text{(V.13)} \\ &+ \frac{1}{8}\frac{(1+v^2)^2}{\varepsilon_2{}^3} - \frac{1}{8}\frac{(1+u^2)^2}{\varepsilon_1{}^3} + \ldots \end{aligned}$$

We note at this point that the expansion parameters for $\frac{d_1}{\varepsilon_1}$ and $\frac{d_2}{\varepsilon_2}$ are $\frac{1+u^2}{\varepsilon_1{}^2}$ and $\frac{1+v^2}{\varepsilon_2{}^2}$. In order to obtain a better appreciation of these expressions, we examine them for high energies and small angles. We then have

$$u \approx \varepsilon_1 \, \theta_1 \quad , \quad v \approx \varepsilon_2 \, \theta_2 \quad .$$

Thus for

$$\theta_1 \lesssim 1/\varepsilon_1 \quad , \qquad \frac{1+u^2}{\varepsilon_1^{\ 2}} \approx \frac{1}{\varepsilon_1^{\ 2}}$$

and for

$$\theta_1 \gtrsim 1/\varepsilon_1 \quad , \qquad \frac{1+u^2}{\varepsilon_1^{\ 2}} \approx \theta_1^{\ 2} \quad .$$

Similarly, for

$$\theta_2 \lesssim 1/\varepsilon_2 \quad , \qquad \frac{1+v^2}{\varepsilon_2^{\ 2}} \approx \frac{1}{\varepsilon_2^{\ 2}}$$

and for

$$\theta_2 \gtrsim 1/\varepsilon_2 \quad , \qquad \frac{1+v^2}{\varepsilon_2^{\ 2}} \approx \theta_2^{\ 2} \quad .$$

For the experimental situation of interest here, $\theta_1 > 1/\varepsilon_1$. Further, we are concerned with small momentum transfers, $q << 1$, for which $v \approx u$, i.e., $\varepsilon_2\theta_2 \approx \varepsilon_1\theta_1$, or

$$\theta_2 \approx \frac{\varepsilon_1\theta_1}{\varepsilon_2} > \theta_1 \quad \text{since} \quad \varepsilon_2 < \varepsilon_1 \quad .$$

(Also $\theta_2 > 1/\varepsilon_2$, since $\varepsilon_1\theta_1 > 1$.)

The largest of the parameters $1/\varepsilon_1{}^2$, $\theta_1{}^2$, $1/\varepsilon_2{}^2$, $\theta_2{}^2$ is, therefore, in the present study, $\theta_2{}^2$. We note further that in both $\frac{d_1}{\varepsilon_1}$ and $\frac{\varepsilon_1}{d_1}$ $\left(\text{and similarly for } \frac{d_2}{\varepsilon_2} \text{ and } \frac{\varepsilon_2}{d_2}\right)$, the first correction has an additional factor $\frac{1}{4}$. Thus the parameter in this expansion is

$$\frac{1}{4}\,\theta_2{}^2 \approx \frac{1}{4}\left(\frac{\varepsilon_1\theta_1}{\varepsilon_2}\right)^2$$

and, where this is dropped, an exact calculation indeed shows the error to be given very accurately by this parameter. Note that for $\theta_1 = 1.2°$ and $\varepsilon_2 = \frac{1}{3}\,\varepsilon_1$,

$$\frac{1}{4}\left(\frac{\varepsilon_1\theta_1}{\varepsilon_2}\right)^2 \approx 10^{-3} \quad .$$

The neglect of these terms in the expression for q_z thus introduces (for these values of θ_1, ε_1 and ε_2) an error of 10^{-3}. In $q^2 = q_\perp{}^2 + q_z{}^2$ this introduces an error of 2×10^{-3}, and in q^4 (which appears in the denominator of the cross section), and thus also in the cross section, an error of 4×10^{-3} ($\approx$ ½%).

We will neglect these terms of order $\theta_2{}^2$ in this analysis. We then have

$$q_z = \frac{1+v^2}{2\varepsilon_2} - \frac{1+u^2}{2\varepsilon_1}$$

We note here that

$$q_\perp^2 = (\underline{u} - \underline{v})^2$$

$$= (u - v)^2 + 2uv(1 - \cos\phi)$$

Thus unless both $(v - u)^2$ and ϕ^2 are small, of order θ_2^2, q_z^2 will be much smaller, of order θ_2^2, relative to $q_\perp^2$. Therefore, in the expression for q_z we may consider $v - u << u$, and write

$$v^2 = (v - u + u)^2 = u^2 + 2u(v - u) + (v - u)^2 \quad ,$$

so that

$$q_z = \frac{1 + u^2 + 2u(v - u) + (v - u)^2}{2\varepsilon_2} - \frac{1 + u^2}{2\varepsilon_1}$$

$$= \frac{1}{2}\left(\frac{1}{\varepsilon_2} - \frac{1}{\varepsilon_1}\right)(1 + u^2) + \frac{u}{\varepsilon_2}(v - u) + \frac{(v - u)^2}{2\varepsilon_2}$$

$$= \frac{k}{2\varepsilon_1\varepsilon_2}(1 + u^2) + \frac{u}{\varepsilon_2}(v - u) + \frac{(v - u)^2}{2\varepsilon_2}$$

$$= \delta(1 + u^2) + \frac{u}{\varepsilon_2}(v - u) + \frac{(v - u)^2}{2\varepsilon_2} ,$$

where

$$\delta \equiv \frac{k}{2\varepsilon_1\varepsilon_2} \quad . \tag{V.14}$$

Therefore

$$q^2 = (u-v)^2 + 2uv(1-\cos\phi) + \left[\delta(1+u^2) + \frac{u}{\varepsilon_2}(v-u) + \frac{(v-u)^2}{2\varepsilon_2}\right]^2$$

Here, in the last term, q_z^2, we have

$$\delta^2(1+u^2)^2 + 2\delta(1+u^2)\frac{u(v-u)}{\varepsilon_2}$$

$$+ \left(\frac{u}{\varepsilon_2}\right)^2 (v-u)^2 + \frac{(v-u)^4}{4\varepsilon_2^2} + \frac{u(v-u)^3}{\varepsilon_2^3} + \frac{\delta(1+u^2)(v-u)^2}{\varepsilon_2} .$$

The terms in the last line here are of order θ_2^2, or smaller, relative to the term $(u - v)^2$ in q^2 and may therefore be neglected:

$$\left(\frac{u}{\varepsilon_2}\right)^2 \approx \left(\frac{\varepsilon_1\theta_1}{\varepsilon_2}\right)^2 \approx \left(\frac{\varepsilon_2\theta_2}{\varepsilon_2}\right)^2 = \theta_2^2$$

$$\frac{\delta(1+u^2)}{2\varepsilon_2} = \frac{k(1+u^2)}{4\varepsilon_1\varepsilon_2^2} \approx \frac{u^2}{4\varepsilon_2^2} \approx \frac{1}{4}\theta_2^2 .$$

We note that the second term in the expression for q_z^2, viz., $\frac{2\delta(1 + u^2)\, u(v - u)}{\varepsilon_2}$ is, for $v - u \approx \delta(1 + u^2)$, of order $\frac{u}{\varepsilon_2}$ relative to $\delta^2(1 + u^2)^2$, i.e., of order θ_2. Since $\theta_2^2 \approx 4 \times 10^{-3}$, we have $\theta_2 \approx 6 \times 10^{-2}$ and therefore we keep this term. As we will see, the cross section achieves its maximum for $|v - u| \approx \delta(1 + u^2)$. Thus in this region neglect of this term would introduce errors of the order of 10% in the cross section.

We thus have, finally, our high energy, small angle approximation for the square of the momentum transfer:

$$q^2 = (v-u)^2 + 2uv(1-\cos\phi) + \frac{2\delta(1+u^2)u(v-u)}{\varepsilon_2} + \delta^2(1+u^2)^2 \qquad (V.15)$$

In figs. 4 and 5 we show q^2 considered as a function of v (i.e., θ_2) for fixed values of ε_1, ε_2, θ_1, and ϕ. Note that the neglected terms are of relative order θ_2^2 (and never larger) for <u>all</u> values of $v - u$ (i.e., for $v - u \equiv 0$, $v - u = O(u\theta^2)$, $v - u = O(u\theta)$, and $v - u = O(u)$). Since we will require derivatives of q^2 with respect to v for our further calculations, we note that

$$(q^2)' = 2(v - u) + 2u(1 - \cos\phi) + 2\delta(1 + u^2)\,\frac{u}{\varepsilon_2}$$

$$(q^2)'' = 2 \qquad (V.16)$$

Again, since we have kept the principal terms multiplying both $(v - u)^2$ and $(v - u)$, the errors in $(q^2)'$ and $(q^2)''$ are also of relative order θ_2^2.

For the minimum value of q^2 considered as a function of v (i.e., θ_2), for fixed values of ε_1, ε_2, θ_1, and ϕ, we have

$$(q^2)' = 2\left[v - u + u(1 - \cos\phi) + \frac{u\,\delta(1 + u^2)}{\varepsilon_2}\right] = 0$$

or

$$v - u = -u\left[(1 - \cos\phi) + \frac{\delta(1 + u^2)}{\varepsilon_2}\right]$$

or

$$v = u \left\{ 1 - \left[(1 - \cos\phi) + \frac{\delta(1 + u^2)}{\varepsilon_2} \right] \right\} , \tag{V.17}$$

and since $(q^2)'' = 2$, this is indeed a minimum.

With $\theta_1 = 1°$, $\varepsilon_1 = 140$ MeV, $\varepsilon_2 = 45$ MeV ($1\ mc^2 = 0.511$ MeV) we get $\theta_2 = 3.1094°$ for the position of the minimum of q^2.

Note that q^2 as given by this approximation is of the form

$$q^2 = (v - u)^2 + 2B(v - u) + C$$

with

$$B = u(1 - \cos\phi) + \delta(1 + u^2) \frac{u}{\varepsilon_2}$$

$$C = \delta^2(1 + u^2)^2 + 2u^2(1 - \cos\phi) .$$

Thus we can write

$$q^2 = (v - u + B)^2 + C - B^2 .$$

The minimum value is thus clearly achieved when $v - u + B = 0$ (as we have just seen), since C and B are independent of v. The minimum value of q^2 is then

$$q_{min}^2 = C - B^2$$

$$= \delta^2(1+u^2)^2 + 2u^2(1-\cos\phi) - u^2\left[(1-\cos\phi) + \frac{\delta(1+u^2)}{\varepsilon_2}\right]^2$$

$$= \delta^2(1+u^2)^2\left(1 - \frac{u^2}{\varepsilon_2^{\,2}}\right) + 2u^2(1-\cos\phi)\left(1 - \frac{\delta(1+u^2)}{\varepsilon_2}\right)$$

$$- u^2(1 - \cos\phi)^2$$

Here $\dfrac{\delta}{\varepsilon_2} = \dfrac{k}{2\varepsilon_1} \cdot \dfrac{1}{\varepsilon_2^{\,2}} < \dfrac{1}{2} \cdot \dfrac{1}{\varepsilon_2^{\,2}}$. Thus the corrections in the two large parentheses, $\dfrac{u^2}{\varepsilon_2^{\,2}}$ and $\dfrac{\delta(1 + u^2)}{\varepsilon_2}$, are of relative order $\dfrac{u^2}{\varepsilon_2^{\,2}} = O(\theta_2^{\,2})$. Since we have neglected terms of this order multiplying C $\left(\text{specifically, multiplying } \delta^2(1 + u^2)^2\right)$ these terms should also be neglected.

Thus we have

$$q_{min}^2 = \delta^2(1 + u^2)^2 + u^2(1 - \cos\phi)(2 - (1 - \cos\phi))$$

Here

$$(1 - \cos\phi)(2 - (1 - \cos\phi)) = (1 - \cos\phi)(1 + \cos\phi)$$

$$= \sin^2\phi \quad .$$

Therefore

$$q_{min}^2 = \delta^2(1 + u^2)^2 + u^2\sin^2\phi \quad . \qquad \text{(V.18)}$$

For $\phi = 0$, $\theta_1 = 1°$, $\varepsilon_1 = 140$ MeV, and $k = 95$ MeV, this gives

$$q_{min}^2 = 8.45222 \times 10^{-3} \quad . \qquad \text{(V.19)}$$

Note that if we keep the factor $\left(1 - \dfrac{u^2}{\varepsilon_2^2}\right)$ we obtain, for the value of q_{min}^2 given directly by our approximation for q^2, we have

$$q_{min}^2 = 8.4273 \times 10^{-3} \quad ,$$

a difference of approximately 3×10^{-3}. However, as we have noted, there are other terms of this order in the exact expression for q_{min}^2.

VI. HIGH ENERGY, SMALL ANGLE APPROXIMATION OF THE DIFFERENTIAL CROSS SECTION FOR BREMSSTRAHLUNG

In this section we examine in detail the expression in the numerator of the cross section, in $\{\ \}$, eq (IV.1). In particular we derive an approximate expression for $\{\ \}$, valid for high energies ε_1 and ε_2, and small angles θ_1 and θ_2: $\varepsilon_1 \gg 1$, $\varepsilon_2 \gg 1$, $\theta_1 \ll 1$, $\theta_2 \ll 1$.

We start with the exact expression for $\{\ \}$, given in (IV.1). With the notation as defined in section III describing the kinematics we then have

$$\{\ \} = \frac{p_1^2 \sin^2\theta_1}{(\varepsilon_1 - p_1\cos\theta_1)^2}(4\varepsilon_2^2 - q^2) + \frac{p_2^2\sin^2\theta_2}{(\varepsilon_2 - p_2\cos\theta_2)^2}(4\varepsilon_1^2 - q^2)$$

$$- \frac{2p_1p_2\sin\theta_1\sin\theta_2\cos\phi(4\varepsilon_1\varepsilon_2 - q^2)}{(\varepsilon_1 - p_1\cos\theta_1)(\varepsilon_2 - p_2\cos\theta_2)}$$

$$+ \frac{2k^2(p_1^2\sin^2\theta_1 + p_2^2\sin^2\theta_2 - 2p_1p_2\sin\theta_1\sin\theta_2\cos\phi)}{(\varepsilon_1 - p_1\cos\theta_1)(\varepsilon_2 - p_2\cos\theta_2)}$$

$$= \left(\frac{2\varepsilon_2\underline{u}}{d_1} - \frac{2\varepsilon_1\underline{v}}{d_2}\right)^2 - q^2\left(\frac{\underline{u}}{d_1} - \frac{\underline{v}}{d_2}\right)^2 + 2k^2\,\frac{(\underline{u}-\underline{v})^2}{d_1d_2} \qquad \text{(VI.1)}$$

where

$$d_1 = \varepsilon_1 - p_1\cos\theta_1$$

$$d_2 = \varepsilon_2 - p_2\cos\theta_2 \quad . \qquad \text{(VI.2)}$$

Next we define

$$\xi = \frac{1}{2\varepsilon_1 d_1} \qquad , \qquad \eta = \frac{1}{2\varepsilon_2 d_2} \quad . \tag{VI.3}$$

The expression for $\left\{ \quad \right\}$ given in eq (VI.1) may be written very simply in terms of ξ and η:

$$\left\{ \quad \right\} = 8k^2\varepsilon_1\varepsilon_2(\underline{u}-\underline{v})^2\xi\eta + 16\varepsilon_1{}^2\varepsilon_2{}^2(\underline{u}\xi-\underline{v}\eta)^2 - 4q^2(\varepsilon_1\underline{u}\xi-\varepsilon_2\underline{v}\eta)^2 \ . \tag{VI.4}$$

We will examine each of the three terms appearing here, obtaining an estimate of their relative magnitudes for high energies and small angles. First, however, let us consider ξ and η in detail. We have

$$\xi = \frac{1}{2\varepsilon_1(\varepsilon_1 - p_1\cos\theta_1)}$$

$$= \frac{\varepsilon_1 + p_1\cos\theta_1}{2\varepsilon_1(\varepsilon_1{}^2 - p_1{}^2\cos^2\theta_1)}$$

$$= \frac{2\varepsilon_1 - (\varepsilon_1 - p_1\cos\theta_1)}{2\varepsilon_1(\varepsilon_1{}^2 - p_1{}^2\cos^2\theta_1)}$$

$$= \frac{1}{1 + u^2} - \frac{1}{2\varepsilon_1(\varepsilon_1 + p_1\cos\theta_1)} \tag{VI.5}$$

since

$$\varepsilon_1^{\,2} - p_1^{\,2}\cos^2\theta_1 = \varepsilon_1^{\,2} - p_1^{\,2} + p_1^{\,2}(1 - \cos^2\theta_1)$$

$$= 1 + p_1^{\,2}\sin^2\theta_1$$

$$= 1 + u^2 \quad .$$

In similar fashion

$$\eta = \frac{1}{1 + v^2} - \frac{1}{2\varepsilon_2(\varepsilon_2 + p_2\cos\theta_2)} \quad . \tag{VI.6}$$

The second term in the expressions (VI.5) and (VI.6) for ξ and η is small, of order θ^2 (or of order $1/\varepsilon^2$ if $\theta < 1/\varepsilon$) relative to the first term:

$$\xi = \frac{1}{1 + u^2} - \frac{1}{2\varepsilon_1(\varepsilon_1 + p_1\cos\theta_1)}$$

$$= \frac{1}{1 + u^2}\left(1 - \frac{1 + u^2}{2\varepsilon_1(\varepsilon_1 + p_1\cos\theta_1)}\right)$$

and

$$\frac{1 + u^2}{2\varepsilon_1(\varepsilon_1 + p_1\cos\theta_1)} \approx \frac{1 + u^2}{4\varepsilon_1^{\,2}} \approx \frac{1}{4\varepsilon_1^{\,2}} + \frac{1}{4}\,\theta_1^{\,2}$$

Thus we may write

$$\xi = \xi_o - \xi_1$$

$$\eta = \eta_o - \eta_1 \qquad \text{(VI.7)}$$

where

$$\xi_o = \frac{1}{1 + u^2} \quad , \quad \xi_1 = \frac{1}{2\varepsilon_1(\varepsilon_1 + p_1\cos\theta_1)}$$

$$\eta_o = \frac{1}{1 + v^2} \quad , \quad \eta_1 = \frac{1}{2\varepsilon_2(\varepsilon_2 + p_2\cos\theta_2)} \quad . \qquad \text{(VI.8)}$$

Then ξ_o and η_o (and hence also ξ and η) are of order $1/u^2$ for $u \gtrsim 1$ $(\theta \gtrsim 1/\varepsilon)$ and of order 1 if $u \lesssim 1$ $(\theta \lesssim 1/\varepsilon)$. (As may be seen from eq (V.15), $q \ll 1$ implies $u \approx v$ and hence also $\xi_o \approx \eta_o$ from eq (VI.8)). Moreover, as we have just shown, ξ_1 and η_1 are of order θ^2 relative to ξ_o and η_o (or of order $1/\varepsilon^2$ if $\theta < 1/\varepsilon$).

Returning now to eq (VI.4) we note that the first term there is

$$8k^2\varepsilon_1\varepsilon_2(\underline{u} - \underline{v})^2\xi\eta = 8k^2\varepsilon_1\varepsilon_2 q_\perp{}^2\xi\eta \quad .$$

Writing

$$\varepsilon_1\varepsilon_2, k = O(\varepsilon)$$

$$u, v = O(u)$$

$$\xi, \eta = O(\xi)$$

$$\theta_1, \theta_2 = O(\theta)$$

we may then say that this first term is $O(\varepsilon^4\xi^2 q_\perp{}^2)$. Now, as we will show, not only this first term, but $\{\ \}$ as well is $O(\varepsilon^4\xi^2 q_\perp{}^2)$ over a very wide range of values of $q_\perp$, viz., for "large" $q_\perp$: $q_\perp = O(u)$, for "small" $q_\perp$: $q_\perp = O(\theta u)$ (or $q_\perp = O(u/\varepsilon)$ if $\theta \lesssim 1/\varepsilon$), and for $q_\perp = O(\theta^2 u)$ (or $q_\perp = O(u/\varepsilon^2)$ if $\theta \lesssim 1/\varepsilon$), this last region, $q_\perp = O(\theta^2 u)$, being the region of the dip in the cross section, indicated in figures 4 through 10. For $q_\perp < O(\theta^2 u)$, the second and third terms in eq (VI.4) do not decrease further, so that $\{\ \}$ remains of the order of magnitude it has for $q_\perp = O(\theta^2 u)$, viz., of order $\varepsilon^4\xi^2\theta^4 u^2$.

Thus a numerical evaluation of eq (IV.1), the initial expression for $\{\ \}$, shows significant cancellations for values of θ_2 and ϕ corresponding to very small $q_\perp$, so that in this dip the value of $\{\ \}$ is of order θ^4 (or $1/\varepsilon^4$ if $\theta < 1/\varepsilon$) relative to its value for $q_\perp = O(u)$. However, in contrast to eq (VI.4), where the nature of this cancellation is manifest, in eq (IV.1) it is quite obscure.

To demonstrate these order of magnitude statements we return to eq (VI.4) and proceed with our examination of the terms there. As we noted, the first term there is $O(\varepsilon^4\xi^2 q_\perp{}^2)$. In the second term we write

$$(\underline{u}\xi - \underline{v}\eta)^2 = (\underline{u} - \underline{v})^2\xi\eta + (u^2\xi - v^2\eta)(\xi - \eta) \qquad \text{(VI.9)}$$

Let us consider the last term here in detail. From eqs (VI.5) and (VI.6) we have

$$\xi = \frac{1}{1 + u^2} - \frac{1}{2\varepsilon_1(2\varepsilon_1 - d_1)}$$

$$= \frac{1}{1 + u^2} - \frac{1}{4\varepsilon_1{}^2 - 2\varepsilon_1 d_1}$$

$$= \frac{1}{1 + u^2} - \frac{1}{4\varepsilon_1{}^2 - \frac{1}{\xi}}$$

$$= \frac{1}{1 + u^2} - \frac{\xi}{4\varepsilon_1{}^2\xi - 1} \quad .$$

Thus

$$\xi\left(1 + \frac{1}{4\varepsilon_1{}^2\xi - 1}\right) = \frac{1}{1 + u^2}$$

or

$$\xi = \frac{1}{1 + u^2}\left(\frac{4\varepsilon_1{}^2\xi - 1}{4\varepsilon_1{}^2\xi}\right)$$

$$= \frac{1}{1 + u^2}\left(1 - \frac{1}{4\varepsilon_1{}^2\xi}\right)$$

Hence

$$1 + u^2 = \frac{1}{\xi}\left(1 - \frac{1}{4\varepsilon_1{}^2\xi}\right)$$

and

$$\xi^2 + u^2\xi^2 = \xi - \frac{1}{4\varepsilon_1{}^2}$$

so that, finally,

$$u^2\xi^2 = \xi(1 - \xi) - \frac{1}{4\varepsilon_1{}^2} \tag{VI.10}$$

and, in similar fashion,

$$v^2\eta^2 = \eta(1 - \eta) - \frac{1}{4\varepsilon_2{}^2} \quad . \tag{VI.11}$$

Thus for one of the factors in eq (VI.9) we have

$$u^2\xi - v^2\eta = u^2(\xi - \eta) + \eta(u^2 - v^2) \quad .$$

Hence, from eq (VI.9),

$$(\underline{u}\xi - \underline{v}\eta)^2 = (\underline{u} - \underline{v})^2\xi\eta + u^2(\xi - \eta)^2 + \eta(u^2 - v^2)(\xi - \eta) \quad . \tag{VI.12}$$

Now from eqs (VI.5) and (VI.6)

$$\xi - \eta = \frac{1}{1 + u^2} - \frac{1}{1 + v^2} + O(1/\varepsilon^2)$$

$$= - \frac{(u^2 - v^2)}{(1 + u^2)(1 + v^2)} + O(1/\varepsilon^2) \quad .$$

Thus from eqs (VI.5) and (VI.6)

$$|\xi - \eta| = O(|u^2 - v^2|\xi\eta) + O(1/\varepsilon^2)$$

$$= O(u|u - v|\xi^2) + O(1/\varepsilon^2) \quad . \tag{VI.13}$$

Further, from

$$q_\perp{}^2 = (\underline{u} - \underline{v})^2$$

$$= (u - v)^2 + 2uv(1 - \cos\phi)$$

$$= (u - v)^2 + 4uv \sin^2 \tfrac{1}{2}\phi$$

we have

$$|u - v| \lesssim O(q_\perp) \quad . \tag{VI.14}$$

Thus

$$|\xi - \eta| \lesssim O(q_\perp u\xi^2) + O(1/\varepsilon^2) \quad . \tag{VI.15}$$

Note here that $q_\perp$ can go to zero; therefore we can not simply neglect the terms denoted here as $O(1/\varepsilon^2)$.

From eq (VI.15) we have

$$(\xi - \eta)^2 = O(q_\perp{}^2 u^2 \xi^4) + O(q_\perp u \xi^2/\varepsilon^2) + O(1/\varepsilon^4) \quad . \tag{VI.16}$$

Referring to the terms in eq (VI.12), we then have

$$(\underline{u} - \underline{v})^2 \xi\eta = O(q_\perp{}^2 \xi^2)$$

$$u^2(\xi - \eta)^2 = O(q_\perp{}^2 u^4 \xi^4) + O(q_\perp u^3 \xi^2/\varepsilon^2) + O(u^2/\varepsilon^4) \quad . \tag{VI.17}$$

Here we note that

$$O(q_\perp{}^2 u^4 \xi^4) = O(q_\perp{}^2 \xi^2 (u^2\xi)^2) \lesssim O(q_\perp{}^2 \xi^2)$$

and

$$O(q_\perp u^3 \xi^2/\varepsilon^2) = O(q_\perp u\xi(u^2\xi)/\varepsilon^2) \lesssim O(q_\perp u\xi/\varepsilon^2)$$

since $u^2\xi \lesssim O(1)$ from eq (VI.5).

