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AN APPROXIMATE EXPRESSION FOR THE PRINCIPAL BEAMWIDTHS OF DIRECTIVE ANTENNAS IN TERMS OF APERTURE FIELDS

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An approximate, but general, formula for the half-power prinicpal beamwidths of directive antennas is derived from the Fourier transform of the aperture electric field. The derivation, which can be accomplished by expanding the transform in a Taylor series, is greatly streamlined by introducing a Gaussian beam approximation. The formula, which also relates beamwidth to directivity indirectly through simple integrations of the aperture field, reduces to a very compact expression in terms of the zero and second order moments of the aperture distribution for linearly polarized, uniformly phased aperture fields.

Key words: Antenna beamwidths; aperture fields; directive antennas.

Because the far field of an arbitrary antenna can be expressed exactly in terms of a double Fourier transform of the transverse electric field on an "aperture" plane in front of the antenna [1], the directive gain function of an antenna can also be expressed simply in terms of the aperture $\underline{\text{transverse}}$ electric field $\bar{E}_a$ by the following formula:

$$D(\theta,\phi) = \frac{2\pi Y_o}{\lambda^2 P_o} \cos^2\theta \left| (1-\sec\theta\ \hat{e}_z\hat{e}_r\cdot)\int_A \bar{E}_a \exp(-ik\hat{e}_r\cdot\bar{R}_a)dA \right|^2. \quad (1a)$$

We will be concentrating on the main beam of directive antennas which are electrically large (in both aperture dimensions) with fairly uniform

phase[†] for each significant polarization in the transverse aperture field, thus producing a narrow main beam. (The term "directive" will be used throughout to denote specifically this class of antennas.) In the main beam of such antennas θ^2 is much less than 1 radian^2, and (1a) is well approximated by the simplified formula,[††]

$$D(\theta,\phi) = \frac{2\pi Y_0}{\lambda^2 P_0} \left| \int_A \bar{E}_a \exp(-ik\hat{e}_r \cdot \bar{R}_a) dA \right|^2 . \tag{1b}$$

The coordinates in (1) are defined in figure 1, $k = 2\pi/\lambda$ is the free-space propagation constant, λ is the wavelength, Y_0 is the wave admittance, A is the designated planar aperture which is truncated to a finite size, and $\exp(-i\omega t)$ time dependence for the fields has been suppressed. If the power P_0 radiated by the antenna is approximated by integrating $.5Y_0|\bar{E}_a|^2$ over the aperture plane, (1) reduces, for its maximum value, to the familiar formula for directivity D_0 in terms of aperture fields ([1a], eq (3.37)),

$$D_0 = \frac{4\pi}{\lambda^2} \frac{\left| \int_A \bar{E}_a \, dA \right|^2}{\int_A |\bar{E}_a|^2 \, dA} . \tag{2}$$

The direction of maximum gain has been chosen to lie along the z-axis ($\theta = 0$), i.e., $D_\theta(0) \equiv \frac{\partial D(0,\phi)}{\partial \theta} = 0$.

The main purpose of this communication is to derive a simple approximate formula, similar to (2) for half-power (3 dB) beamwidths of directive antennas in terms of aperture fields. We can accomplish this by expanding (1) in a Taylor series about $\theta = 0$, thereby generating a generalized moment expansion, and discarding terms which are negligible for directive antennas. However, it turns out that the same basic

[†]This restriction can be relaxed for many aperture illuminations (e.g., see (16)), and the central result (10) remains a good approximation.

[††]Equation (1b) is a decent approximation to (1a) for fairly large θ; e.g., when $\theta = 15$ degrees, $D(\theta,\phi)$ determined from (1a) and (1b) differ by only a few percent.

approximate formula (10) which results for principal beamwidths can be derived in a much less complicated way by assuming the directive gain within -3 dB of the main-beam axis obeys a Gaussian curve with respect to θ; specifically,

$$D(\theta,\phi) = D_0 \, e^{-g(\phi)\theta^2}. \tag{3}$$

(Far-field patterns determined from measurements indicate that (3) is satisfied down to -3 dB to a high degree of accuracy for directive antennas.)

