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Probabilities of Vertical Overlap: A Sensitivity Analysis

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Operations Research Division
Center for Applied Mathematics
National Bureau of Standards
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U.S. DEPARTMENT OF COMMERCE, Philip M. Klutznick, *Secretary*

Luther H. Hodges, Jr., *Deputy Secretary*
Jordan J. Baruch, *Assistant Secretary for Science and Technology*

NATIONAL BUREAU OF STANDARDS, Ernest Ambler, *Director*

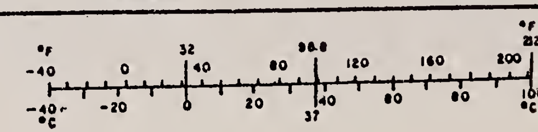
METRIC CONVERSION FACTORS

Approximate Conversions to Metric Measures

Symbol	When You Know	Multiply by	To Find	Symbol
		LENGTH		
in	inches	2.5	centimeters	cm
ft	feet	30	centimeters	cm
yd	yards	0.9	meters	m
mi	miles	1.6	kilometers	km
		AREA		
in^2	square inches	6.5	square centimeters	cm^2
ft^2	square feet	0.09	square meters	m^2
yd^2	square yards	0.8	square meters	m^2
mi^2	square miles	2.6	square kilometers	km^2
	acres	0.4	hectares	ha
		MASS (weight)		
oz	ounces	28	grams	g
lb	pounds	0.45	kilograms	kg
	short tons (2000 lb)	0.9	tonnes	t
		VOLUME		
tsp	teaspoons	5	milliliters	ml
Tbsp	tablespoons	15	milliliters	ml
fl oz	fluid ounces	30	milliliters	ml
c	cups	0.24	liters	l
pt	pints	0.47	liters	l
qt	quarts	0.95	liters	l
gal	gallons	3.8	liters	l
ft^3	cubic feet	0.03	cubic meters	m^3
yd^3	cubic yards	0.76	cubic meters	m^3
		TEMPERATURE (exact)		
°F	Fahrenheit temperature	5/9 (after subtracting 32)	Celsius temperature	°C

Approximate Conversions from Metric Measures

Symbol	When You Know	Multiply by	To Find	Symbol
		LENGTH		
mm	millimeters	0.04	inches	in
cm	centimeters	0.4	inches	in
m	meters	3.3	feet	ft
m	meters	1.1	yards	yd
km	kilometers	0.6	miles	mi
		AREA		
cm^2	square centimeters	0.16	square inches	in^2
m^2	square meters	1.2	square yards	yd^2
km^2	square kilometers	0.4	square miles	mi^2
ha	hectares (10,000 m^2)	2.5	acres	
		MASS (weight)		
g	grams	0.035	ounces	oz
kg	kilograms	2.2	pounds	lb
t	tonnes (1000 kg)	1.1	short tons	
		VOLUME		
ml	milliliters	0.03	fluid ounces	fl oz
l	liters	2.1	pints	pt
l	liters	1.06	quarts	qt
l	liters	0.26	gallons	gal
m^3	cubic meters	35	cubic feet	ft^3
m^3	cubic meters	1.3	cubic yards	yd^3
		TEMPERATURE (exact)		
°C	Celsius temperature	9/5 (then add 32)	Fahrenheit temperature	°F



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Abstract

Because of the potential increase in traffic at FL 290 and above, both current and alternative vertical separation standards are being reviewed. A plan to collect data on the vertical navigational performance of aircraft is also contemplated. This report documents a sensitivity analysis carried out to assess how different assumptions about the probability distribution of "total vertical error" affect estimates of the probability of vertical overlap. The four factors affecting the probability of vertical overlap which are examined in this study are: shape of the vertical-error distribution; the standard deviation (and other parameters, if any) of this probability distribution; the vertical dimensions of the aircraft; and the vertical separation standard. Probabilities of vertical overlap were computed over a range of possibilities for each of these four factors in order to discern the effect of each factor. A final section discusses the findings of this study and draws some conclusions.

Key Words: Air safety; collision probability; probability distribution; sensitivity analysis; separation standard; vertical error; vertical overlap.

1. Introduction

In its continuing effort to improve the National Airspace System, the Federal Aviation Administration (FAA) is evaluating the existing standards for separation of designated parallel flight tracks in the airspace over conterminous United States (CONUS). As part of this effort, the FAA intends to study the implications of both current and alternative vertical separation standards for aircraft operating over CONUS at flight levels of 29,000 feet (FL290) and above. The paucity of existing data on the vertical component of navigational performance necessitates initiating a data collection program to determine the altitude-keeping capabilities of aircraft flying at FL290 and above. The FAA is in the process of designing a plan for the collection and analysis of these data.

To support the FAA in this effort, the Operations Research Division of the National Bureau of Standards has been investigating analytical problems relating to aircraft vertical separation standards [1].[1] This report presents the results of one such study, a sensitivity analysis to assess how different assumed probability distributions of "total vertical error" affect the probability of vertical overlap.

The total vertical error[2] is defined as the difference between the assigned pressure altitude[3] of an aircraft and its true pressure altitude. A vertical

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[1]Numbers in the square brackets refer to the References listed at the end of this report.

[2]Total vertical error, as defined in Reference [1], is the sum of the pilot response error, the altimeter instrument error, and the static pressure system error.

[3]The altitude measured by an altimeter is termed pressure altitude here because it is first measured barometrically in terms of atmospheric pressure, and then converted to altitude.

overlap occurs when the vertical separation between the centroids of the rectangular envelope of a pair of aircraft is less than the average of the vertical dimensions of the aircraft. It should be emphasized that an overlap is not a collision, and that the probability of vertical overlap is not the probability of collision. A collision occurs only when the pair of aircraft overlap in all three dimensions at the same time. This report will examine the likelihood of overlap in only one dimension--the vertical dimension.

The four factors affecting the probability of vertical overlap which are examined in this study are: the distributional form of the probability density function for total vertical error, the standard deviation of this probability density function, the vertical dimensions of the aircraft, and the vertical separation standard. Probabilities of vertical overlap were computed over a range of possibilities for each of these four factors in order to discern the effect of each factor. The computational results have been plotted to permit visual inspection.

The remainder of this report is organized as follows. Section 2 reviews the definition of probability of vertical overlap between two aircraft, and Section 3 discusses the four factors affecting this probability which were studied. Section 4 briefly outlines the approach employed to compute the probability of vertical overlap. Section 5 reports the computational results. Finally, Section 6 discusses various findings of this study and draws some conclusions.

The report did not use metric units because in the fields of air traffic control and airspace management currently conducted in the U. S., the aircraft altitude is expressed either in feet or in hundreds of feet. A conversion of the altitude measurement system to metric units is not likely in the near future [9].

2. Probability of Vertical Overlap

The approach employed here to define the probability of vertical overlap is based on the collision risk methodology pioneered by P. Reich [2], which assumes that all aircraft in the track system are of the same size,[4] and which represents each aircraft as a rectangular slab with longitudinal, lateral, and vertical dimensions of λ_x, λ_y, and λ_z corresponding to the length, width, and height of the aircraft, respectively. Two aircraft are said to overlap in the rth dimension (where r can be x, y, or z) when the separation between the centroids of the two aircraft in the rth dimension becomes less than λ_r. A collision is considered to occur when a pair of aircraft overlap in three dimensions simultaneously; in other words, when the distances between the centroids of the two aircraft longitudinally, laterally, and vertically are less than λ_x, λ_y, and λ_z, respectively.

Since the probability of vertical overlap is the main subject of this report, we will explore this concept in detail.

Assume two aircraft are flying over the same geographic point on the earth. The vertical overlap probability between them is the probability that the centroid of the second aircraft ("2") is separated vertically by less than one aircraft height (the vertical dimension) from that of the first aircraft ("1"). Formally, we have

---

[4]If the sizes of a pair of aircraft differ in some dimension, the average of their sizes in that dimension can be used in the pertinent mathematics without loss of validity.

$$\int_v P\{\text{"1" is centered at } v \text{ and "2" is in } (v-\lambda_z, v+\lambda_z)\}$$

where λ_z is the aircraft vertical dimension, v ranges over the possible vertical positions of the first aircraft, and P{ } is the probability of the event in brackets.

If (as is assumed here) the two aircraft are flying independently, the above equation can be restated as follows:

$$P\{\text{Vertical Overlap}\} = \int_v P\{\text{"1" is at } v\}\ P\{\text{"2" is in } (v-\lambda_z, v+\lambda_z)\}$$

Assume $f_1(v)$ and $f_2(w)$ are the probability density function of total vertical error for aircraft 1 and aircraft 2, respectively. The probability that the centroid of aircraft 1 lies between altitudes v and v+dv, relative to its assigned altitudes, is $f_1(v)dv$. Since the two aircraft are nominally separated by a separation standard S in the vertical dimension, the probability that the altitude u of aircraft "2" (relative to aircraft 1's assigned altitude) lies in the interval between $v-\lambda_z$ and $v+\lambda_z$ is the same as the probability that its vertical error w=u-S lies between $(v-\lambda_z)-S$ and $(v+\lambda_z)-S$.

$$\int_{v-\lambda_z-S}^{v+\lambda_z-S} f_2(w)dw = \int_{v-\lambda_z}^{v+\lambda_z} f_2(u-S)du.$$

The probability that "1" is at v and the two aircraft overlap is

$$f_1(v)dv \int_{v-\lambda_z}^{v+\lambda_z} f_2(u-S)du.$$

By integrating over all values of v, the probability of vertical overlap can be represented as follows

$$P\{\text{Vertical Overlap}\} = P_z(S) = \int_{-\infty}^{\infty} f_1(v)\left[\int_{v-\lambda_z}^{v+\lambda_z} f_2(u-S)du\right]dv;$$

cf. References [1] - [2].

Throughout this paper we will assume that the probability distributions of vertical flight error are the same for both aircraft, i.e., $f_1 = f_2$. Although it is possible to relax this assumption, the analysis becomes considerably more complicated, and there is no reason to believe that different distributions are likely.

3. Factors Affecting Probability of Vertical Overlap

From the above discussion, it is clear that the probability of vertical overlap of a pair of aircraft depends on the following factors:

(1) The shape of the probability density function ($f_1 = f_2 = f$) for total vertical error of an aircraft.

(2) The parameter(s) that specify the probability density function (e.g. the standard deviation).

(3) The vertical dimension (λ_z) of the aircraft.

(4) The vertical separation standard (S).

We shall discuss each of these factors in the following pages, and present computed results on their effects upon overlap probabilities.

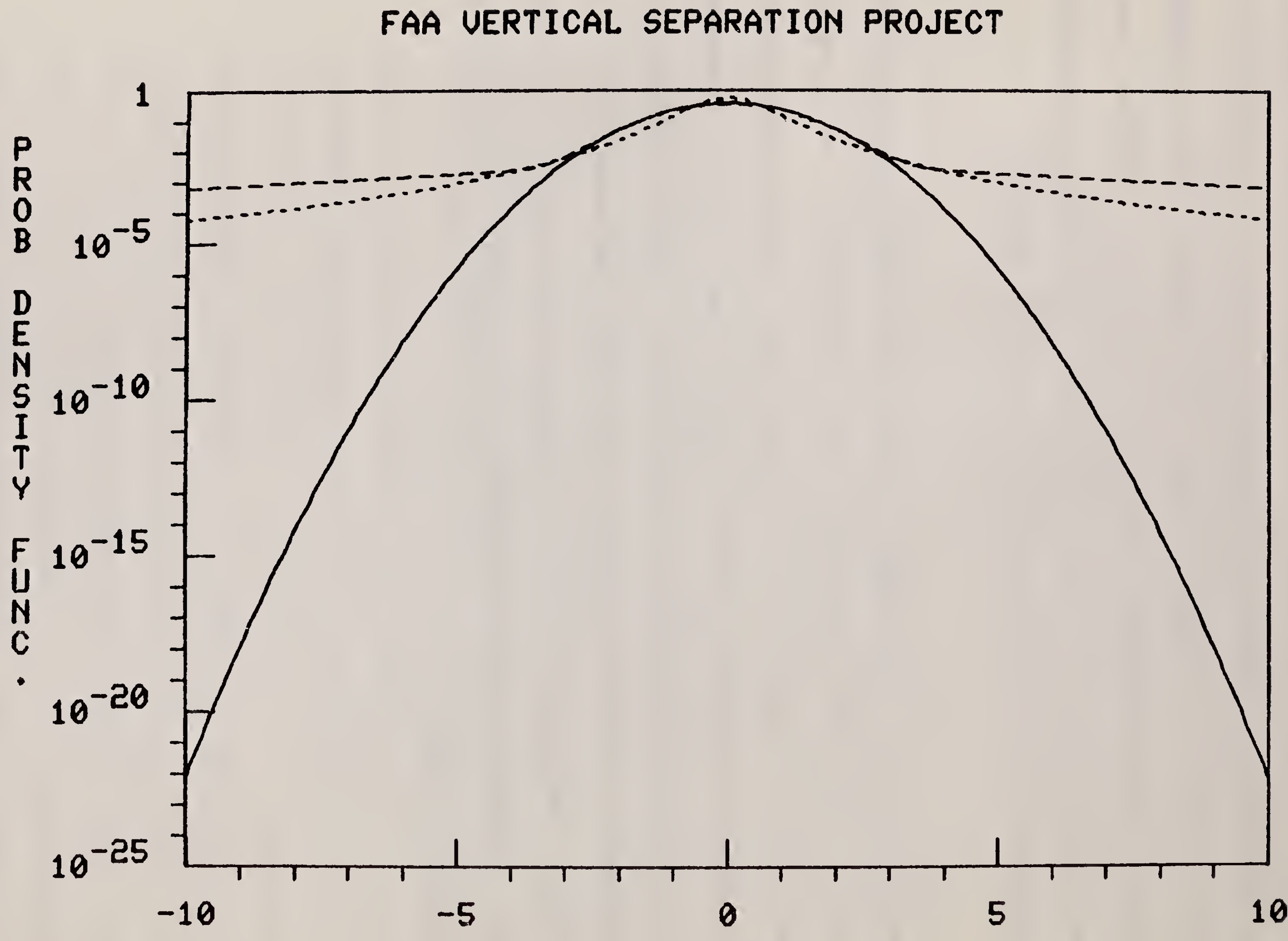
3.1 Probability Distributions

Because of the lack of empirical data, the mathematical form of the probability distribution for total vertical errors is not well established. Although it is generally accepted that this probability distribution is unimodal* and symmetric [3], debate over its shape continues. The distributional form is expected to affect significantly the probability of overlap. Employing the assumption that the probability of total vertical error is unimodal and symmetric, we limit our investigation to such probability distributions, differing only in "tail shape." It is hypothesized by many in the air traffic safety field and supported by some empirical data on navigational performance that small vertical flight errors (the majority of the errors) are normally distributed, while the larger errors (rare events) follow a double exponential distribution [4]. The resultant probability distribution is a weighted combination of a normal distribution and a double exponential distribution.

*A unimodal function has exactly one peak or "bump."

Another probability distribution family which we will include in our analysis is the t distribution. A t distribution is totally described by one parameter, the number of degrees of freedom. A t-distribution with one degree of freedom is a Cauchy distribution, while a t-distribution with infinite degrees of freedom is a normal distribution. Varying the degrees of freedom of this distribution while keeping the standard deviation fixed, alters the shape of the density function.

Therefore, we will investigate three different probability distribution families: the normal distribution, the t distribution, and the weighted normal/double-exponential distribution. These include a wide range of unimodal and symmetric probability forms. Regarding them as possible probability distributions of total vertical errors, we will examine how these functional forms and corresponding parameter values affect the probability of vertical overlap. The probability density functions of the normal distribution, the t distribution, and the normal/double-exponential distribution are plotted in Figure 1.



STANDARD DEVIATIONS (SIGMA=1)*
SOLID: NORMAL DIST., DOT: T-DIST(3 DF)
DASH: NORMAL/D EXP DIST.(GAMMA=0.95,RHO=7)
FIGURE 1

*The first line in all figure captions is the label of the horizontal axis.

(A) Normal Distribution

The normal distribution is the most widely known and used of all distributions. It arises as the distribution for measurement or control errors in many physical processes.

The normal distribution, symmetric about 0 (i.e., positive and negative errors are equally likely), has a bell-shaped probability density function given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{x^2}{2\sigma^2}\right] \qquad -\infty < x < \infty ,$$

where σ is the standard deviation of the distribution. Note that log f(x) is a quadratic function of (x/σ), explaining the parabolic shape of the solid curve in Figure 1.

In the air traffic environment, large flight errors are of great concern despite their rarity, since these rare errors may lead to midair collision or near-collision. For a normal distribution, the probabilities that the random variable deviates from its mean by less than one standard deviation, two standard deviations, and three standard deviations are respectively 0.6826, 0.9544, and 0.9974. The probability of deviating from the mean by more than three standard deviations is only 0.0026.

Normal probability density functions with standard deviations of 100 feet, 150 feet, and 200 feet are plotted in Figure 2 and Figure 3. To help display the characteristics of the tail portion of normal probability distributions, a logarithmic scale is used for the vertical axis in Figure 3.

Figure 4 illustrates the vertical overlap between two aircraft, each having a normally distributed total vertical error, when the vertical separation is set at 1000 feet. Their assigned altitudes are 30,000 feet and 31,000 feet, respectively. The dotted lines represent normal distributions with a standard deviation of 150 feet and the solid lines represent normal distributions with a standard deviation of 200 feet. The overlaps occur in the shaded area centered at 30500 feet on the horizontal axis. The shaded area under the solid line is the area of overlap when the standard deviation is 200 feet. One can see in Figure 4 that the probability of overlap when the standard deviation is 150 (the dotted line) is much smaller. All these overlaps occur in the tails of normal probability distributions.