Thus

$$u^2(\xi - \eta)^2 \lesssim O(q_\perp{}^2 \xi^2) + O(q_\perp u\xi/\varepsilon^2) + O(u^2/\varepsilon^4) \quad . \tag{VI.18}$$

Finally, from eq (VI.14) and (VI.15),

$$\eta(u^2 - v^2)(\xi - \eta) = O(q_\perp{}^2 u^2 \xi^3) + O(q_\perp u\xi/\varepsilon^2)$$

$$= O(q_\perp{}^2 \xi^2 (u^2\xi)) + O(q_\perp u\xi/\varepsilon^2)$$

$$\lesssim O(q_\perp{}^2 \xi^2) + O(q_\perp u\xi/\varepsilon^2) \qquad \text{(VI.19)}$$

Substituting eqs (VI.17), (VI.18), and (VI.19) in eq (VI.12), we have, for the second term in eq (VI.4),

$$16\varepsilon_1{}^2\varepsilon_2{}^2(\underline{u}\xi - \underline{v}\eta)^2 = O(\varepsilon^4\xi^2 q_\perp{}^2) + O(\varepsilon^2 u\xi q_\perp) + O(u^2) \quad . \qquad \text{(VI.20)}$$

Now, as we will show shortly, the third term in eq (VI.4) is $O(u^2)$, and does not cancel the terms of $O(u^2)$ in eq (VI.20). Thus the terms in eq (VI.4) give contributions of $O(\varepsilon^4\xi^2 q_\perp{}^2)$, $O(\varepsilon^2 u\xi q_\perp)$, and $O(u^2)$ to $\{\ \}$. We may write these as

$$O\left(u^2\left(\varepsilon^2\xi\,\frac{q_\perp}{u}\right)^2\right), \quad O\left(u^2\left(\varepsilon^2\xi\,\frac{q_\perp}{u}\right)\right), \quad \text{and} \quad O(u^2) \ . \qquad \text{(VI.21)}$$

Note here that

$$\varepsilon^2\xi = O\left(\frac{\varepsilon^2}{1 + u^2}\right) = \begin{cases} O\left(\dfrac{1}{\theta^2}\right) & \theta \gtrsim 1/\varepsilon \\ O(\varepsilon^2) & \theta \lesssim 1/\varepsilon \end{cases} \ .$$

We may thus distinguish three regions of q:

1) "large" q, for which $q_\perp = O(u)$

2) "small" q, with $q_\perp = O(u\theta) = O(q_z)$ for $\theta \gtrsim 1/\varepsilon$
(or $q_\perp = O(u/\varepsilon)$ for $\theta \lesssim 1/\varepsilon$).

3) small q with $q_\perp = O(u\theta^2) \ll q_z$ for $\theta \gtrsim 1/\varepsilon$
(or $q_\perp = O(u/\varepsilon^2)$ for $\theta \lesssim 1/\varepsilon$).

For "large" q, the first term in eq (VI.21) is the largest; the other two can be neglected since they are of relative order θ^2 and θ^4 (for $\theta \gtrsim 1/\varepsilon$) or of relative order $1/\varepsilon^2$ and $1/\varepsilon^4$ (for $\theta \lesssim 1/\varepsilon$). For "small" q as given in 2) the first term is still the largest, but the second term in eq (VI.21) must also be kept since it is of relative order θ (for $\theta \gtrsim 1/\varepsilon$) or of relative order $1/\varepsilon$ (for $\theta \lesssim 1/\varepsilon$). The third term may still be neglected, since it is of order θ^2 (or $1/\varepsilon^2$) relative to the first term in eq (VI.21). Finally, for small q as specified in 3) all three terms in eq (VI.21) are of the same order of magnitude and must all be retained.

We have thus demonstrated the statement made earlier, that over a very wide range of values of $q_\perp$, specifically for $\theta^2 u \lesssim q_\perp \lesssim u$, the order of magnitude of $\{\ \}$ is that of the first term in eq (VI.4), namely $O(\varepsilon^4 \xi^2 q_\perp{}^2)$. Since we want an approximate expression for $\{\ \}$ in which neglected terms are of <u>relative</u> order θ^2 or $1/\varepsilon^2$ (relative to the terms retained) for <u>all</u> $q_\perp$, we must retain the small terms of order u^2, which are the only ones that remain when $q_\perp \to 0$. Only terms of <u>relative</u> order θ^2 (or $1/\varepsilon^2$) for <u>all</u> $q_\perp$ will be neglected.

From eq (VI.21) the terms to be neglected are of order

$$u^2\left(\frac{\varepsilon^2\xi q_\perp}{u}\right)^2\theta^2 \quad , \quad u^2\left(\frac{\varepsilon^2\xi q_\perp}{u}\right)\theta^2 \quad , \quad \text{and} \quad u^2\theta^2 \ ,$$

or of order

$$u^2\left(\frac{\varepsilon^2\xi q_\perp}{u}\right)^2\Big/\varepsilon^2 \quad , \quad u^2\left(\frac{\varepsilon^2\xi q_\perp}{u}\right)\Big/\varepsilon^2 \quad , \quad \text{and} \quad u^2/\varepsilon^2 \quad .$$

We therefore begin our approximation of $\left\{\ \right\}$ by neglecting, in eq (VI.4), these terms of relative order θ^2 or $1/\varepsilon^2$. We consider first the last term in eq (VI.4):

$$-\ 4q^2(\varepsilon_1\underline{u}\xi-\varepsilon_2\underline{v}\eta)^2 = -\ 4(q_\perp{}^2+q_z{}^2)(\varepsilon_1\underline{u}\xi-\varepsilon_2\underline{v}\eta)^2 \tag{VI.22}$$

and note that

$$(\varepsilon_1\underline{u}\xi-\varepsilon_2\underline{v}\eta)^2 = O(\varepsilon^2u^2\xi^2) = O(\varepsilon^4\theta^2\xi^2) \quad .$$

Thus for the term in eq (VI.22) with factor $q_\perp{}^2$ we have

$$-\ 4q_\perp{}^2(\varepsilon_1\underline{u}\xi-\varepsilon_2\underline{v}\eta)^2 = O(\varepsilon^4\xi^2q_\perp{}^2\theta^2) \quad ,$$

which is of order θ^2 relative to the first term in eq (VI.4), and hence may be neglected. We are then left with

$$- 4q_z^{\,2}(\varepsilon_1\underline{u}\xi-\varepsilon_2\underline{v}\eta)^2 = - 4\varepsilon_1\varepsilon_2 q_z^{\,2}(\underline{u}\xi-\underline{v}\eta)^2$$

$$- 4q_z^{\,2}\left[(\varepsilon_1^{\,2}-\varepsilon_1\varepsilon_2)u^2\xi^2 + (\varepsilon_2^{\,2}-\varepsilon_1\varepsilon_2)v^2\eta^2\right] \quad \text{(VI.23)}$$

The first term on the right-hand side of eq (VI.23) may be compared with the second term in eq (VI.4); it is of relative order

$$\frac{q_z^{\,2}}{4\varepsilon_1\varepsilon_2} = 0\left(\frac{1}{\varepsilon^4\xi^2}\right) = 0\left(\frac{(1+u^2)^2}{\varepsilon^4}\right) = 0(\theta^4) + 0(1/\varepsilon^4)$$

and hence may be neglected, since

$$q_z^{\,2} = \left(\frac{1}{2\varepsilon_2\eta} - \frac{1}{2\varepsilon_1\xi}\right)^2 = 0\left(\frac{1}{\varepsilon^2\xi^2}\right) \quad , \qquad \text{(VI.24)}$$

We are thus left with the remaining term on the right-hand side of eq (VI.23) which may be written as

$$- 4q_z^{\,2}k(\varepsilon_1 u^2\xi^2 - \varepsilon_2 v^2\eta^2)$$

$$= - 4kq_z^{\,2}\left[(\varepsilon_1-\varepsilon_2)u^2\xi^2 + \varepsilon_2(u^2\xi^2-v^2\eta^2)\right]$$

$$= - 4kq_z^{\,2}\left[ku^2\xi^2 + \varepsilon_2\xi^2(u^2-v^2) + \varepsilon_2 v^2(\xi^2-\eta^2)\right] . \qquad \text{(VI.25)}$$

Consider next the second term in eq (VI.25):

$$- 4kq_z^{\,2}\varepsilon_2\xi^2(u^2-v^2) = 0(\varepsilon^2\xi u q_\perp \bullet (q_z^{\,2}\xi))$$

from eq (VI.14). But from eq (VI.24)

$$q_z^{\,2}\xi = O\left(\frac{1}{\varepsilon^2\xi}\right) = O\left(\frac{1+u^2}{\varepsilon^2}\right) = O\left(\frac{1}{\varepsilon^2}\right) + O(\theta^2) \quad .$$

Thus the second term in eq (VI.25) has terms of $O(\theta^2)$ and $O(1/\varepsilon^2)$ times the second term in eq (VI.21), and hence may be neglected. In the third term in eq (VI.25) we have, from eq (VI.13),

$$\varepsilon_2 v^2(\xi^2-\eta^2) = \varepsilon_2 v^2(\xi+\eta)(\xi-\eta)$$

$$= \varepsilon_2 v^2(\xi+\eta)\left[-\frac{(u^2-v^2)}{(1+v^2)(1+v^2)} + O(1/\varepsilon^2)\right] \quad . \qquad \text{(VI.26)}$$

The first term here is thus of $O(\varepsilon u^2\xi \cdot |u^2-v^2|\xi^2) \lesssim O(\varepsilon\xi^2|u^2-v^2|)$; it is therefore of the same order as the second term in eq (VI.25) and may also be neglected. We are thus left with the remaining term in eq (VI.26), of order $u^2\xi/\varepsilon$. Compared with the first term in eq (VI.25) this is of relative $O(1/(\varepsilon^2\xi)) = O\left(\frac{1+u^2}{\varepsilon^2}\right) = O(\theta^2) + O(1/\varepsilon^2)$ and thus may also be neglected. We are thus left with the first term in (VI.25), viz.,

$$-\ 4k^2u^2\xi^2q_z^{\,2} = -\left[ku\xi\left(\frac{1}{\varepsilon_2\eta} - \frac{1}{\varepsilon_1\xi}\right)\right]^2$$

$$= -\left[\frac{ku}{\varepsilon_1\varepsilon_2\eta}\,(\varepsilon_1\xi-\varepsilon_2\eta)\right]^2 \quad .$$

Here we write

$$\varepsilon_1\xi - \varepsilon_2\eta = (\varepsilon_1-\varepsilon_2)\eta + \varepsilon_1(\xi-\eta)$$

giving

$$- 4k^2u^2\xi^2q_z{}^2 = - \left[\frac{k^2u}{\varepsilon_1\varepsilon_2} + \frac{ku}{\varepsilon_2}\left(\frac{\xi-\eta}{\eta}\right)\right]^2$$

$$= - \left(\frac{k^2u}{\varepsilon_1\varepsilon_2}\right)^2 \left[1 + \frac{\varepsilon_1}{k}\left(\frac{\varepsilon-\eta}{\eta}\right)\right]^2 \quad . \qquad \text{(VI.27)}$$

The second term here gives, on expanding the square, contributions of $O\left(u^2\left(\frac{\xi-\eta}{\eta}\right)\right)$ and $O\left(u^2\left(\frac{\xi-\eta}{\eta}\right)^2\right) < O\left(u^2\left(\frac{\xi-\eta}{\eta}\right)\right)$ since $\frac{\xi-\eta}{\eta} = O\left(\frac{u-v}{u}\right) < O(1)$. From eq (VI.15) we have

$$u^2\left(\frac{\xi-\eta}{\eta}\right) = O(u^3\xi q_\perp) + O\left(\frac{u^2}{\varepsilon^2\xi}\right)$$

$$= O\left(u^2\left(\frac{\varepsilon^2\xi q_\perp}{u}\right)\theta^2\right) + O\left((1+u^2)\theta^2\right) \quad .$$

The first term on the right-hand side here is $O(\theta^2)$ relative to the second term in eq (VI.21) and may thus be neglected. The last term here may be written

$$O((1+u^2)\theta^2) = O(u^2\theta^2) + O(u^2/\varepsilon^2)$$

and thus makes contributions of $O(\theta^2)$ and $O(1/\varepsilon^2)$ relative to the last term in eq (VI.21); it may therefore also be neglected.

Finally, therefore, neglecting terms of relative order θ^2 and $1/\varepsilon^2$ in $\left\{ \right\}$, we see that the last term in eq (VI.4) may be replaced by

$$-\left(\frac{k^2 u}{\varepsilon_1 \varepsilon_2}\right)^2 \quad . \tag{VI.28}$$

We continue our high energy, small angle approximation, considering next the first term in eq (VI.4),

$$8k^2\varepsilon_1\varepsilon_2(\underline{u} - \underline{v})^2\xi\eta = 8k^2\varepsilon_1\varepsilon_2 q_\perp{}^2\xi\eta \quad .$$

From eq (VI.7) we substitute $\xi = \xi_o - \xi_1$ and $\eta = \eta_o - \eta_1$ in this term, and recall from the discussion surrounding eq (VI.7) that ξ_1 and η_1 are of order θ^2 relative to ξ_o and η_o (or $O(1/\varepsilon^2)$ for $\theta \lesssim 1/\varepsilon$). We may therefore neglect ξ_1 and η_1 in this term, and replace the first term in eq (VI.4) by

$$8k^2\varepsilon_1\varepsilon_2(\underline{u} - \underline{v})^2\xi_o\eta_o \quad . \tag{VI.29}$$

Finally, we obtain the high energy, small angle approximation for the second term in eq (VI.4),

$$16\varepsilon_1{}^2\varepsilon_2{}^2(\underline{u}\xi-\underline{v}\eta)^2 = 16\varepsilon_1{}^2\varepsilon_2{}^2[(\underline{u}-\underline{v})^2\xi\eta + u^2(\xi-\eta)^2 + \eta(u^2-v^2)(\xi-\eta)] \tag{VI.30}$$

from eq (VI.12). In the first term in eq (VI.30) we may replace $\xi\eta$ by $\xi_0\eta_0$, as we just did in arriving at (VI.29), again with neglect of terms of relative order θ^2 and $1/\varepsilon^2$. In the remaining two terms in eq (VI.30) we substitute $\xi = \xi_0 - \xi_1$ and $\eta = \eta_0 - \eta_1$ from eq (VI.7). The terms with subscript zero are then

$$u^2(\xi_0-\eta_0)^2 + \eta_0(u^2-v^2)(\xi_0-\eta_0) = [u^2(\xi_0-\eta_0) + \eta_0(u^2-v^2)](\xi_0-\eta_0)$$

$$= (u^2\xi_0-v^2\eta_0)(\xi_0-\eta_0) \quad .$$

Now from eqs (VI.8), (VI.10), and (VI.11),

$$u^2\xi_0 - v^2\eta_0 = (1-\xi_0) - (1-\eta_0)$$

$$= -(\xi_0-\eta_0) \quad .$$

Thus the terms with subscript zero are simply,

$$-(\xi_0-\eta_0)^2 = -\left(\frac{1}{1+u^2} - \frac{1}{1+v^2}\right)^2$$

$$= -\left(\frac{(u^2-v^2)}{(1+u^2)(1+v^2)}\right)^2$$

$$= -(u^2-v^2)^2 \, \xi_0{}^2\eta_0{}^2 \quad . \qquad \text{(VI.31)}$$

The terms with subscripts zero and one are, since from eq (VI.7) $\xi = \xi_0 - \xi_1$, $\eta = \eta_0 - \eta_1$,

$$- 2u^2(\xi_o-\eta_o)(\xi_1-\eta_1) - \eta_o(u^2-v^2)(\xi_1-\eta_1) - \eta_1(u^2-v^2)(\xi_o-\eta_o) \quad .$$

Writing $\xi_o - \eta_o = -(u^2 - v^2)\xi_o\eta_o$, these terms can be written as

$$(\xi_1-\eta_1)(u^2-v^2)\,\eta_o(2u^2\xi_o-1) + (u^2-v^2)^2\,\xi_o\eta_o\eta_1 \quad . \tag{VI.32}$$

From eq (VI.8) ε_1 and η_1 are $O(1/\varepsilon^2)$; from eq (VI.14) $u^2-v^2 = O(uq_\perp)$. Therefore, for the second term here we have

$$(u^2-v^2)^2\,\xi_o\eta_o\eta_1 = O(u^2\xi^2 q_\perp{}^2/\varepsilon^2) = O(\theta^2\xi^2 q_\perp{}^2) \quad .$$

It is thus of $O(\theta^2)$ relative to the first term in eq (VI.30) and may thus be neglected.

In the first term in eq (VI.32) we write

$$\begin{aligned} 2u^2\xi_o - 1 &= \frac{2u^2}{1+u^2} - 1 \\ &= \frac{u^2-1}{u^2+1} \\ &= (u^2-1)\xi_o \quad . \end{aligned}$$

We thus retain, of the terms with subscripts zero and one,

$$(\xi_1-\eta_1)(u^2-v^2)\xi_o\eta_o(u^2-1) \quad . \tag{VI.33}$$

Finally, the terms with subscript one are

$$u^2(\xi_1-\eta_1)^2 + \eta_1(u^2-v^2)(\xi_1-\eta_1) \quad . \tag{VI.34}$$

The first term in eq (VI.34) is $O(u^2/\varepsilon^4)$. It therefore gives a contribution of $O(u^2)$ to eq (VI.30) and must be retained. Together with the first term in eq (VI.30), of $O(\varepsilon^4\xi^2q_\perp{}^2)$, we thus have contributions from these two positive terms of

$$O(\varepsilon^4\xi^2q_\perp{}^2) + O(u^2) \gtrsim O(\varepsilon^2\xi q_\perp u) \quad . \tag{VI.35}$$

(Note that $a^2x^2 + b^2y^2 \geqq c^2(x^2 + y^2) \geqq 2c^2xy$ where c^2 = smaller of a^2 and b^2.) The second term in eq (VI.34) is, from eqs (VI.14) and (VI.37) of $O(uq_\perp/\varepsilon^4)$. It therefore gives a contribution to eq (VI.30) of $O(uq_\perp) = O(u^2\xi uq_\perp) = O(\varepsilon^2\xi q_\perp u)\cdot O(\theta^2)$, i.e., a contribution of $O(\theta^2)$ relative to that from the first term in eq (VI.34) together with the first term in eq (VI.30), shown above in eq (VI.35). We may therefore neglect the second term in eq (VI.34). Thus of the terms with subscript one we retain only

$$u^2(\xi_1-\eta_1)^2 \quad . \tag{VI.36}$$

In eq (VI.33) and (VI.36) the factor $\xi_1-\eta_1$ may be simplified further, again neglecting terms of relative order $1/\varepsilon^2$ and θ^2. From eq (VI.8)

$$\xi_1 = \frac{1}{2\varepsilon_1(\varepsilon_1+p_1\cos\theta_1)}$$

$$= \frac{1}{2\varepsilon_1[2\varepsilon_1 - (\varepsilon_1-p_1\cos\theta_1)]}$$

$$= \frac{1}{4\varepsilon_1^{\,2}(1 - d_1/2\varepsilon_1)}$$

$$= \frac{1}{4\varepsilon_1^{\,2}\left(1 - 1/(4\varepsilon_1^{\,2}\xi)\right)} \quad .$$

Neglecting the term

$$\frac{1}{4\varepsilon_1^{\,2}\xi} = 0\left(\frac{1+u^2}{\varepsilon^2}\right) = 0(1/\varepsilon^2) + 0(\theta^2)$$

we have

$$\xi_1 \approx \frac{1}{4\varepsilon_1^{\,2}}$$

and, similarly,

$$\eta_1 \approx \frac{1}{4\varepsilon_2^{\,2}} \quad ,$$

from which

$$\xi_1 - \eta_1 \approx \frac{1}{4}\left(\frac{1}{\varepsilon_1{}^2} - \frac{1}{\varepsilon_2{}^2}\right)$$

$$= - \frac{k(\varepsilon_1 + \varepsilon_2)}{4\varepsilon_1{}^2\varepsilon_2{}^2} \quad . \tag{VI.37}$$

Substituting this in (VI.33) and (VI.36) gives, together with eqs (VI.30) and (VI.31), the high energy, small angle approximation for the second term in eq (VI.4):

$$16\varepsilon_1{}^2\varepsilon_2{}^2(\underline{u}\xi-\underline{v}\eta)^2 \approx 16\varepsilon_1{}^2\varepsilon_2{}^2\left[(\underline{u}-\underline{v})^2\xi_o\eta_o - (u^2-v^2)^2\xi_o{}^2\eta_o{}^2\right]$$

$$- 4k(\varepsilon_1+\varepsilon_2)(u^2-v^2)\xi_o\eta_o(u^2-1)$$

$$+ \left(\frac{k(\varepsilon_1+\varepsilon_2)}{\varepsilon_1\varepsilon_2}\right)^2 u^2 \quad . \tag{VI.38}$$

We now have, in eqs (VI.29), (VI.38), and (VI.28), the high energy, small angle approximations to the first, second, and third terms in eq (VI.4), respectively. The last term in eq (VI.38) may be added to (VI.28) to give

$$\left(\frac{k(\varepsilon_1+\varepsilon_2)}{\varepsilon_1\varepsilon_2}\right)^2 u^2 - \left(\frac{k^2u}{\varepsilon_1\varepsilon_2}\right)^2 = \frac{4k^2u^2}{\varepsilon_1\varepsilon_2} \quad . \tag{VI.39}$$

The first term in eq (VI.38) may be added to (VI.29) to give

$$8\varepsilon_1\varepsilon_2(\varepsilon_1{}^2+\varepsilon_2{}^2)(\underline{u}-\underline{v})^2\xi_o\eta_o \quad . \tag{VI.40}$$

We thus have

$$\left\{\quad\right\} \approx 8\varepsilon_1\varepsilon_2(\varepsilon_1{}^2+\varepsilon_2{}^2)(\underline{u}-\underline{v})^2\xi_o\eta_o - 16\varepsilon_1{}^2\varepsilon_2{}^2(u^2-v^2)^2\xi_o{}^2\eta_o{}^2$$

$$- 4k(\varepsilon_1+\varepsilon_2)(u^2-v^2)\xi_o\eta_o(u^2-1) + \frac{4k^2u^2}{\varepsilon_1\varepsilon_2} \quad . \tag{VI.41}$$

These four terms are indeed of the order of magnitude given in eq (VI.21): The first is

$$O(\varepsilon^4 q_\perp{}^2\xi^2) = O\left(u^2\left(\varepsilon^2\xi\,\frac{q_\perp}{u}\right)^2\right) \quad ,$$

the second is

$$O(\varepsilon^4u^2q_\perp{}^2\xi^4) = O(\varepsilon^4q_\perp{}^2\xi^2(u\xi)^2) \lesssim O(\varepsilon^4q_\perp{}^2\xi^2)$$

since $u\xi \lesssim O(1)$ for all u, the third is

$$O(\varepsilon^2uq_\perp\xi^2(u^2-1)) \lesssim O(\varepsilon^2uq_\perp\xi) = O\left(u^2\left(\varepsilon^2\xi\,\frac{q_\perp}{u}\right)\right) \quad ,$$

since $\xi(u^2-1) \lesssim O(1)$ for all u, and the fourth term is $O(u^2)$.

We next write eq (VI.41) in a form that is manifestly positive. To this end we substitute

$$(\underline{u}-\underline{v})^2 = (u-v)^2 + 2uv(1-\cos\phi)$$

$$= (u-v)^2 + 4uv\sin^2\frac{\phi}{2}$$

in eq (VI.41) and obtain, separating terms with factors $(u-v)^2$ and $(u-v)$,

$$\{\ \} \approx 8\varepsilon_1\varepsilon_2\xi_o\eta_o\left[\varepsilon_1{}^2 + \varepsilon_2{}^2 - 2\varepsilon_1\,\varepsilon_2\,(u+v)^2\xi_o\eta_o\right](u-v)^2$$

$$- 4k(\varepsilon_1+\varepsilon_2)\xi_o\eta_o(u+v)(u^2-1)(u-v) + \frac{4k^2u^2}{\varepsilon_1\varepsilon_2}$$

$$+ 32\varepsilon_1\varepsilon_2(\varepsilon_1{}^2+\varepsilon_2{}^2)\xi_o\eta_o uv\sin^2\frac{\phi}{2}\ . \qquad \text{(VI.42)}$$

In the first term here we may write the expression in square brackets so that it is clearly positive:

$$\varepsilon_1{}^2 + \varepsilon_2{}^2 - 2\varepsilon_1\varepsilon_2(u+v)^2\xi_o\eta_o = k^2 + 2\varepsilon_1\varepsilon_2\left(1 - \frac{(u+v)^2}{(1+u^2)(1+v^2)}\right)$$

$$= k^2 + 2\varepsilon_1\varepsilon_2\,\frac{(uv-1)^2}{(1+u^2)(1+v^2)}\quad .$$

Keeping in mind that we will wish to consider the cross section as a function of the angles of the final electron, θ_2 and ϕ, or, alternatively, as a function of the variables v and ϕ, we may now write eq (VI.42) in the form

$$\Big\{\ \Big\} \approx f(v)(v-u)^2 + 4k(\varepsilon_1+\varepsilon_2)\xi_o\eta_o(u+v)(u^2-1)(v-u) + \frac{4k^2u^2}{\varepsilon_1\varepsilon_2} + g(v)\sin^2\frac{\phi}{2} \tag{VI.43}$$

where

$$f(v) = \frac{8\varepsilon_1^{\,2}\varepsilon_2^{\,2}}{(1+u^2)(1+v^2)}\left[\frac{k^2}{\varepsilon_1\varepsilon_2} + 2\frac{(uv-1)^2}{(1+u^2)(1+v^2)}\right] \tag{VI.44}$$

and

$$g(v) = \frac{32\varepsilon_1\varepsilon_2(\varepsilon_1^{\,2}+\varepsilon_2^{\,2})uv}{(1+u^2)(1+v^2)} \quad . \tag{VI.45}$$

Completing the square in eq (VI.43) now gives

$$\Big\{\ \Big\} \approx f(v)\left(v-u + \frac{2k(\varepsilon_1+\varepsilon_2)\xi_o\eta_o(u+v)(u^2-1)}{f(v)}\right)^2 - \frac{[2k(\varepsilon_1+\varepsilon_2)\xi_o\eta_o(u+v)(u^2-1)]^2}{f(v)} + \frac{4k^2u^2}{\varepsilon_1\varepsilon_2} + g(v)\sin^2\frac{\phi}{2} \quad . \tag{VI.46}$$

Now from eq (VI.44), $f(v) = O(\epsilon^4\xi^2)$. The second term above is thus $O(u^2)$, of the same order as the third term. Furthermore, if we consider the second term above as a function of v, then setting $v = u$ everywhere in this term implies neglect of terms of relative order $(v - u)/u$:

$$v = u + (v - u)$$

$$= u\left(1 + \frac{v - u}{u}\right) .$$

Thus, setting $v = u$ in this term, which is of order u^2, introduces errors of order $u^2 \cdot (v - u)/u = O(uq_\perp)$ in $\{\ \}$. These terms may be neglected, however, since they are

$$O(1/(\epsilon^2\xi)) = O\left(\frac{1+u^2}{\epsilon^2}\right) = O\left(\frac{1}{\epsilon^2}\right) + O(\theta^2)$$

relative to the third term, of $O(\epsilon^2\xi uq_\perp)$, in the expression (VI.41) for $\{\ \}$. We therefore set $v = u$ in the second term in eq (VI.46), which then becomes

$$-\frac{16k^2(\epsilon_1+\epsilon_2)^2\xi_0{}^4u^2(u^2-1)^2}{f(u)} = -\frac{\dfrac{2k^2(\epsilon_1+\epsilon_2)^2u^2}{\epsilon_1{}^2\epsilon_2{}^2}\left(\dfrac{u^2-1}{u^2+1}\right)^2}{\left[\dfrac{k^2}{\epsilon_1\epsilon_2} + 2\left(\dfrac{u^2-1}{u^2+1}\right)^2\right]}$$

from eq (VI.44).