Observe in (3) that the beam is not assumed to be circularly symmetric but can vary azimuthally with φ. However, the azimuthal function $g(\phi)$ is restricted by the fact that for finite-sized apertures (1) demands all derivatives of D with respect to x and y be continuous. Since $x = r \sin\theta \cos\phi$ and $y = r \sin\theta \sin\phi$, a straight-forward application of the chain rule to any function $D(\theta,\phi)$ with continuous second derivatives proves that

$$D_{\theta\theta}(0,\phi) = D_{xx}^0 \cos^2\phi + D_{xy}^0 \sin2\phi + D_{yy}^0 \sin^2\phi, \tag{4}$$

where subscripts indicate their respective partial derivatives, and the superscript o denotes the values of the partial derivatives D_{xx}, D_{xy}, and D_{yy} at $(x,y,z) = (0,0,1)$. D_{xx}^0, D_{xy}^0, and D_{yy}^0 are constants for chosen xy axes; however, $-D_{\theta\theta}(0,\phi)$, the curvature of $D(\theta,\phi)$ at $\theta = 0$, varies, in general, as a function of φ.

For the sake of some simplification, we can choose the xy axes along principal directions defined so that $D_{xy}^0 = 0$. (A straightforward rotation of coordinates proves such a choice is always possible [2]. For many linearly polarized antennas, these principal directions will coincide with "E-plane" and "H-plane" cuts.) Noting from (3) that

$$D_{\theta\theta}(0,\phi) = -2 D_0 \, g(\phi), \tag{5}$$

we can then express $g(\phi)$ more specifically in terms of curvature as

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$$g(\phi) = \frac{-1}{2D_0} \left[D_{xx}^0 \cos^2\phi + D_{yy}^0 \sin^2\phi \right]. \tag{6}$$

A formal expression for the half-power beamwidth θ_B, which is a function of ϕ, follows directly from (3);

$$\theta_B = 2 \sqrt{\ln 2} \sqrt{\frac{1}{g(\phi)}}, \tag{7}$$

with, of course, $g(\phi)$ given by (6). By inspection of (4) or (6) we obtain the elementary theorem of differential geometry [2], that the principal curvatures along the principal directions ($\phi = 0$ or π) and ($\phi = \frac{\pi}{2}$ or $\frac{3\pi}{2}$) are the minimum and maximum curvatures (or vice versa). Hence from (7) and (6) the "principal beamwidths" along the principal directions are the minimum and maximum beamwidths given by

$$\theta_B^x = 2 \sqrt{2\ln 2} \sqrt{\frac{-D_0}{D_{xx}^0}}, \tag{8a}$$

$$\theta_B^y = 2 \sqrt{2\ln 2} \sqrt{\frac{-D_0}{D_{yy}^0}}. \tag{8b}$$

Moveover, θ_B from (7) can be rewritten for all ϕ in terms of the principal beamwidths from (8);

$$\theta_B(\phi) = \frac{\theta_B^x \theta_B^y}{\sqrt{(\theta_B^x)^2 \sin^2\phi + (\theta_B^y)^2 \cos^2\phi}}. \tag{9}$$

Equation (9) is the representation of an ellipse in polar coordinates, and thus expresses explicitly that the -3 dB contour for narrow main beams approximates an ellipse.

All that remains to finding the beamwidth in terms of aperture fields is to evaluate the partial derivatives D_{xx}^0 and D_{yy}^0 from (1). This straightforward exercise in calculus finally yields for directive

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antennas,

$$
\begin{bmatrix} \theta^x_B \\ \theta^y_B \end{bmatrix} = 2 \sqrt{\ln 2} \;\; \frac{\left| \int\limits_A \bar{E}_a \, dA \right|}{k \sqrt{ \mathrm{Re}\left[\int\limits_A \bar{E}_a \begin{bmatrix} x^2_a \\ y^2_a \end{bmatrix} dA \cdot \int\limits_A \bar{E}^*_a \, dA \right] - \left| \int\limits_A \bar{E}_a \begin{bmatrix} x_a \\ y_a \end{bmatrix} dA \right|^2 }} \;,
$$

$$(10)$$

For most directive antennas used in practice, $\bar{E}_a$ exhibits a reasonable degree of even symmetry with respect to x and y axes whose origin is located at a central point in the aperture. For these antennas the second term in the denominator of (10) is negligible because x_a and y_a are odd functions with respect to the origin. For a perfectly symmetric (even with respect to x and y) aperture field, or for a linearly polarized, uniformly phased aperture field, the origin can always be chosen to make this second term exactly zero, i.e.