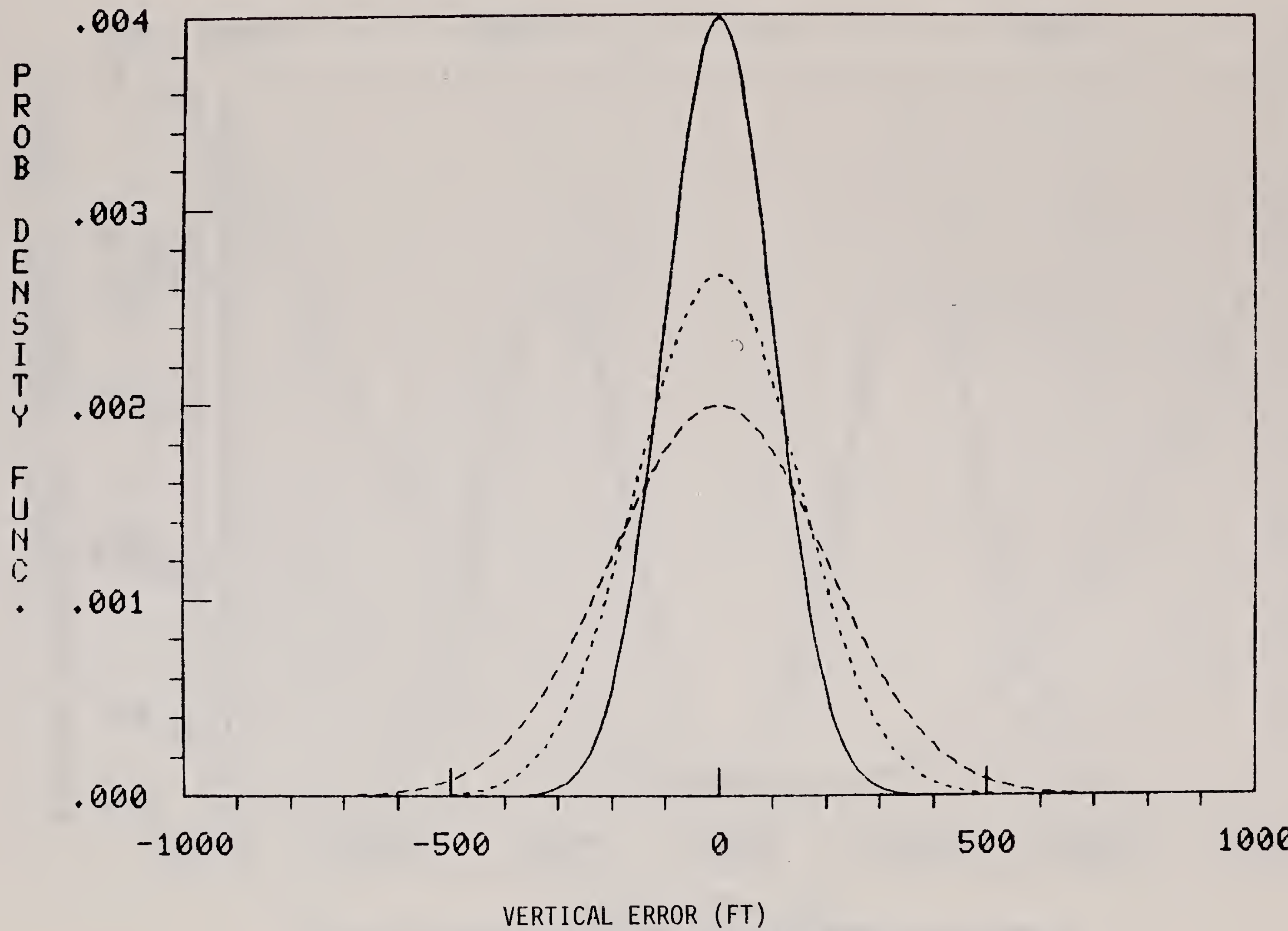
(B) t Distribution

The density function of the simplest t distribution with n degrees of freedom, is given by

$$f(t) = \frac{\Gamma[(n+1)/2]}{n\pi\ \Gamma(n/2)} \left(1 + \frac{t^2}{n}\right)^{-\left(\frac{n+1}{2}\right)} , \quad -\infty < t < \infty$$

where $\Gamma(\cdot)$ is the gamma function. For non-negative integers m, $\Gamma(m+1) = m!$ and $\Gamma(\frac{m+1}{2}) = \frac{m-1}{2} \cdot \frac{m-3}{2} \cdot \cdot \cdot \frac{1}{2} \cdot \sqrt{\pi}$. This probability density function is symmetric about its mean of zero, and its standard deviation is $(\frac{n}{n-2})^{\frac{1}{2}}$ for $n>2$. It is customary to describe the characteristics of this distribution in terms of the number of degrees of freedom. If t is replaced by $\frac{t-\mu}{k}$, where μ and k are

FAA VERTICAL SEPARATION PROJECT



NORMAL DISTRIBUTION

SOLID:SIGMA=100, DOT:SIGMA=150, DASH:SIGMA=200 FT

FIGURE 2

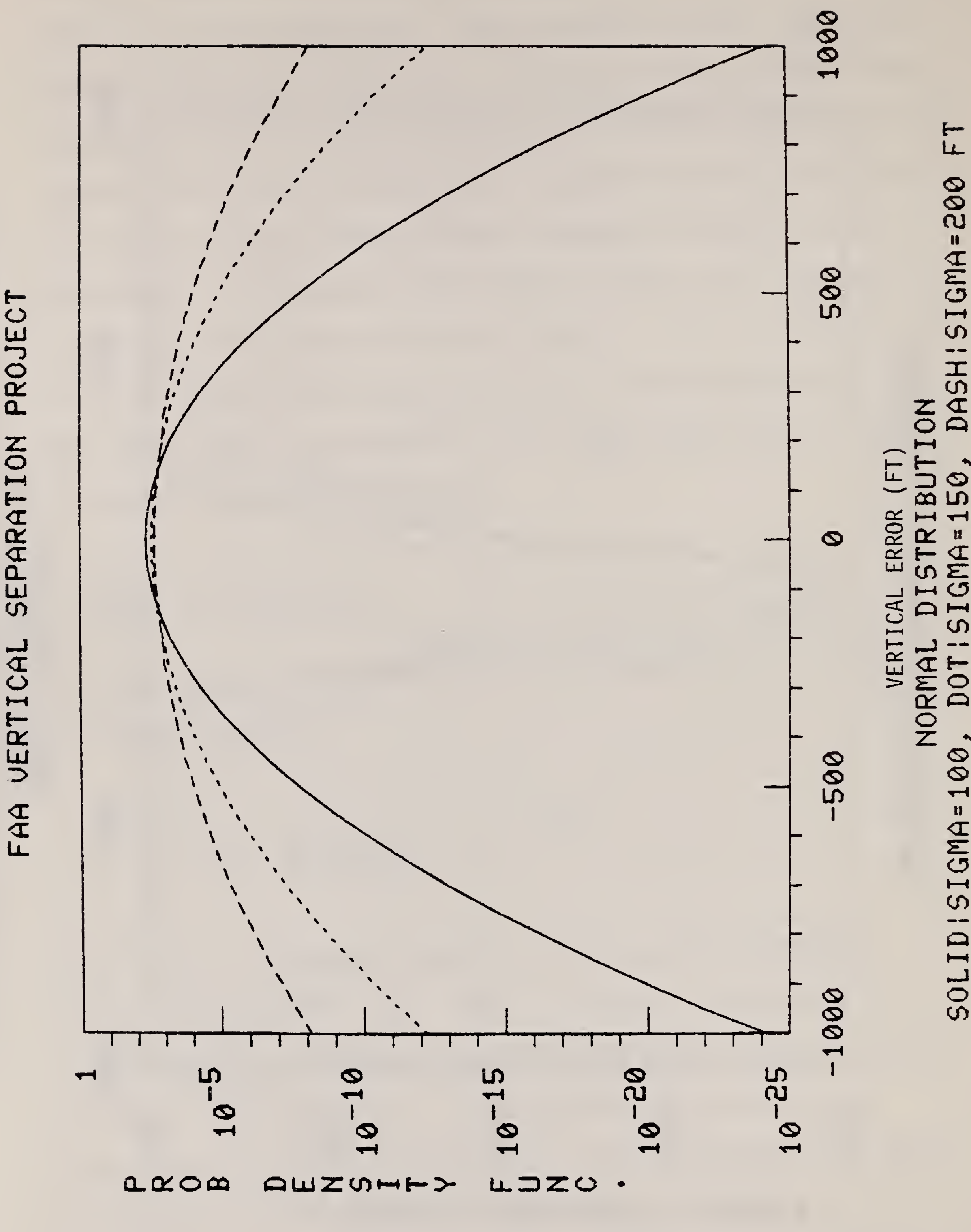
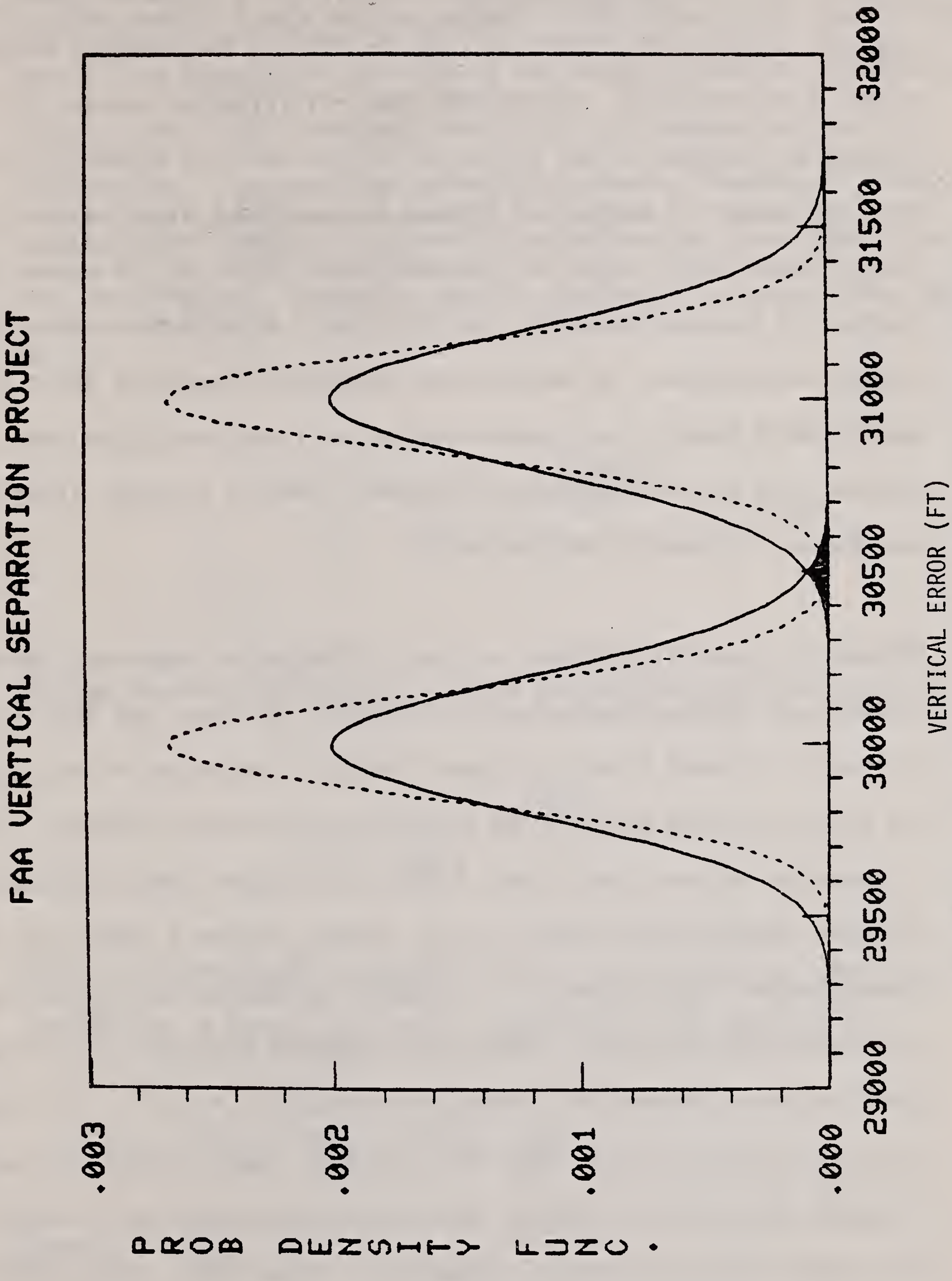

FAA VERTICAL SEPARATION PROJECT
PROB DENSITY FUNC.
1
10^{-5}
10^{-10}
10^{-15}
10^{-20}
10^{-25}
-1000
-500
0
500
1000
VERTICAL ERROR (FT)
NORMAL DISTRIBUTION
SOLID:SIGMA=100, DOT:SIGMA=150, DASH:SIGMA=200 FT

FIGURE 3



NORMAL DISTRIBUTION, 1000 FT SEPARATION
DOT! SIGMA=150 FT, SOLID! SIGMA=200 FT

FIGURE 4

constants, then the resulting transformed t distribution would have a mean of μ and standard deviation of $\frac{(n/n-2)^{\frac{1}{2}}}{k}$. In the context of aircraft vertical navigational performance, the μ is the assigned altitude and k is a conversion factor to show the size of the standard deviation of the t distribution. We consider the t distribution because for small degrees of freedom, it has larger tails than does the normal distribution. When the number of degrees of freedom is one, the t distribution is the Cauchy distribution which has extremely long tails; as the number of degrees of freedom approaches infinity, the t distribution approaches a normal distribution. By varying the degrees of freedom of the t distribution while keeping the standard deviation fixed, we obtain many distributions with tail characteristics between those of a normal distribution and those of a Cauchy distribution.

Probability density functions of the t distribution with nine degrees of freedom and standard deviations of 100 feet, 150 feet, and 200 feet are plotted in Figures 5 and 6. Figure 5 depicts the shapes of the graphs of these functions while Figure 6 magnifies their tail portions. It is interesting to note from Figure 3 that on log paper normal probability density functions are concave curves, whereas Figure 6 shows that the corresponding "log" graph of a t probability density function can be dissected into two convex curves if one ignores that part of the graph near the mean, between the inflection points at $t = \pm\ n^{-\frac{1}{2}}$. This suggests that as one moves farther away from the mean, the t probability density function decreases more slowly than that of the normal distribution with the same standard deviation, indicating a longer tail. This relationship suggested by Figures 3 and 6 can be proven analytically; we omit the details.

(c) Combination of Normal and Double Exponential Distributions

It is hypothesized by many in the air traffic safety field, and supported by some empirical data on navigational performance, that small flight errors (the majority of the errors) are normally distributed while the larger errors (i.e., the rare events) follow a double exponential distribution. The resultant probability density function is a weighted combination of those for a normal distribution and for a double exponential distribution. Since the density function of a normal distribution is

$$f_1(x) = \frac{1}{\sqrt{2\pi}\ \sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad -\infty < x < \infty ,$$

and the density function of a double exponential distribution is

$$f_2(x) = \frac{1}{2\beta} e^{-\frac{|x-\alpha|}{\beta}} \qquad -\infty < x < \infty$$

where α = mean and $\sqrt{2}\beta$ = standard deviation, the density function of a combination of normal and double exponential distributions is

$$g(x) = \gamma\ f_1(x) + (1-\gamma)\ f_2(x), \qquad -\infty < x < \infty$$

where $0 \leq \gamma \leq 1$.