Adding this to the third term in eq (VI.46) then gives

$$\frac{\dfrac{2k^2u^2}{\varepsilon_1\varepsilon_2}\left[\dfrac{2k^2}{\varepsilon_1\varepsilon_2} + 4\left(\dfrac{u^2-1}{u^2+1}\right)^2 - \dfrac{(\varepsilon_1+\varepsilon_2)^2}{\varepsilon_1\varepsilon_2}\left(\dfrac{u^2-1}{u^2+1}\right)^2\right]}{\dfrac{k^2}{\varepsilon_1\varepsilon_2} + 2\left(\dfrac{u^2-1}{u^2+1}\right)^2} \cdot$$

Here in the numerator we have, multiplying $\left(\frac{u^2-1}{u^2+1}\right)^2$,

$$4 - \frac{(\varepsilon_1+\varepsilon_2)^2}{\varepsilon_1\varepsilon_2} = - \frac{(\varepsilon_1-\varepsilon_2)^2}{\varepsilon_1\varepsilon_2}$$

$$= - \frac{k^2}{\varepsilon_1\varepsilon_2} \quad .$$

The expression in the numerator above thus becomes

$$2\left(\frac{k^2}{\varepsilon_1\varepsilon_2}\right)^2 u^2\left[2 - \left(\frac{u^2-1}{u^2+1}\right)^2\right] = 2\left(\frac{k^2}{\varepsilon_1\varepsilon_2}\right)^2 u^2\left[1 + 1 - \left(\frac{u^2-1}{u^2+1}\right)^2\right]$$

$$= 2\left(\frac{k^2}{\varepsilon_1\varepsilon_2}\right)^2 u^2\left[1 + \left(\frac{2u}{1+u^2}\right)^2\right]$$

For the sum of the second and third terms in eq (VI.46) we now have

$$\frac{2\left(\dfrac{k^2}{\varepsilon_1\varepsilon_2}\right)^2 u^2\left[1+\left(\dfrac{2u}{1+u^2}\right)^2\right]}{\dfrac{k^2}{\varepsilon_1\varepsilon_2}+2\left(\dfrac{u^2-1}{u^2+1}\right)^2} \equiv \rho \quad . \tag{VI.47}$$

Finally, we consider in eq (VI.46) the last term in $(\quad)^2$, which is of

$$O\left(\frac{u(u^2-1)}{\varepsilon^2}\right) \lesssim O\left(\frac{u(u^2+1)}{\varepsilon^2}\right) = O\left(\frac{u}{\varepsilon^2\xi}\right)$$

since $f(v) = O(\varepsilon^4\xi^2)$. After taking the square, this term gives contributions to $\{\ \}$ of $O(\varepsilon^2 u\xi q_\perp)$ from the cross term and of $O(u^2)$ from its square; they are therefore both retained, as we have discussed. Now setting $v = u$ in the last term in $(\quad)^2$ introduces errors of relative order $(v - u)/u \lesssim O(q_\perp/u)$, as we noted following eq (VI.46); these errors are thus $O\left(\dfrac{u}{\varepsilon^2\xi}\cdot\dfrac{q_\perp}{u}\right) = O\left(\dfrac{q_\perp}{\varepsilon^2\xi}\right)$ and thus give, from the cross term with $v - u$, contributions to $\{\ \}$ of $O\left(\varepsilon^4\xi^2\cdot q_\perp\cdot\dfrac{q_\perp}{\varepsilon^2\xi}\right) = O(\varepsilon^2\xi q_\perp{}^2)$, and, from the cross term with the last term itself, evaluated at $v = u$, contributions of $O\left(\varepsilon^4\xi^2\cdot\dfrac{u}{\varepsilon^2\xi}\cdot\dfrac{q_\perp}{\varepsilon^2\xi}\right) = O(uq_\perp)$. These errors thus give contributions of $O(1/(\varepsilon^2\xi)) = O\left(\dfrac{1+u^2}{\varepsilon^2}\right) = O(1/\varepsilon^2) + O(\theta^2)$ relative to the retained terms of $O(\varepsilon^4\xi^2 q_\perp{}^2)$ and $O(\varepsilon^2 u\xi q_\perp)$, and may therefore be neglected. Thus in the last term in $(\quad)^2$ in eq (VI.46) we may set $v = u$, giving, for this term,

$$\frac{2k(\varepsilon_1+\varepsilon_2)\xi_o{}^2 2u(u^2-1)}{f(u)} = \frac{k(\varepsilon_1+\varepsilon_2)(u^2-1)u}{2\varepsilon_1{}^2\varepsilon_1{}^2\left[\dfrac{k^2}{\varepsilon_1\varepsilon_2} + 2\left(\dfrac{u^2-1}{u^2+1}\right)^2\right]} \equiv \sigma \quad . \qquad \text{(VI.48)}$$

We thus have, finally, our high energy, small angle approximation to $\{\ \}$, defined initially by eq (IV.1):

$$\{\ \} = f(v)(v-u+\sigma)^2 + g(v)\sin^2\tfrac{1}{2}\phi + \rho \qquad \text{(VI.49)}$$

where $f(v) > 0$ and $g(v) > 0$ are given by eqs (VI.44) and (VI.45) and $\rho = \rho(u) > 0$ and $\sigma = \sigma(u)$ are given by eqs (VI.47) and (VI.48).

At this point a few observations concerning this approximate expression for $\{\ \}$ are in order. We note first that all the terms here are positive. Thus our approximate expression for $\{\ \}$ is positive, as it should be since the cross section is, apart from a few simple factors, equal to $\{\ \}/q^4$.

Next, we note that $\{\ \}$ as given without approximation in eq (VI.4), simplifies directly to one given some time ago, [15], if we consider either the region of "large" momentum transfers, $q \approx q_\perp = O(u)$, or that region of small momentum transfers in which $q_\perp \sim q_z = O(u\theta)$, and if in addition we drop terms of relative order θ as well as those of relative order θ^2. In that case, as we have seen, we can set $\xi = \xi_o$, $\eta = \eta_o$ in the first two terms and drop the last term in eq (IV.4) entirely, giving

$$\{\ \} = 8\varepsilon_1\varepsilon_2\left[k^2(\underline{u}-\underline{v})\xi_o\eta_o + 2\varepsilon_1\varepsilon_2(\underline{u}\xi_o-\underline{v}\eta_o)^2\right] \quad . \qquad \text{(VI.50)}$$

This expression also provides a satisfactory starting point if one wishes to integrate over the angles of either the final electron or the emitted photon, since the significant contributions to the cross section integrated over angles come from the regions of momentum transfers in which $q_\perp = O(u)$ and $q_\perp \sim q_z = O(u\theta)$. The expression given in eq (VI.50) does not, however, provide a useful approximation to the cross section in the region of the sharp dip, i.e., for small momentum transfers with $q_\perp \lesssim O(u\theta^2) \ll q_z$. In this region the accurate approximation, eq (VI.49) is, as we have shown, of $O(u^2)$. Since eq (VI.50) is of $O(\varepsilon^4\xi^2 q_\perp{}^2)$, the relative errors in eq (VI.50) in this region are of $O(u^2/(\varepsilon^4\xi^2 q_\perp{}^2)) = O(1/(q_\perp/u\theta^2))$ for $\theta \gtrsim 1/\varepsilon$. The relative errors in eq (VI.50) are thus of $O(1)$ for $q_\perp = O(u\theta^2)$, and get larger for smaller $q_\perp$.

We now consider $\left\{ \ \right\}$ as a function of v (i.e., θ_2) for fixed values of the other variables (ε_1, ε_2, θ_1 and ϕ) and determine the value of v for which $\left\{ \ \right\}$ achieves its minimum as well as the value of $\left\{ \ \right\}$ at the minimum. In figs. 4 and 5 we show $\left\{ \ \right\}$ as a function of θ_2 for $\varepsilon_1 = 140$ MeV, $k = 95$ MeV, $\theta_1 = 1°$ and $\phi = 0°$. We use our approximate expression, (VI.49), which differs from the exact expression only by neglect of terms of <u>relative</u> order $1/\varepsilon^2$ and θ^2, for <u>all</u> values of the variables. From eq (VI.49) we have

$$\frac{\partial}{\partial v}\left\{ \ \right\} = \left\{ \ \right\}' = f'(v)(v-u+\sigma)^2 + 2f(v)(v-u+\sigma) + g'(v)\sin^2\tfrac{1}{2}\phi = 0 \quad ,$$

which we write in the form

$$uf'(v)\left(\frac{v-u+\sigma}{u}\right)^2 + 2f(v)\left(\frac{v-u+\sigma}{u}\right) + \frac{g'(v)}{u}\sin^2\tfrac{1}{2}\phi = 0 \quad . \qquad \text{(VI.51)}$$

First, we observe from eqs (VI.44) and (VI.45) that $uf'(v)$, $f(v)$, and $g'(v)/u$ are all of the same order, viz., $O(\varepsilon^4\xi^2)$. Now we are looking for a solution to eq (VI.51) for small q, i.e., $q_\perp \ll u$, which implies $\frac{v-u}{u} \ll 1$ and $\phi \ll 1$ since $|v-u| \lesssim O(q_\perp)$ and $u\phi \lesssim O(q_\perp)$. But from eq (VI.48), σ is small, of $O(u\theta^2)$ for $\theta \gtrsim 1/\varepsilon$ (or $O(u/\varepsilon^2)$ for $\theta \lesssim 1/\varepsilon$). Thus we are looking for a solution to eq (VI.51) such that $\frac{v-u+\sigma}{u} << 1$. Assuming $q_\perp \lesssim O(u\theta)$, we have $\phi \lesssim O(\theta)$. Then from eq (VI.51) the solution is such that $\frac{v-u+\sigma}{u} = O(\phi^2) = O(\theta^2)$. And from this it follows that the first term in eq (VI.51) is of order $uf'(v) \cdot \phi^4$, i.e., that it is of order θ^2 relative to the second term in eq (VI.51) and hence may be neglected. We then have, neglecting terms of relative order ϕ^2,

$$\frac{v-u+\sigma}{u} \approx \frac{-g'(v)\phi^2}{8uf(v)} \quad .$$

Here too we may expand the right hand side about the point $v = u$. Neglecting terms of relative order $\frac{v-u}{u} = O(\theta^2)$ is thus equivalent to setting $v = u$ there. We then obtain

$$v-u+\sigma = \frac{-g'(u)\phi^2}{8f(u)}$$

or

$$(v-u)_{min} = -\sigma - \frac{g'(u)\phi^2}{8f(u)} \quad . \tag{VI.52}$$

From eq (VI.48) we have σ, from eq (VI.45) we have

$$g'(u) = \frac{32\varepsilon_1\varepsilon_2(\varepsilon_1{}^2+\varepsilon_2{}^2)u(1-u^2)}{(1+u^2)^3} \quad ,$$

and from eq (VI.44)

$$f(u) = \frac{8\varepsilon_1{}^2\varepsilon_1{}^2}{(1+u^2)^2}\left[\frac{k^2}{\varepsilon_1\varepsilon_2} + 2\left(\frac{u^2-1}{u^2+1}\right)^2\right] .$$

Thus

$$(v-u)_{min} = \frac{\left[\dfrac{-k(\varepsilon_1+\varepsilon_2)u(u^2-1)}{2\varepsilon_1{}^2\varepsilon_1{}^2} + \dfrac{(\varepsilon_1{}^2+\varepsilon_2{}^2)}{2\varepsilon_1\varepsilon_2}\, u\left(\dfrac{u^2-1}{u^2+1}\right)\phi^2\right]}{\dfrac{k^2}{\varepsilon_1\varepsilon_2} + 2\left(\dfrac{u^2-1}{u^2+1}\right)^2}$$

or

$$v_{min} = u\left[1 + \left(\frac{u^2-1}{u^2+1}\right)\left(\frac{\dfrac{-k(\varepsilon_1+\varepsilon_2)(u^2+1)}{2\varepsilon_1{}^2\varepsilon_1{}^2} + \dfrac{(\varepsilon_1{}^2+\varepsilon_2{}^2)}{2\varepsilon_1\varepsilon_2}\,\phi^2}{\dfrac{k^2}{\varepsilon_1\varepsilon_2} + 2\left(\dfrac{u^2-1}{u^2+1}\right)^2}\right)\right] . \tag{VI.53}$$

The value of $\left\{ \ \right\}$ at this minimum is given by eqs (VI.52) and (VI.49). From eq (VI.52), recalling that $\sigma = O(u\theta^2)$ and that $g'(u) = O(uf(u))$, we note that $(v-u)_{min} = O(u\theta^2) + O(u\phi^2) = O(u\theta^2)$ since we have assumed that $\phi \lesssim O(\theta)$. Thus, at the minimum, the first term in eq (VI.49) is $O(f(u)\cdot(u\theta^2)^2) = O(u^2 f(u)\cdot\theta^4)$. This is of $O(\theta^2)$ relative to the second term in eq (VI.49), which is $O(g(v)\phi^2) = O(u^2 f(u)\theta^2)$, since from eqs (VI.44) and (VI.45) $g(v) = O(u^2 f(u))$. We may therefore neglect the first term in eq (VI.49), and in the second term set $v = u$ (since $v - u = O(u\theta^2)$ at the minimum) and neglect terms of relative order ϕ^2, thus again neglecting terms of relative order θ^2. We thus obtain, for the value of $\left\{ \ \right\}$ at its minimum,

$$\left\{ \ \right\}_{min} = \frac{1}{4} g(u)\phi^2 + \rho$$

$$= \frac{8\varepsilon_1\varepsilon_2(\varepsilon_1{}^2+\varepsilon_2{}^2)u^2\phi^2}{(1+u^2)^2} + \frac{2\left(\frac{k^2}{\varepsilon_1\varepsilon_2}\right)^2 u^2 \left[1 + \left(\frac{2u}{1+u^2}\right)^2\right]}{\frac{k^2}{\varepsilon_1\varepsilon_2} + 2\left(\frac{u^2-1}{u^2+1}\right)^2} \quad . \qquad \text{(VI.54)}$$

It may appear that the second term here, being of $O(u^2)$, should be neglected since the first term is $O(\varepsilon^4 u^2 \xi^2 \phi^2) = O\left(u^2\left(\frac{\phi}{\theta}\right)^2 \cdot \frac{1}{\theta^2}\right)$ (for $\theta \gtrsim 1/\varepsilon$). Thus for $\phi = O(\theta)$ the second term in eq (VI.54) is indeed of order θ^2 relative to the first term. However, since we want this expression to be valid for all small ϕ, provided only that $\phi \lesssim O(\theta)$, we must retain the second term, that being all that remains when $\phi \to 0$.

It is worth noting that the minimum for $\left\{ \right\}$ does not occur for precisely the same value of v as the minimum for q^2, as is seen by comparing eq (VI.53) with eq (V.17). In both cases the value of v_{min} differs from u by terms of order ϕ^2, θ^2, and $1/\varepsilon^2$, but the terms are not the same for $\left\{ \right\}$ and for q^2.

VII. DETERMINATION OF THE MAXIMA AND MINIMUM OF THE BREMSSTRAHLUNG CROSS SECTION

In this section we consider the cross section as a function of v (i.e., θ_2) for fixed values of the other variables (ε_1, ε_2, θ_1, and ϕ) and determine the value of v for which the cross section achieves its maximum and minimum values. In order to locate the extrema of the cross section and evaluate the cross section at these extrema we use the high energy small angle approximations for q^2 and $\{\ \}$ given in eqs (V.15) and (VI.49). However, we begin our considerations with the cross section as given in eq (IV.1):

$$\frac{d^3\sigma}{d\Omega_k d\Omega_{p_2} dk} = Z^2 \frac{e^2}{\hbar c}\left(\frac{e^2}{mc^2}\right)^2 [1-F(q)]^2 \frac{1}{k} \frac{p_2}{p_1} \frac{1}{(2\pi)^2} \frac{\{\ \}}{q^4} \quad . \tag{VII.1}$$

Clearly the location of the extrema of the cross section depends to some extent on the atomic form factor, $F(q)$, of the particular atom in question. This dependence is very small, however, for the energies and angles of experimental interest here, for which the minimum momentum transfer, q_{min} in eq (V.18), is relatively large compared with the inverse of the screening radius, $\beta \approx Z^{1/3}/121$ in the Thomas-Fermi model for screening, [16]. Specifically, screening is important only if $q_{min}^2 \lesssim \beta^2$. Even if we take the case which gives the largest screening (q_{min} as small as possible, β as large as possible), we have, from eq (V.19), for $\phi = 0$, $\theta_1 = 1°$, $\varepsilon_1 = 140$ MeV and $k = 95$ MeV,

$$q^2_{min} = 8.5 \times 10^{-3}$$

whereas, for $Z = 92$,

$$\beta^2 = 1.4 \times 10^{-3} \quad .$$

Thus in this case we have $q^2_{min} > \beta^2$. For smaller values of Z, and for $\phi > 0$, q^2_{min} will be even larger, and β^2 will be even smaller, thus reducing further the effect of screening. We therefore neglect screening in determining the location of the extrema of the cross section, setting $F(q) = 0$ in eq (VII.1). The extrema are then determined from

$$\begin{aligned} \frac{\partial}{\partial v}\left[(q^2)^{-2}\left\{\quad\right\}\right] &= \left[(q^2)^{-2}\left\{\quad\right\}\right]' \\ &= (q^2)^{-2}\left\{\quad\right\}' - 2(q^2)^{-3}(q^2)'\left\{\quad\right\} \\ &= (q^2)^{-3}\left[q^2\left\{\quad\right\}' - 2(q^2)'\left\{\quad\right\}\right] \\ &= 0 \quad . \end{aligned} \tag{VII.2}$$

Our determining equation is thus

$$q^2\left\{\quad\right\}' = 2(q^2)'\left\{\quad\right\} \quad . \tag{VII.2a}$$

However, we are looking for maxima and minima in the region of small q, i.e., $q \lesssim O(u\theta)$ and hence also $q_\perp \lesssim O(u\theta)$. Now from $q_\perp^2 = (\underline{u}-\underline{v})^2 = (u-v)^2 + 4uv\sin^2\tfrac{1}{2}\phi$ we have $|u-v| \leqslant q_\perp$ and $u\phi \lesssim O(q_\perp)$. Thus for $q_\perp \lesssim O(u\theta)$ we have $\frac{v-u}{u} \lesssim O(\theta)$ and $\phi \lesssim O(\theta)$. We therefore assume $\phi \lesssim O(\theta)$, neglecting terms of relative order $\phi^2 \lesssim O(\theta^2)$ in our high energy small angle approximations for q^2, $(q^2)'$, $\{\ \}$ and $\{\ \}'$ (eqs (V.15), (V.16), (VI.49) and (VI.51)) and substitute them in eq (VII.2a). Defining

$$z = \frac{v-u}{u} \tag{VII.3}$$

we then obtain

$$\left[z^2 + 2z\left(\frac{\phi^2}{2} + \frac{\delta(1+u^2)}{\varepsilon_2}\right) + \phi^2 + \frac{\delta^2(1+u^2)^2}{u^2}\right]$$

$$\cdot \left[uf'(v)\left(z + \frac{\sigma}{u}\right)^2 + 2f(v)\left(z + \frac{\sigma}{u}\right) + \frac{g'(v)}{u}\frac{\phi^2}{4}\right]$$

$$= 4\left[z + \frac{\phi^2}{2} + \frac{\delta(1+u^2)}{\varepsilon_2}\right] \cdot \left[f(v)\left(z + \frac{\sigma}{u}\right)^2 + \frac{\rho}{u^2} + \frac{g(v)}{u^2}\frac{\phi^2}{4}\right] . \tag{VII.4}$$

We then look for solutions of eq (VII.4), first in the region $z \lesssim O(\theta^2)$ and then in the region $z = O(\theta)$. We find that for sufficiently small ϕ, the equation (VII.4), considered as a function of z (or v) has one solution in the region $z \lesssim O(\theta^2)$, corresponding to the minimum of the cross section, and two solutions in the region $z = O(\theta)$, corresponding to maxima of the cross section, one for $z < 0$ $(v < u)$ and one for $z > 0$ $(v > u)$. However, as ϕ increases the position of the minimum moves to larger values of v, and the position of the second maximum (the one for $z > 0$) moves to smaller values of v. When ϕ reaches a critical value, ϕ_c, the minimum and the second maximum occur at the same point, which is then an inflection point of the cross section. This point occurs for $z = O(\theta^{4/3})$. For $\phi > \phi_c$, there is only one extremum, the maximum which occurs for $v < u$. This behavior of the extrema as a function of ϕ may be seen clearly in figures 11 and 12.

To guide us in the solution of eq (VII.4), we note, from eqs (VI.44) and (VI.45), that $uf'(v)$, $f(v)$, $\frac{g'(v)}{u}$ and $\frac{g(v)}{u^2}$ are all of the same order of magnitude, of $O(\varepsilon^4\xi^2)$. For the other terms in eq (VII.4) we note that ϕ^2, $\frac{\delta(1+u^2)}{\varepsilon_2}$, $\frac{\delta^2(1+u^2)^2}{u^2}$, and $\frac{\sigma}{u}$ are all of $O(\theta^2)$. (In these estimates we assume, for ease of presentation, that $\theta \gtrsim 1/\varepsilon$, so that $u \gtrsim O(1)$.) Thus on the left hand side of eq (VII.4), in the first square bracket the term $2z\left(\frac{\phi^2}{2} + \frac{\delta(1+u^2)}{\varepsilon_2}\right)$ is smaller by $O(z)$ than the remaining terms, $\phi^2 + \frac{\delta^2(1+u^2)^2}{u^2}$. And in the second bracket, the term $uf'(v)(z+\sigma)^2$ is smaller by $O\left(z + \frac{\sigma}{u}\right)$ than the next term, $2f(v)\left(z + \frac{\sigma}{u}\right)$.

We first look for solutions of eq (VII.4) for which $z = O(\theta^2)$, but make no assumption as to whether $\phi = O(\theta)$ or $\phi \lesssim O(\theta^2)$. Thus if $\phi = O(\theta)$ the solution corresponds to $q_\perp = O(u\theta)$, whereas if $\phi \lesssim O(\theta^2)$ the solution corresponds to $q_\perp = O(u\theta^2)$. With $z = O(\theta^2)$, the equation (VII.4) simplifies considerably. Dropping terms of relative order θ^2, in the first bracket on the left hand side only the term $\phi^2 + \frac{\delta^2(1+u^2)^2}{u^2}$ remains. And in the second bracket on the left hand side the term $uf'(v)\left(z + \frac{\sigma}{u}\right)^2$ may be neglected. The left hand side of eq (VII.4) thus simplifies to

$$\left[\phi^2 + \frac{\delta^2(1+u^2)^2}{u^2}\right] \cdot \left[2f(v)\left(z + \frac{\sigma}{u}\right) + \frac{g'(v)}{4u}\,\phi^2\right] = O(f(v)\theta^4) \quad .$$

Now on the right hand side of eq (VII.4) the first bracket is of order θ^2. In the second bracket, the first term, $f(v)\left(z + \frac{\sigma}{u}\right)^2$ is $O(f(v)\theta^4)$, and thus, with the first bracket as factor, gives a contribution of $O(f(v)\theta^6)$, and thus may be neglected. In similar fashion, the second term in the second bracket, $\frac{\rho}{u^2}$, is of $O(1)$. This therefore gives a contribution to the right hand side of $O(\theta^2) = O\left(f(v)\,\frac{\theta^2}{\varepsilon^4\xi^2}\right) = O\left(f(v)\,\frac{\theta^2(1+u^2)^2}{\varepsilon^4}\right) = O(f(v)\theta^6)$, and may therefore also be neglected. We are therefore left with

$$4\left[z + \frac{\phi^2}{2} + \frac{\delta(1+u^2)}{\varepsilon_2}\right]\frac{g(v)}{u^2}\,\frac{\phi^2}{4}$$

from the right hand side of eq (VII.4). We note that in fact only if $\phi = O(\theta)$ need we keep this term. For $\phi = O(\theta^2)$ it too may be neglected.

Thus, for $z = O(\theta^2)$, eq (VII.4) simplifies, after neglect of terms of relative order θ^2, to

$$\left[2f(v)\left(z+\frac{\sigma}{u}\right)+\frac{g'(v)}{u}\frac{\phi^2}{4}\right]\left[\phi^2+\frac{\delta^2(1+u^2)^2}{u^2}\right]=4\left[z+\frac{\phi^2}{2}+\frac{\delta(1+u^2)}{\varepsilon_2}\right]\frac{g(v)\phi^2}{4u} \quad .$$

This equation may be simplified further: Since we have assumed $z = O(\theta^2)$, the functions of v appearing here, $f(v)$, $g'(v)$, and $g(v)$, may each be expanded about the point $v = u$. Corrections to their value at $v = u$, being of relative order $\frac{v-u}{u} \approx z = O(\theta^2)$, can thus be neglected. We then obtain

$$\left[2f(u)\left(z+\frac{\sigma}{u}\right)+\frac{g'(u)}{u}\frac{\phi^2}{4}\right]\left[\phi^2+\frac{\delta^2(1+u^2)^2}{u^2}\right]=4\left[z+\frac{\phi^2}{2}+\frac{\delta(1+u^2)}{\varepsilon_2}\right]\frac{g(u)\phi^2}{4u} \quad . \tag{VII.5}$$

We have here a linear equation for z. Thus for $z = O(\theta^2)$ there is only one solution. We will show later, in the footnote following eq (VII.33), that this corresponds to a minimum of the cross section. For the case $\phi \lesssim O(\theta^2)$, in which case $q_\perp = O(u\theta^2)$, we have $\phi^2 \lesssim O(\theta^4)$ so that all terms with factor ϕ^2 may be neglected. The equation for the minimum of the cross section then becomes extremely simple, viz.,

$$z + \frac{\sigma}{u} = 0 \quad .$$

From eq (VI.48) we then have, for $\phi \lesssim O(\theta^2)$,

$$z_{min} = - \frac{k(\varepsilon_1+\varepsilon_2)(u^2-1)}{2\varepsilon_1{}^2\varepsilon_1{}^2\left[\frac{k^2}{\varepsilon_1\varepsilon_2} + 2\left(\frac{u^2-1}{u^2+1}\right)^2\right]} \quad . \tag{VII.6}$$

We note that z_{min} as given here is indeed $O(\theta^2)$, as initially assumed.