$$
\int\limits_A \bar{E}_a \begin{vmatrix} x_a \\ y_a \end{vmatrix} dA = 0.
$$

$$(11a)$$

Recall also that the orientations of the xy axes have been chosen along principal directions defined such that $D^o_{xy} = 0$, which for the above stated aperture fields satisfying (11a), translates from (1b) to

$$
\int\limits_A \bar{E}_a x_a y_a \, dA = 0 \; .
$$

$$(11b)$$

If the transverse electric field is given or measured, an estimate of the beamwidths in (8) and (9) is provided through evaluation of the four (and usually three) independent aperture area integrations in (10). For linearly polarized, uniformly phased aperture fields, (10) and (11) reduce to a more compact form which allows θ^x_B and θ^y_B from (8) to be written in convenient "moment" terminology,

$$
\theta^x_B = \frac{2 \sqrt{\ln 2}}{k} \sqrt{\frac{\mu_{00}}{\mu_{20}}}
$$

$$(12a)$$

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$$\theta_B^y = \frac{2 \sqrt{\ell n\ 2}}{k} \sqrt{\frac{\mu_{00}}{\mu_{02}}}\ , \tag{12b}$$

$$(\mu_{10} = \mu_{01} = \mu_{11} = 0)$$

where μ_{mn} is the joint moment defined by

$$\mu_{mn} = \int_A \left|\bar{E}_a\right| x^m y^n\ dx\ dy\ . \tag{13}$$

For a circularly symmetric beam, $\theta_B^x = \theta_B^y = \theta_B^c$ and (12) combine with (9) to produce

$$\theta_B^c = \frac{2 \sqrt{2\ell n\ 2}}{k} \sqrt{\frac{\int_A \left|E_a\right|\ dA}{\int_A \left|E_a\right| R_a^2\ dA}}. \tag{14}$$

Equations (10), (12), and (14) are approximate expressions for the beamwidth which approach in simplicity of form and ease of evaluation the corresponding approximate expression (2) for directivity. Although (10), (12), and (14) were derived by assuming a Gaussian main beam pattern down to -3 dB, it is emphasized that this is an unnecessary artifact introduced to greatly streamline their derivation. The same equations except for a factor $\sqrt{2 - \sqrt{2}} = 0.77$ replacing $\sqrt{\ell n\ 2} = 0.83$ emerge directly from (1) by expanding $\sqrt{D(\theta,\phi)}$ in a Taylor series about $\theta = 0$, retaining terms out to the fourth power in θ, setting $D_\theta(0) = 0$, and showing that the third and fourth terms are negligible for directive antennas. If the $\sqrt{\ell n\ 2}$ factor in (12) is replaced by $\sqrt{2 - \sqrt{2}}$, (12) becomes identical to Spencer's result for two-dimensional scalar patterns from uniform-phase, symmetric aperture distributions ([3], Sec. 6.6).

Table I compares beamwidths computed from the approximate expression (14) with the exact values obtained by Hansen [4] and Silver ([3], Table 6.2) for circular aperture distributions with different amplitude tapers. The maximum discrepancy of 3 percent between exact and approximate values occurs for the uniform distribution (no taper).

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Finally, we estimate θ_B^C for the same circular aperture distributions using the common, rule-of-thumb relationship [5] which emerges between directivity and principal beamwidths when the total radiated power is roughly approximated by the power in the "equivalent" Gaussian beam (3). Specifically, insert $D(\theta,\phi)$ from (3) into the identity,

$$1 = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} D(\theta,\phi) \sin\theta \; d\theta \; d\phi,$$

perform the integration for large $g(\phi)$ (narrow beams), and obtain after substitution from (8)

$$D_0 = \frac{16 \; \ell n \; 2}{\theta_B^x \; \theta_B^y} = \frac{11.09}{\theta_B^x \; \theta_B^y} \quad . \tag{15}$$

This relationship, which has been derived by Ko [6] for circular beams $(\theta_B^x = \theta_B^y)$, represents somewhat of a compromise between Kraus's formula [5], $D_0 = 4\pi / \theta_B^x \; \theta_B^y$, and Tai and Pereira's formula [7], $D_0 = 16 \; \ell n \; 2 \; / \; \frac{1}{2}(\theta_B^{x2} + \theta_B^{y2})$.

The values of θ_B^C calculated from (15) (with D_0 evaluated from (2)) are included in table I. On the average they do not lie as close to the exact beamwidths as the values calculated from (14). But this would be expected since the derivation of (15), unlike (14), depends strongly upon the crude approximation that the total radiated power is contained in an equivalent Gaussian beam. For antennas with pattern shape factors or beam efficiencies differing appreciably from unity [5b], the relationship (15) between directivity and principal beamwidths must be modified by these parameters to remain useful, whereas (10) continues to yield a reliable estimate of beamwidths because it inherently accounts for variations in beam efficiencies and pattern shape factors. This is a distinct advantage of (10) because often pattern shape factors and beam efficiencies are not readily available.