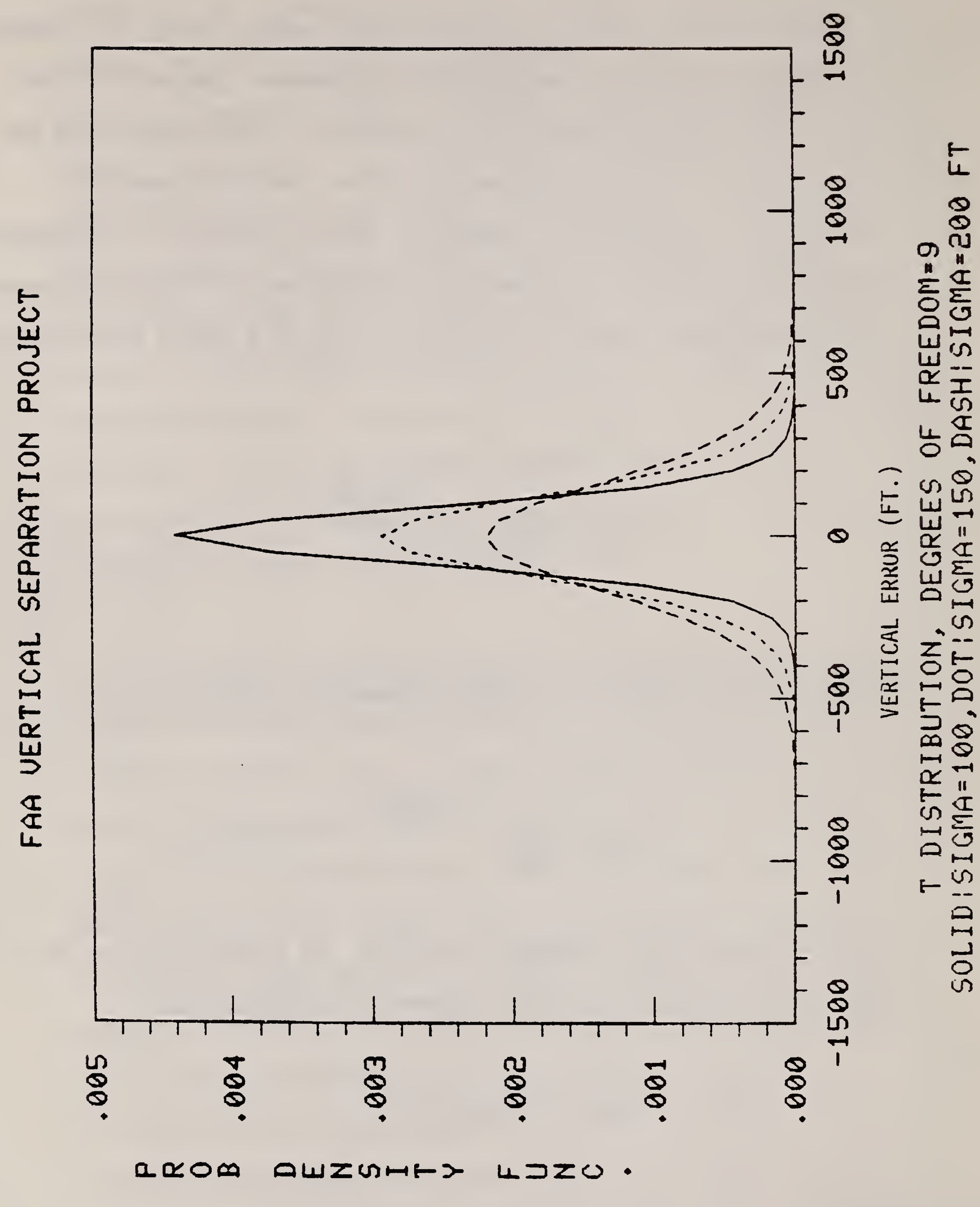
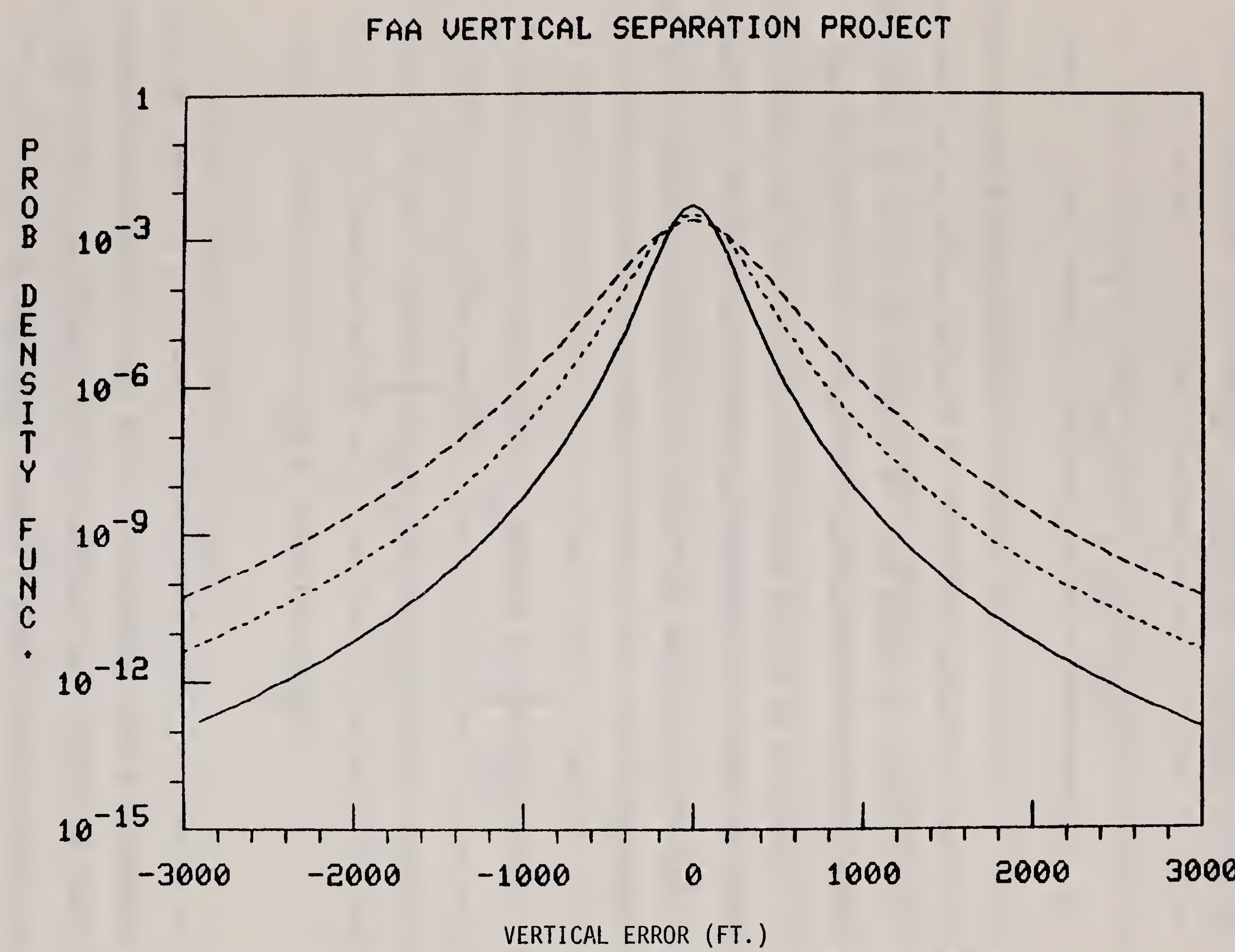

FAA VERTICAL SEPARATION PROJECT
PROB DENSITY FUNC.
.005
.004
.003
.002
.001
.000
-1500
-1000
-500
0
500
1000
1500
VERTICAL ERROR (FT.)
T DISTRIBUTION, DEGREES OF FREEDOM=9
SOLID:SIGMA=100,DOT:SIGMA=150,DASH:SIGMA=200 FT

FIGURE 5



T DISTRIBUTION, DEGREES OF FREEDOM=9
SOLID|SIGMA=100,DOT|SIGMA=150,DASH|SIGMA=200 FT

FIGURE 6

Let σ_N and σ_{DE} be the standard deviations of the normal and the double exponential distributions respectively, i.e., the σ and $\sqrt{2\beta}$ of the preceding equations. Then the standard deviation σ_{NDE} of a combination of normal and double exponential distributions is

$$\sigma_{NDE} = (\gamma\sigma_N^2 + (1-\gamma)\sigma_{DE})^{\frac{1}{2}} , \; 0 \leq \gamma \leq 1 .$$

Since it is likely that the sources of vertical flight errors reflected in the normal distribution are different from the sources of the rare events, it is reasonable to assume that the standard deviations for normal and double exponential distributions are generally of different values. $\rho = \sigma_{DE}/\sigma_N$ will be used to denote the ratio of these two standard deviations. Hence, the relationship between the standard deviation of the normal/double-exponential distribution and that of the normal distribution can be represented as follows

$$\begin{aligned}
\sigma_{NDE} &= \left[\gamma\sigma_N^2 + (1-\gamma)\sigma_{DE}^2 \quad ^{\frac{1}{2}}\right] \\
&= \left[\gamma\sigma_N^2 + (1-\gamma)\rho^2\sigma_N^2 \quad ^{\frac{1}{2}}\right] \\
&= \left[\; \gamma + (1-\gamma)\rho^2 \;\right]^{\frac{1}{2}} \sigma_N
\end{aligned}$$

The probability density functions of a combination of normal and double exponential distributions with $\gamma = 0.95$ and $\rho = 7.0$, and with respective values of 100, 150, and 200 feet for the standard deviation of the normal distribution, are plotted in Figure 7 and Figure 8.

The probability distributions of normal/double-exponential type plotted in Figures 7 and 8 are combinations of 95% normal distribution and 5% double exponential distribution. The body of the combined distribution is close to that of a normal distribution, while its tail is close to that of a double exponential distribution. This phenomenon is explicitly shown in Figure 8 where a logarithmic scale is used on the vertical axis.

3.2 Standard Deviations

It is generally true that the standard deviation of a probability distribution largely determines its shape "within" the over-all functional form involved. The shape of the probability distribution of the total vertical errors is important in estimating the probability of vertical overlap, because most overlaps are supposed to occur in the tail of the probability distribution. Also, the standard deviation has a significant interpretation as a measure of the precision of altitude control. Therefore, it is important to examine the potential impact of the standard deviation on the probability of vertical overlap. Standard deviations of three values; namely, 100 feet, 150 feet, and 200 feet are used in most computer runs in this study. These values are within the range of values of standard deviations of vertical errors found in an early study [5].

3.3 Aircraft Vertical Dimensions (Heights)

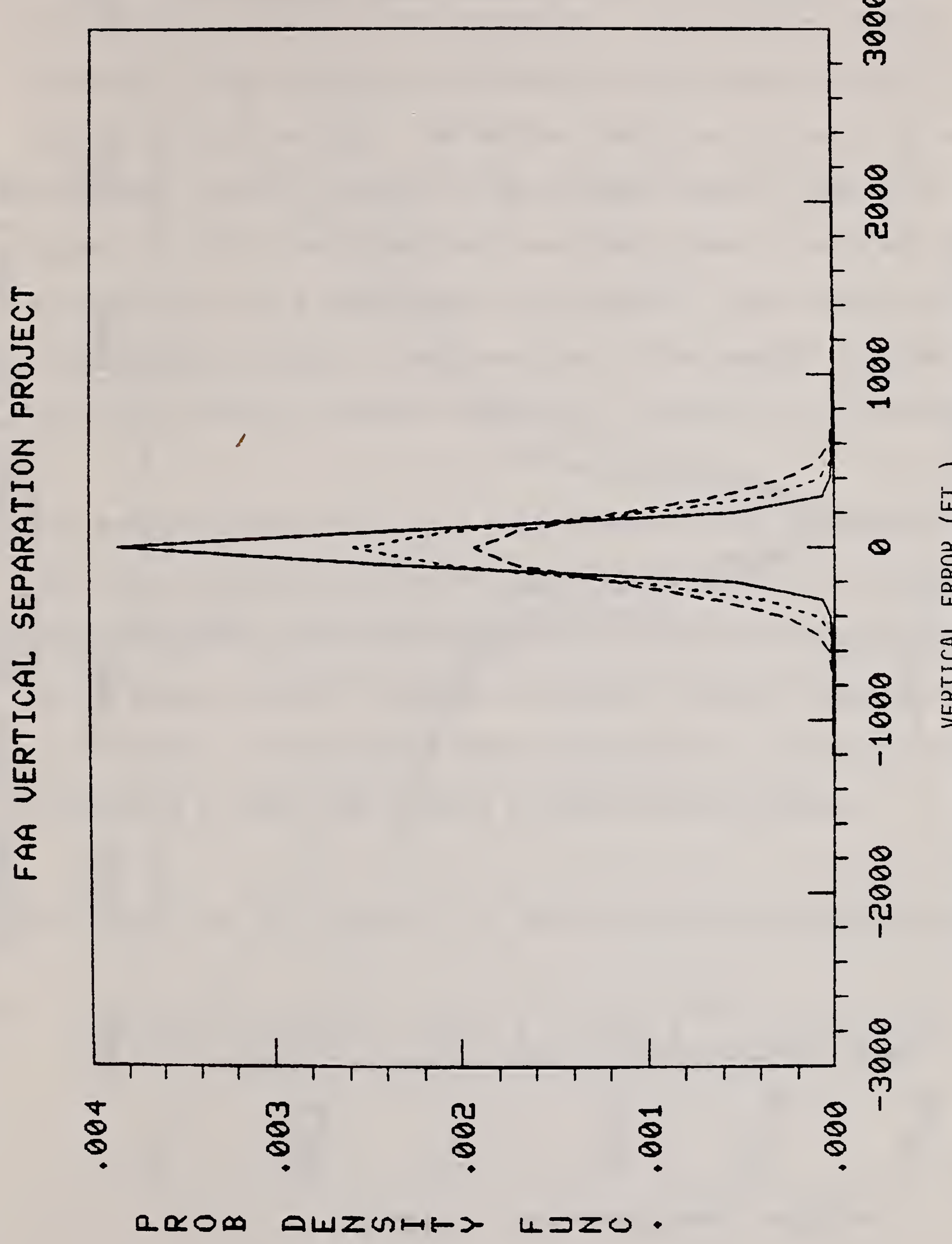
A vertical overlap of a pair of aircraft occurs when the centroid of the second craft is within one aircraft height of the centroid of the first. Thus, the height of the two aircraft will affect the vertical overlap probability. There are many different types of aircraft flying above 29,000 feet over CONUS. In this study, we examined two representative aircraft heights: one was λ_z = 30 ft. which is roughly that of a medium-sized jet airliner, and the other was λ_z = 50 ft., the approximate height of a large-sized jet airliner [6].

3.4 Vertical Separation Standards

One of the most crucial factors influencing the vertical overlap probability is the vertical separation standard. As discussed in Section 2, the probability of vertical overlap can be expressed as follows:

$$P\{\text{Vertical Overlap}\} = P_z(S) = \int_{-\infty}^{\infty} f_1(v) \int_{v-\lambda_z}^{v+\lambda_z} f_2(u-S)\, du\, dv$$

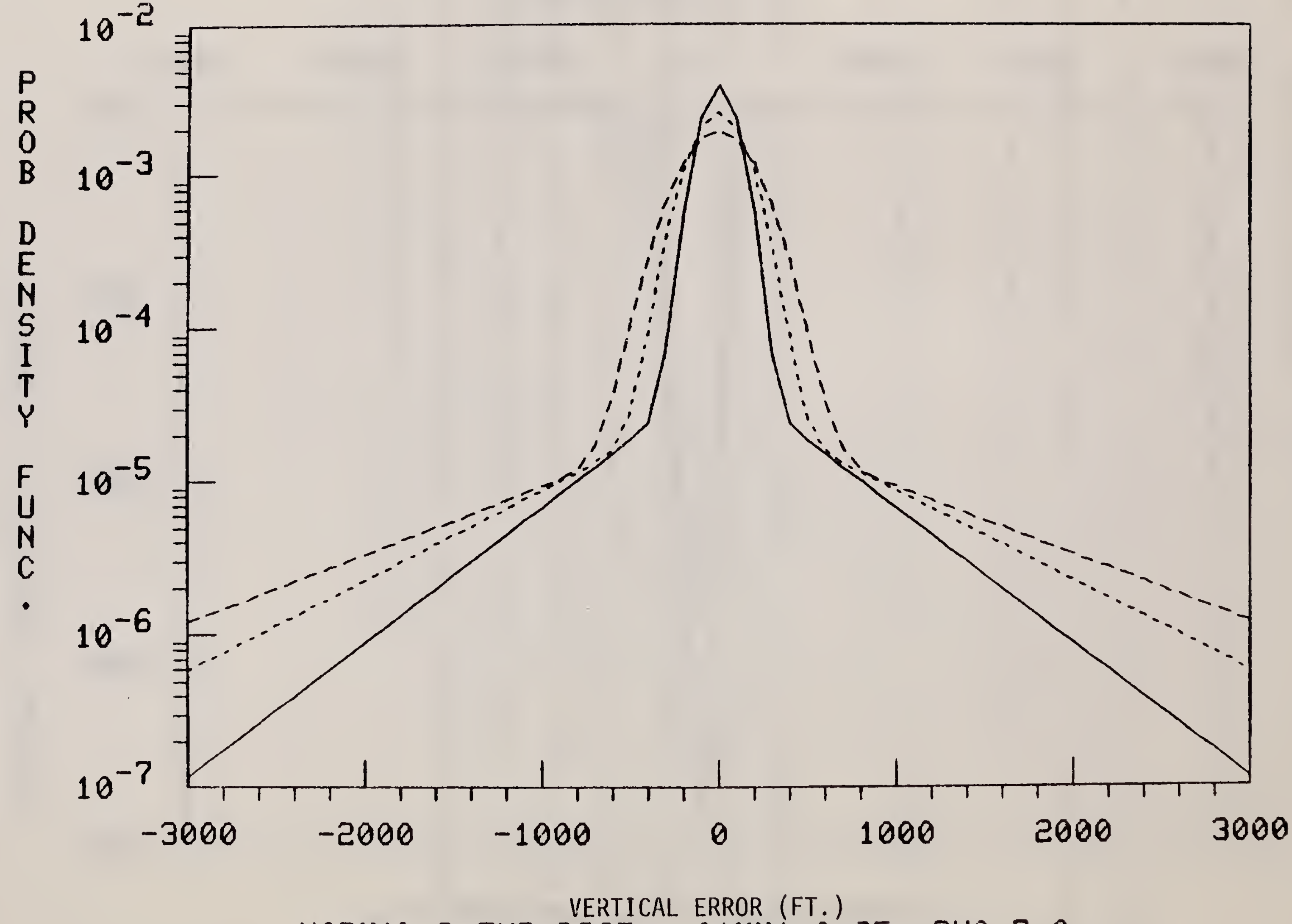
where S is an assigned separation standard. The vertical overlap probability can be made infinitesimal by assigning an extremely large separation standard in the air traffic track system. However, in so doing, the traffic capacity of the track system might be reduced to an intolerable level. The present vertical separation standard for flight levels above 29,000 feet over the CONUS is 2000 feet. In this study, we shall calculate vertical overlap probabilities between a pair of aircraft for separation standards ranging from zero to two thousand feet.