For ϕ arbitrary (but still assuming $\phi \lesssim O(\theta)$), we have, from eq (VII.5),

$$z_{min} = - \frac{\left(\frac{\sigma f(u)}{u} + \frac{g'(u)}{u}\frac{\phi^2}{8}\right)\left(\phi^2 + \frac{\delta^2(1+u^2)^2}{u^2}\right) - \frac{g(u)}{u^2}\frac{\phi^2}{2}\left(\frac{\phi^2}{2} + \frac{\delta(1+u^2)}{\varepsilon_2}\right)}{f(u)\left(\phi^2 + \frac{\delta^2(1+u^2)^2}{u^2}\right) - \frac{g(u)}{u^2}\frac{\phi^2}{2}} \quad . \tag{VII.7}$$

The functions appearing here may all be obtained directly from eqs (VI.44), (VI.45), and (VI.48):

$$\frac{\sigma f(u)}{u} = \frac{4k(\varepsilon_1+\varepsilon_2)(u^2-1)}{(1+u^2)^2}$$

$$= \frac{8\varepsilon_1\varepsilon_2}{(1+u^2)^2}(\varepsilon_1+\varepsilon_2)\,\delta(1+u^2)\left(\frac{u^2-1}{u^2+1}\right)$$

$$\frac{g'(u)}{u} = \frac{1}{u}\,\frac{\partial g(v)}{\partial v}\bigg|_{v=u} = -\frac{32\varepsilon_1\varepsilon_2}{(1+u^2)^2}\,(\varepsilon_1{}^2+\varepsilon_2{}^2)\left(\frac{u^2-1}{u^2+1}\right)$$

$$f(u) = \frac{8\varepsilon_1{}^2\varepsilon_2{}^2}{(1+u^2)^2}\left[\frac{k^2}{\varepsilon_1\varepsilon_2} + 2\left(\frac{u^2-1}{u^2+1}\right)^2\right]$$

$$= \frac{8\varepsilon_1\varepsilon_2}{(1+u^2)^2}\left[\varepsilon_1{}^2+\varepsilon_2{}^2 - 2\varepsilon_1\varepsilon_2\left(1 - \left(\frac{u^2-1}{u^2+1}\right)^2\right)\right]$$

$$= \frac{8\varepsilon_1\varepsilon_2}{(1+u^2)^2}\left[\varepsilon_1{}^2+\varepsilon_2{}^2 - 2\varepsilon_1\varepsilon_2\left(\frac{2u}{1+u^2}\right)^2\right] \qquad \text{(VII.8)}$$

$$\frac{g(u)}{u^2} = \frac{32\varepsilon_1\varepsilon_2}{(1+u^2)^2}\,(\varepsilon_1{}^2+\varepsilon_2{}^2) \quad .$$

Substituting these in eq (VII.7) we have, in the denominator there,

$$f(u)\left(\phi^2 + \frac{\delta^2(1+u^2)^2}{u^2}\right) - \frac{g(u)}{u^2}\,\frac{\phi^2}{2}$$

$$= \frac{8\varepsilon_1\varepsilon_2}{(1+u^2)^2}\left\{\left[\varepsilon_1{}^2+\varepsilon_2{}^2-2\varepsilon_1\varepsilon_2\left(\frac{2u}{1+u^2}\right)^2\right]\left(\phi^2 + \frac{\delta^2(1+u^2)^2}{u^2}\right) - 2(\varepsilon_1{}^2+\varepsilon_2{}^2)\phi^2\right\}$$

$$= \frac{8\varepsilon_1\varepsilon_2}{(1+u^2)^2}\left\{\left[\varepsilon_1{}^2+\varepsilon_2{}^2-2\varepsilon_1\varepsilon_2\left(\frac{2u}{1+u^2}\right)^2\right]\frac{\delta^2(1+u^2)^2}{u^2} - \left[\varepsilon_1{}^2+\varepsilon_2{}^2+2\varepsilon_1\varepsilon_2\left(\frac{2u}{1+u^2}\right)^2\right]\phi^2\right\} \quad .$$

We thus have

$$z_{\min} = -\frac{\left(\phi^2 + \frac{\delta^2(1+u^2)^2}{u^2}\right)\left[(\varepsilon_1+\varepsilon_2)\delta(1+u^2)-(\varepsilon_1{}^2+\varepsilon_2{}^2)\frac{\phi^2}{2}\right]\left(\frac{u^2-1}{u^2+1}\right)-2(\varepsilon_1{}^2+\varepsilon_2{}^2)\phi^2\left(\frac{\phi^2}{2}+\frac{\delta(1+u^2)}{\varepsilon_2}\right)}{\left[\varepsilon_1{}^2+\varepsilon_2{}^2-2\varepsilon_1\varepsilon_2\left(\frac{2u}{1+u^2}\right)^2\right]\frac{\delta^2(1+u^2)^2}{u^2} - \left[\varepsilon_1{}^2+\varepsilon_2{}^2+2\varepsilon_1\varepsilon_2\left(\frac{2u}{1+u^2}\right)^2\right]\phi^2} \qquad \text{(VII.9)}$$

We note that for $\phi \lesssim O(\theta)$ the numerator here is $O(\varepsilon^2\theta^4)$ and the denominator is, in general, $O(\varepsilon^2\theta^2)$. Thus indeed $z_{\min}$ is $O(\theta^2)$ as originally assumed. However, the denominator in eq (VII.9) is zero for

$$\phi = \phi_o \equiv \frac{\delta(1+u^2)}{u}\left[\frac{\varepsilon_1{}^2+\varepsilon_2{}^2-2\varepsilon_1\varepsilon_2\left(\frac{2u}{1+u^2}\right)^2}{\varepsilon_1{}^2+\varepsilon_2{}^2+2\varepsilon_1\varepsilon_2\left(\frac{2u}{1+u^2}\right)^2}\right]^{1/2} \qquad \text{(VII.10)}$$

so that eq (VII.9) is clearly invalid for $\phi \geqq \phi_o$. (Note, however, that $\phi_o = O(\theta)$.) In fact eq (VII.9) becomes invalid for ϕ somewhat less than ϕ_o. Once ϕ is sufficiently close to ϕ_o that $\phi_o{}^2 - \phi^2$ is less than $O(\theta^2)$ then $z_{\min}$ is larger than $O(\theta^2)$ and the assumptions made in deriving eq (VII.9) no longer hold. As we show shortly, as ϕ approaches ϕ_o (from below), the position of the minimum, $z_{\min}$, increases from $O(\theta^2)$ to $O(\theta^{4/3})$. When ϕ reaches the critical value ϕ_c ($\phi_c < \phi_o$) the minimum fuses with the second maximum of the cross section, becoming an inflection point. For $\phi > \phi_c$ the cross section no longer has a minimum. For $\theta_1 = 1°$, $\varepsilon_1 = 140$ MeV, $k = 95$ MeV, we find, from a careful computer analysis of the cross section as a function of v and ϕ (see figs. 11 and 12),

$$\phi_c = 0.836492\ldots°$$

From eq (VII.10) we have, for these same values of θ_1, ε_1 and k,

$$\phi_o = 1.002958\ldots°$$

We next look for solutions of eq (VII.4) under the assumption that $z = O(\theta)$. In contrast to the previous assumption that $z = O(\theta^2)$, in which case the small terms in eq (VII.4) were of relative order θ^2 and hence could be neglected, now the various terms in eq (VII.4) are of relative order θ and hence must be kept. However, the nature of the solution for $z = O(\theta)$ may be seen most easily if we first examine eq (VII.4) and neglect the terms of $O(\theta)$ as well. Then in the first square bracket on the left-hand side of eq (VII.4) we neglect the term $2z\left(\frac{\phi^2}{2} + \frac{\delta(1+u^2)}{\varepsilon_2}\right)$, since it is $O(\theta^3)$ while the other terms in this bracket are $O(\theta^2)$. In the second square bracket on the left-hand side we then keep only the single term $2f(v)z$, all other terms in this bracket being of relative order θ. In similar fashion, on the right-hand side of eq (VII.4), in the first square bracket we keep only the first term, z. And in the second square bracket on the right-hand side we keep only the terms $f(v)z^2 + \frac{g(v)}{u^2}\frac{\phi^2}{4}$. We then have, as a zeroth-order approximation to eq (VII.4) for $z = O(\theta)$,

$$\left[z^2 + \phi^2 + \frac{\delta^2(1+u^2)^2}{u^2}\right] \cdot 2f(v)z = 4z\left[f(v)z^2 + \frac{g(v)}{u^2}\frac{\phi^2}{4}\right]$$

or

$$z^2 + \phi^2 + \frac{\delta^2(1+u^2)}{u^2} = 2z^2 + \frac{g(v)}{u^2 f(v)} \frac{\phi^2}{2} \quad .$$

Again neglecting terms of relative order θ for this zeroth-order equation, we may set $v = u$ in the last term, thus giving

$$z^2 = \frac{\delta^2(1+u^2)}{u^2} + \phi^2\left(1 - \frac{g(u)}{2u^2 f(u)}\right) \quad , \tag{VII.11}$$

where from eq (VII.8) we have

$$1 - \frac{g(u)}{2u^2 f(u)} = 1 - \frac{2(\varepsilon_1{}^2+\varepsilon_2{}^2)}{k^2+2\varepsilon_1\varepsilon_2\left(\frac{u^2-1}{u^2+1}\right)^2}$$

$$= - \frac{\left[\varepsilon_1{}^2+\varepsilon_2{}^2+2\varepsilon_1\varepsilon_2\left(\frac{2u}{1+u^2}\right)^2\right]}{\left[\varepsilon_1{}^2+\varepsilon_2{}^2-2\varepsilon_1\varepsilon_2\left(\frac{2u}{1+u^2}\right)^2\right]} \quad . \tag{VII.11a}$$

From eqs (VII.10) and (VII.11a) we may write the solution to the zeroth-order equation, (VII.11), as

$$z^2 = \left[\frac{\varepsilon_1{}^2+\varepsilon_2{}^2+2\varepsilon_1\varepsilon_2\left(\frac{2u}{1+u^2}\right)^2}{\varepsilon_1{}^2+\varepsilon_2{}^2-2\varepsilon_1\varepsilon_2\left(\frac{2u}{1+u^2}\right)^2}\right]\left[\phi_o{}^2 - \phi^2\right] \tag{VII.12}$$

or

$$z^2 = \frac{\delta^2(1+u^2)}{u^2}\left[1 - \left(\frac{\phi}{\phi_o}\right)^2\right] \quad . \tag{VII.13}$$

As for the case $z = 0(\theta^2)$, we note that here too the solution is valid only for $\phi < \phi_o$. And since we assume now that $z = 0(\theta)$, we must again have

$$\phi_o{}^2 - \phi^2 = 0(\theta^2)$$

or

$$\phi_o - \phi = 0(\theta) \quad .$$

Again we see that the solution breaks down as ϕ approaches ϕ_o. In contrast to the case $z = 0(\theta^2)$, however, we see from eqs (VII.12) or (VII.13) that there are now two solutions; in zeroth order they are

$$z_{max}^{(o)} = \pm \frac{\delta(1+u^2)}{u}\left[1 - \left(\frac{\phi}{\phi_o}\right)\right]^{1/2} \tag{VII.14}$$

or

$$(v-u)_{max}^{(o)} = \pm\, \delta(1+u^2)\left[1 - \left(\frac{\phi}{\phi_o}\right)^2\right]^{1/2} \quad . \tag{VII.15}$$

These correspond to the two maxima of the cross section.

We have now found, for sufficiently small ϕ, the minimum of the cross section (given by eq (VII.7)) and the two maxima (given to zeroth order by eq (VII.14)). We could, at this point, continue with the determination of the maxima of the cross section, obtaining the first order corrections, of order θ relative to the values given in eq (VII.14). However, having noted that the solutions of eq (VII.4) determined thus far under the assumption that <u>either</u> $z = 0(\theta^2)$ <u>or</u> $z = 0(\theta)$, both break down for ϕ sufficiently close to ϕ_0, we now return to eq (VII.4) and solve it with no assumptions concerning either z or ϕ other than $z \lesssim 0(\theta)$ and $\phi \lesssim 0(\theta)$, i.e., assuming only that neither z nor ϕ exceed $0(\theta)$. To that end, let us first see what terms in eq (VII.4) are of relative order θ^2 for all z and hence may be neglected. In the second bracket on the left-hand side of this equation, the first term, $uf'(v)\left(z + \frac{\sigma}{u}\right)^2$, is of $0\left(z + \frac{\sigma}{u}\right) \lesssim 0(\theta)$ relative to the second term, $2f(v)\left(z + \frac{\sigma}{u}\right)$. In the first term we may therefore neglect the terms $uf'(v) \cdot 2z\frac{\sigma}{u}$ and $uf'(v)\left(\frac{\sigma}{u}\right)^2$ since they are of $0\left(\frac{\sigma}{u}\right) = 0(\theta^2)$ relative to the terms $2f(v)z$ and $2f(v)\frac{\sigma}{u}$, respectively, in the second term. For the second square bracket on the left-hand side of eq (VII.4) we now have

$$\left[uf'(v)z^2 + 2f(v)\left(z + \frac{\sigma}{u}\right) + \frac{g'(v)}{u}\frac{\phi^2}{4}\right] . \qquad \text{(VII.16)}$$

Consider next the second square bracket on the right-hand side of eq (VII.4). In that bracket we have the term $f(v)\left(\frac{\sigma}{u}\right)^2$, which is then multiplied with the terms in the first square bracket on the right-hand side of eq (VII.4), giving terms of $O\left(zf(v)\left(\frac{\sigma}{u}\right)^2\right)$, $O\left(\phi^2 f(v)\left(\frac{\sigma}{u}\right)^2\right)$, and $O\left(\theta^2 f(v)\left(\frac{\sigma}{u}\right)^2\right)$. These can all be neglected, since they are of $O(\theta^2)$ relative to terms of $O(zf(v)\theta^2)$, $O\left(\phi^2 f(v)\left(\frac{\sigma}{u}\right)\right)$, and $O\left(\theta^2 f(v)\left(\frac{\sigma}{u}\right)\right)$. We may thus neglect the term $f(v)\left(\frac{\sigma}{u}\right)^2$. Similarly, in this same bracket, the term $\frac{\rho}{u^2} = O(1)$ may be neglected, since it is of the same order as $f(v)\left(\frac{\sigma}{u}\right)^2 = O(\varepsilon^4\xi^2\theta^4) = O\left(\frac{u^4}{(1+u^2)^2}\right) = O(1)$. Thus for the second square bracket on the right-hand side of eq (VII.4) we now have

$$\left[f(v)z^2 + 2f(v)z\,\frac{\sigma}{u} + \frac{g(v)}{u^2}\,\frac{\phi^2}{4}\right] \quad . \tag{VII.17}$$

Substituting eqs (VII.16) and (VII.17) in eq (VII.4) and dividing both sides of the resulting equation by $2f(v)$ we now have

$$\left[z^2 + 2z\left(\frac{\phi^2}{2} + \frac{\delta(1+u^2)}{\varepsilon_2}\right) + \phi^2 + \frac{\delta^2(1+u^2)^2}{u^2}\right]\left[\frac{uf'(v)}{2f(v)}\,z^2 + z + \frac{\sigma}{u} + \frac{g'(v)}{8uf(v)}\,\phi^2\right]$$

$$= 2\left[z + \frac{\phi^2}{2} + \frac{\delta(1+u^2)}{\varepsilon_2}\right]\left[z^2 + 2z\frac{\sigma}{u} + \frac{g(v)}{4u^2 f(v)}\,\phi^2\right] \quad . \tag{VII.18}$$

Referring now to the second bracket on the left-hand side of eq (VII.18) we note again that the first term is $O(z^2)$, i.e., that it is smaller than the second term by $O(z) \lesssim O(\theta)$. Therefore in the first term we may expand $f'(v)/f(v)$ about the point $v = u$, and neglect terms in this expansion of relative $O((v-u)/u) = O(z) \lesssim O(\theta)$, since these give a contribution of $O(\theta^2)$ relative to the second term in the bracket. For the first term we therefore now write $\frac{uf'(u)}{2f(u)} z^2$. In similar fashion, in this same bracket, we may expand the last term, $\frac{g'(v)}{8uf(v)} \phi^2$, about the point $v = u$. In this expansion, the terms of relative $O((v-u)/u) = O(z)$ are of $O(z\phi^2) \lesssim O(z\theta^2)$. They are thus of order θ^2 relative to the second term, z, in the bracket, and may thus be neglected. For the last term we therefore now write $\frac{g'(u)}{8uf(u)} \phi^2$. Next consider the last term in the second bracket on the right-hand side of eq (VII.18), $\frac{g(v)}{4u^2 f(v)} \phi^2 = O(\phi^2) \lesssim O(\theta^2)$. If we expand the factor $\frac{g(v)}{f(v)}$ in this term about the point $v = u$, then the terms in this expansion, having successively higher powers of $(v - u)$, are of order ϕ^2, $z\phi^2$, $z^2\phi^2$, ... The term of order $z^2\phi^2 \lesssim O(z^2\theta^2)$ may thus be neglected, being of $O(\theta^2)$ (or less) relative to the first term in this bracket, z^2. However, the term of order $z\phi^2 \lesssim O(z\theta^2)$ must be kept, being of the same order as the term $2z\frac{\sigma}{u}$, also in this bracket, if $\phi = O(\theta)$. We thus expand

$$\frac{g(v)}{f(v)} = \frac{g(u)}{f(u)} + (v-u)\,\frac{[f(u)g'(u) - g(u)f'(u)]}{f^2(u)} + \ldots$$

$$= \frac{g(u)}{f(u)} + zu\,\frac{g'(u)}{f(u)} - zu\,\frac{g(u)f'(u)}{f^2(u)} + \ldots$$

and keep only the terms shown here explicitely. Thus for the last term we write

$$\frac{g(u)}{4u^2f(u)}\,\phi^2 + \frac{zg'(u)}{4uf(u)}\,\phi^2 - \frac{zg(u)f'(u)}{4uf^2(u)}\,\phi^2 \quad .$$

We may thus write, neglecting terms of relative order θ^2 in eq (VII.18),

$$\left[z^2 + 2z\left(\frac{\phi^2}{2} + \frac{\delta(1+u^2)}{\varepsilon_2}\right) + \phi^2 + \frac{\delta^2(1+u^2)^2}{u^2}\right]\left[\frac{uf'(u)}{2f(u)}\,z^2 + z + \frac{\sigma}{u} + \frac{g'(u)}{8uf(u)}\,\phi^2\right]$$

$$= 2\left[z + \frac{\phi^2}{2} + \frac{\delta(1+u^2)}{\varepsilon_2}\right]\left[z^2 + 2z\,\frac{\sigma}{u} + \frac{g(u)}{4u^2f(u)}\,\phi^2 + z\,\frac{g'(u)}{4uf(u)}\,\phi^2 - z\,\frac{g(u)f'(u)}{4uf^2(u)}\,\phi^2\right] \quad . \tag{VII.19}$$

We have here a fourth-order algebraic equation for z. We multiply out the terms appearing here, noting that the cross term coming from the second term in the first bracket and the first term in the second bracket, viz.,

$$2z\left(\frac{\phi^2}{2} + \frac{\delta(1+u^2)}{\varepsilon_2}\right)\cdot\frac{uf'(u)}{2f(u)}\,z^2 \quad ,$$

may be neglected since it is of $O(\theta^2)$ relative to the cross term z^3. Similarly, cross terms of $O(z\phi^4)$ or $O(z\phi^2\theta^2)$ may be neglected since they are of $O(\theta^2)$ relative to the cross term $z\phi^2$. We then have

$$
- \frac{uf'(u)}{2f(u)} z^4 + z^3
$$

$$
+ \left[\frac{3\sigma}{u} + \frac{3g'(u)}{8uf(u)} \phi^2 - \frac{g(u)f'(u)}{2uf^2(u)} \phi^2 - \frac{uf'(u)}{2f(u)} \left(\phi^2 + \frac{\delta^2(1+u^2)^2}{u^2} \right) \right] z^2
$$

$$
- \left[\frac{\delta^2(1+u^2)^2}{u^2} + \left(1 - \frac{g(u)}{2u^2 f(u)} \right) \phi^2 \right] z
$$

$$
+ \left(\frac{\phi^2}{2} + \frac{\delta(1+u^2)}{\varepsilon_2} \right) \frac{g(u)}{2u^2 f(u)} \phi^2 - \left(\phi^2 + \frac{\delta^2(1+u^2)^2}{u^2} \right) \left(\frac{\sigma}{u} + \frac{g'(u)}{8uf(u)} \phi^2 \right)
$$

$$
= 0 \quad . \qquad \text{(VII.20)}
$$

We note here that the terms with z^4 and z^3 have coefficients of $O(1)$, the terms with z^2 and z have coefficients of $O(\theta^2)$, and the term without z is of $O(\theta^4)$. We note further that the coefficient of the term with z is given by eq (VII.10) and (VII.11a). We may therefore write eq (VII.20) in the form

$$
a_o z^4 + z^3 + \lambda^{(2)} z^2 - b_o(\phi_o^{\,2} - \phi^2) z + \mu^{(4)} = 0 \qquad \text{(VII.21)}
$$

where a_o and b_o are of order unity, $\lambda^{(2)}$ is $O(\theta^2)$, and $\mu^{(4)}$ is $O(\theta^4)$. Specifically, from eq (VII.20)

$$
a_o = - \frac{uf'(u)}{2f(u)} \qquad \text{(VII.22)}
$$

and from eq (VII.11a)

$$b_o = \frac{\left[\varepsilon_1{}^2+\varepsilon_2{}^2+2\varepsilon_1\varepsilon_2\left(\frac{2u}{1+u^2}\right)^2\right]}{\left[\varepsilon_1{}^2+\varepsilon_2{}^2-2\varepsilon_1\varepsilon_2\left(\frac{2u}{1+u^2}\right)^2\right]} \geqq 1 \quad . \tag{VII.23}$$

In eq (VII.21), $\lambda^{(2)}$ is the coefficient of z^2 in eq (VII.20) and $\mu^{(4)}$ is the term without z in eq (VII.20).

Let us first recapitulate our previous findings, and assume therefore, for the moment, that $\phi_o{}^2 - \phi^2 = O(\theta^2)$. Then if we look for a solution with $z = O(\theta^2)$ we note that the first three terms in eq (VII.21) are, respectively, of $O(\theta^8)$, $O(\theta^6)$ and $O(\theta^6)$, and hence may be neglected relative to the last two terms, which are each of $O(\theta^4)$. The solution is then

$$z = \frac{\mu^{(4)}}{b_o(\phi_o{}^2 - \phi^2)} \quad ,$$

which we gave earlier in eqs (VII.7) and (VII.9). On the other hand, if we look for a solution with $z = O(\theta)$ then we note that the successive terms in eq (VII.21) are $O(\theta^4)$, $O(\theta^3)$, $O(\theta^4)$, $O(\theta^3)$, and $O(\theta^4)$. Thus if we desire the zeroth-order solution (i.e., neglect terms of relative order θ), we need keep only the second and fourth terms in eq (VII.21), and find

$$z^3 - b_o(\phi_o{}^2 - \phi^2)z = 0$$

or

$$z^{(o)} = \pm\left[b_o(\phi_o{}^2 - \phi^2)\right]^{\frac{1}{2}} \quad , \tag{VII.24}$$

which we gave earlier in eq (VII.12).

We note first that both of these solutions are included in the third-order equation obtained from eq (VII.21) by neglect of the z^4 and z^2 terms there:

$$z^3 - b_o(\phi_o{}^2 - \phi^2)z + \mu^{(4)} = 0 \quad . \tag{VII.25}$$

Next we observe that in fact the z^4 and z^2 terms are smaller than the remaining terms for all $z \lesssim O(\theta)$, whether or not $\phi_o{}^2 - \phi^2 = O(\theta^2)$. That this is true for the z^4 term is clear, since it is smaller than z^3 by a factor $z \lesssim O(\theta)$. With regard to the z^2 term we note that it is of $O(z^2\theta^2)$. Thus for $z \lesssim O(\theta^{4/3})$, it is of $O(\theta^{8/3} \cdot \theta^2) = O(\theta^4 \cdot \theta^{2/3})$. It is therefore smaller by a factor of $O(\theta^{2/3})$ than the term $\mu^{(4)}$, which is of $O(\theta^4)$. On the other hand, for $z \gtrsim O(\theta^{4/3})$, it is, relative to the z^3 term, of $O(z^2\theta^2/z^3) = O(\theta^2/z) \lesssim O(\theta^{2/3})$. Thus for all $z \lesssim O(\theta)$, the term in eq (VII.21) with factor z^2 is smaller by $O(\theta^{2/3})$ than either the term $\mu^{(4)}$ or the term z^3. Therefore this term with factor z^2, as well as the term with z^4, may be considered a perturbation, independently of the order of magnitude of $\phi_o{}^2 - \phi^2$. We therefore assume a solution of the form

$$z = z_o + z_1 \tag{VII.26}$$

and choose z_o so that it satisfies the third-order equation (VII.25):

$$z_0^3 - b_0(\phi_0^2 - \phi^2)z_0 + \mu^{(4)} = 0 \quad . \tag{VII.27}$$

Substituting eq (VII.26) in the full equation, (VII.21), we have

$$a_0(z_0+z_1)^4 + (z_0+z_1)^3 + \lambda^{(2)}(z_0+z_1)^2 - b_0(\phi_0^2-\phi^2)(z_0+z_1) + \mu^{(4)} = 0 \quad ,$$

which, in view of eq (VII.27), we may write in the form

$$[3z_0^2-b_0(\phi_0^2-\phi^2)]z_1 + 3z_0z_1^2\left(1 + \frac{z_1}{3z_0}\right) = -a_0z_0^4\left(1 + \frac{z_1}{z_0}\right)^4 - \lambda^{(2)}z_0^2\left(1 + \frac{z_1}{z_0}\right)^2 \quad . \tag{VII.28}$$

Since we assume z_1 small relative to z_0, we will assume that in this equation for z_1 the terms which are manifestly of relative $O(z_1/z_0)$ can be dropped. By solving the equation with these terms neglected, we obtain the first correction, z_1, to the main term, z_0 given by eq (VII.27):

$$\left[3z_0^2 - b_0(\phi_0^2 - \phi^2)\right]z_1 + 3z_0z_1^2 = - a_0z_0^4 - \lambda^{(2)}z_0^2 \quad , \tag{VII.29}$$

but we will have neglected corrections to z_1 of $O(z_1/z_0)^2$ relative to the main term.

We begin our consideration of eq (VII.29) by examining again the cases already considered, assuming for the moment that $\phi_0^2 - \phi^2 = O(\theta^2)$. Thus if $z_0 = O(\theta^2)$ then the largest term on the left-hand side of eq (VII.29) is $- b_0(\phi_0^2 - \phi^2)z_1 = O(\theta^2 z_1)$, the other terms there being smaller: $3z_0^2z_1 = O(\theta^4 z_1)$ and $3z_0z_1^2 = O(\theta^2 z_1^2)$. On the right-hand side of eq (VII.29) the largest term is

$$- \lambda^{(2)} z_o^{\ 2} = O(\theta^6) \quad ,$$

since the other term is $- a_o z_o^{\ 4} = O(\theta^8)$. Thus for z_1 we have

$$- b_o(\phi_o^{\ 2} - \phi^2) z_1 = - \lambda^{(2)} z_o^{\ 2}$$

so that

$$z_1 = \frac{\lambda^{(2)} z_o^{\ 2}}{b_o(\phi_o^{\ 2} - \phi^2)} = O(\theta^4) \quad .$$

Thus for $z_o = O(\theta^2)$, the first correction, $z_1 = O(\theta^4)$, is of relative order θ^2: $z_1/z_o = O(\theta^2)$. In this case the correction may therefore be neglected, as we noted earlier in deriving eq (VII.7).