As a simple yet graphic example of an antenna aperture distribution

for which (15) fails but (10) gives an accurate approximation to the beamwidths, consider the electrically-large square aperture of side S illuminated with a linearly y-polarized transverse electric field

$$E_y = 1 + \beta e^{ikx\sin\alpha}, \quad (0<\beta<1, \frac{2\lambda}{S} <\alpha< \frac{\pi}{2}). \tag{16}$$

Such an antenna would have its mainbeam near $\theta = 0$ and a grating lobe near $\theta = \alpha$. Its mainbeam is nearly circular with beamwidth and directivity given approximately by $.89\lambda/S$ and $4\pi S^2/\lambda^2(1 + \beta^2 \cos\alpha)$, respectively. Thus, the beamwidth and directivity of this antenna greatly deviate from the relationship (15) for non-negligible $\beta^2\cos\alpha$. However, application of the formula (10) yields approximate beamwidths,
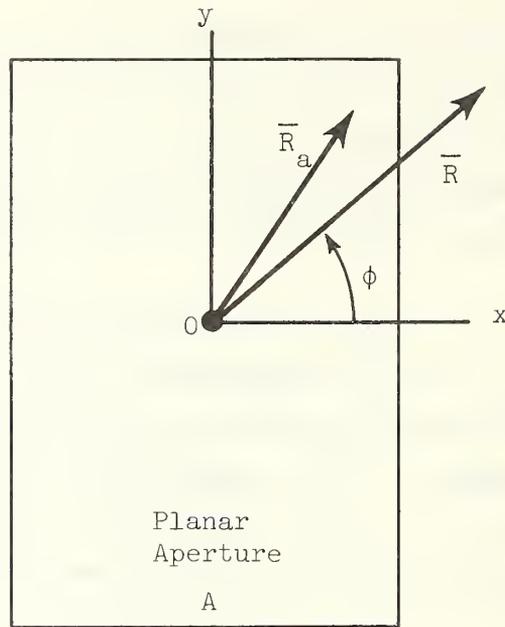
$$\theta_B^x \approx \theta_B^y \approx .92 \, \lambda/S,$$

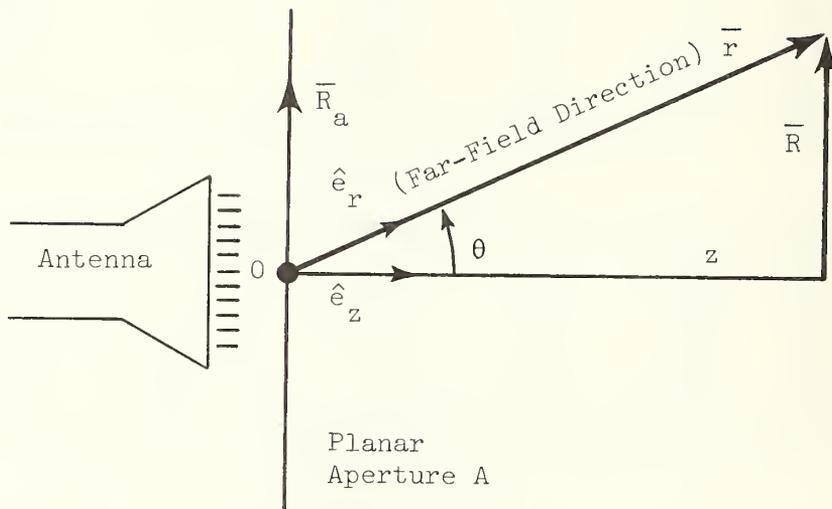which are within a few percent of the actual beamwidths.

TABLE 1

Exact and approximate values of beamwidth for circular aperture
distributions $\{b + [1 - (R_a/a)^2]^n\}$. (a is the radius of the aperture.)

	Beamwidth $\theta_B^c \times \frac{2a}{\lambda}$		
n	exact	approximate eq. (14)	approximate eq. (15)
b = 0			
0	1.03	1.06	1.06
1	1.27	1.30	1.22
2	1.47	1.50	1.43
3	1.65	1.68	1.58
4	1.81	1.84	1.77
b = 1/4			
1	1.17	1.20	1.14
2	1.23	1.25	1.18
3	1.25	1.27	1.19
b = 1/2			
1	1.13	1.162	1.105
2	1.16	1.186	1.130
3	1.16	1.186	1.136

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Figure 1. Definition of coordinates for
far-field and aperture integration;
$$\bar{R}_a = x_a \hat{e}_x + y_a \hat{e}_y.$$

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An approximate, but general, formula for the half-power principal beamwidths of directive antennas is derived from the Fourier transform of the aperture electric field. The derivation, which can be accomplished by expanding the transform in a Taylor series, is greatly streamlined by introducing a Gaussian beam approximation. The formula which also relates beamwidth to directivity indirectly through simple integrations of the aperture field, reduces to a very compact expression in terms of the zero and second order moments of the aperture distribution for linearly polarized uniformly phased aperture fields.

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