NORMAL/D EXP DIST., GAMMA=0.95, RHO=7.0
SOLID:SIGMA=100, DOT:SIGMA=150, DASH:SIGMA=200 FT

FIGURE 7

FAA VERTICAL SEPARATION PROJECT



NORMAL/D EXP DIST., GAMMA=0.95, RHO=7.0
SOLID:SIGMA=100, DOT:SIGMA=150, DASH:SIGMA=200 FT

FIGURE 8

4. Description of Computer Programs

Three computer programs, one for each of the three probability distributions discussed earlier, were developed. It was assumed that the probability distributions of total vertical errors were the same for the two aircraft involved. The vertical locations for the first and the second aircraft, relative to the assigned height of the first, are assumed to be at v and u=S+w feet respectively where v and w are random variables and S is a given separation standard. Since standardized distributions are used in the subroutines of the computer programs, the necessary scaling of standard deviations is carried out in the main program.

The computer program first finds the probability that the centroid of the first aircraft is at a particular altitude relative to its assigned vertical location, and then calculates the probability that the centroid of the second aircraft is within one vertical aircraft dimension of that of the first. The product of these probabilities is the vertical overlap probability of these two aircraft at that vertical location.

The formula for the probability of vertical overlap was derived earlier as

$$P\{\text{Vertical Overlap}\} = P_z(S) = \int_{-\infty}^{\infty} f_1(v) \int_{v-\lambda_z}^{v+\lambda_z} f_2(u-S)\, du\, dv$$

This can be rewritten as

$$P_z(S) = \int_{-\infty}^{\infty} f_1\ (v)\ \left[F_2(v+\lambda_z-S)\ -\ F_2(v-\lambda_z-S)\right]\ dv$$

where $F_2(*)$ is the cumulative distribution for the probability density function $f_2(*)$.

Numerical integration was performed using the trapezoid rule to obtain the probability of vertical overlap. Instead of integrating from minus infinity to plus infinity, the computer program uses only the locations between minus and plus six standard deviations from the center, and the mesh size for the trapezoid rule is $\frac{S+12\sigma}{1000}$. The summation of overlap probabilities at these locations is a good approximation to the total probability of vertical overlap of the two aircraft.

5. Results of Computation

5.1 Normal Distribution

Probabilities of vertical overlap were calculated under the assumption that the total vertical errors were normally distributed for each aircraft. Two vertical dimensions of aircraft, 30 feet and 50 feet, and three sets of standard deviations, 100 feet, 150 feet, and 200 feet, are used in the computation. The separation standards examined in this study ranged from 0 to 2000 feet with an increment of 50 feet. The calculated results are plotted in Figures 9 through 11. Figures 9 and 10 refer to the same set of input data but use different scales on their vertical axes in order to highlight respectively the vertical overlap probabilities below and above the separation standard of 500 feet. Figure 11 shows the plots of probability of vertical overlap for two values of the vertical dimension of the aircraft.

As observed in Figure 9 and 10, with increasing separation standards, the probabilities of vertical overlap drop more rapidly under a normal distribution with a smaller standard deviation than under one with a larger standard deviation. It is also observed that the curves of probabilities of vertical overlap cross each other somewhere between the separation standards of 200 feet and 300 feet. This phenomenon will be discussed later in Section 6.

Since the convolution of two normal distributions is a normal distribution, the probability of overlap of the two aircraft can be determined analytically by integrating the convolved normal distribution. Since a normal probability

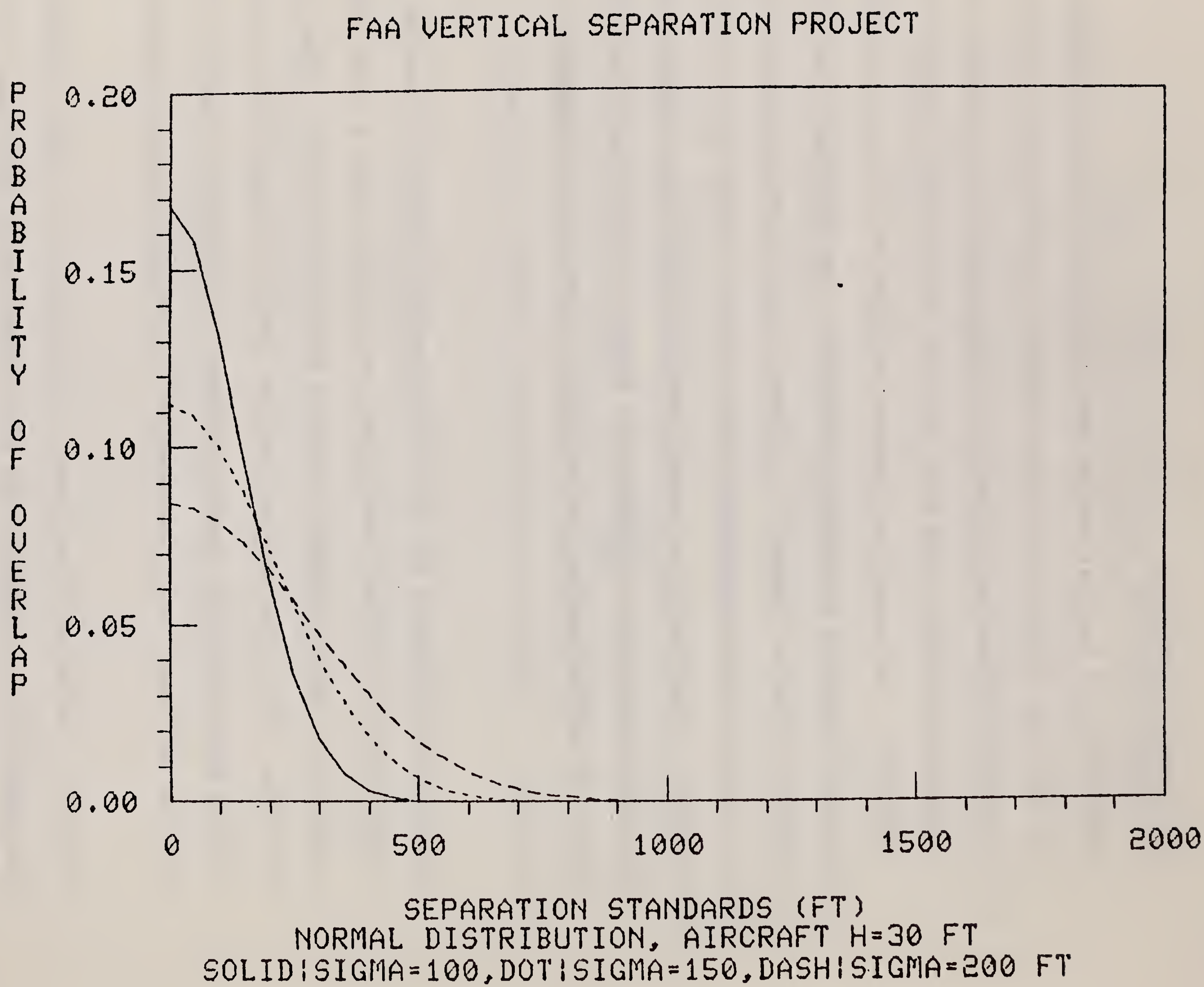

FAA VERTICAL SEPARATION PROJECT
PROBABILITY OF OVERLAP
0.20
0.15
0.10
0.05
0.00
0
500
1000
1500
2000
SEPARATION STANDARDS (FT)
NORMAL DISTRIBUTION, AIRCRAFT H=30 FT
SOLID:SIGMA=100,DOT:SIGMA=150,DASH:SIGMA=200 FT

FIGURE 9

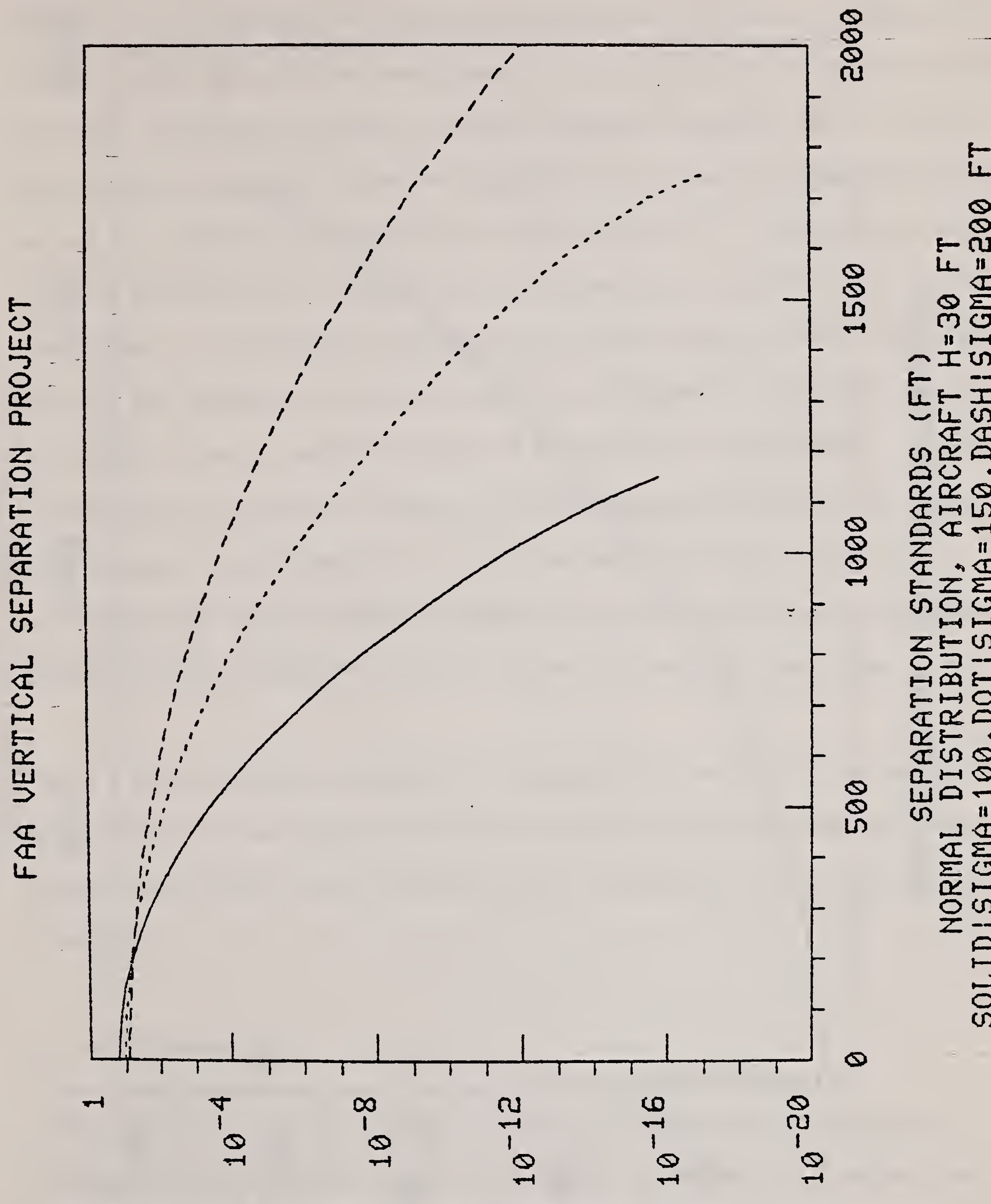

FAA VERTICAL SEPARATION PROJECT
PROBABILITY OF OVERLAP
1
10^{-4}
10^{-8}
10^{-12}
10^{-16}
10^{-20}
0
500
1000
1500
2000
SEPARATION STANDARDS (FT)
NORMAL DISTRIBUTION, AIRCRAFT H=30 FT
SOLID:SIGMA=100,DOT:SIGMA=150,DASH:SIGMA=200 FT

FIGURE 10

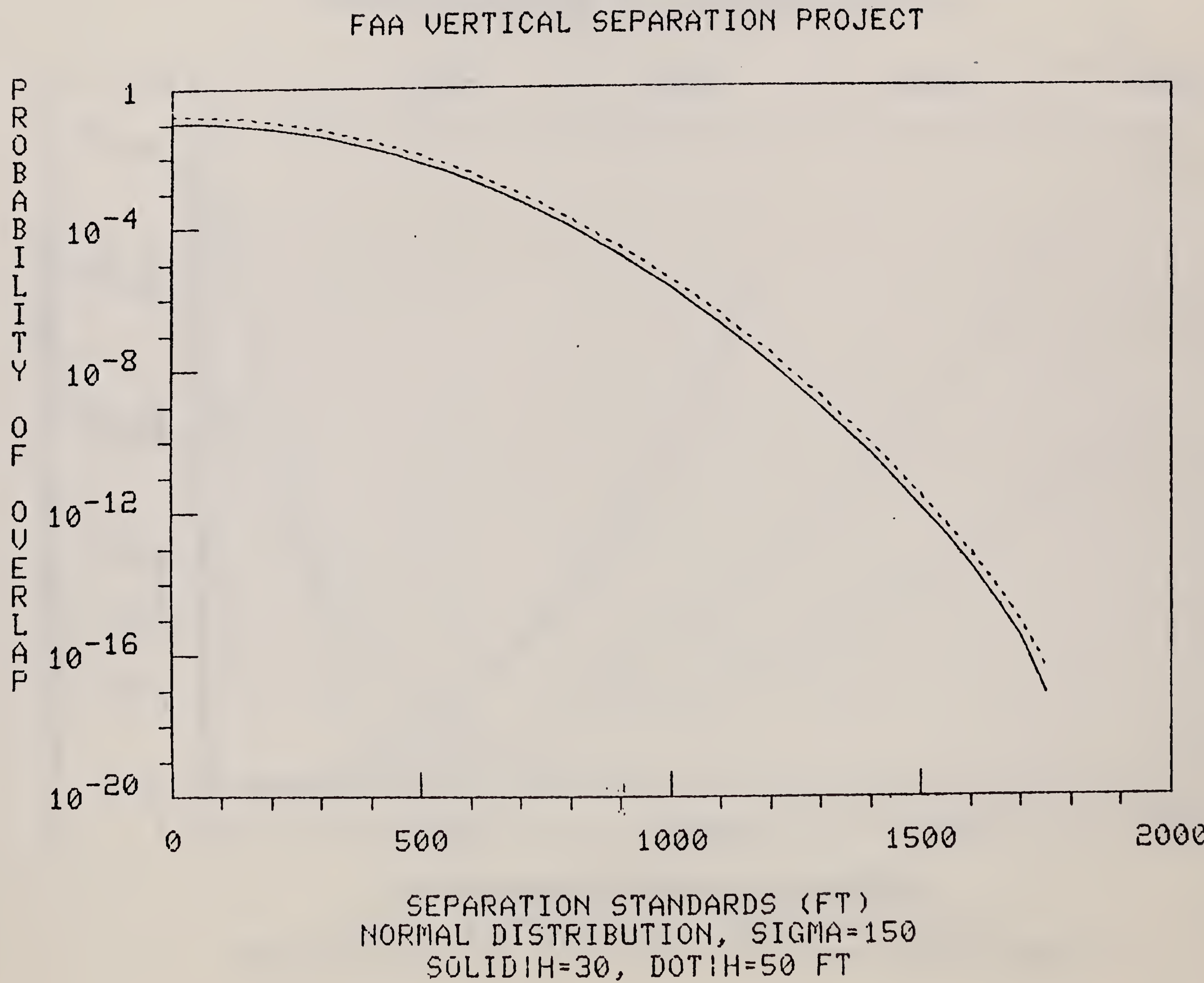

FAA VERTICAL SEPARATION PROJECT
PROBABILITY OF OVERLAP
1
10^-4
10^-8
10^-12
10^-16
10^-20
0
500
1000
1500
2000
SEPARATION STANDARDS (FT)
NORMAL DISTRIBUTION, SIGMA=150
SOLID:H=30, DOT:H=50 FT

FIGURE 11

density function cannot be integrated in closed form, tables of Normal Probability Functions [7] were used to obtain the numerical results of the integration. Computational results obtained by the numerical method and by the computer program are tabulated in Table 1. Column C/A in Table 1 shows the relative magnitudes of the probability of vertical overlap obtained by computer program (C) and by numerical methods (A). It can be seen that the computed values are generally less than the numerically derived values for probabilities equal to or less than 10^{-5}. It is also observed that as the values for probabilities decrease, the values for C/A also deviate away from 1. Two possible explanations for the causes of the above phenomena are (1) the computer program is limited to integrate to ± 6 sigma and (2) for very small values, computer round-off errors can be significant. It should be mentioned here that the Tables of Normal Probability Functions carry only fifteen significant digits after the decimal point. This also contributes to the discrepancy between the values obtained from the computer program and that from the Tables of Normal Probability Functions.

Table 1 also serves as a summary of computational results of probabilities of vertical overlap based upon normal distributions with three different standard deviations, two aircraft vertical dimensions, and three separation standards.

5.2 t Distribution

Probabilities of vertical overlap derived from the t distribution are affected by four factors; namely, the number of degrees of freedom, the standard deviation, the vertical dimension of the aircraft, and the separation standard. Sensitivity studies have been carried out to assess the affect of each of these four factors on the probability of vertical overlap; the computational results are plotted in Figures 12 through 17, which are discussed below.

	σ	S=1000 Analytic (A)	S=1000 Comp. (C)	C/A	S=1500 Analytic (A)	S=1500 Comp. (C)	C/A	S=2000 Analytic (A)	S=2000 Comp. (C)	C/A
$\lambda_z = 30$	100	.33066-11	.25626-11	.775	0	0	--	0	0	--
	150	.18081-5	.18071-5	.999	.18360-11	.13949-11	.760	0	0	--
	200	.16687-3	.16691-3	1.000	.69538-7	.69258-7	.996	.12855-11	.96531-12	.751
$\lambda_z = 50$	100	.91870-11	.75101-11	.817	0	0	--	0	0	--
	150	.33903-5	.33884-5	.999	.39522-11	.31036-11	.785	0	0	--
	200	.28872-3	.28878-3	1.000	.12593-6	.12587-6	1.000	.25010-11	.19180-11	.767

Table 1: The Probability of Vertical Overlap Obtained by Analytical Method and Computer Runs Under Normal Distributions

Holding the number of degrees of freedom at 20 and 40, and the vertical dimension of aircraft at 50 feet, probabilities of vertical overlap are plotted for each of the three values (100, 150, and 200 feet) of the standard deviation. Probabilities of vertical overlap derived from the t distribution with 20 degrees of freedom are plotted in Figure 12; those for 40 degrees of freedom are plotted in Figure 13. The difference between a curve in Figure 12 and the corresponding curve in Figure 13 demonstrates the effect of the functional form of the t distribution, which in turn is determined by the number of degrees of freedom. The differences of the three curves in each figure reflect the effect of the standard deviation on the probability of vertical overlap. Another potential use of plots like these is that if a t distribution with the "correct" degrees of freedom can be fitted to the empirical probability distribution of the total vertical errors, then one can use these curves to estimate probabilities of vertical overlap based on the empirical distribution under different separation standards.