Next, if $z_o = O(\theta)$ then on the left-hand side of eq (VII.29) we must retain

$$\left[3z_o^{\ 2} - b_o(\phi_o^{\ 2} - \phi^2)\right] z_1 = O(\theta^2 z_1) \quad ,$$

the other term there being $3z_o z_1^{\ 2}$ which is of $O(z_1/z_o)$ relative to the term $3z_o^{\ 2} z_1$. On the right-hand side of eq (VII.29) we now keep all the terms,

$$-a_o z_o^{\ 4} - \lambda^{(2)} z_o^{\ 2} = O(\theta^4) \quad .$$

We now have the equation for z_1,

$$\left[3z_o^2 - b_o(\phi_o^2 - \phi^2)\right] z_1 = - a_o z_o^4 - \lambda^{(2)} z_o^2 \quad . \tag{VII.30}$$

From eqs (VII.24) or (VII.14) we may substitute the zeroth-order solution, z_o, given there as $z^{(o)}$. From eq (VII.24) we have

$$z_o^2 = b_o(\phi_o^2 - \phi^2) \quad ,$$

which, substituted in eq (VII.30), gives

$$z_1 = - \frac{1}{2} (a_o z_o^2 + \lambda^{(2)}) \quad . \tag{VII.31}$$

Thus for $z_o = O(\theta)$ the first correction, z_1, is of $O(\theta^2)$, i.e., it is of relative order θ and is therefore retained: $z_1/z_o = O(\theta)$. Here a_o is given by eq (VII.22) and $\lambda^{(2)}$ is the coefficient of z^2 in eq (VII.20). With z_o as given in eq (VII.14), we then have the full solution, $z = z_o + z_1$, for the maxima of the cross section (i.e., with neglect only of terms of relative order θ^2) for $\phi < \phi_c < \phi_o$ and $\phi_o^2 - \phi^2 = O(\theta^2)$:

$$z_{max} = \pm \frac{\delta(1+u^2)}{u}\left[1 - \left(\frac{\phi}{\phi_o}\right)^2\right]^{\frac{1}{2}} + \frac{uf'(u)}{4f(u)}\,\frac{\delta^2(1+u^2)^2}{u^2}\left[1 - \left(\frac{\phi}{\phi_o}\right)^2\right] - \tfrac{1}{2}\lambda^{(2)} \quad . \tag{VII.32}$$

Having considered the extrema of the cross section for $\phi < \phi_c < \phi_o$ (and $\phi_o{}^2 - \phi^2 = O(\theta^2)$), we now consider the case in which ϕ is sufficiently close to ϕ_o that $\phi_o{}^2 - \phi^2 < O(\theta^2)$. This is the condition for which the minimum of the cross section fuses with the second maximum and becomes an inflection point. To determine the inflection point we turn to eqs (VII.2) and (VII.2a), and require not only that eq (VII.2a) be satisfied, but that the second derivative of the cross section be zero as well. From eq (VII.2) we have

$$\frac{\partial^2}{\partial v^2}\left[(q^2)^{-2}\{\ \}\right] = \left[(q^2)^{-3}\right]'\left[q^2\{\ \}' - 2(q^2)'\{\ \}\right]$$

$$+ (q^2)^{-3}\left[q^2\{\ \}' - 2(q^2)'\{\ \}\right]'$$

$$= 0$$

From eq (VII.2a) the first term on the right-hand side above is zero. Thus we have as our added condition

$$\left[q^2\{\ \}' - 2(q^2)'\{\ \}\right]' = 0 \ . \qquad \text{(VII.33)}$$

If we now follow the analysis from eqs (VII.2a) through (VII.4), (VII.18), (VII.19), (VII.20), and arriving finally at eq (VII.21), we see that, apart from a factor independent of v, the polynomial on the left-hand side of eq (VII.21) is the negative of the expression in square brackets in eq (VII.33).[2] Thus the inflection point occurs when both eq (VII.21) and its first derivative with respect to z are zero (recall, from eq (VII.3), that $z = (v - u)/u$):

$$a_o z^4 + \lambda^{(2)} z^3 - b_o(\phi_o{}^2 - \phi^2)z + \mu^{(4)} = 0$$

and

$$4a_o z^3 + 3z^2 + 2\lambda^{(2)} z - b_o(\phi_o{}^2 - \phi^2) = 0 \quad . \tag{VII.34}$$

Earlier we showed that the z^4 and z^2 terms in the first equation here were smaller than the z^2 or $\mu^{(4)}$ terms. Similarly, in the second equation here, the z^3 and z are smaller than the z^2 term for $z > O(\theta^2)$. We then have, in first approximation, at the inflection point,

$$z^3 - b_o(\phi_o{}^2 - \phi^2)z + \mu^{(4)} = 0 \tag{VII.35a}$$

and

$$3z^2 - b_o(\phi_o{}^2 - \phi^2) = 0 \quad , \tag{VII.35b}$$

from which

$$2z^3 = \mu^{(4)}$$

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[2] It follows from this observation that, for the extremum which occurs for $z = O(\theta^2)$ when $\phi < \phi_c < \phi_o$, the second derivative of the cross section at this extremum is given simply from eq (VII.21) by $b_o(\phi_o{}^2 - \phi^2) > 0$ (apart from a factor which is positive), since the terms in eq (VII.21) with z^4, z^3 and z^2 are all negligible, of relative order θ^2. Thus the point given earlier in eqs (VII.6) and (VII.7) is indeed a minimum.

or

$$z = \left(\frac{\mu^{(4)}}{2}\right)^{1/3} = O(\theta^{4/3}) \quad . \tag{VII.36}$$

Then from eq (VII.35) it follows that

$$\phi_o^{\ 2} - \phi^2 = O(\theta^{8/3}) \tag{VII.37}$$

at the inflection point, since $b_o = O(1)$ (eq (VII.23)). Although in principle we have in eqs (VII.35a) and (VII.35b) two equations for the two unknowns, z and ϕ, the critical value of ϕ for which the inflection point occurs is most simply obtained directly from the first of these two equations. It has [17] three real and unequal roots (the minimum and two maxima already determined) when

$$\left(\frac{\mu^{(4)}}{2}\right)^2 < \left[\frac{b_o(\phi_o^{\ 2} - \phi^2)}{3}\right]^3 \quad . \tag{VII.38a}$$

It has three real roots, of which two (viz., the inflection point and the second maximum) are equal when

$$\left(\frac{\mu^{(4)}}{2}\right)^2 = \left[\frac{b_o(\phi_o^{\ 2} - \phi^2)}{3}\right]^3 \quad . \tag{VII.38b}$$

And it has only one real root (a maximum) when

$$\left(\frac{\mu^{(4)}}{2}\right)^2 > \left[\frac{b_o(\phi_o^{\ 2} - \phi^2)}{3}\right]^3 \quad . \tag{VII.38c}$$

We note from eq (VII.37) that the right-hand side of eq (VII.38b) is $O(\theta^8)$. This indeed checks, since $\mu^{(4)}$, on the left-hand side of this equation, is $O(\theta^4)$. In solving eq (VII.38b) we recall that $\mu^{(4)}$ contains all the terms in eq (VII.20) without a z-dependent factor:

$$\mu^{(4)} = \left(\frac{\phi^2}{2} + \frac{\delta(1+u^2)}{\varepsilon_2}\right)\frac{g(u)\phi^2}{2u^2 f(u)} - \left(\phi^2 + \frac{\delta^2(1+u^2)^2}{u^2}\right)\left(\frac{\sigma}{u} + \frac{g'(u)\phi^2}{8uf(u)}\right) \quad . \tag{VII.39}$$

Now in substituting this in eq (VII.38b) we note from eq (VII.37) that $\phi_o^2 - \phi^2 = O(\theta^{8/3}) < O(\theta^2)$. We will therefore set $\phi^2 = \phi_o^2$ in eq (VII.39), thus neglecting terms of relative order $(\phi_o^2 - \phi^2)/\phi_o^2 = O(\theta^{2/3})$. With $\mu_o^{(4)} \equiv \mu^{(4)}(\phi^2 = \phi_o^2)$, the critical value of ϕ at which the minimum disappears is then given from eq (VII.38b) by

$$\left[\frac{b_o(\phi_o^2 - \phi_c^2)}{3}\right]^3 = \left(\frac{\mu_o^{(4)}}{2}\right)^2 \tag{VII.40}$$

or

$$\phi_c^2 = \phi_o^2 - \frac{3}{b_o}\left(\frac{\mu_o^{(4)}}{2}\right)^{2/3} \quad .$$

Now from eqs (VII.10) and (VII.23) we have $\dfrac{1}{b_o} = \dfrac{\mu^2 \phi_o{}^2}{\delta^2(1+u^2)^2}$, so that

$$\phi_c = \phi_o \left\{ 1 - \frac{3u^2}{\delta^2(1+u^2)^2} \left(\frac{\bar{\mu}_o{}^{(4)}}{2} \right)^{2/3} \right\}^{1/2} \quad . \qquad \text{(VII.41)}$$

We thus indeed find $\phi_c < \phi_o$, as noted earlier from our computer determination of ϕ_c for the specific values $\theta_1 = 1°$, $\varepsilon_1 = 140$ MeV, and $k = 95$ MeV.

The value of z at the inflection point is given at once by eqs (VII.35b) and (VII.40), from which

$$z^2 = z_c{}^2 = \frac{b_o(\phi_o{}^2 - \phi_c{}^2)}{3} = \left(\frac{\mu_o{}^{(4)}}{2} \right)^{2/3}$$

or

$$z = z_c = \left(\frac{\mu_o{}^{(4)}}{2} \right)^{1/3} \quad . \qquad \text{(VII.42)}$$

It follows at once, from eqs (VII.42) and (VII.40), that this value of z_c also satisfies eq (VII.35a), if we replace $\mu^{(4)}$ there by $\mu_o{}^{(4)}$, as it should. The expression $\mu_o{}^{(4)}$, which appears in eqs (VII.41) and (VII.42) may be simplified somewhat. From eqs (VII.10) and (VII.11a) we have

$$\phi_o^{\,2} + \frac{\delta^2(1+u^2)^2}{u^2} = \frac{g(u)\phi_o^{\,2}}{2u^2 f(u)}$$

and from eqs (VII.10) and (VII.23),

$$b_o\phi_o^{\,2} = \frac{\delta^2(1+u^2)^2}{u^2} \quad , \tag{VII.43}$$

from which

$$\phi_o^{\,2} + \frac{\delta^2(1+u^2)^2}{u^2} = \frac{g(u)\phi_o^{\,2}}{2u^2 f(u)} = \left(\frac{b_o+1}{b_o}\right)\frac{\delta^2(1+u^2)^2}{u^2} \quad . \tag{VII.44}$$

Substituting this in eq (VII.39) with $\phi = \phi_o$ we have

$$\mu_o^{(4)} = \left(\frac{b_o+1}{b_o}\right)\frac{\delta^2(1+u^2)^2}{u^2}\left[\frac{\phi_o^{\,2}}{2} + \frac{\delta(1+u^2)}{\varepsilon_2} - \frac{\sigma}{u} - \frac{g'(u)}{8uf(u)}\,\phi_o^{\,2}\right] \quad . \tag{VII.45}$$

Here, from eq (VII.8),

$$\frac{g'(u)}{u} = -\left(\frac{u^2-1}{u^2+1}\right)\frac{g(u)}{u^2} \quad ,$$

from which, with eq (VII.44),

$$\frac{g'(u)\phi_o^{\,2}}{8uf(u)} = -\frac{1}{4}\left(\frac{u^2-1}{u^2+1}\right)\frac{g(u)\phi_o^{\,2}}{2u^2f(u)}$$

$$= -\frac{1}{4}\left(\frac{u^2-1}{u^2+1}\right)\left(\frac{b_o+1}{b_o}\right)\frac{\delta^2(1+u^2)^2}{u^2} \quad . \tag{VII.46}$$

And from eq (VI.48)

$$\frac{\sigma}{u} = \frac{k(\varepsilon_1+\varepsilon_2)(u^2-1)}{2\varepsilon_1\varepsilon_2\left[k^2+2\varepsilon_1\varepsilon_2\left(\frac{u^2-1}{u^2+1}\right)^2\right]}$$

$$= \frac{\delta(\varepsilon_1+\varepsilon_2)(u^2-1)}{\varepsilon_1^{\,2}+\varepsilon_2^{\,2}-2\varepsilon_1\varepsilon_2\left(\frac{2u}{1+u^2}\right)^2} \quad . \tag{VII.47}$$

Substituting eqs (VII.43), (VII.46), and (VII.47) in eq (VII.45) we have

$$\mu_o^{(4)} = \left(\frac{b_o+1}{b_o}\right)\frac{\delta^3(1+u^2)^3}{u^3}\left[\frac{\delta(1+u^2)}{2b_ou} + \frac{u}{\varepsilon_2} - \frac{(\varepsilon_1+\varepsilon_2)u}{\left[\varepsilon_1^{\,2}+\varepsilon_2^{\,2}-2\varepsilon_1\varepsilon_2\left(\frac{2u}{1+u^2}\right)^2\right]}\left(\frac{u^2-1}{u^2+1}\right) + \frac{1}{4}\left(\frac{b_o+1}{b_o}\right)\frac{\delta(u^2-1)}{u}\right] \quad . \tag{VII.48}$$

Defining

$$\varepsilon_\pm \equiv \varepsilon_1{}^2+\varepsilon_2{}^2\pm 2\varepsilon_1\varepsilon_2\left(\frac{2u}{1+u^2}\right)^2 \tag{VII.49}$$

we have, from eq (VII.23),

$$b_o = \frac{\varepsilon_+}{\varepsilon_-} \tag{VII.50}$$

and

$$\frac{b_o+1}{b_o} = \frac{2(\varepsilon_1{}^2+\varepsilon_2{}^2)}{\varepsilon_+}$$

From eqs (VII.48) and (VII.50) we then have

$$\mu_o{}^{(4)} = \frac{2(\varepsilon_1{}^2+\varepsilon_2{}^2)}{\varepsilon_+}\,\frac{\delta^3(1+u^2)^3}{u^2}\left[\frac{\delta(1+u^2)\varepsilon_-}{2u^2\varepsilon_+}+\frac{1}{\varepsilon_2}-\frac{(\varepsilon_1+\varepsilon_2)}{\varepsilon_-}\left(\frac{u^2-1}{u^2+1}\right)\right.$$
$$\left.+\frac{(\varepsilon_1{}^2+\varepsilon_2{}^2)\delta(u^2-1)}{2u^2\varepsilon_+}\right] .$$

Adding the first and last terms in the square brackets here gives

$$\frac{\delta}{2u^2\varepsilon_+}\left(\varepsilon_-(1+u^2) + (\varepsilon_1{}^2+\varepsilon_2{}^2)(u^2-1)\right) = \frac{\delta}{2u^2\varepsilon_+}\left(2(\varepsilon_1{}^2+\varepsilon_2{}^2)u^2 - 2\varepsilon_1+\varepsilon_2\,\frac{4u^2}{1+u^2}\right)$$

$$= \frac{\delta}{\varepsilon_+}\left(\varepsilon_1{}^2+\varepsilon_2{}^2 - \frac{4\varepsilon_1\varepsilon_2}{1+u^2}\right)$$

$$= \frac{\delta}{\varepsilon_+}\left(k^2 + 2\varepsilon_1\varepsilon_2\left(1 - \frac{2}{1+u^2}\right)\right)$$

$$= \frac{\delta}{\varepsilon_+}\left(k^2 + 2\varepsilon_1\varepsilon_2\left(\frac{u^2-1}{u^2+1}\right)\right)$$

$$= \frac{\delta k^2}{\varepsilon_+} + \frac{k}{\varepsilon_+}\left(\frac{u^2-1}{u^2+1}\right) \quad ,$$

from which

$$\mu_o{}^{(4)} = \frac{2(\varepsilon_1{}^2+\varepsilon_2{}^2)}{\varepsilon_+}\;\frac{\delta^3(1+u^2)^3}{u^2}\left[\frac{\delta k^2}{\varepsilon_+} + \frac{1}{\varepsilon_2} - \left(\frac{\varepsilon_1+\varepsilon_2}{\varepsilon_-} - \frac{k}{\varepsilon_+}\right)\left(\frac{u^2-1}{u^2+1}\right)\right]. \qquad \text{(VII.51)}$$

Substituting eq (VII.51) in eqs (VII.41) and (VII.42) gives, finally,

$$\phi_c = \phi_o(1 - 3\zeta^{2/3}) \qquad \text{(VII.52)}$$

and

$$z_c = \frac{\delta(1+u^2)}{u}\,\zeta^{1/3} \tag{VII.53}$$

where

$$\zeta = u\left(\frac{\varepsilon_1{}^2+\varepsilon_2{}^2}{\varepsilon_+}\right)\left[\frac{\delta k^2}{\varepsilon_+} + \frac{1}{\varepsilon_2} - \left(\frac{\varepsilon_1+\varepsilon_2}{\varepsilon_-} - \frac{k}{\varepsilon_+}\right)\left(\frac{u^2-1}{u^2+1}\right)\right] \quad . \tag{VII.54}$$

It should be emphasized that eq (VII.52) provides only a zeroth-order approximation for $\phi_o - \phi_c = 3\phi_o\zeta^{2/3}$, and eq (VII.53) provides only a zeroth-order approximation for z_c. As mentioned in the discussion following eq (VII.39), we have neglected terms of relative order $\theta^{2/3}$:

$$\frac{\phi_o - \phi_c}{\phi_o} = 3\zeta^{2/3}(1 + O(\theta^{2/3})) \tag{VII.52a}$$

$$z_c = \frac{\delta(1+u^2)}{u}\,\zeta^{1/3}(1 + O(\theta^{2/3})) \quad . \tag{VII.53a}$$

These approximations are thus much less accurate than expressions (VII.9) and (VII.32) for z_{min} and z_{max}, respectively, in which the neglected terms are of relative order θ^2. The higher order corrections, indicated above as $O(\theta^{2/3})$ could be obtained in a straightforward manner by continuing our perturbative approach for the region in which $\phi_o{}^2 - \phi^2 = O(\theta^{8/3})$, $z = O(\theta^{4/3})$, but we shall not pursue this further in this report.

Finally, we determine the value of the cross section at its minimum and maxima.

The cross section is given by eq (VII.1) in which $\{\ \}$ is given by eq (VI.49) and q^2 by eq (V.15). At the minimum of the cross section $z = z_{min} = O(\theta^2)$, given in eq (VII.9), provided that ϕ is not too close to ϕ_c; specifically provided that $\phi_0^2 - \phi^2 = O(\theta^2)$. With this assumption we may set $v = u$ in the expression (V.15) for q^2, thus neglecting terms of relative order θ^2, and obtain

$$q^2 = u^2\phi^2 + \delta^2(1+u^2)^2 \quad . \tag{VII.55a}$$

Next, in eq (VI.49) for $\{\ \}$, consider the first term, which is

$$f(v)(v- u + \sigma)^2 = u^2 f(v)\left(z + \frac{\sigma}{u}\right)^2 \quad .$$

Here, $z = z_{min}$ is given by eq (VII.5), from which it is seen that $z + \frac{\sigma}{u}$ is $O(\phi^2)$ at the minimum. Thus, at the minimum, the first term in eq (VII.49) is $O(u^2 f(v)\phi^4)$, which is $O(\phi^2)$ relative to the second term in eq (VII.49) and may therefore be neglected. Similarly, in the second term itself we may neglect terms of higher order in ϕ^2, and, in addition, set $v = u$, again neglecting terms of relative order θ^2. We thus have, at the minimum of the cross section,

$$\{\ \} = \frac{1}{4}\, g(u)\phi^2 + \rho \quad . \tag{VII.55b}$$

It may be noted, on referring to eq (VI.54), that the value just given above in eq (VII.55b) for $\{\ \}$ at the minimum of the cross section is the same (to within terms of relative order θ^2) as the value of $\{\ \}$ at the minimum of $\{\ \}$. This follows from the fact that both minima occur for $z + \frac{\sigma}{u} = O(\phi^2)$. (The minimum of $\{\ \}$ is given by eq (VI.52), and that of the cross section by eq (VII.5).) However, they do not occur at precisely the same value of z, and indeed cannot. The minimum of the cross section is determined from eq (VII.2a), and there the minimum of q^2 does not occur at the same point as the minimum of $\{\ \}$.

From eqs (VII.55a), (VI.54), (VII.55b), and (VII.1) we then have, for the cross section at the minimum (with $F(q) = 0$),

$$\frac{d^3\sigma}{d\Omega_k d\Omega_{p_2} dk} = Z^2 \frac{e^2}{\hbar c}\left(\frac{e^2}{mc^2}\right)^2 \frac{1}{k}\frac{p_2}{p_1}\frac{1}{(2\pi)^2}\frac{1}{[u^2\phi^2 + \delta^2(1+u^2)^2]^2}$$

$$\times \frac{8\varepsilon_1\varepsilon_2}{(1+u^2)^2}\left\{(\varepsilon_1{}^2+\varepsilon_2{}^2)u^2\phi^2 + \delta^2(1+u^2)^2 \cdot \frac{k^2u^2\left[1+\left(\frac{2u}{1+u^2}\right)^2\right]}{\left[\varepsilon_1{}^2+\varepsilon_2{}^2-2\varepsilon_1\varepsilon_2\left(\frac{2u}{1+u^2}\right)^2\right]}\right\} . \tag{VII.56}$$

It should be noted here that unless ϕ is very small, i.e., unless $\phi \lesssim O(\theta^2)$, the first term in eq (VII.56) is much larger than the second term there. Thus for $O(\theta^2) < \phi \lesssim O(\theta)$ we can write eq (VII.56) in the much simpler form (recalling that here $q_\perp{}^2 = u^2\phi^2$),

$$\frac{d^3\sigma}{d\Omega_k d\Omega_{p_2} dk} = Z^2 \frac{e^2}{\hbar c}\left(\frac{e^2}{mc^2}\right)^2 \frac{1}{k}\frac{p_2}{p_1}\frac{1}{(2\pi)^2} \cdot \frac{8\varepsilon_1\varepsilon_2}{(1+u^2)^2} \cdot (\varepsilon_1{}^2+\varepsilon_2{}^2)\frac{q_\perp{}^2}{q^4} \quad . \quad \text{(VII.57)}$$

(See comment following reference [14] in the list of references.)

With regard to the cross section at the maxima, these values may be obtained by substituting eq (VII.32) in eq (VI.49) (for $\{\ \}$) and in eq (V.15) (for q^2), and then substituting these in eq (VII.1). However, the expression (VII.32) for z_{max} is rather complicated. On the other hand, the average value of the cross section at the two maxima is rather simple, and from the experimental viewpoint this value, as well as the ratio of this average value to the value of the cross section at the minimum is of almost equal interest. We therefore determine this average value rather than the cross section at each of the maxima. For this we need retain only the first term, of $O(\theta)$, in eq (VII.32). Since it has opposite sign for the two maxima, the contributions from the remaining terms in eq (VII.32) (of order θ relative to the first term) cancel in the average. In the same manner, all terms of relative order θ cancel in the expression for the average of the cross section at the maxima. We may thus use only the first term in eq (VII.32) and set $v = u$ throughout, neglecting terms of relative order θ since they give contributions of relative order θ^2 to the average. From eqs (VII.49) and (V.15) we then have

$$\left\{\ \right\} = u^2 f(u) z^2 + \frac{1}{4}\, g(u)\phi^2$$

$$= f(u)\left[u^2 z^2 + \frac{g(u){\phi_o}^2}{2f(u)} \cdot \frac{1}{2}\left(\frac{\phi}{\phi_o}\right)^2\right] \tag{VII.58a}$$

and

$$q^2 = u^2 z^2 + u^2 \phi^2 + \delta^2 (1+u^2)^2 \quad . \tag{VII.58b}$$

Here, from eq (VII.32) (retaining only the first term there),

$$u^2 z^2 = \delta^2 (1+u^2)^2 \left[1 - \left(\frac{\phi}{\phi_o}\right)^2\right] \tag{VII.59}$$

and from eq (VII.44)

$$\frac{g(u){\phi_o}^2}{2f(u)} = u^2 {\phi_o}^2 + \delta^2 (1+u^2)^2 \quad .$$

Thus in eq (VII.58a) we have

$$u^2 z^2 + \frac{g(u){\phi_o}^2}{2u^2 f(u)}\, \frac{1}{2}\left(\frac{\phi}{\phi_o}\right)^2 = \delta^2 (1+u^2)^2 \left[1 - \left(\frac{\phi}{\phi_o}\right)^2\right] + \frac{1}{2}\left(\frac{\phi}{\phi_o}\right)^2 \left[u^2 {\phi_o}^2 + \delta^2 (1+u^2)^2\right]$$

$$= \frac{1}{2}\,\delta^2 (1+u^2)^2 \left[1 - \left(\frac{\phi}{\phi_o}\right)^2\right] + \frac{1}{2}\, u^2 \phi^2 + \frac{1}{2}\,\delta^2 (1+u^2)^2$$

$$= \frac{1}{2}\, q^2 \quad .$$

The last line here follows from eq (VII.58b). We thus have, at the maxima, neglecting terms of relative order θ,

$$\left\{\quad\right\} = \frac{1}{2} f(u) q^2 \tag{VII.60a}$$

so that

$$\frac{\left\{\quad\right\}}{q^4} = \frac{f(u)}{2q^2} \quad . \tag{VII.60b}$$

Substituting eq (VII.59) in eq (VII.58b) we have

$$q^2 = 2\delta^2(1+u^2)^2 - u^2\phi^2 \left[\frac{\delta^2(1+u^2)^2}{u^2\phi_o^2} - 1\right] \quad .$$

Here, from eqs (VII.43), (VII.49), and (VII.50),

$$\begin{aligned} \frac{\delta^2(1+u^2)^2}{u^2\phi_o^2} - 1 &= b_o - 1 \\ &= \frac{\varepsilon_+ - \varepsilon_-}{\varepsilon_+} \\ &= \frac{4\varepsilon_1\varepsilon_2\left(\frac{2u}{1+u^2}\right)^2}{\varepsilon_1^2+\varepsilon_2^2+2\varepsilon_1\varepsilon_2\left(\frac{2u}{1+u^2}\right)^2} \quad . \end{aligned}$$

Thus

$$q^2 = 2\left[\delta^2(1+u^2)^2 - u^2\phi^2 \frac{2\varepsilon_1\varepsilon_2\left(\frac{2u}{1+u^2}\right)^2}{\varepsilon_1{}^2+\varepsilon_2{}^2+2\varepsilon_1\varepsilon_2\left(\frac{2u}{1+u^2}\right)^2}\right] . \tag{VII.61}$$

Finally, from eqs (VII.8), (VII.60b), (VII.61), and (VII.1), we have the average of the cross sections at the two maxima, neglecting terms of relative order θ^2,

$$\frac{d^3\sigma}{d\Omega_k d\Omega_{p_2} dk} = Z^2 \frac{e^2}{\hbar c}\left(\frac{e^2}{mc^2}\right)^2 \frac{1}{k}\frac{p_2}{p_1}\frac{1}{(2\pi)^2}$$

$$\times\left[\frac{\frac{2\varepsilon_1\varepsilon_2}{(1+u^2)^2}\left[\varepsilon_1{}^2+\varepsilon_2{}^2-2\varepsilon_1\varepsilon_2\left(\frac{2u}{1+u^2}\right)^2\right]}{\delta^2(1+u^2)^2 - u^2\phi^2\left[\frac{2\varepsilon_1\varepsilon_2\left(\frac{2u}{1+u^2}\right)^2}{\varepsilon_1{}^2+\varepsilon_2{}^2-2\varepsilon_1\varepsilon_2\left(\frac{2u}{1+u^2}\right)^2}\right]}\right] . \tag{VII.62}$$

VIII. SUMMARY OF ESSENTIAL FORMULAS DERIVED IN SECTIONS V, VI, AND VII

In this section we present a summary of the formulas of sections V, VI, and VII, which constitute the essence of this report. The details of their derivation are given in those sections, but for the reader more concerned with applying them than with their derivation, we have abstracted them from those sections, and present them here with their original equation numbers. One may thereby locate them easily in their original context if one wishes to consider them in greater detail. However, for the definition of the essential variables of the problem one should see section III.