Probabilities of vertical overlap are plotted in Figure 14 for three, six, and nine degrees of freedom, and in Figure 15 for 20, 30, and 40 degrees of freedom, while the vertical dimension is set at 30 feet and standard deviation at 100 feet. These figures illustrate the effects of the number of degrees of freedom.

In Figure 16, the effect of aircraft vertical dimension on the probability of vertical overlap under a t distribution is illustrated.

In Figure 17, probabilities of vertical overlap for three values (1000 feet, 1500 feet, and 2000 feet) of the separation standard are plotted against degrees of freedom.

Table 2 provides a summary of computational results of probabilities of vertical overlap under t distributions and three separation standards.

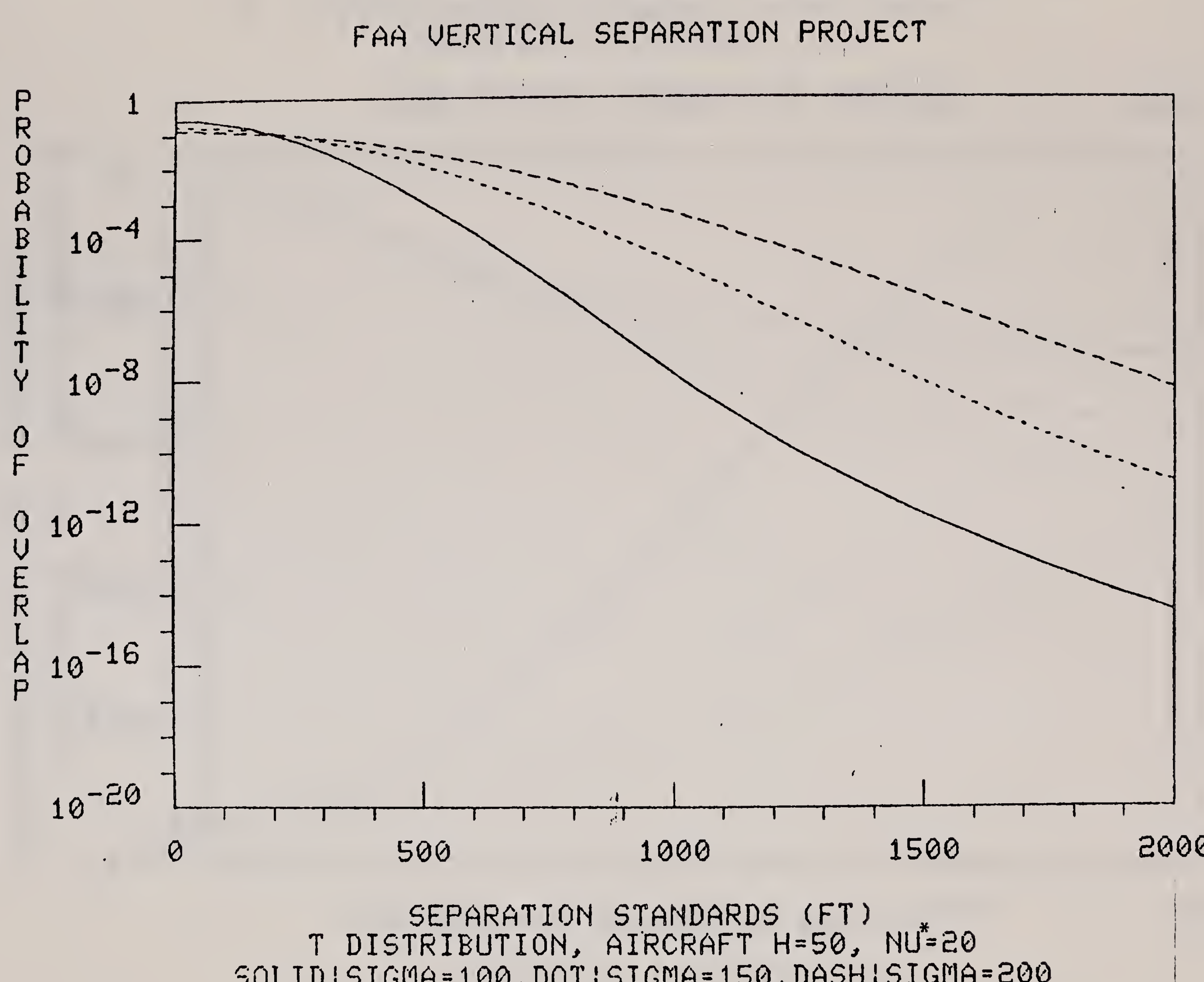


FIGURE 12

*NU is the number of degrees of freedom

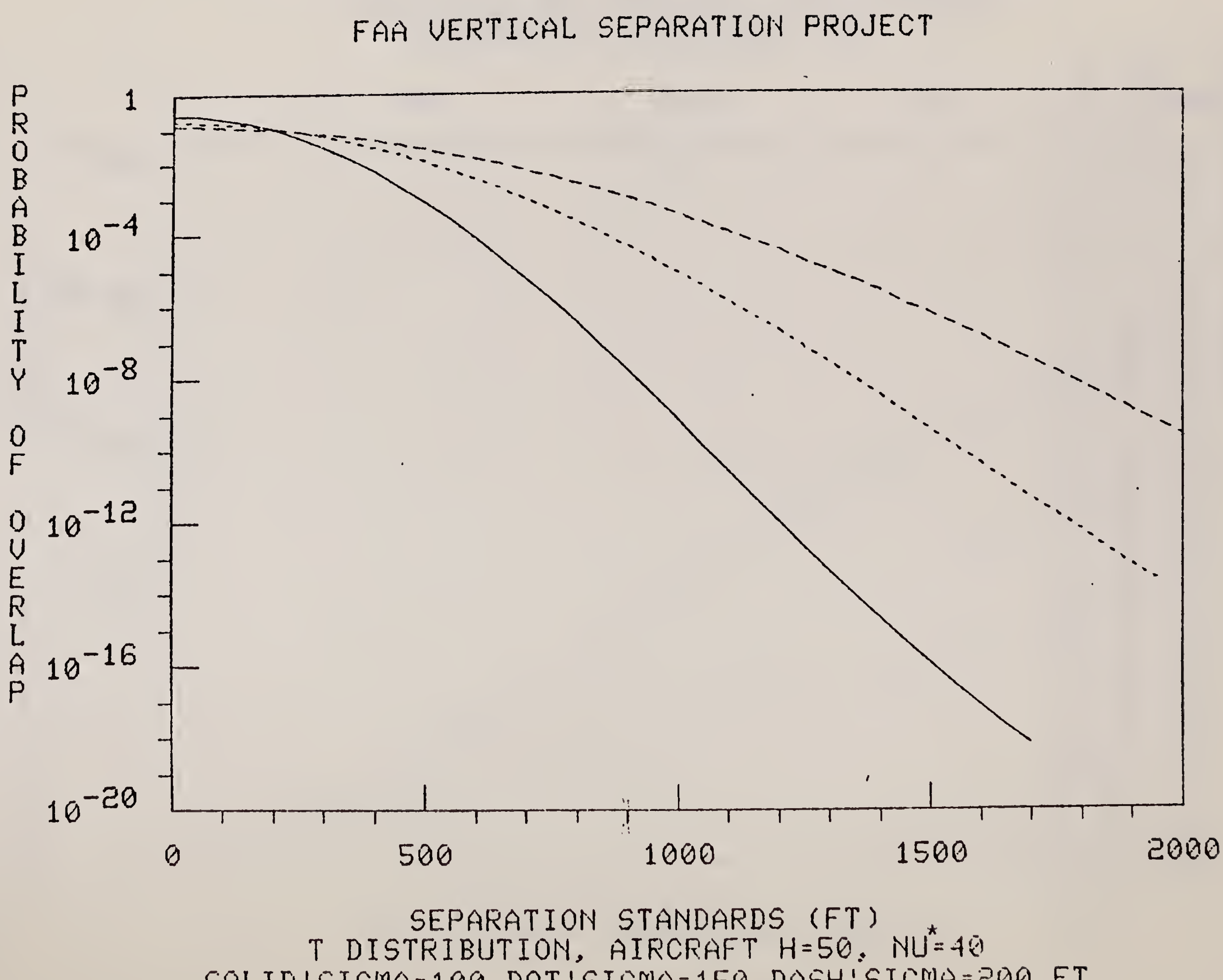
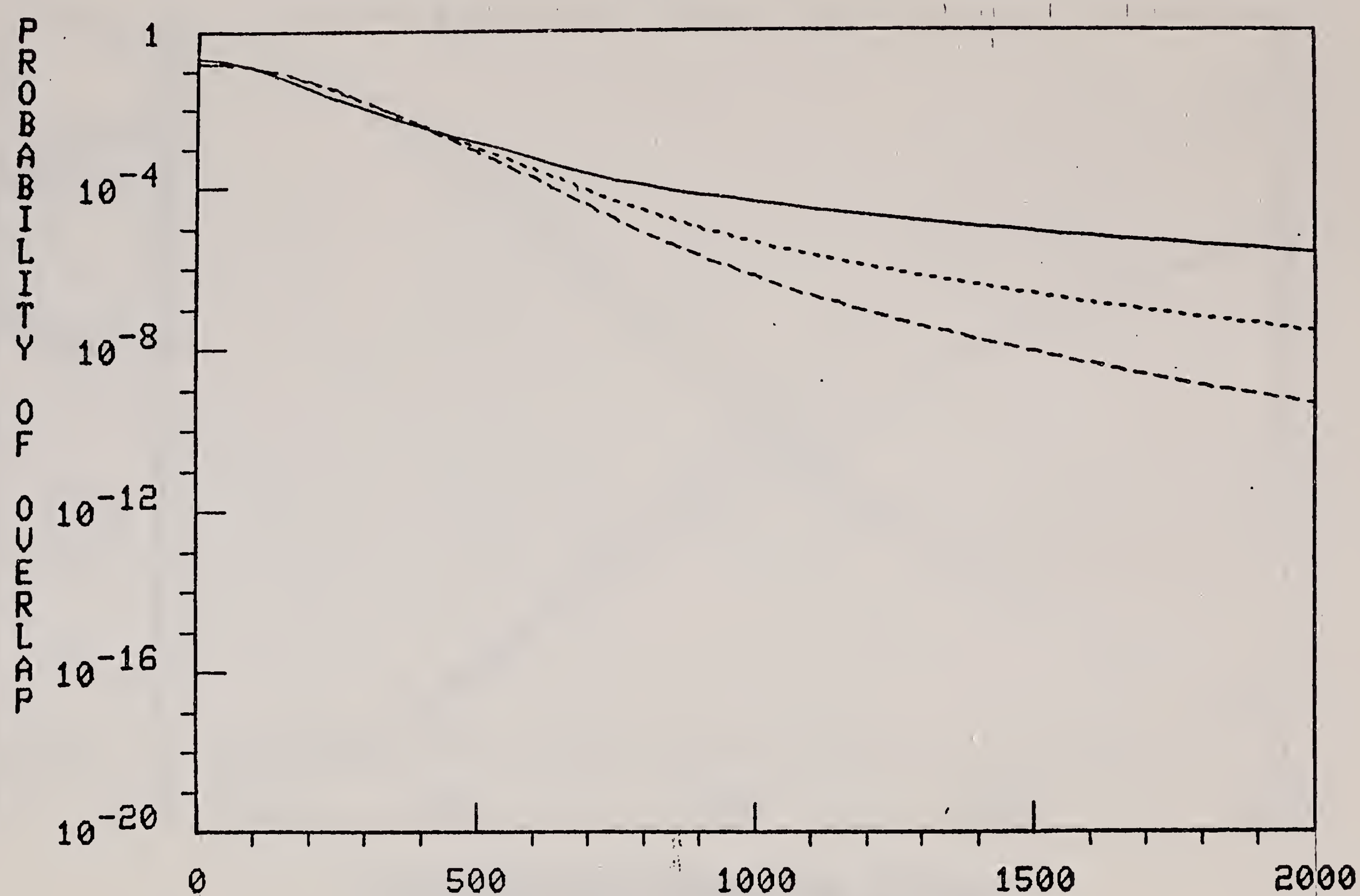