In section V we derive an approximation for q^2, the momentum transfer squared, for high energies and small angles. In that section we start with the exact expression for q^2,

$$q^2 = (\underline{p}_1 - \underline{p}_2 - \underline{k})^2 \quad , \tag{V.1}$$

and write

$$q^2 = q_\perp{}^2 + q_z{}^2 \tag{V.2}$$

where

$$\begin{aligned} \underline{q}_\perp &= \underline{p}_{1\perp} - \underline{p}_{2\perp} \\ &= \underline{u} - \underline{v} \end{aligned} \tag{V.3}$$

is the component of $\underline{q}$ perpendicular to $\underline{k}$

and

$$q_z = p_1\cos\theta_1 - p_2\cos\theta_2 - k \tag{V.4}$$

is the component of $\underline{q}$ in the direction of $\underline{k}$,

$$u = |\underline{u}| = p_1\sin\theta_1$$

$$v = |\underline{v}| = p_2\sin\theta_2 \quad . \tag{V.5}$$

We define the energy denominators

$$d_1 = \varepsilon_1 - p_1\cos\theta_1$$

$$d_2 = \varepsilon_2 - p_2\cos\theta_2 \quad . \tag{V.7}$$

We obtain expansions for d_1 and d_2 which are useful for high energies and small angles:

$$\frac{d_1}{\varepsilon_1} = \frac{1}{2}\left(\frac{1+u^2}{\varepsilon_1^{\,2}}\right) + \frac{1}{8}\left(\frac{1+u^2}{\varepsilon_1^{\,2}}\right)^2 + \frac{1}{16}\left(\frac{1+u^2}{\varepsilon_1^{\,2}}\right)^3 + \ldots \tag{V.9}$$

and

$$\frac{\varepsilon_1}{d_1} = \frac{2\varepsilon_1^{\,2}}{1+u^2} - \frac{1}{2} - \frac{1}{8}\left(\frac{1+u^2}{\varepsilon_1^{\,2}}\right) - \frac{1}{16}\left(\frac{1+u^2}{\varepsilon_1^{\,2}}\right)^2 - \ldots \quad . \tag{V.10}$$

In identical fashion we have

$$\frac{d_2}{\varepsilon_2} = \frac{1}{2}\left(\frac{1+v^2}{\varepsilon_2^{\,2}}\right) + \frac{1}{8}\left(\frac{1+v^2}{\varepsilon_2^{\,2}}\right)^2 + \frac{1}{16}\left(\frac{1+v^2}{\varepsilon_2^{\,2}}\right)^3 + \dots \tag{V.11}$$

and

$$\frac{\varepsilon_2}{d_2} = \frac{2\varepsilon_2^{\,2}}{1+v^2} - \frac{1}{2} - \frac{1}{8}\left(\frac{1+v^2}{\varepsilon_2^{\,2}}\right) - \frac{1}{16}\left(\frac{1+v^2}{\varepsilon_2^{\,2}}\right)^2 - \dots \quad . \tag{V.12}$$

We then have

$$\begin{aligned} q_z &= d_2 - d_1 \\ &= \frac{1+v^2}{2\varepsilon_2} - \frac{(1+u^2)}{2\varepsilon_1} \\ &= \frac{1}{8}\,\frac{(1+v^2)^2}{\varepsilon_2^{\,3}} - \frac{1}{8}\,\frac{(1+u^2)^2}{\varepsilon_1^{\,3}} + \dots \quad . \end{aligned} \tag{V.13}$$

For high energies and small angles we obtain the following approximation for q^2:

$$q^2 = (v-u)^2 + 2uv(1-\cos\phi) + \frac{2\delta(1+u^2)\,u(v-u)}{\varepsilon_2} + \delta^2(1+u^2)^2 \quad . \tag{V.15}$$

In this approximation neglected terms are of relative order θ_2^2 (and never larger) for <u>all</u> values of $v - u$ (i.e., for $v - u \equiv 0$, $v - u = O(u\theta^2)$, $v - u = O(u\theta)$, and $v - u = O(u)$).

We show that the minimum value of q^2 considered as a function of v (i.e., θ_2), for fixed values of ε_1, ε_2, θ and ϕ, occurs for

$$v = u \left\{ 1 - \left[(1 - \cos\phi) + \frac{\delta(1 + u^2)}{\varepsilon_2} \right] \right\} \quad . \tag{V.17}$$

At this value of v, the minimum value of q^2 is, neglecting terms of relative order θ_2^2,

$$q^2_{min} = \delta^2(1 + u^2)^2 + u^2\sin^2\phi \quad . \tag{V.18}$$

In section VI we derive the high energy, small angle approximation of the differential cross section given in eq (IV.1). We define the variables ξ and η in terms of the energy denominators d_1 and d_2 given in eq (V.7):

$$\xi = \frac{1}{2\varepsilon_1 d_1} \quad , \quad \eta = \frac{1}{2\varepsilon_2 d_2} \quad . \tag{VI.3}$$

We note that the expression for $\{ \ \}$ given in eq (VI.1) may be written very simply in terms of ξ and η:

$$\{ \ \} = 8k^2\varepsilon_1\varepsilon_2(\underline{u}-\underline{v})^2\xi\eta + 16\varepsilon_1{}^2\varepsilon_2{}^2(\underline{u}\xi-\underline{v}\eta)^2 - 4q^2(\varepsilon_1\underline{u}\xi-\varepsilon_2\underline{v}\eta)^2 \quad . \tag{VI.4}$$

We note that in terms of the variables defined earlier, we may write

$$\xi = \xi_0 - \xi_1$$

$$\eta = \eta_0 - \eta_1 \tag{VI.7}$$

where

$$\xi_0 = \frac{1}{1+u^2} \quad , \quad \xi_1 = \frac{1}{2\varepsilon_1(\varepsilon_1 + p_1\cos\theta_1)} \approx \frac{1}{4\varepsilon_1^{\,2}}$$

$$\eta_0 = \frac{1}{1+v^2} \quad , \quad \eta_1 = \frac{1}{2\varepsilon_2(\varepsilon_2 + p_2\cos\theta_2)} \approx \frac{1}{4\varepsilon_2^{\,2}} \quad . \tag{VI.8}$$

We derive the high energy, small angle approximation to $\left\{\ \right\}$, defined initially by eq (IV.1):

$$\left\{\ \right\} = f(v)(v-u+\sigma)^2 + g(v)\sin^2\tfrac{1}{2}\phi + \rho \tag{VI.49}$$

where

$$f(v) = \frac{8\varepsilon_1^{\,2}\varepsilon_2^{\,2}}{(1+u^2)(1+v^2)}\left[\frac{k^2}{\varepsilon_1\varepsilon_2} + 2\,\frac{(uv-1)^2}{(1+u^2)(1+v^2)}\right] \tag{VI.44}$$

$$g(v) = \frac{32\varepsilon_1\varepsilon_2(\varepsilon_1^{\,2}+\varepsilon_2^{\,2})uv}{(1+u^2)(1+v^2)} \quad . \tag{VI.45}$$

$$\rho = \frac{2\left(\frac{k^2}{\varepsilon_1\varepsilon_2}\right)^2 u^2\left[1 + \left(\frac{2u}{1+u^2}\right)^2\right]}{\frac{k^2}{\varepsilon_1\varepsilon_2} + 2\left(\frac{u^2-1}{u^2+1}\right)^2} \tag{VI.47}$$

and

$$\sigma = \frac{k(\varepsilon_1+\varepsilon_2)(u^2-1)u}{2\varepsilon_1{}^2\varepsilon_1{}^2\left[\frac{k^2}{\varepsilon_1\varepsilon_2} + 2\left(\frac{u^2-1}{u^2+1}\right)^2\right]} \quad . \tag{VI.48}$$

We consider $\left\{\ \right\}$ as a function of v (i.e., θ_2) for fixed values of the other variables (ε_1, ε_2, θ_1 and ϕ) and determine the value of v for which $\left\{\ \right\}$ achieves its minimum as well as the value of $\left\{\ \right\}$ at the minimum. From eq (VI.49) we find

$$v_{min} = u\left[1 + \left(\frac{u^2-1}{u^2+1}\right)\left(\frac{\frac{-k(\varepsilon_1+\varepsilon_2)(u^2+1)}{2\varepsilon_1{}^2\varepsilon_1{}^2} + \frac{(\varepsilon_1{}^2+\varepsilon_2{}^2)}{2\varepsilon_1\varepsilon_2}\,\phi^2}{\frac{k^2}{\varepsilon_1\varepsilon_2} + 2\left(\frac{u^2-1}{u^2+1}\right)^2}\right)\right] . \tag{VI.53}$$

We obtain, for the value of $\left\{ \right\}$ at its minimum,

$$\left\{ \right\}_{\min} = \frac{1}{4} g(u)\phi^2 + \rho$$

$$= \frac{8\varepsilon_1\varepsilon_2(\varepsilon_1{}^2+\varepsilon_2{}^2)u^2\phi^2}{(1+u^2)^2} + \frac{2\left(\frac{k^2}{\varepsilon_1\varepsilon_2}\right)^2 u^2 \left[1 + \left(\frac{2u}{1+u^2}\right)^2\right]}{\frac{k^2}{\varepsilon_1\varepsilon_2} + 2\left(\frac{u^2-1}{u^2+1}\right)^2} \quad . \quad \text{(VI.54)}$$

We note that the minimum for $\left\{ \right\}$ does not occur for precisely the same value of v as the minimum for q^2, as is seen by comparing eq (VI.53) with eq (V.17).

In section VII we consider the cross section as a function of v (i.e., θ_2) for fixed values of the other variables (ε_1, ε_2, θ_1, and ϕ) and determine the value of v for which the cross section achieves its maximum and minimum values. In this determination we neglect atomic screening, since its effect is rather small for the energies and angles of experimental interest. We show that for ϕ less than a critical value, ϕ_c, the cross section has two maxima, which occur for $z \equiv (v - u)/u = O(\theta)$, one for $v < u$ and one for $v > u$, and in between these a very sharp minimum which occurs for $z = O(\theta^2)$. For $\phi = \phi_c$ the minimum of the cross section occurs at the same point ($z = z_c$) as the second maximum (the one for $v > u$), and we then have a point of inflection at $z = z_c$. For $\phi > \phi_c$ the cross section has only one extremum, the maximum which occurs for $v < u$. We derive a zeroth-order approximation for ϕ_c and z_c:

$$\phi_c = \phi_o(1 - 3\zeta^{2/3}) \tag{VII.52}$$

and

$$z_c = \frac{\delta(1+u^2)}{u}\,\zeta^{1/3} \tag{VII.53}$$

where

$$\phi_o = \frac{\delta(1+u^2)}{u}\left(\frac{\varepsilon_-}{\varepsilon_+}\right)^{\frac{1}{2}} \tag{VII.10}$$

and

$$\zeta = u\left(\frac{\varepsilon_1{}^2+\varepsilon_2{}^2}{\varepsilon_+}\right)\left[\frac{\delta k^2}{\varepsilon_+} + \frac{1}{\varepsilon_2} - \left(\frac{\varepsilon_1+\varepsilon_2}{\varepsilon_-} - \frac{k}{\varepsilon_+}\right)\left(\frac{u^2-1}{u^2+1}\right)\right] \quad . \tag{VII.54}$$

with

$$\varepsilon_\pm \equiv \varepsilon_1{}^2+\varepsilon_2{}^2\pm 2\varepsilon_1\varepsilon_2\left(\frac{2u}{1+u^2}\right)^2 \quad . \tag{VII.49}$$

The relative errors in these zeroth-order approximations for ϕ_c and z_c are $O(\theta^{2/3})$, as noted in eqs (VII.52a) and (VII.53a). For $\phi_c < \phi_o$ (specifically, for $\phi_o - \phi_c = O(\theta)$), we show that the position of the minimum is given by

$$z_{min} = -\frac{\left(\phi^2 + \frac{\delta^2(1+u^2)^2}{u^2}\right)\left[(\varepsilon_1+\varepsilon_2)\delta(1+u^2)-(\varepsilon_1{}^2+\varepsilon_2{}^2)\frac{\phi^2}{2}\right]\left(\frac{u^2-1}{u^2+1}\right)-2(\varepsilon_1{}^2+\varepsilon_2{}^2)\phi^2\left(\frac{\phi^2}{2} + \frac{\delta(1+u^2)}{\varepsilon_2}\right)}{\left[\varepsilon_1{}^2+\varepsilon_2{}^2-2\varepsilon_1\varepsilon_2\left(\frac{2u}{1+u^2}\right)^2\right]\frac{\delta^2(1+u^2)^2}{u^2} - \left[\varepsilon_1{}^2+\varepsilon_2{}^2+2\varepsilon_1\varepsilon_2\left(\frac{2u}{1+u^2}\right)^2\right]\phi^2}$$

(VII.9)

and the position of the maxima by

$$z_{max} = \pm \frac{\delta(1+u^2)}{u}\left[1 - \left(\frac{\phi}{\phi_o}\right)^2\right]^{\frac{1}{2}} + \frac{uf'(u)}{4f(u)}\,\frac{\delta^2(1+u^2)^2}{u^2}\left[1 - \left(\frac{\phi}{\phi_o}\right)^2\right] - \tfrac{1}{2}\lambda^{(2)} \quad . \qquad \text{(VII.32)}$$

where $\lambda^{(2)}$ is the coefficient of z^2 in eq (VII.20), viz.,

$$\lambda^{(2)} = \left[\frac{3\sigma}{u} + \frac{3g'(u)}{8uf(u)}\,\phi^2 - \frac{g(u)f'(u)}{2uf^2(u)}\,\phi^2 - \frac{uf'(u)}{2f(u)}\left(\phi^2 + \frac{\delta^2(1+u^2)^2}{u^2}\right)\right] \qquad \text{see (VII.20)}$$

and $f(u)$, $f'(u)$, $g(u)$, and $g'(u)$ are given from eqs (VI.44), (VI.45) and (VII.8) by

$$f(u) = \frac{8\varepsilon_1\varepsilon_2}{(1+u^2)^2}\left[\varepsilon_1{}^2+\varepsilon_2{}^2-2\varepsilon_1\varepsilon_2\left(\frac{2u}{1+u^2}\right)^2\right] \qquad \text{see (VI.44) and (VII.8)}$$

$$f'(u) = \left.\frac{\partial f(v)}{\partial v}\right|_{v=u} = \frac{64\varepsilon_1{}^2\varepsilon_2{}^2u(u^2-1)}{(1+u^2)^5} - \frac{16\varepsilon_1\varepsilon_2 u}{(1+u^2)^3}\left[\varepsilon_1{}^2+\varepsilon_2{}^2 - 2\varepsilon_1\varepsilon_2\left(\frac{2u}{1+u^2}\right)^2\right]$$

$$g(u) = \frac{32\varepsilon_1\varepsilon_2 u^2}{(1+u^2)^2}\,(\varepsilon_1{}^2+\varepsilon_2{}^2)$$ see (VI.45) and (VII.8)

$$g'(u) = \left.\frac{\partial g(v)}{\partial v}\right|_{v=u} = -\frac{32\varepsilon_1\varepsilon_2 u}{(1+u^2)^2}\,(\varepsilon_1{}^2+\varepsilon_2{}^2)\left(\frac{u^2-1}{u^2+1}\right) .$$

The errors in the expressions just given for z_{min} and z_{max} are of relative order θ^2.

For the value of the cross section at the minimum (with $F(q) = 0$) we find

$$\frac{d^3\sigma}{d\Omega_k d\Omega_{p_2} dk} = Z^2\,\frac{e^2}{\hbar c}\left(\frac{e^2}{mc^2}\right)^2 \frac{1}{k}\,\frac{p_2}{p_1}\,\frac{1}{(2\pi)^2}\,\frac{1}{[u^2\phi^2 + \delta^2(1+u^2)^2]^2}$$
$$\times\,\frac{8\varepsilon_1\varepsilon_2}{(1+u^2)^2}\left\{(\varepsilon_1{}^2+\varepsilon_2{}^2)u^2\phi^2 + \delta^2(1+u^2)^2 \cdot \frac{k^2u^2\left[1+\left(\frac{2u}{1+u^2}\right)^2\right]}{\left[\varepsilon_1{}^2+\varepsilon_2{}^2-2\varepsilon_1\varepsilon_2\left(\frac{2u}{1+u^2}\right)^2\right]}\right\} . \tag{VII.56}$$

We note here that unless ϕ is very small, i.e., unless $\phi \lesssim O(\theta^2)$, the first term in eq (VII.56) is much larger than the second term there. Thus for $O(\theta^2) < \phi \lesssim O(\theta)$ we can write eq (VII.56) in the much simpler form (recalling that here $q_\perp{}^2 = u^2\phi^2$),

$$\frac{d^3\sigma}{d\Omega_k d\Omega_{p_2} dk} = Z^2 \frac{e^2}{\hbar c}\left(\frac{e^2}{mc^2}\right)^2 \frac{1}{k}\frac{p_2}{p_1}\frac{1}{(2\pi)^2} \cdot \frac{8\varepsilon_1\varepsilon_2}{(1+u^2)^2} \cdot (\varepsilon_1{}^2+\varepsilon_2{}^2)\frac{q_\perp{}^2}{q^4} \quad . \quad \text{(VII.57)}$$

With regard to the cross section at the maxima, these values may be obtained by substituting eq (VII.32) in eq (VI.49) (for $\{\ \}$) and in eq (V.15) (for q^2), and then substituting these in eq (VII.1). However, the expression (VII.32) for z_{max} is rather complicated. On the other hand, the average value of the cross section at the two maxima is rather simple, and from the experimental viewpoint this value, as well as the ratio of this average value to the value of the cross section at the minimum is of almost equal interest. We therefore determine this average value rather than the cross section at each of the maxima. For the average of the cross sections at the two maxima we find

$$\frac{d^3\sigma}{d\Omega_k d\Omega_{p_2} dk} = Z^2 \frac{e^2}{\hbar c}\left(\frac{e^2}{mc^2}\right)^2 \frac{1}{k}\frac{p_2}{p_1}\frac{1}{(2\pi)^2}$$

$$\times \left[\frac{\dfrac{2\varepsilon_1\varepsilon_2}{(1+u^2)^2}\left[\varepsilon_1{}^2+\varepsilon_2{}^2-2\varepsilon_1\varepsilon_2\left(\dfrac{2u}{1+u^2}\right)^2\right]}{\delta^2(1+u^2)^2 - u^2\phi^2\left[\dfrac{2\varepsilon_1\varepsilon_2\left(\dfrac{2u}{1+u^2}\right)^2}{\varepsilon_1{}^2+\varepsilon_2{}^2-2\varepsilon_1\varepsilon_2\left(\dfrac{2u}{1+u^2}\right)^2}\right]}\right] \quad . \quad \text{(VII.62)}$$

The errors in the expressions (VII.56) and (VII.62) are of relative order θ^2.

A simple picture of the overall behavior of the differential cross section for bremsstrahlung follows from the analysis presented in sections V, VI, and VII. For high energies and small angles we may distinguish three regions which are characterized by the momentum transfer vector, $\vec{q}$, and illustrated in fig. 13. In the first region $|\vec{q}| = O(u)$. We call this the "large" momentum transfer region. Here $q_\perp \gg q_z$, so that $\vec{q}$ is essentially perpendicular to the photon momentum, $\vec{k}$. In the second region $q_\perp \sim q_z$. Thus in this region the vector $\vec{q}$ may have any direction. We call this the region of "small" momentum transfers. In this region, in general the differential cross section is larger by a factor of $O(u^2/q_{min}^2) = O(1/\theta^2)$ than the differential cross section in the region of large momentum transfers. However, in the middle of the region of small momentum transfers the differential cross section has a very sharp dip (for $\phi < \phi_c$). The region of this dip (the third region) is characterized by $q_\perp \ll q_z$. Thus the magnitude of q is of the same order of magnitude as in the region in which $q_\perp \sim q_z$, but now the vector $\vec{q}$ is essentially parallel to the photon momentum $\vec{k}$. In the region of the dip the differential cross section is smaller by a factor of $O(\theta^2)$ than it is in the small q region where $q_\perp \sim q_z$, i.e., it is of the same order of magnitude as in the large q region.

IX. COULOMB CORRECTIONS TO THE DIFFERENTIAL CROSS SECTION FOR BREMSSTRAHLUNG

Thus far, all of the analysis presented in this report has dealt exclusively with the first Born approximation differential cross section for bremsstrahlung, the Bethe-Heitler cross section. We consider now the question of the Coulomb corrections to this cross section. Having noted and considered in detail the sharp dip in the cross section in the region of small momentum transfers, the question of whether the Coulomb correction might not fill in this dip arises in particular. In fact the Coulomb correction to the differential cross section for bremsstrahlung at high energies has been calculated specifically in this region of small momentum transfers. It was first given for a pure Coulomb field (no screening) by Bethe and Maximon [18], and later for arbitrary screening by Olsen, Maximon, and Wergeland [8]. In both cases it was found that the entire effect of the Coulomb corrections is simply to <u>multiply</u> the Born approximation cross section by a factor which, although it varies throughout the region of small momentum transfers, is always of order unity. Thus the Coulomb corrections do not fill in the sharp dip, as they might if they were additive. Expressions for this factor are given in reference [8], both in terms of the potential representing the screened atom and in terms of the atomic form factor. It is denoted by $|A|^2$ in [8] and by R in [18], p. 781, Eq. (8.20). For the case of no screening, it is given in both [18] and [8]:

$$R = |A|^2 = \frac{V^2(x) + a^2y^2W^2(x)}{V^2(1)}$$

where

$$a = Z\,\frac{e^2}{\hbar c}$$

$$x = \frac{q_\perp^{\;2}}{q^2}$$

$$y = 1 - x = \frac{q_z^{\;2}}{q^2}$$

$$V(x) = F(ia,-ia;1;x)$$

$$W = F(1+ia,1-ia;2;x)$$

$$= \frac{1}{a^2}\,\frac{dV}{dx}\ ,$$

F being the hypergeometric function. In particular,

$$V(1) = \frac{\sinh \pi a}{\pi a}\ .$$

In the absence of screening it is shown in reference [18] that R is a monotonically increasing function of x, that $R = 1$ for $x = 1$ and $R < 1$ otherwise.

Thus the effect of the Coulomb corrections is to deepen the dip observed in the Born approximation. However, it must be noted that in the analysis in reference [8], as well as that in [18], there have been neglected throughout, not only terms of relative order $1/\varepsilon^2$, as in the present report, but terms of relative order $1/\varepsilon$ as well. Thus if one wishes to evaluate the Coulomb corrections to the differential cross section to the same level of accuracy as that pursued here for the Born approximation, the analyses given earlier in [18] and [8] must be extended to include the terms of relative order $1/\varepsilon$.

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Note, in this paper, that after summing eq (4) over photon polarizations, we have, from eqs (3) and (5),

$$d\sigma \sim \frac{(\epsilon_1{}^2+\epsilon_2{}^2)}{(1+u^2)^2}\,\frac{q_\perp{}^2}{q^4} \quad .$$

Although this paper deals with pair production, the same result follows in bremsstrahlung, as may be seen in our eq. (VII.57). Note, however, from the discussion following eq (VII.56), that this result presupposes that $O(\theta^2) < \phi \lesssim O(\theta)$. The more correct result, valid also when $\phi \lesssim O(\theta^2)$, is given in our eq. (VII.56).

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From eqs (3.8), (3.8a), (4.4), and (4.9), after summing over electron spins and photon polarization, we have, for our expression $\{\ \}$, in Born approximation,

$$8\epsilon_1\epsilon_2\, k^2[(\underline{u}\xi_o-\underline{v}\eta_o)^2 + (\xi_o-\eta_o)^2] + 2\epsilon_1\epsilon_2(\underline{u}\xi_o-\underline{v}\eta_o)^2 \quad .$$

With a bit of algebra, we find $(\underline{u}\xi_o-\underline{v}\eta_o)^2 + (\xi_o-\eta_o)^2 = (\underline{u}-\underline{v})\xi_o\eta_o$, and thus arrive at eq. (VI.50).

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See also, Handbook of Mathematical Tables (The Chemical Rubber Co., Cleveland, Ohio (1964))2nd ed. Note p. 464.

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FIGURE CAPTIONS

Fig. 1. Theoretical integrated-over-angles thin-target bremsstrahlung cross section multiplied by the photon energy. (The formula used is listed as 3BS(e) in reference [13], with $Z = 78$.)

Fig. 2. Schematic representation of different partial cross sections for a 100 MeV positron beam incident on a hydrogen target. (a) Represents the "monochromatic" annihilation line, (b) the positron-electron bremsstrahlung contribution, and (b') the nuclear bremsstrahlung part. The spectra are observed along a line making an angle of 4° with the line of arrival of the incoming positrons. (Figure taken from reference [3].)

Fig. 3. Momenta of the initial and final electrons ($\underline{p}_1$ and $\underline{p}_2$) and photon ($\underline{k}$), (a) in the system with z-axis in the direction of $\underline{k}$; and with $\underline{p}_1$ in the x-z plane, (b) in the system with z-axis in the direction of $\underline{p}_1$; and with $\underline{k}$ in the x-z plane.

Fig. 4. q^2, $\{\ \}$, and the cross section as a function of θ_2 for $\varepsilon_1 = 140$ MeV, $k = 95$ MeV, $\theta_1 = 1°$, $\phi = 0°$, for θ_2 ranging between 0° and 4°.

Fig. 5. q^2, $\{\ \}$, and the cross section as a function of θ_2 for $\varepsilon_1 = 140$ MeV, $k = 95$ MeV, $\theta_1 = 1°$, $\phi = 0°$, for θ_2 ranging between 3° and 3.2°.

Fig. 6. The cross section as a function of θ_2 for $\varepsilon_1 = 140$ MeV, $k = 95$ MeV, $\phi = 0°$ and for $\theta_1 = 0.5°$, 1°, 1.5°, and 2°.

Fig. 7. The cross section as a function of θ_2 for $\varepsilon_1 = 140$ MeV, $k = 95$ MeV, $\theta_1 = 1°$ and for $\phi = 0°$, 0.5°, and 1°.

Fig. 8. The cross section as a function of θ_2 for $\varepsilon_1 = 140$ MeV, $k = 95$ MeV, $\theta_1 = 2°$ and for $\phi = 0°$, 0.5°, 1°, and 1.5°.

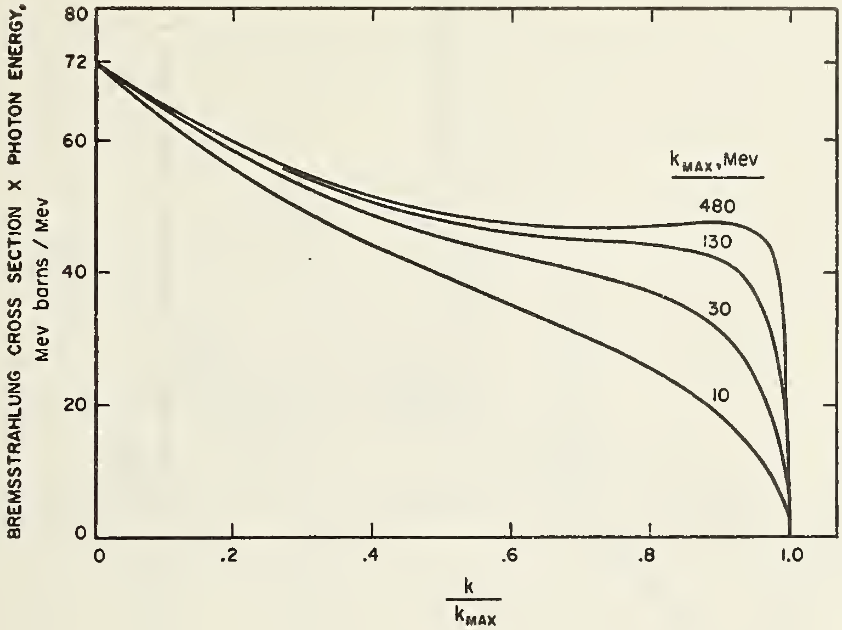
Fig. 9. The cross section as a function of θ_2 for $\varepsilon_1 = 140$ MeV, $k = 95$ MeV, $\theta_1 = 0.5°$, $1°$, $1.5°$, and $2°$, and $\phi = 0°$, $0.5°$, and $1°$.

Fig. 10. The cross section as a function of θ_2 for $\varepsilon_1 = 140$ MeV, $k = 95$ MeV, $\theta_1 = 0.5°$, $1°$, $1.5°$, and $2°$, and $\phi = 0°$, $0.5°$, $1°$, and $1.5°$; linear scale.

Fig. 11. The positions, $z = (v-u)/u$, of the maxima, minimum, and points of inflection as a function of the azimuthal angle ϕ, for $\varepsilon_1 = 140$ MeV, $k = 95$ MeV, and $\theta_1 = 1°$.

Fig. 12. The positions, $z = (v-u)/u$, of the minimum and the second maximum as a function of the azimuthal angle ϕ, for $\varepsilon_1 = 140$ MeV, $k = 95$ MeV, and $\theta_1 = 1°$.

Fig. 13. The momentum transfer $\vec{q}$ corresponding to the different regions of the cross section. See p. 118 at the end of section VIII for a discussion of this figure.



BREMSSTRAHLUNG CROSS SECTION X PHOTON ENERGY,
Mev barns / Mev
80
72
60
40
20
0
0
.2
.4
.6
.8
1.0
k / k_MAX
k_MAX, Mev
480
130
30
10

Fig. 1

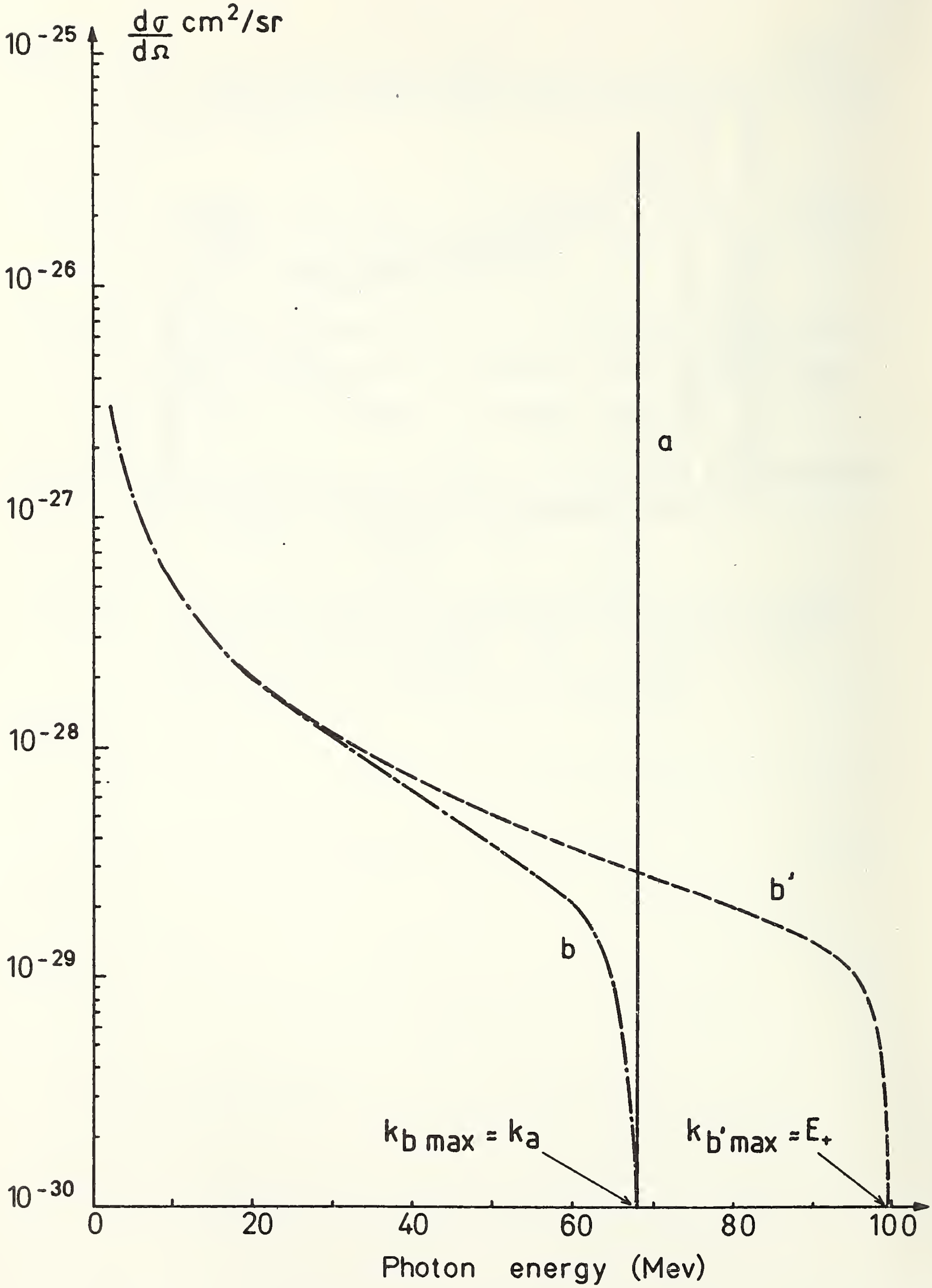

$\frac{d\sigma}{d\Omega}$ cm^2/sr
10^{-25}
10^{-26}
10^{-27}
10^{-28}
10^{-29}
10^{-30}
a
b
b'
$k_{b\,max} = k_a$
$k_{b'max} = E_+$
0
20
40
60
80
100
Photon energy (Mev)

Fig. 2

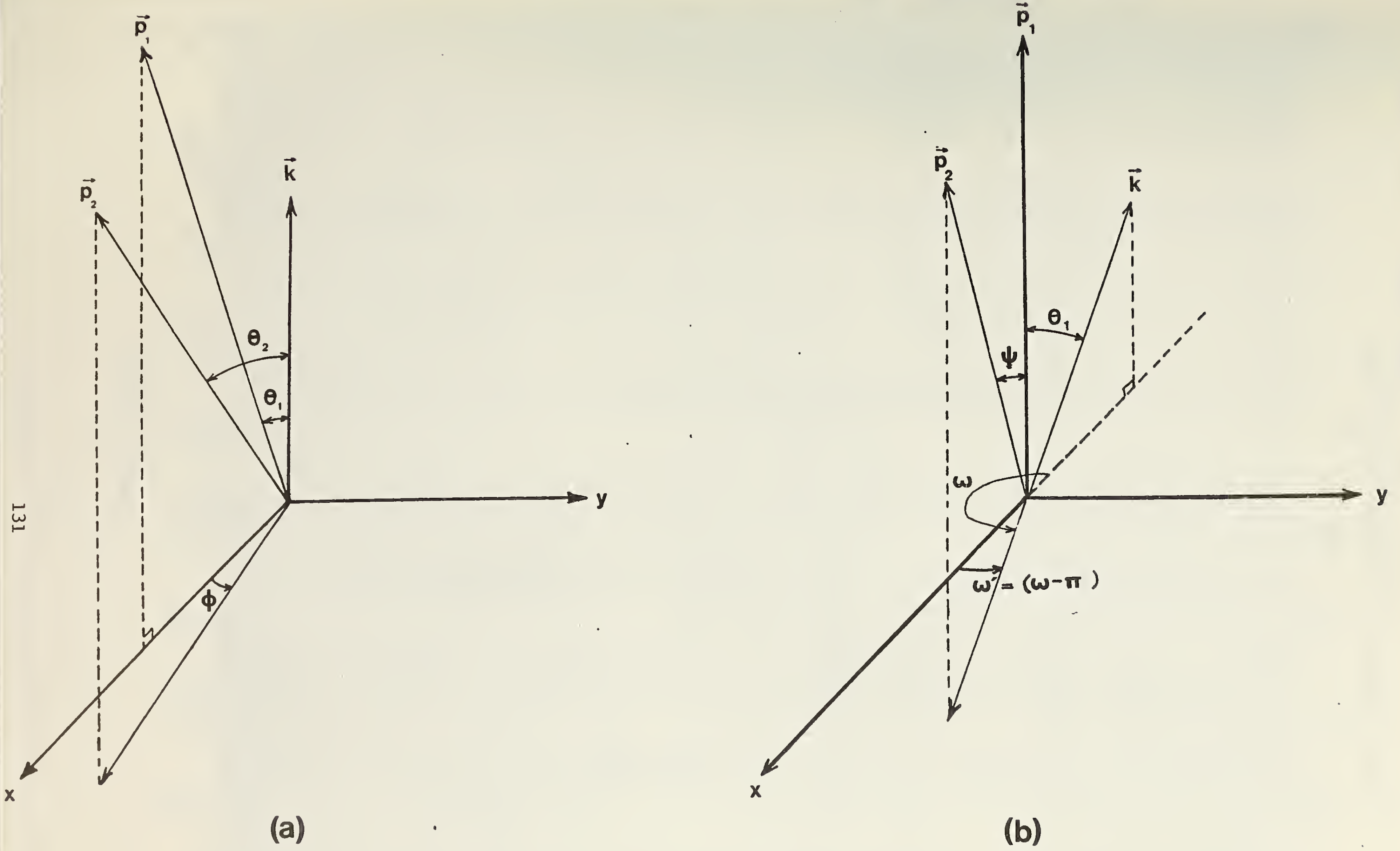
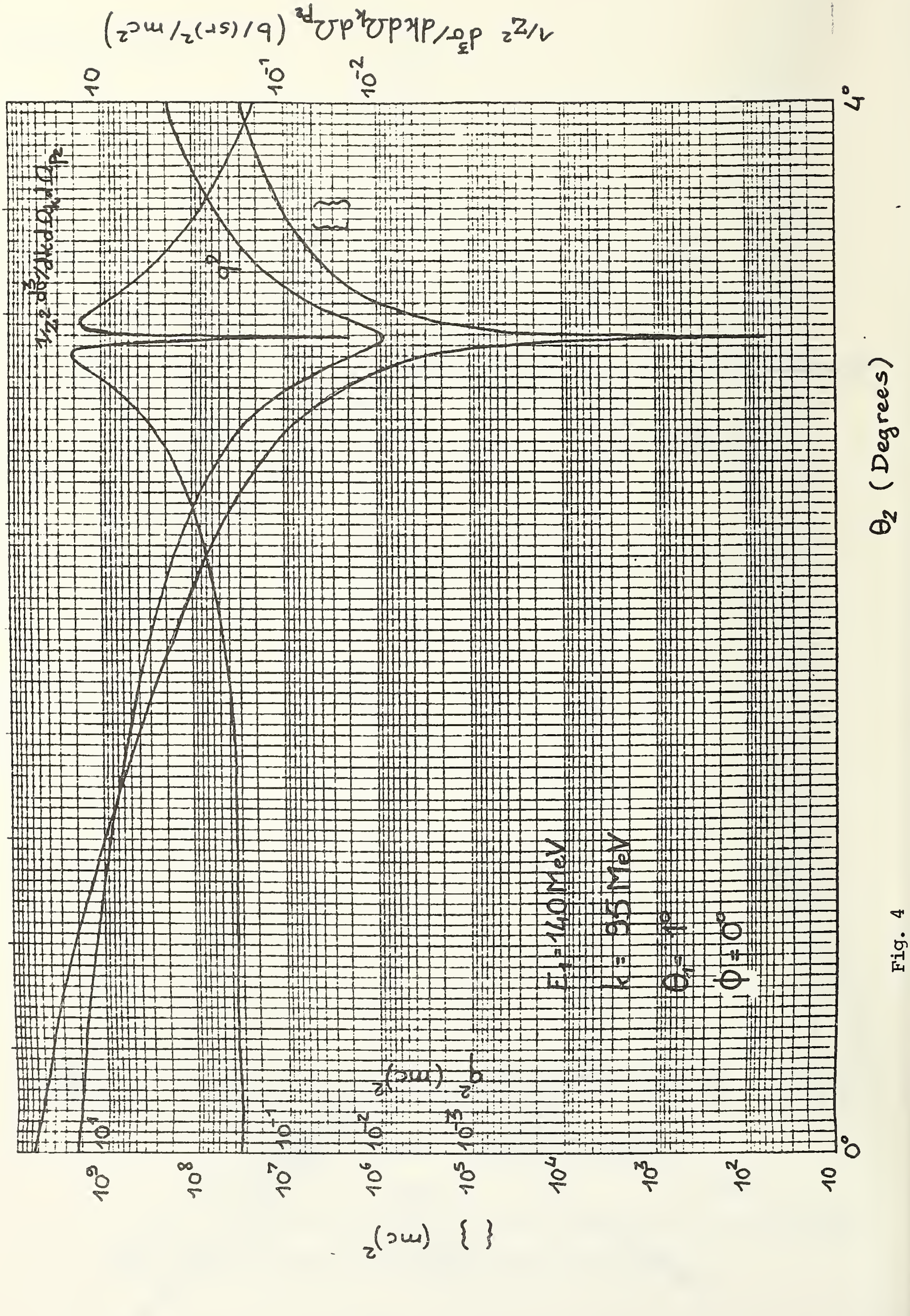
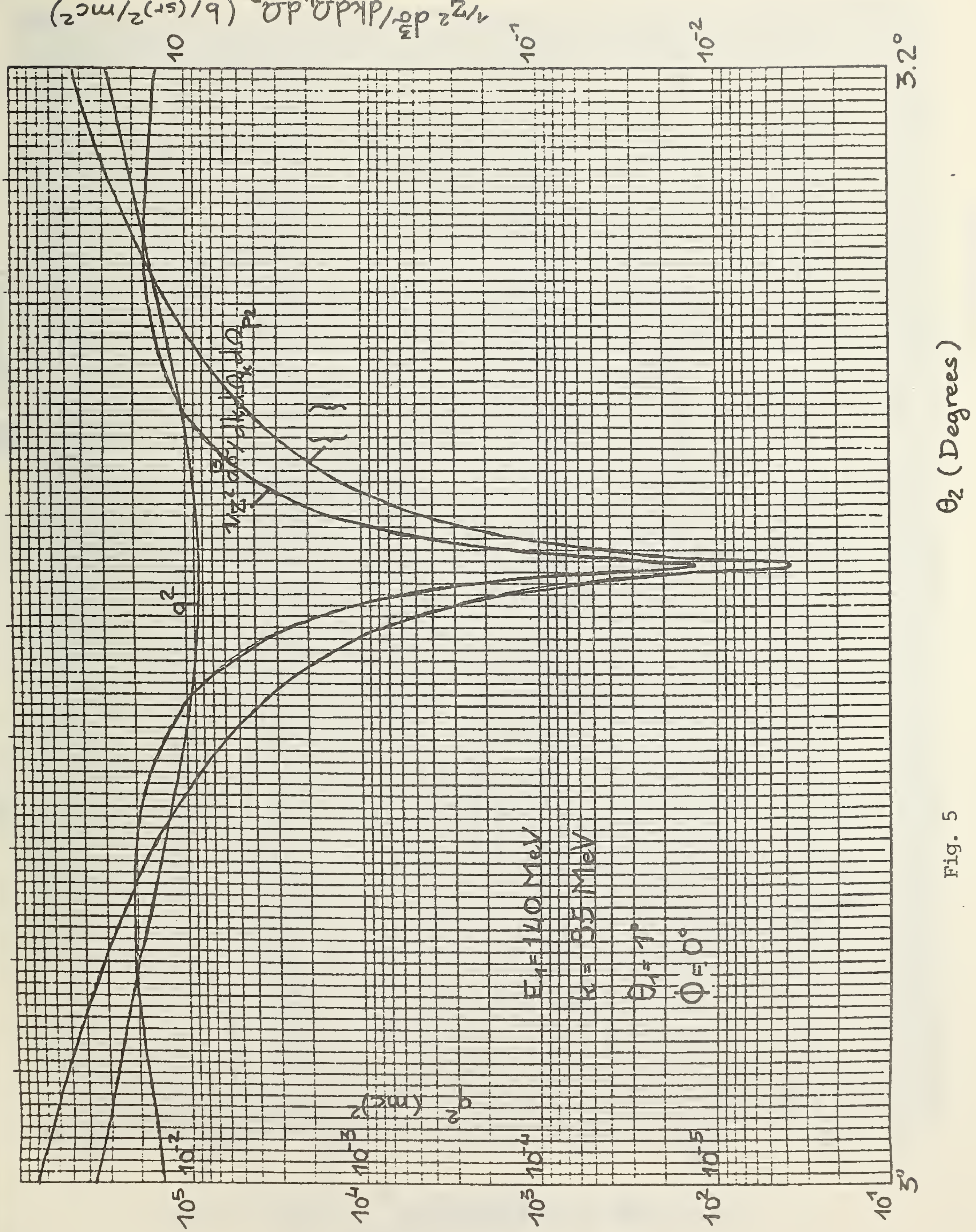

$\vec{p}_1$
$\vec{p}_2$
$\vec{k}$
θ_2
θ_1
ϕ
y
x
(a)
$\vec{p}_1$
$\vec{p}_2$
$\vec{k}$
θ_1
ψ
ω
$\omega' = (\omega - \pi)$
y
x
(b)

Fig. 3



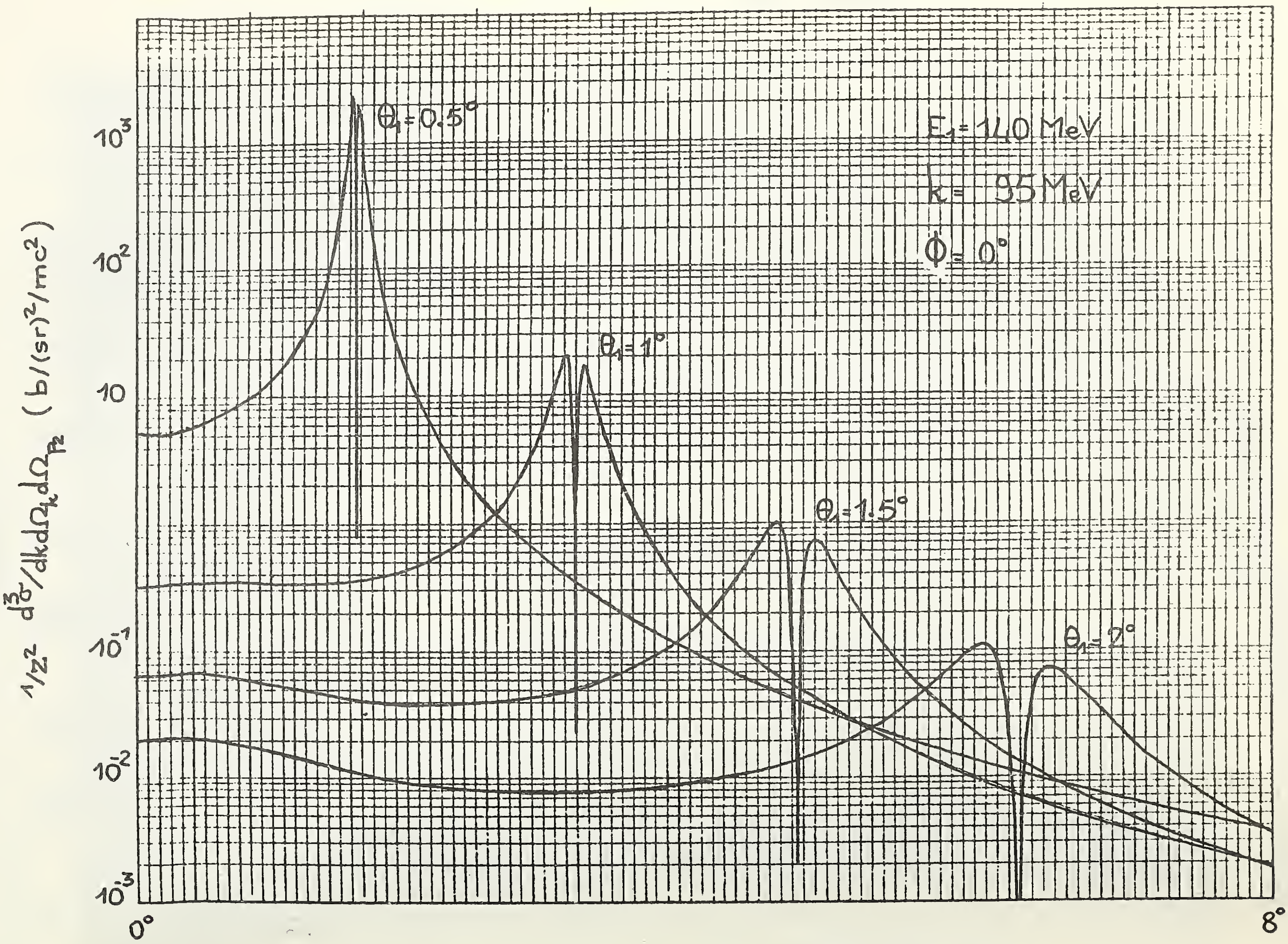
$E_1 = 140$ MeV
$k = 95$ MeV
$\theta_1 = 1°$
$\phi = 0°$
θ_2 (Degrees)
q^2
q^2 $(mc)^2$
{ } $(mc)^2$
$1/Z^2\ d^3\sigma/dk\,d\Omega_k\,d\Omega_{p_2}$ $(b/(sr)^2/mc^2)$

Fig. 4



$1/Z^2\, d^3\sigma/dk\,d\Omega_k\,d\Omega_{p_2}$ (b/(sr)^2/mc^2)
10
10^{-1}
10^{-2}
{ } (mc)^2
10^5
10^4
10^3
10^2
10^1
q^2 (mc)^2
10^{-2}
10^{-3}
10^{-4}
10^{-5}
q^2
{ }
$E_1 = 140$ MeV
$k = 35$ MeV
$\theta_1 = 1°$
$\Phi = 0°$
5°
3.2°
θ_2 (Degrees)

Fig. 5



$\theta_1 = 0.5°$
$\theta_1 = 1°$
$\theta_1 = 1.5°$
$\theta_1 = 2°$
$E_1 = 140$ MeV
$k = 95$ MeV
$\phi = 0°$
10^3
10^2
10
10^{-1}
10^{-2}
10^{-3}
0°
8°
$1/Z^2 \; d^3\sigma/dk\,d\Omega_k\,d\Omega_{p_2}$ $(b/(sr)^2/mc^2)$
θ_2 (Degrees)

Fig. 6

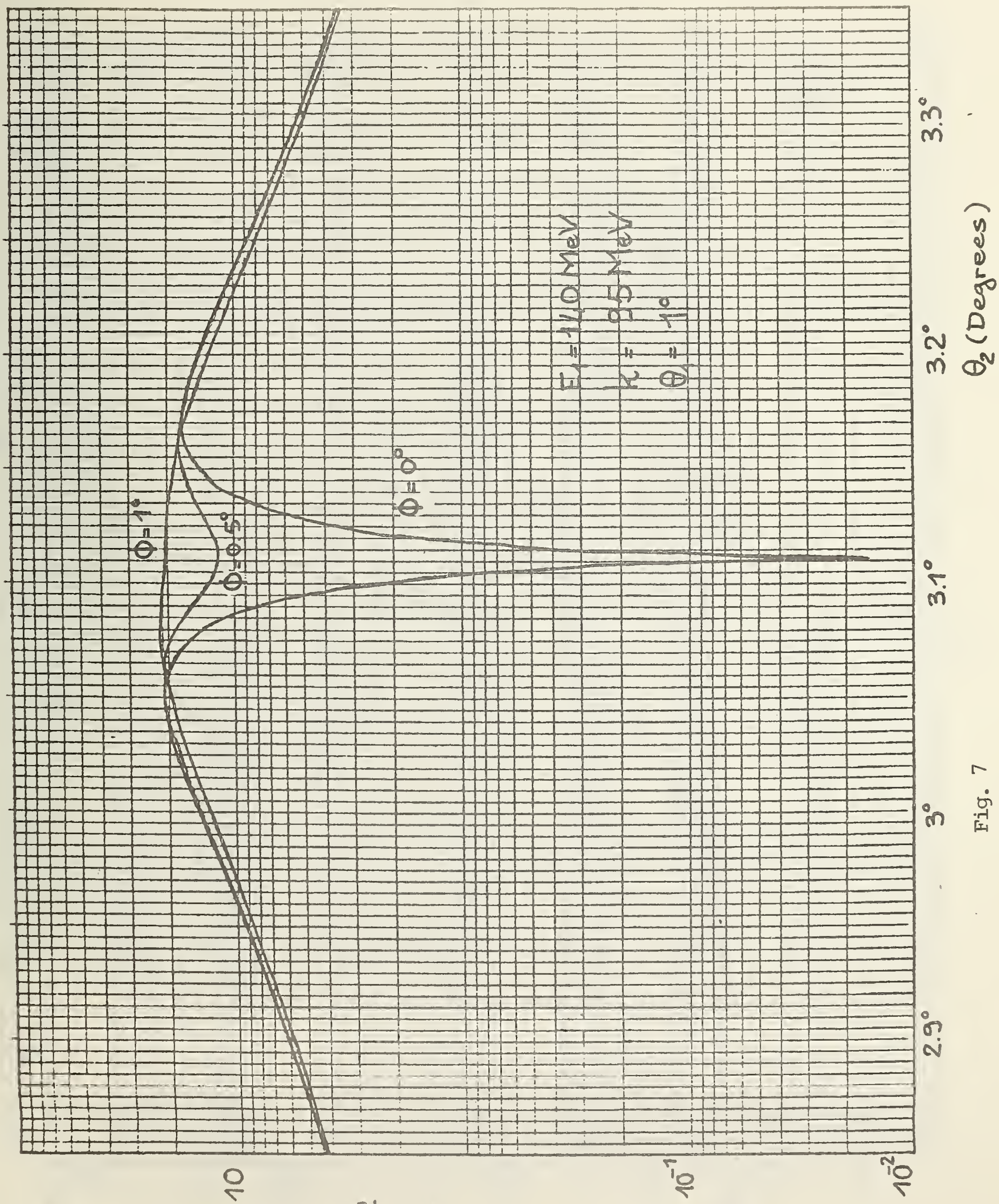

$E_1 = 140$ MeV
$k = 85$ MeV
$\theta_1 = 1°$
$\Phi = 1°$
$\Phi = 0.5°$
$\Phi = 0°$
2.9°
3°
3.1°
3.2°
3.3°
θ_2 (Degrees)
10
10^{-1}
10^{-2}
$1/Z^2 \; d^3\sigma/dk\, d\Omega_k\, d\Omega_{p_2}$ $(b/(sr)^2/mc^2)$