FIGURE 13

*NU is the number of degrees of freedom

FAA VERTICAL SEPARATION PROJECT



SEPARATION STANDARDS (FT)
T DISTRIBUTION, AIRCRAFT H=30, SIGMA=100 FT
SOLID:NU*=3, DOT:NU=6, DASH:NU=9

FIGURE 14

*NU is the number of degrees of freedom

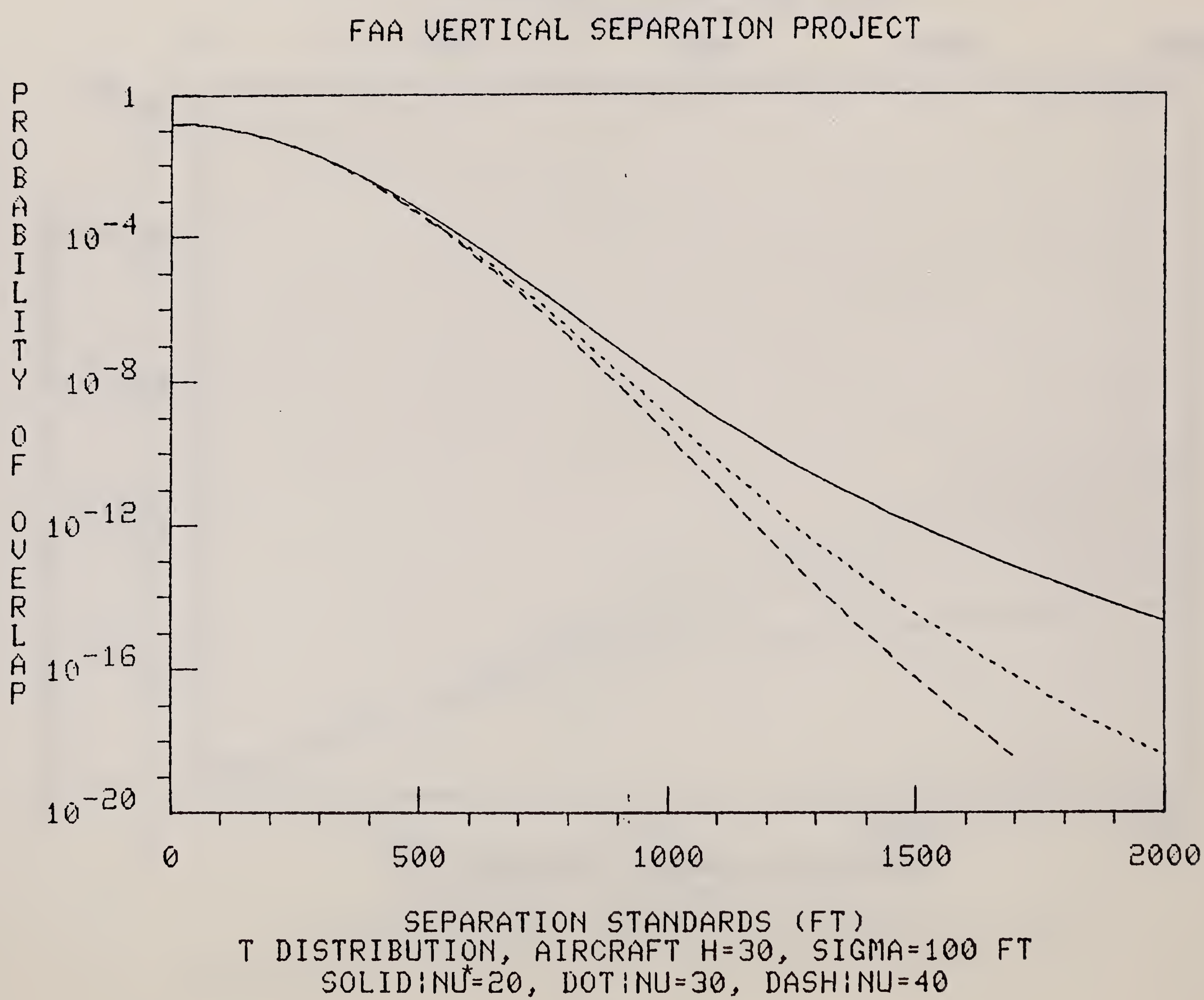


FIGURE 15

*NU is the number of degrees of freedom

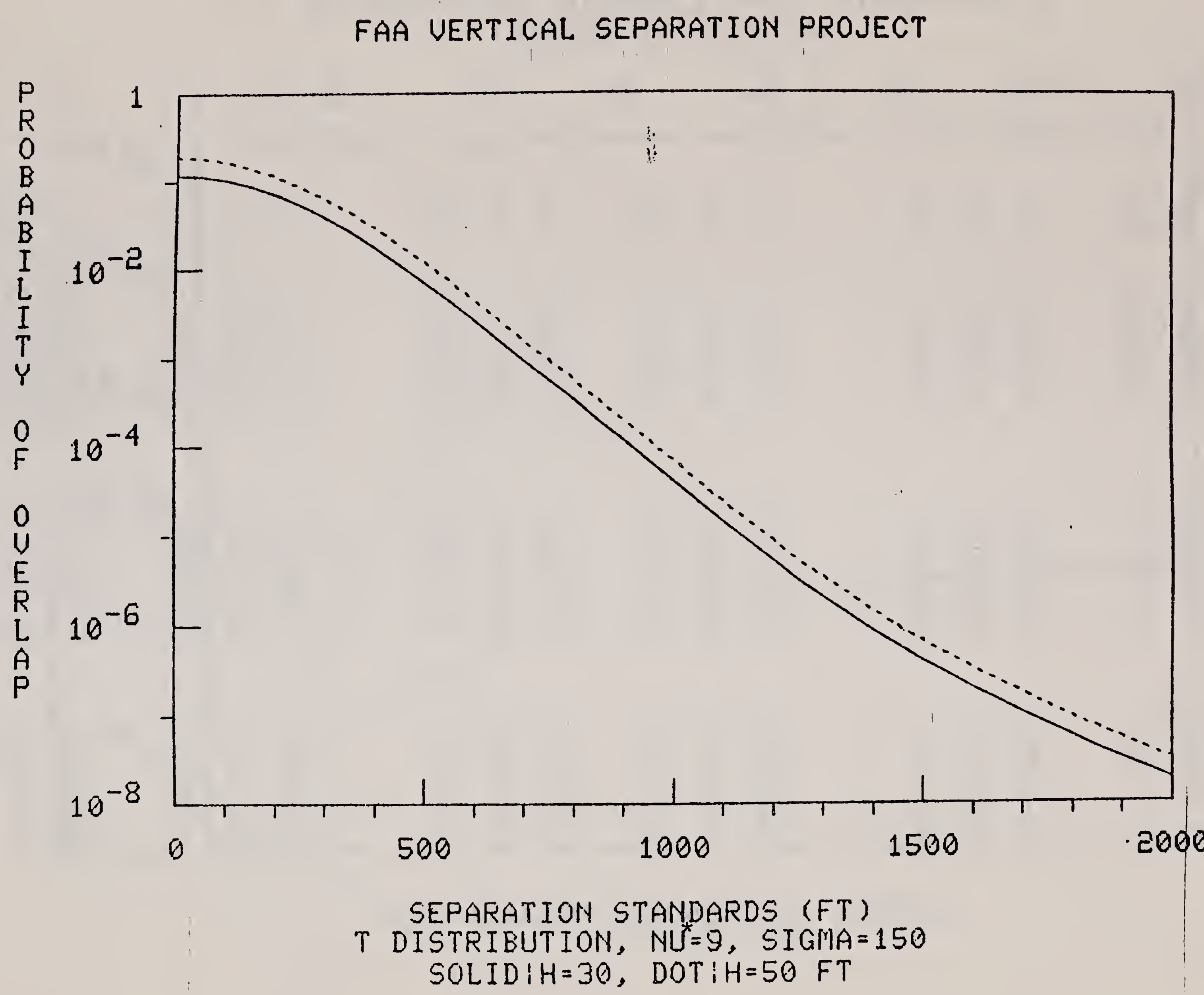


FIGURE 16

*NU is the number of degrees of freedom

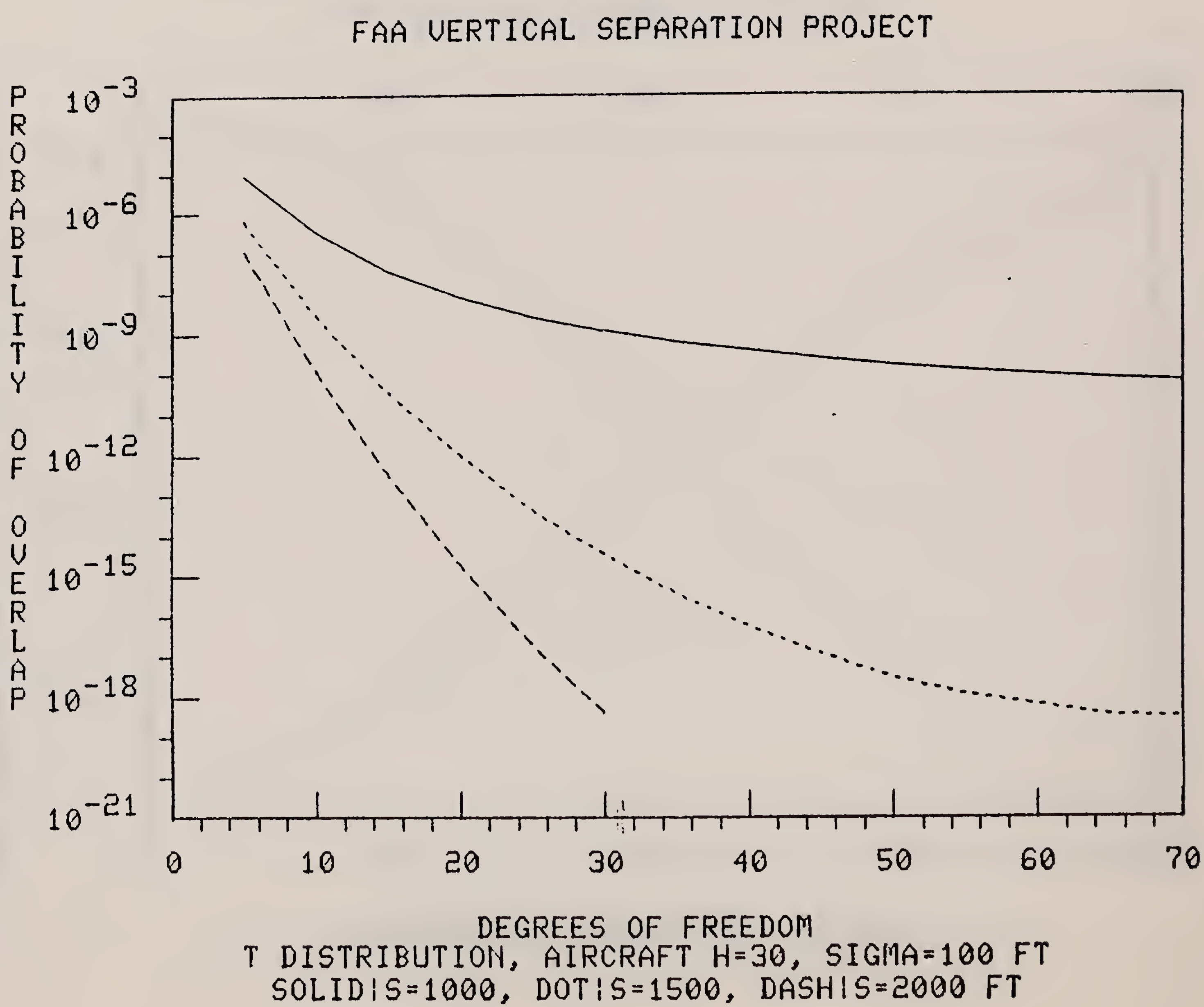

FAA VERTICAL SEPARATION PROJECT
PROBABILITY OF OVERLAP
10^{-3}
10^{-6}
10^{-9}
10^{-12}
10^{-15}
10^{-18}
10^{-21}
0
10
20
30
40
50
60
70
DEGREES OF FREEDOM
T DISTRIBUTION, AIRCRAFT H=30, SIGMA=100 FT
SOLID|S=1000, DOT|S=1500, DASH|S=2000 FT

FIGURE 17

nu* = 3	σ	S = 1000	S = 1500	S = 2000
λ_z = 30	100	0.41443-04	0.77663-05	0.24224-05
	150	0.19458-03	0.27575-04	0.83643-05
	200	0.69759-03	0.75358-04	0.20668-04
λ_z = 50	100	0.69497-04	0.12976-04	0.40428-05
	150	0.32899-03	0.46085-04	0.13960-04
	200	0.11686-02	0.12616-03	0.34499-04
nu = 6				
λ_z = 30	100	0.44697-05	0.22667-06	0.29225-07
	150	0.87719-04	0.29478-05	0.35197-06
	200	0.55314-03	0.23118-04	0.22065-05
λ_z = 50	100	0.75476-05	0.38044-06	0.48891-07
	150	0.14922-03	0.49529-05	0.58897-06
	200	0.93220-03	0.38939-04	0.36942-05
nu = 9				
λ_z = 30	100	0.62310-06	0.86148-08	0.45941-09
	150	0.41759-04	0.41075-06	0.19270-07
	200	0.41774-03	0.83468-05	0.30685-06
λ_z = 50	100	0.10766-05	0.14557-07	0.77145-09
	150	0.71666-04	0.69563-06	0.32371-07
	200	0.70729-03	0.14145-04	0.51604-06

Table 2: Probabilities of Vertical Overlap Under t Distributions and Three Different Separation Standards

*nu is the number of degrees of freedom

nu = 20	σ	S = 1000	S = 1500	S = 2000
λ_z = 30	100	0.79466-08	0.10053-11	0.20716-14
	150	0.10748-04	0.50862-08	0.86057-11
	200	0.26730-03	0.12054-05	0.37601-08
λ_z = 50	100	0.15037-07	0.17745-11	0.35626-14
	150	0.18934-04	0.89814-08	0.14829-10
	200	0.45667-03	0.20792-05	0.64758-08
nu = 30				
λ_z = 30	100	0.11389-08	0.36517-14	0.44370-18
	150	0.64429-05	0.71119-09	0.10170-12
	200	0.23079-03	0.55640-06	0.52110-09
λ_z = 50	100	0.23165-08	0.68603-14	0.71646-18
	150	0.11512-04	0.13003-08	0.18147-12
	200	0.39570-03	0.96967-06	0.91574-09
nu = 40				
λ_z = 30	100	0.35899-09	0.63754-16	--
	150	0.48489-05	0.22011-09	0.50713-14
	200	0.21364-03	0.35694-06	0.16021-09
λ_z = 50	100	0.76878-09	0.12920-15	--
	150	0.87427-05	0.41289-09	0.93996-14
	200	0.36703-03	0.62645-06	0.28580-09

Table 2 (continued)

5.3 The Normal/Double-Exponential Distribution

In addition to the standard deviation and the vertical dimension of aircraft, two new factors appear in the normal/double-exponential probability distribution. They are ρ, which is defined as σ_{DE}/σ_N, where σ_N and σ_{DE} are the standard deviations of the normal and the double exponential distributions; and γ, which is the weighting factor between the two probability distributions. Computer runs were made where ρ was set equal to one, four, and seven and γ equal to 0.95 and 0.99. Values of the standard deviation for the normal distribution used in the computer runs were 100, 150, and 200 feet, consistent with the standard deviations for the normal distribution used earlier. The values of β for the double exponential distribution are obtained from the values of ρ and σ_N. Values of σ_{DE} and σ_{NDE} are computed in Table 3 to illustrate the effects of γ and ρ on σ_{DE} and σ_{NDE}. Computational results are plotted in Figure 18 to Figure 29, and a summary of the values of input parameters for these figures is given in Table 4. Table 5 provides a summary of our computational results on probabilities of vertical overlap under normal/double-exponential distributions and three separation standards.

Since the normal/double-exponential distribution is the combination of a normal distribution and a double exponential distribution, σ_{NDE} is generally less than or equal to σ_{DE}. The probability of vertical overlap is more sensitive to the value of σ_{DE} than to the value of σ_{NDE}.

γ	ρ	$[\gamma + (1-\gamma)\rho^2]^{\frac{1}{2}}$	σ_N	σ_{DE}	σ_{NDE}
0.95	1	1.0000	100	100	100.00
0.95	4	1.3229	100	400	132.29
0.95	7	1.8439	100	700	184.39
0.99	1	1.0000	100	100	100.00
0.99	4	1.0724	100	400	107.24
0.99	7	1.2166	100	700	121.66

Table 3: Values of σ_{DE} and σ_{NDE} for Certain Values of γ, ρ, and σ_N

NORMAL/DOUBLE-EXPONENTIAL DISTRIBUTIONS

FIGURES	γ	λ_Z	ρ	σ_N
18	0.95	30	1	100,150,200
19	0.95	30	4	100,150,200
20	0.95	30	7	100,150,200
21	0.95	50	1	100,150,200
22	0.95	50	4	100,150,200
23	0.95	50	7	100,150,200
24	0.99	30	1	100,150,200
25	0.99	30	4	100,150,200
26	0.99	30	7	100,150,200
27	0.99	50	1	100,150,200
28	0.99	50	4	100,150,200
29	0.99	50	7	100,150,200

Table 4: Values of Input Parameters for Figure 18 to Figure 29

γ	ρ	λ_z	σ_N	S = 1000	S = 1500	S = 2000
0.95	1	30	100	0.58099-09	0.48942-12	0.41568-15
0.95	1	30	150	0.16743-05	0.25404-09	0.22586-11
0.95	1	30	200	0.15094-03	0.68935-07	0.14219-09
0.95	1	50	100	0.10245-08	0.85940-12	0.72991-15
0.95	1	50	150	0.31325-05	0.43440-09	0.38538-11
0.95	1	50	200	0.26115-03	0.12450-06	0.24050-09
0.95	4	30	100	0.55425-06	0.94573-07	0.16145-07
0.95	4	30	150	0.32826-05	0.24601-06	0.75675-07
0.95	4	30	200	0.15367-03	0.51634-06	0.13831-06
0.95	4	50	100	0.92702-06	0.15815-06	0.26998-07
0.95	4	50	150	0.58321-05	0.41656-06	0.12631-06
0.95	4	50	200	0.26569-03	0.87462-06	0.23072-06
0.95	7	30	100	0.78841-06	0.28698-06	0.10451-06
0.95	7	30	150	0.34079-05	0.35022-06	0.17852-06
0.95	7	30	200	0.15370-03	0.54571-06	0.19696-06
0.95	7	50	100	0.13157-05	0.47882-06	0.17437-06
0.95	7	50	150	0.60418-05	0.58402-06	0.29768-06
0.95	7	50	200	0.26574-03	0.92436-06	0.32836-06

Table 5: Probabilities of Vertical Overlap
Under Normal/Double-Exponential Distributions
and Three Different Separation Standards

γ	ρ	λ_z	σ_N	S = 1000	S = 1500	S = 2000
0.99	1	30	100	0.12304-09	0.10194-12	0.86577-16
0.99	1	30	150	0.17802-05	0.54028-10	0.47053-12
0.99	1	30	200	0.16365-03	0.69219-07	0.30391-10
0.99	1	50	100	0.21933-09	0.17899-12	0.15202-15
0.99	1	50	150	0.33365-05	0.92954-10	0.80283-12
0.99	1	50	200	0.28315-03	0.12564-06	0.51627-10
0.99	4	30	100	0.11546-06	0.19701-07	0.33632-08
0.99	4	30	150	0.21153-05	0.51257-07	0.15767-07
0.99	4	30	200	0.16420-03	0.16245-06	0.28820-07
0.99	4	50	100	0.19312-06	0.32944-07	0.56240-08
0.99	4	50	150	0.38991-05	0.85562-07	0.26317-07
0.99	4	50	200	0.28406-03	0.28195-06	0.48077-07
0.99	7	30	100	0.16427-06	0.59792-07	0.21774-07
0.99	7	30	150	0.21414-05	0.72977-07	0.37199-07
0.99	7	30	200	0.16420-03	0.16858-06	0.41043-07
0.99	7	50	100	0.27413-06	0.99767-07	0.36330-07
0.99	7	50	150	0.39428-05	0.12170-06	0.62029-07
0.99	7	50	200	0.28407-03	0.29232-06	0.68427-07

Table 5 (continued)