Fig. 7

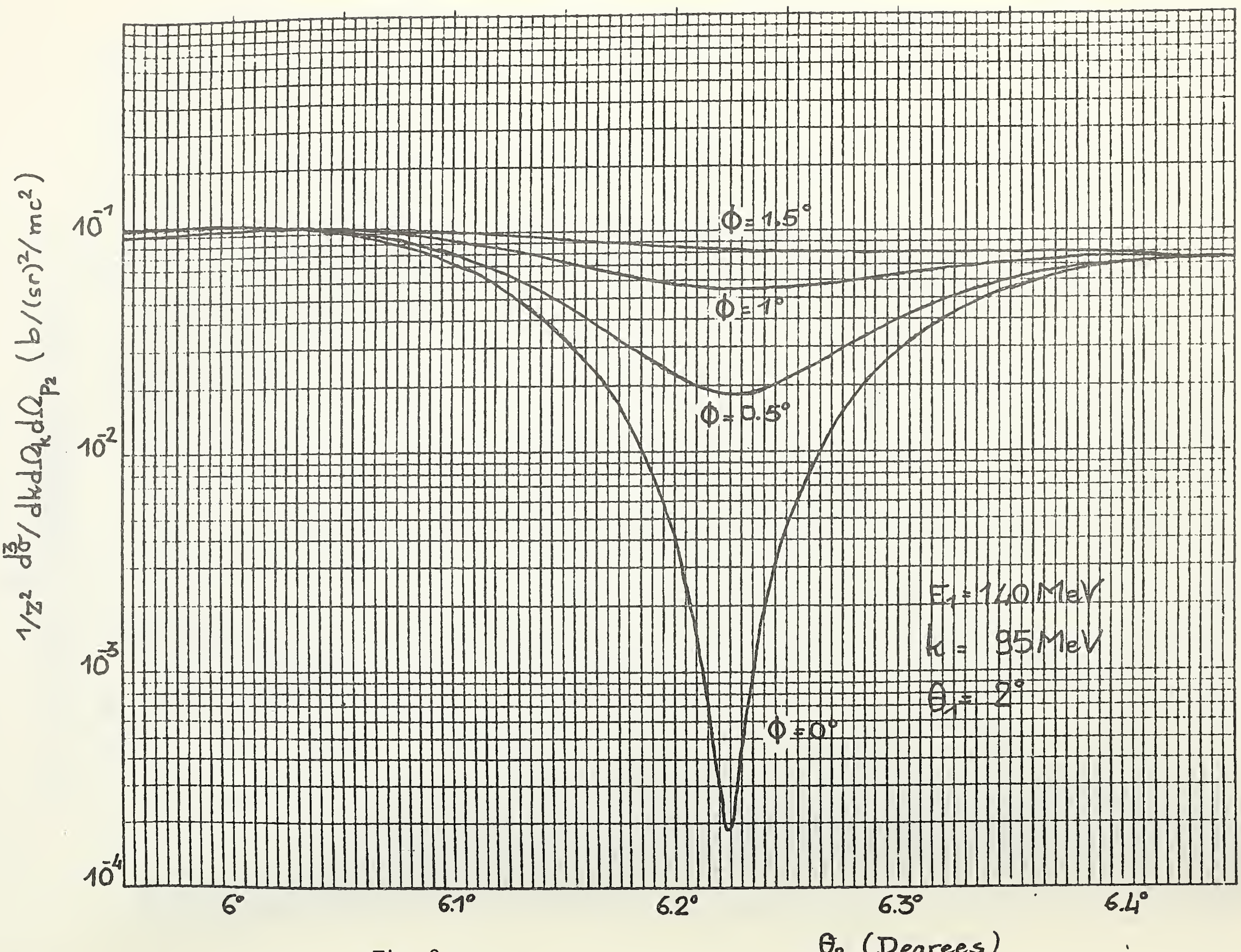

10^{-1}
10^{-2}
10^{-3}
10^{-4}
$1/Z^2 \; d^3\sigma / dk\, d\Omega_k\, d\Omega_{p_2}$ $(b/(sr)^2/mc^2)$
$\Phi = 1.5°$
$\Phi = 1°$
$\Phi = 0.5°$
$\Phi = 0°$
$E_1 = 140$ MeV
$k = 95$ MeV
$\theta_1 = 2°$
6°
6.1°
6.2°
6.3°
6.4°
θ_2 (Degrees)

Fig. 8

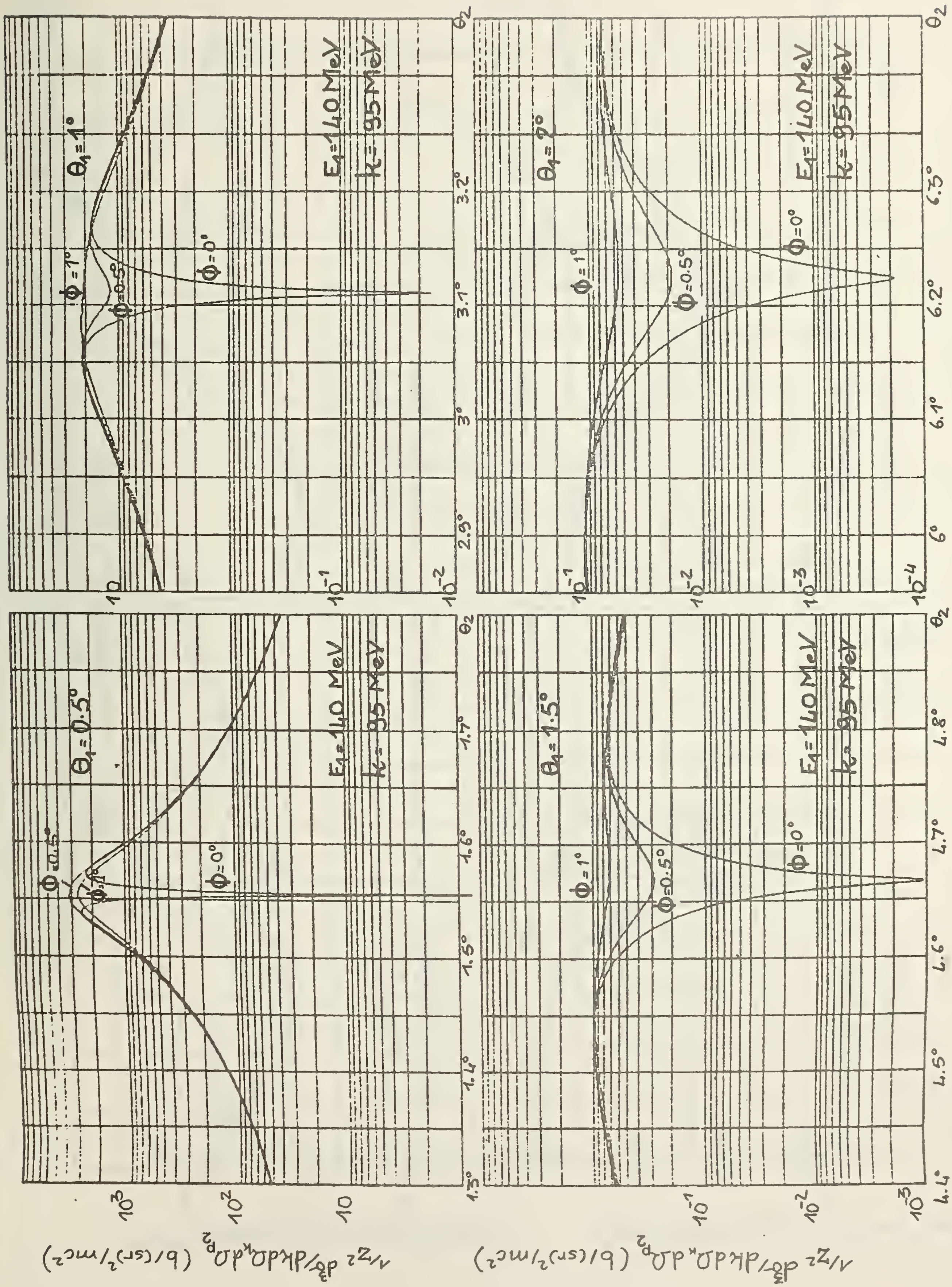

θ1 = 0.5°
E1 = 140 MeV
k = 95 MeV
Φ = 0.5°
Φ = 1°
Φ = 0°
θ1 = 1°
E1 = 140 MeV
k = 95 MeV
Φ = 1°
Φ = 0.5°
Φ = 0°
θ1 = 1.5°
E1 = 140 MeV
k = 95 MeV
Φ = 1°
Φ = 0.5°
Φ = 0°
θ1 = 2°
E1 = 140 MeV
k = 95 MeV
Φ = 1°
Φ = 0.5°
Φ = 0°
θ2
1/Z² d³σ/dk dΩk dΩp2 (b/(sr)²/mc²)

Fig. 9

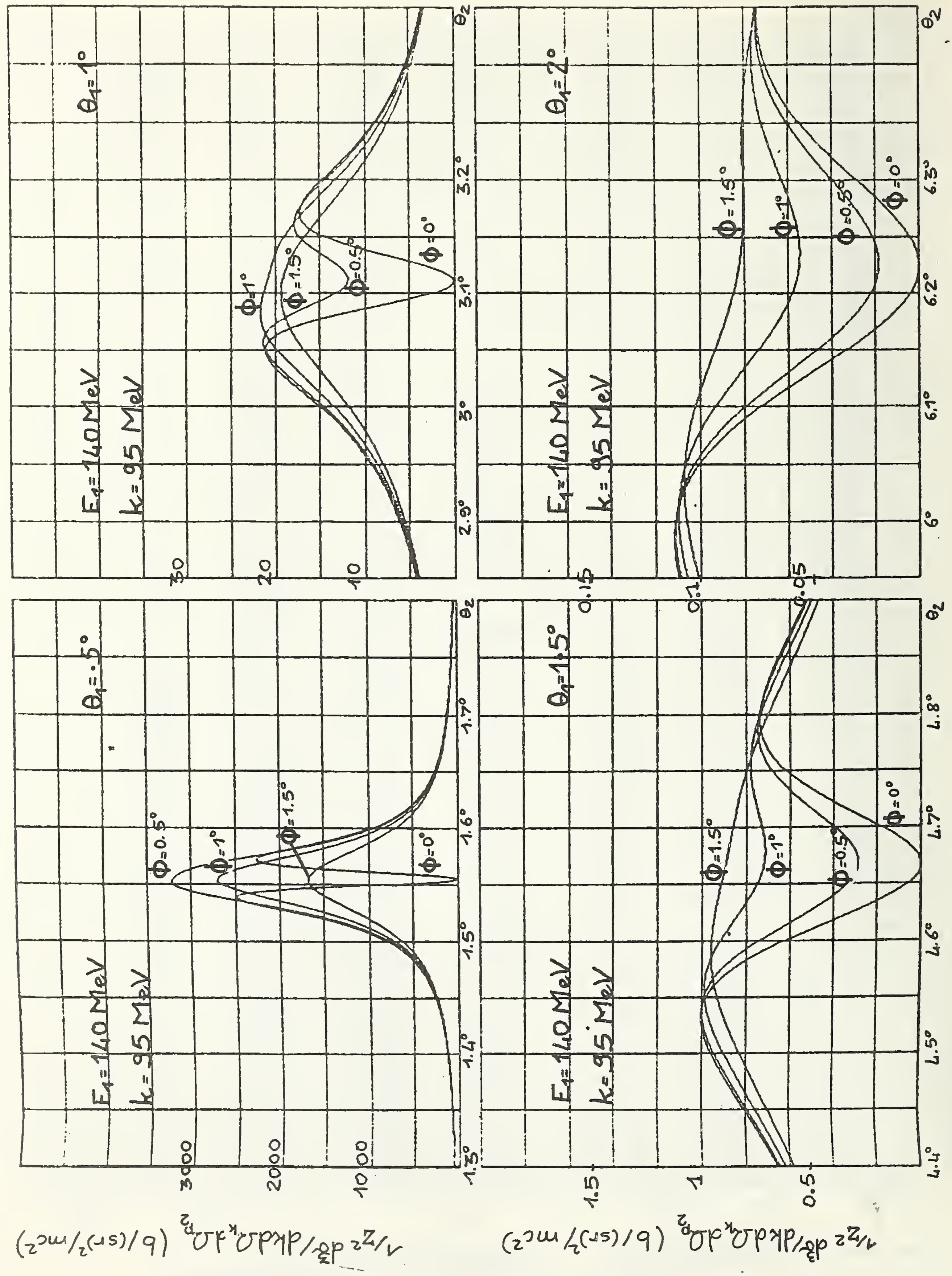
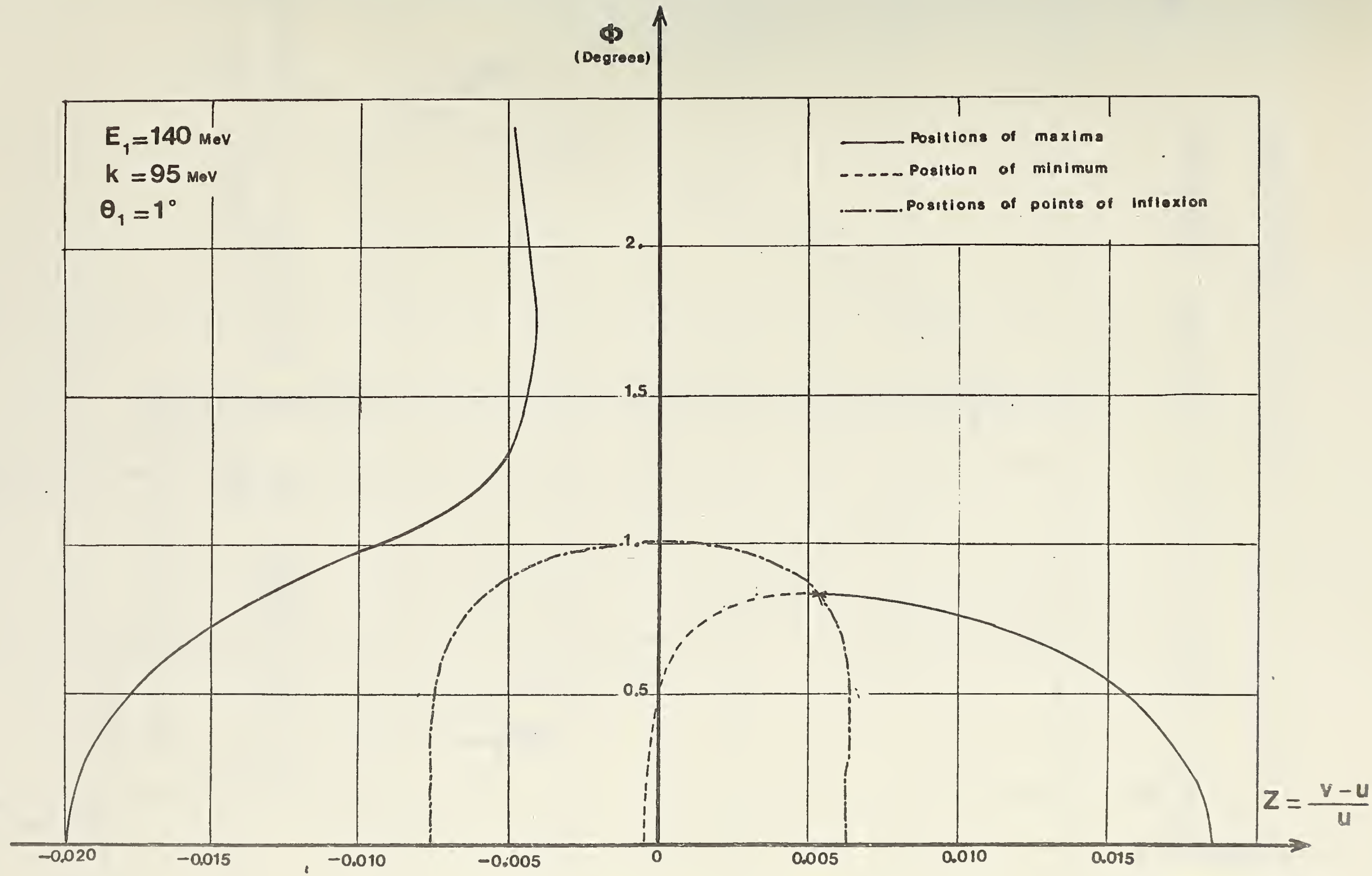

$E_1 = 140$ MeV
$k = 95$ MeV
$\theta_1 = .5°$
$\theta_1 = 1°$
$\theta_1 = 1.5°$
$\theta_1 = 2°$
$\Phi = 0°$
$\Phi = 0.5°$
$\Phi = 1°$
$\Phi = 1.5°$
θ_2
$1/Z^2 \, d\sigma^3/dk\, d\Omega_k\, d\Omega_p$ $(b/(sr)^2/mc^2)$

Fig. 10



Φ
(Degrees)
E1=140 MeV
k =95 MeV
θ1 =1°
Positions of maxima
Position of minimum
Positions of points of inflexion
2
1.5
1
0.5
-0.020
-0.015
-0.010
-0.005
0
0.005
0.010
0.015
Z = (v - u)/u

Fig. 11

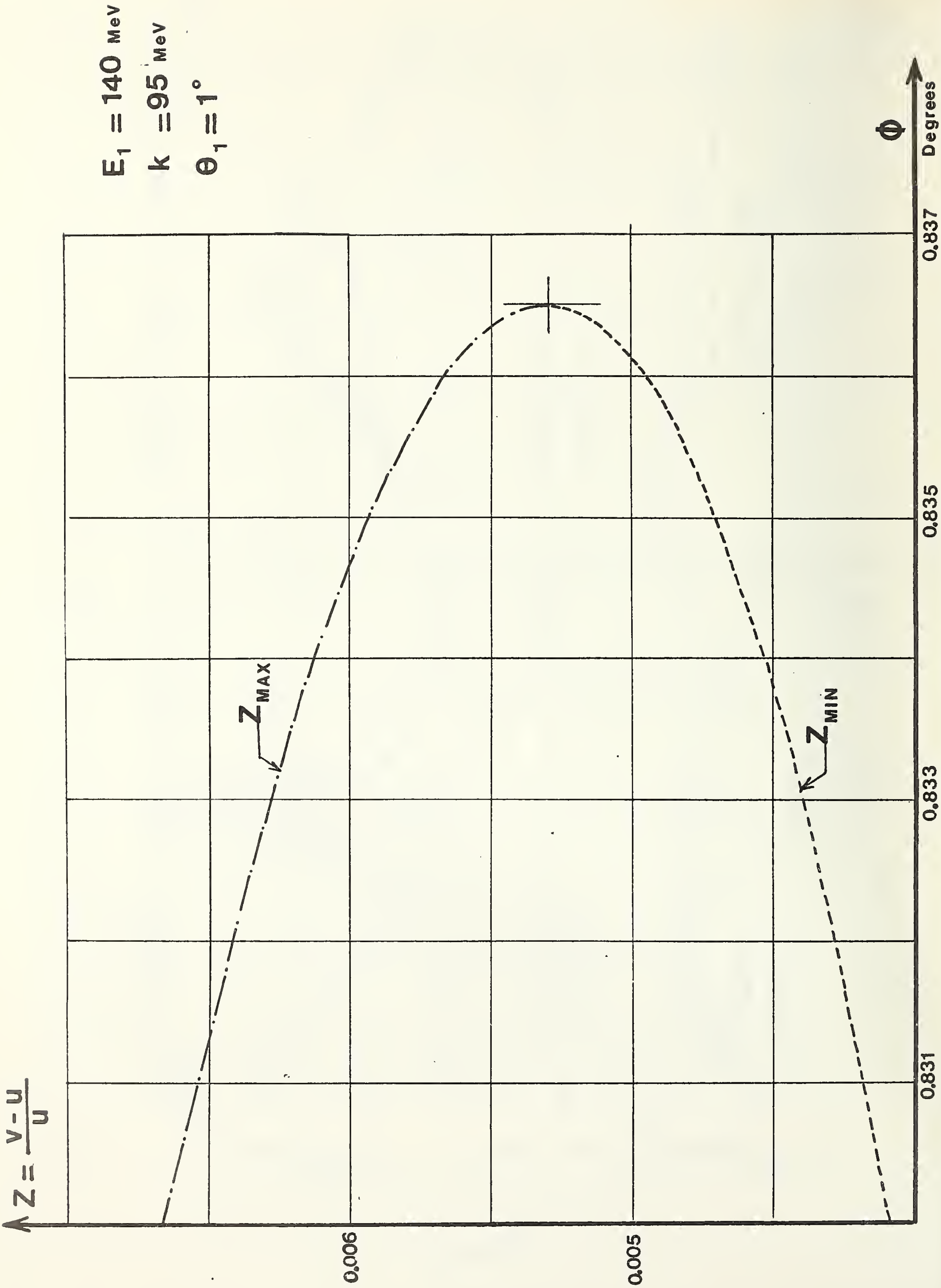

$E_1 = 140$ MeV
$k = 95$ MeV
$\theta_1 = 1°$
Φ
Degrees
0.837
0.835
0.833
0.831
$Z = \frac{V-U}{U}$
0.006
0.005
Z_{MAX}
Z_{MIN}

Fig. 12

Momentum transfer $\vec{q}$ corresponding to different regions of the cross section

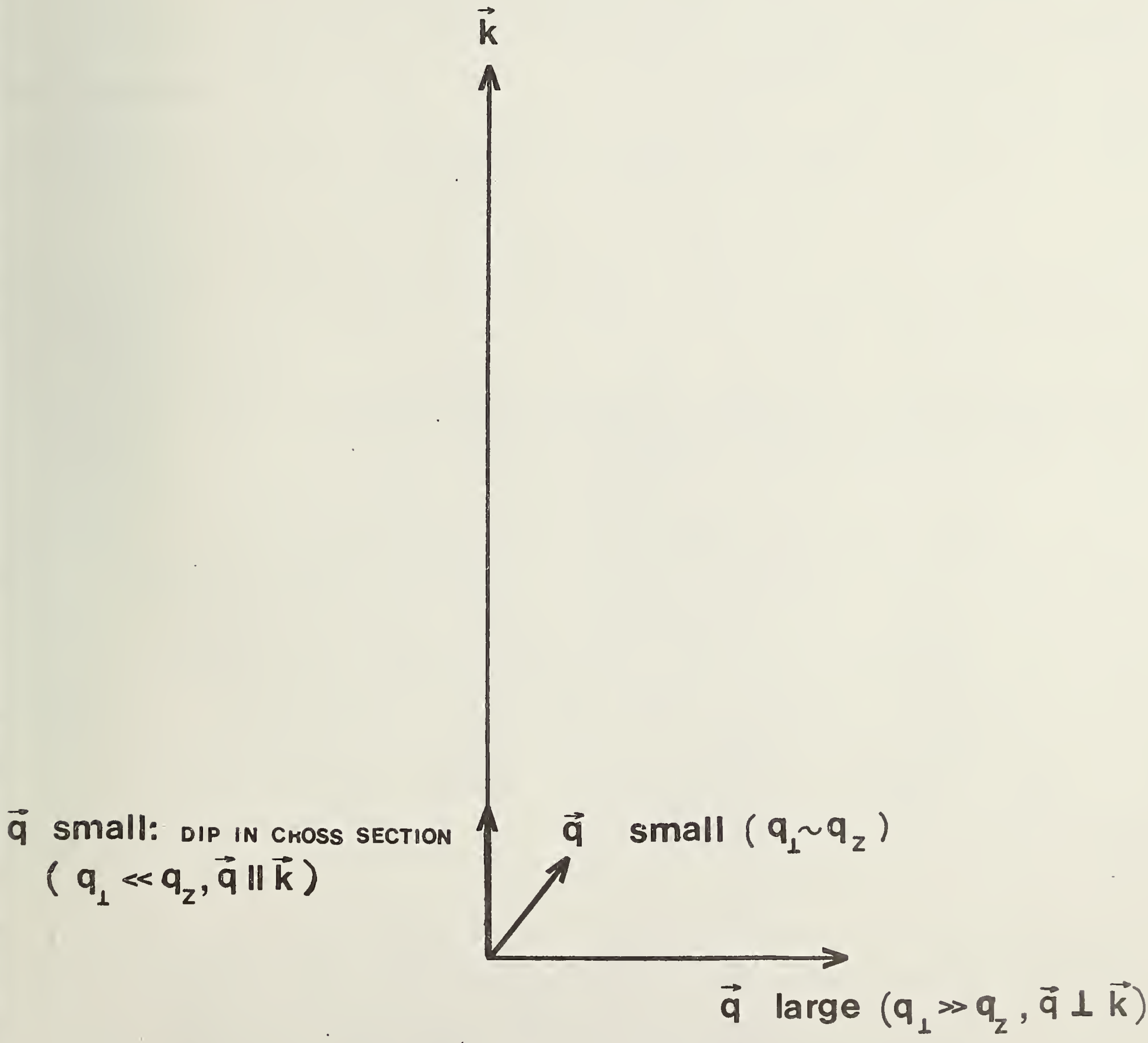


Fig. 13

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Leonard C. Maximon, Arlette de Miniac, and Thierry Aniel

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We consider in detail the differential cross section for bremsstrahlung for angles and energies in the range of interest for a tagging system. We derive a high energy, small angle approximation for the differential cross section for bremsstrahlung, eq (I.1). We use this approximation to determine the maxima and minimum of the cross section and to evaluate it at these extrema. It is shown that the differential cross section has a very sharp dip in the region of small momentum transfers. Coulomb corrections to the Born approximation are considered, and do not fill in this dip.

12. KEY WORDS (Six to twelve entries; alphabetical order; capitalize only proper names; and separate key words by semicolons)

Bethe-Heitler cross section; bremsstrahlung differential cross section; bremsstrahlung monochromator; photon beams; photonuclear research; tagged photon method

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