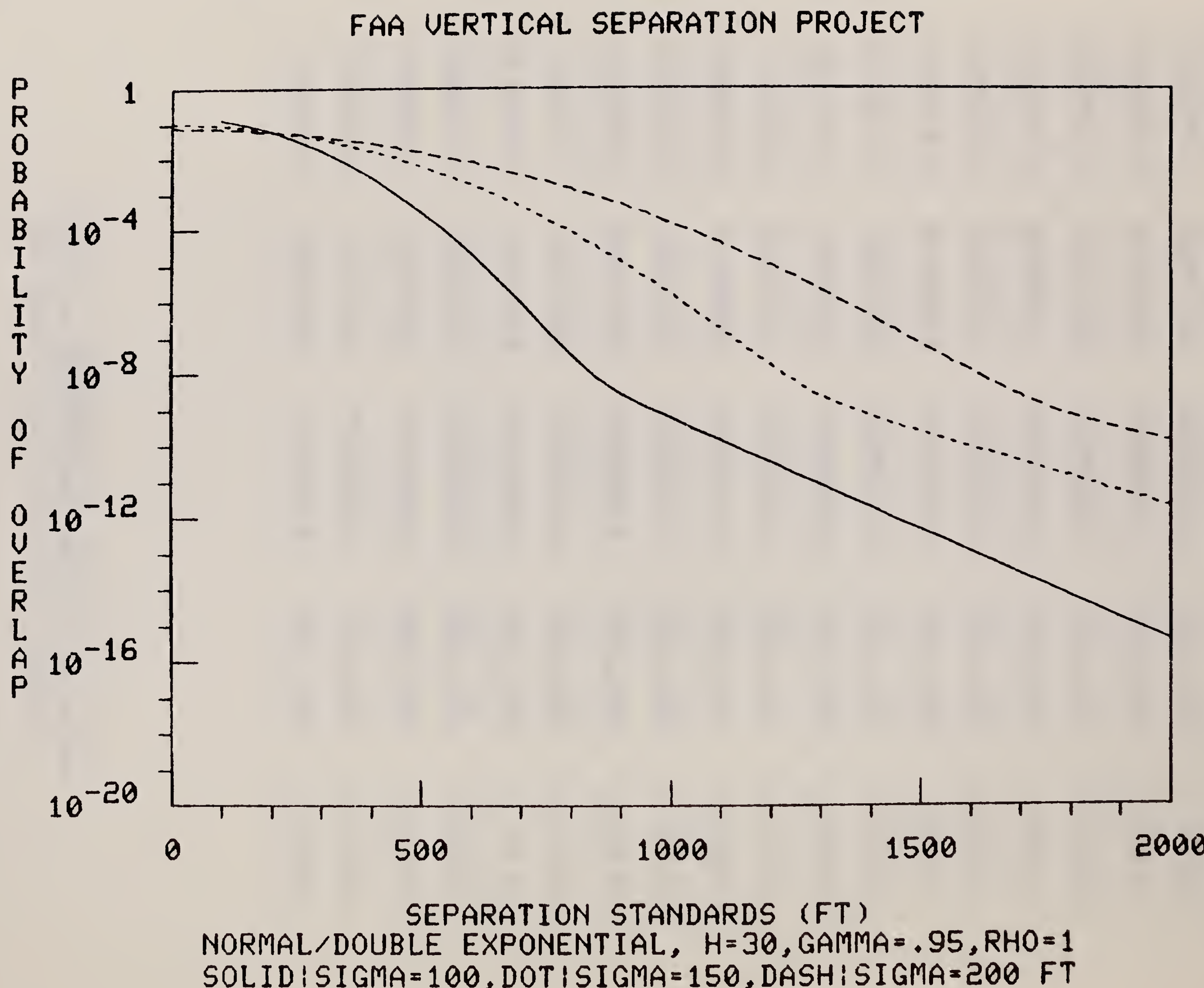

FAA VERTICAL SEPARATION PROJECT
PROBABILITY OF OVERLAP
1
10^{-4}
10^{-8}
10^{-12}
10^{-16}
10^{-20}
0
500
1000
1500
2000
SEPARATION STANDARDS (FT)
NORMAL/DOUBLE EXPONENTIAL, H=30,GAMMA=.95,RHO=1
SOLID:SIGMA=100,DOT:SIGMA=150,DASH:SIGMA=200 FT

FIGURE 18

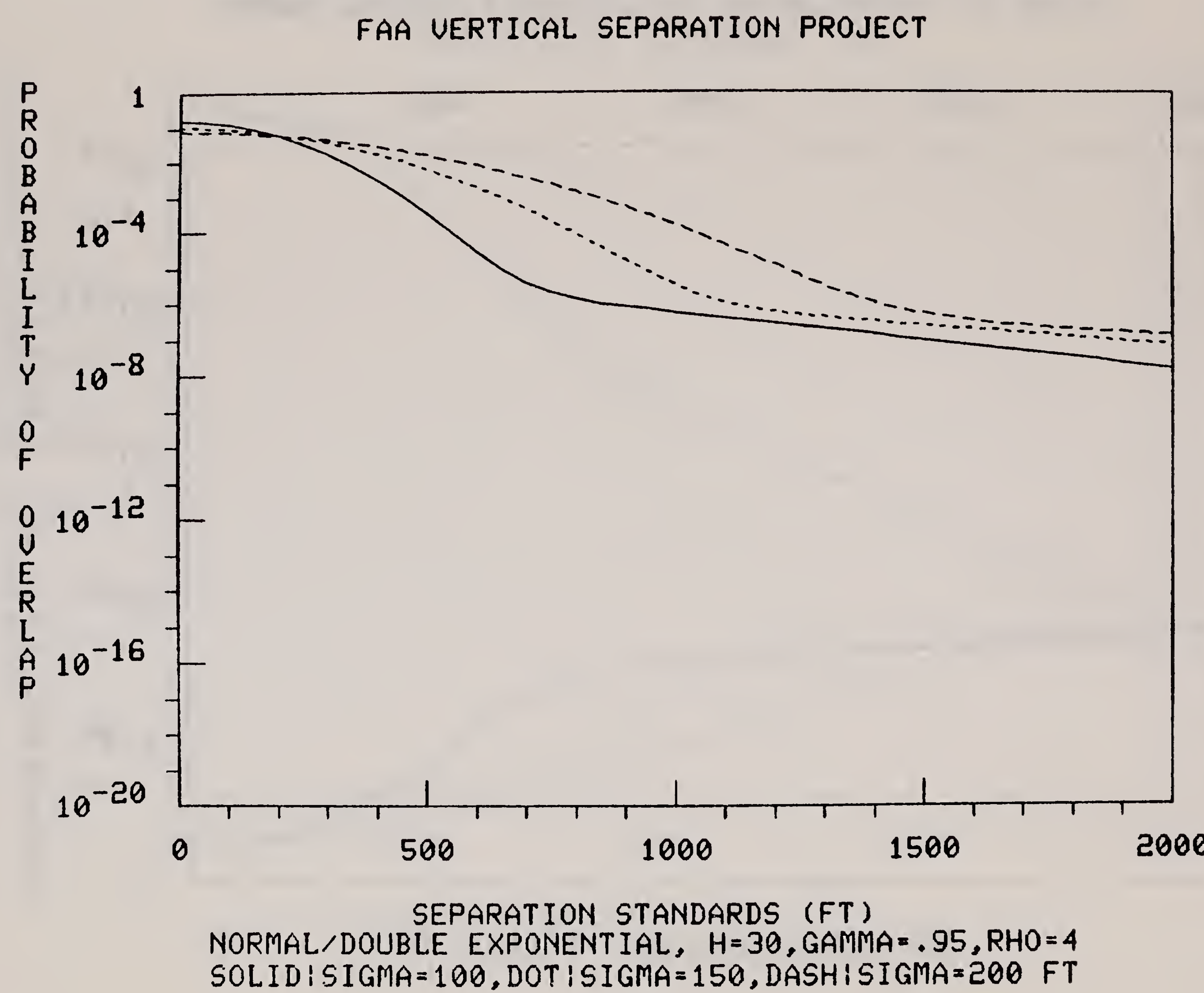

FAA VERTICAL SEPARATION PROJECT
PROBABILITY OF OVERLAP
1
10^-4
10^-8
10^-12
10^-16
10^-20
0
500
1000
1500
2000
SEPARATION STANDARDS (FT)
NORMAL/DOUBLE EXPONENTIAL, H=30,GAMMA=.95,RHO=4
SOLID:SIGMA=100,DOT:SIGMA=150,DASH:SIGMA=200 FT

FIGURE 19

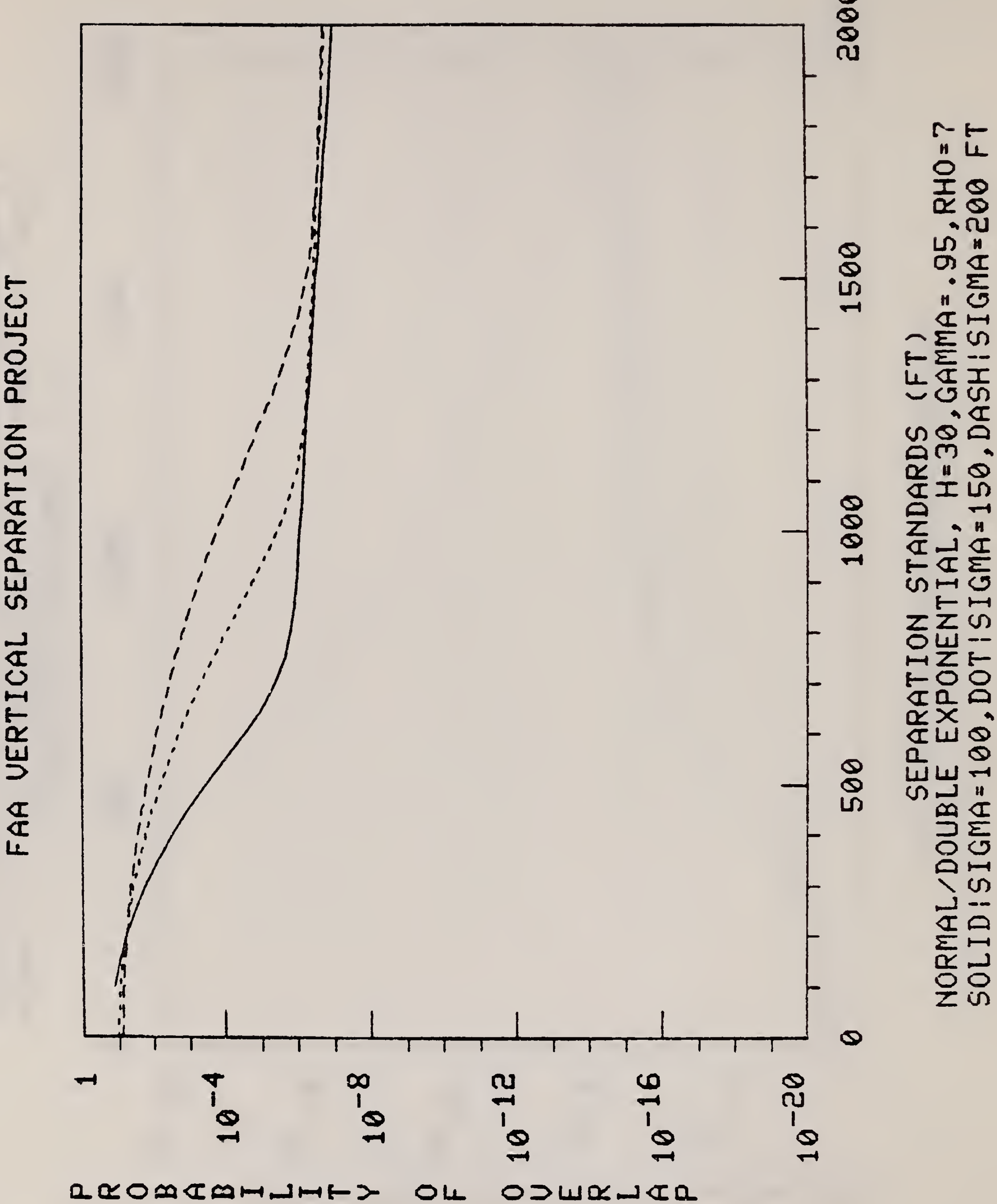
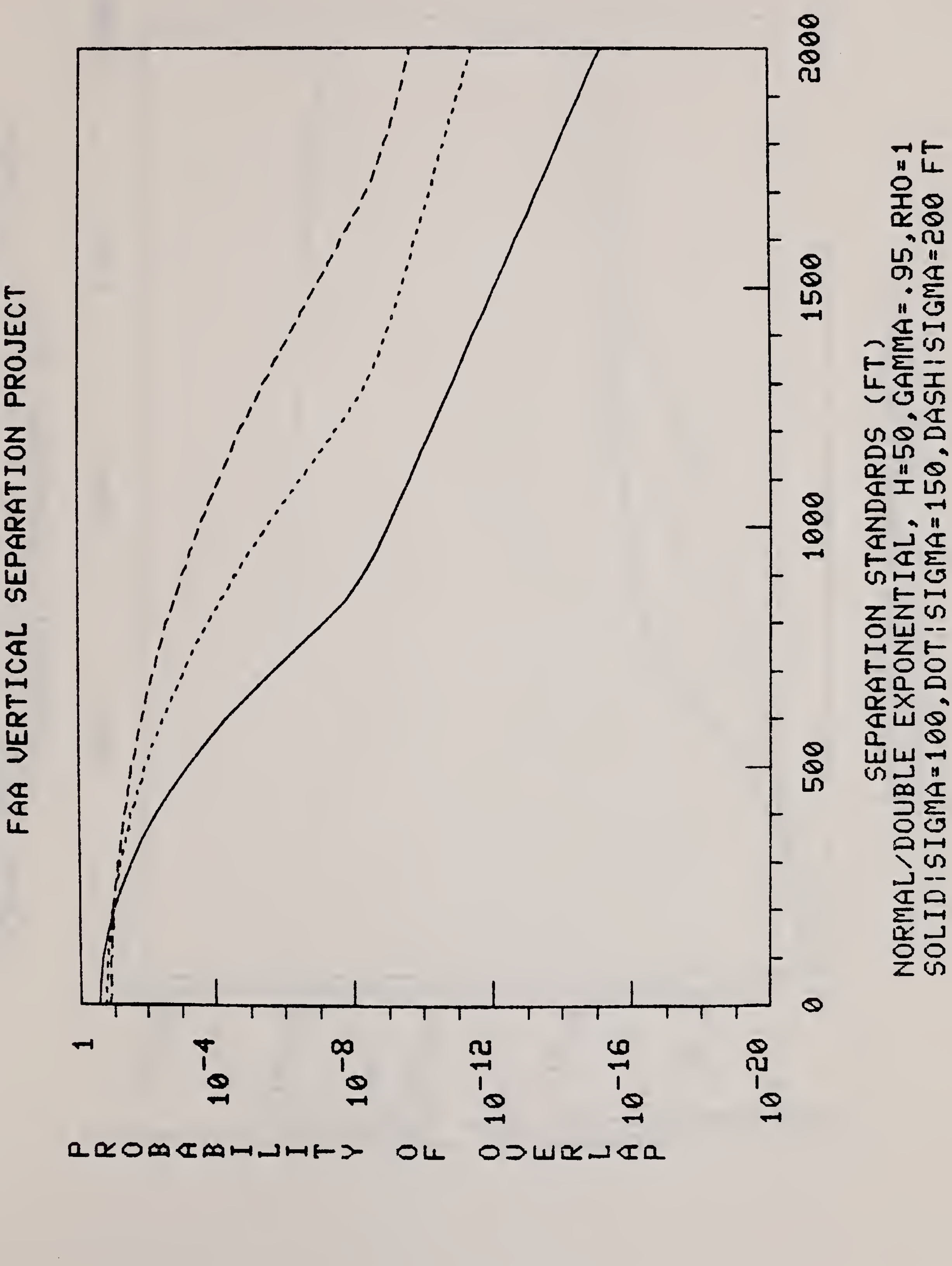

FAA VERTICAL SEPARATION PROJECT
PROBABILITY OF OVERLAP
1
10^-4
10^-8
10^-12
10^-16
10^-20
0
500
1000
1500
2000
SEPARATION STANDARDS (FT)
NORMAL/DOUBLE EXPONENTIAL, H=30,GAMMA=.95,RHO=7
SOLID:SIGMA=100,DOT:SIGMA=150,DASH:SIGMA=200 FT

FIGURE 20



FAA VERTICAL SEPARATION PROJECT
PROBABILITY OF OVERLAP
1
10-4
10-8
10-12
10-16
10-20
0
500
1000
1500
2000
SEPARATION STANDARDS (FT)
NORMAL/DOUBLE EXPONENTIAL, H=50,GAMMA=.95,RHO=1
SOLID:SIGMA=100,DOT:SIGMA=150,DASH:SIGMA=200 FT

FIGURE 21

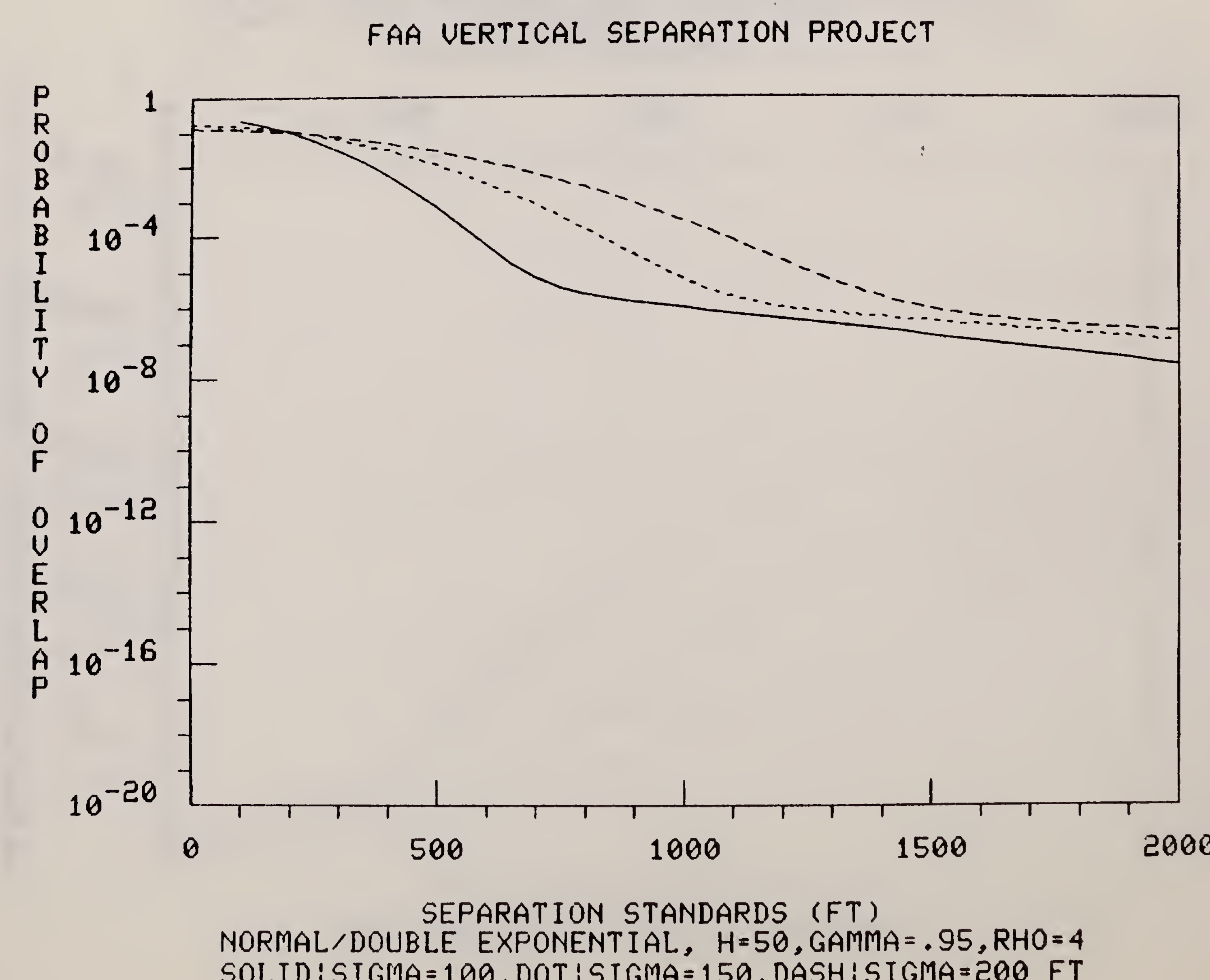

FAA VERTICAL SEPARATION PROJECT
PROBABILITY OF OVERLAP
1
10-4
10-8
10-12
10-16
10-20
0
500
1000
1500
2000
SEPARATION STANDARDS (FT)
NORMAL/DOUBLE EXPONENTIAL, H=50,GAMMA=.95,RHO=4
SOLID:SIGMA=100,DOT:SIGMA=150,DASH:SIGMA=200 FT

FIGURE 22

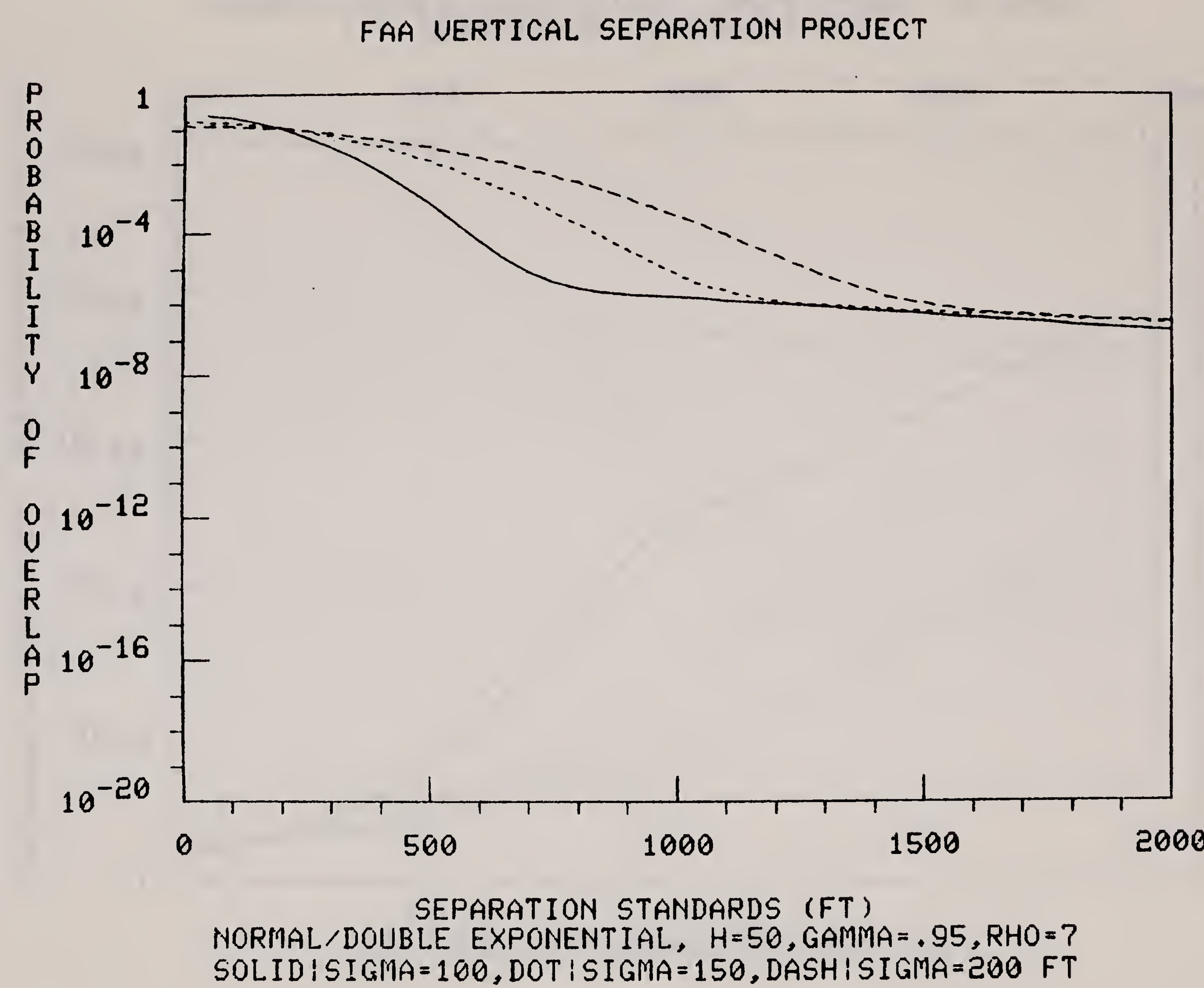
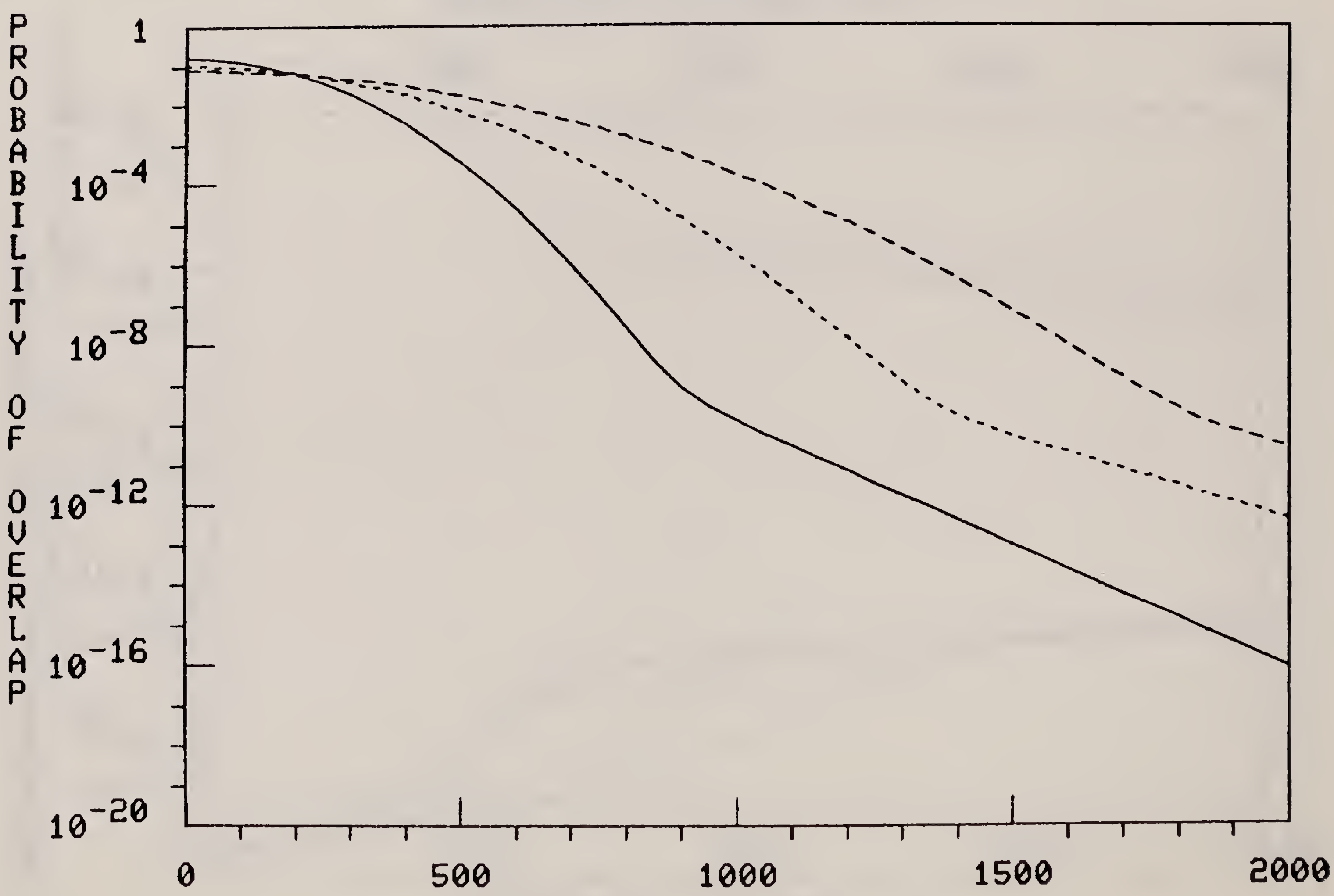

FAA VERTICAL SEPARATION PROJECT
PROBABILITY OF OVERLAP
1
10^{-4}
10^{-8}
10^{-12}
10^{-16}
10^{-20}
0
500
1000
1500
2000
SEPARATION STANDARDS (FT)
NORMAL/DOUBLE EXPONENTIAL, H=50,GAMMA=.95,RHO=7
SOLID:SIGMA=100,DOT:SIGMA=150,DASH:SIGMA=200 FT

FIGURE 23

FAA VERTICAL SEPARATION PROJECT



SEPARATION STANDARDS (FT)
NORMAL/DOUBLE EXPONENTIAL, H=30,GAMMA=.99,RHO=1
SOLID:SIGMA=100,DOT:SIGMA=150,DASH:SIGMA=200 FT

FIGURE 24

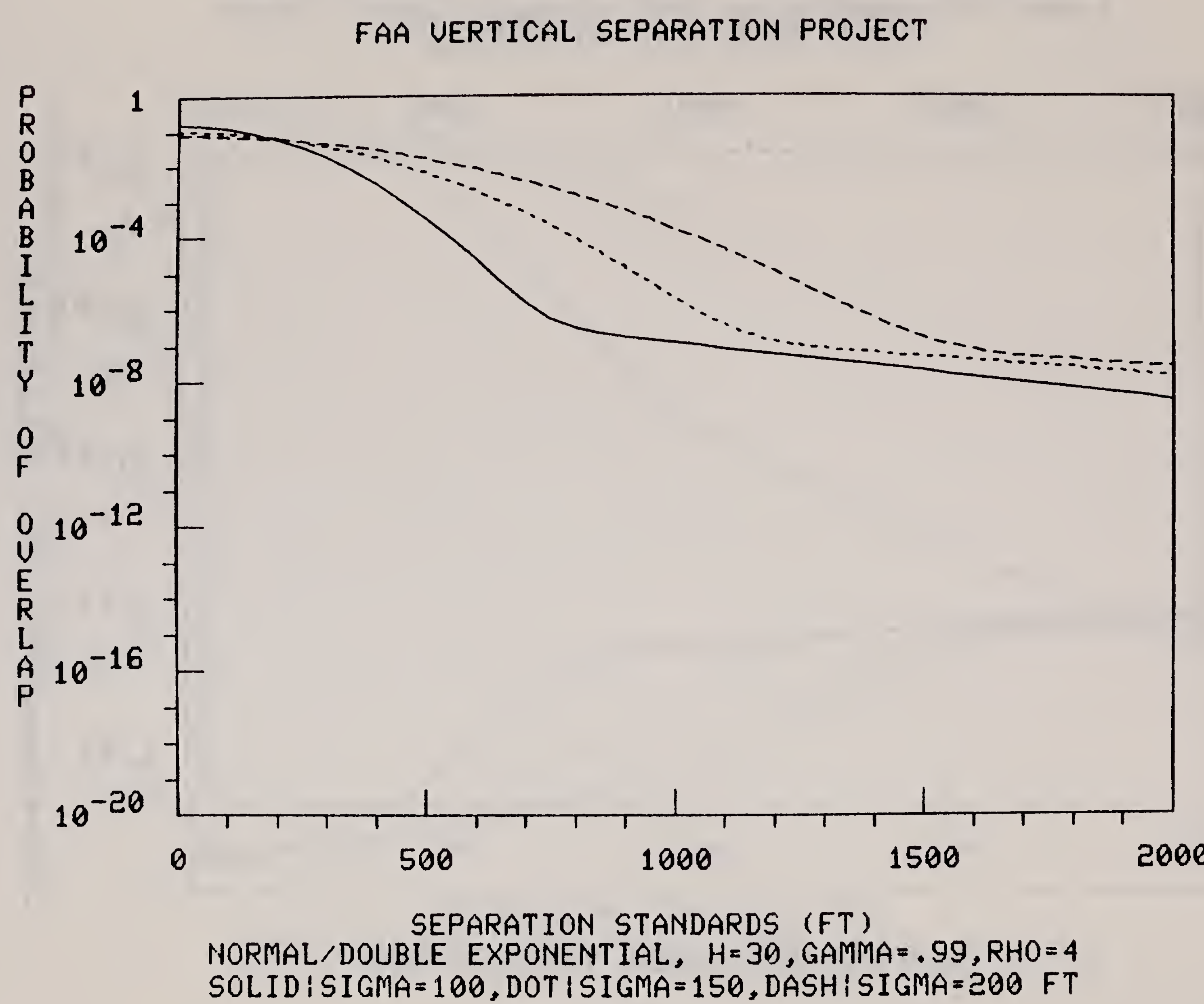
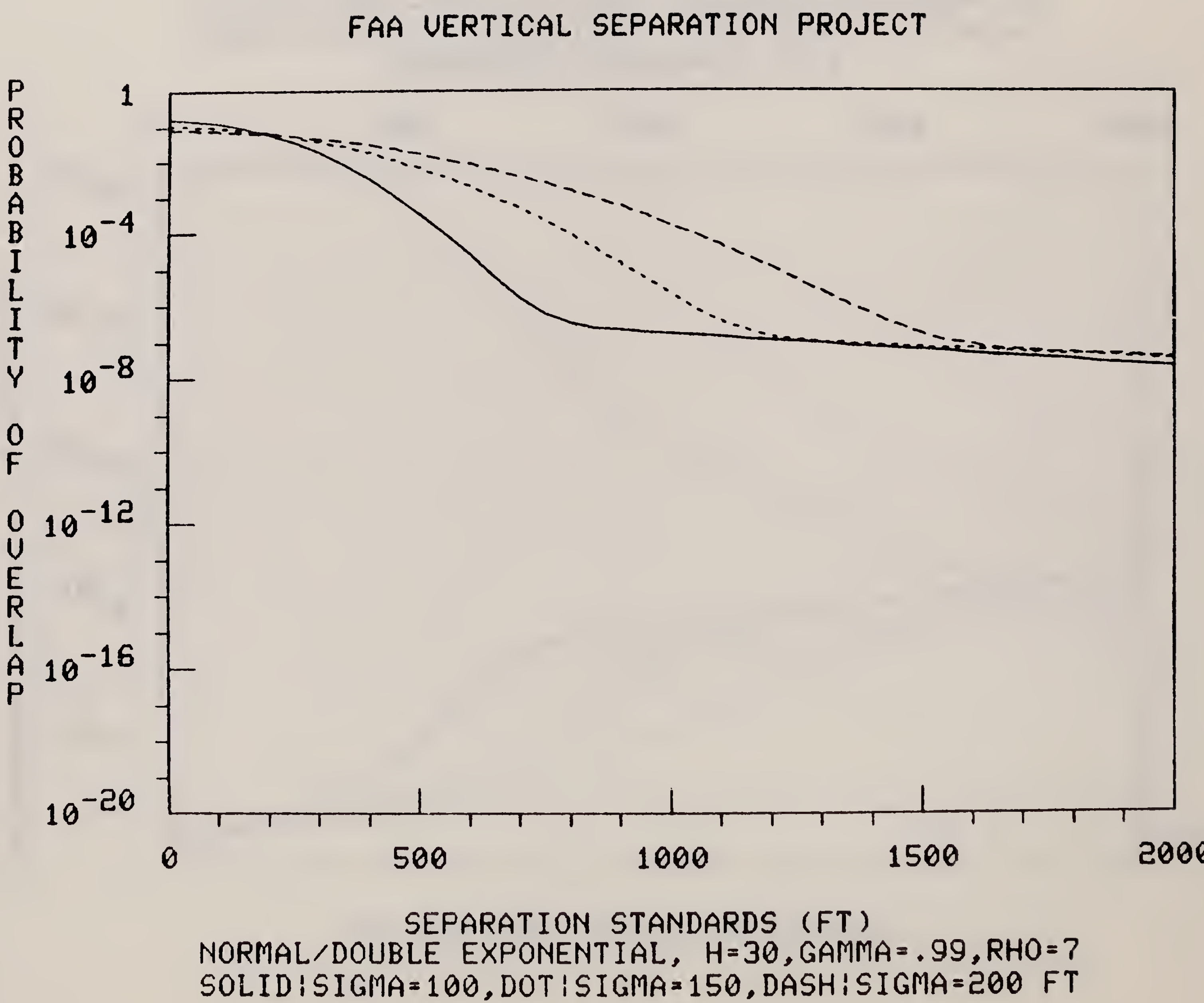

FAA VERTICAL SEPARATION PROJECT
PROBABILITY OF OVERLAP
1
10^-4
10^-8
10^-12
10^-16
10^-20
0
500
1000
1500
2000
SEPARATION STANDARDS (FT)
NORMAL/DOUBLE EXPONENTIAL, H=30,GAMMA=.99,RHO=4
SOLID!SIGMA=100,DOT!SIGMA=150,DASH!SIGMA=200 FT

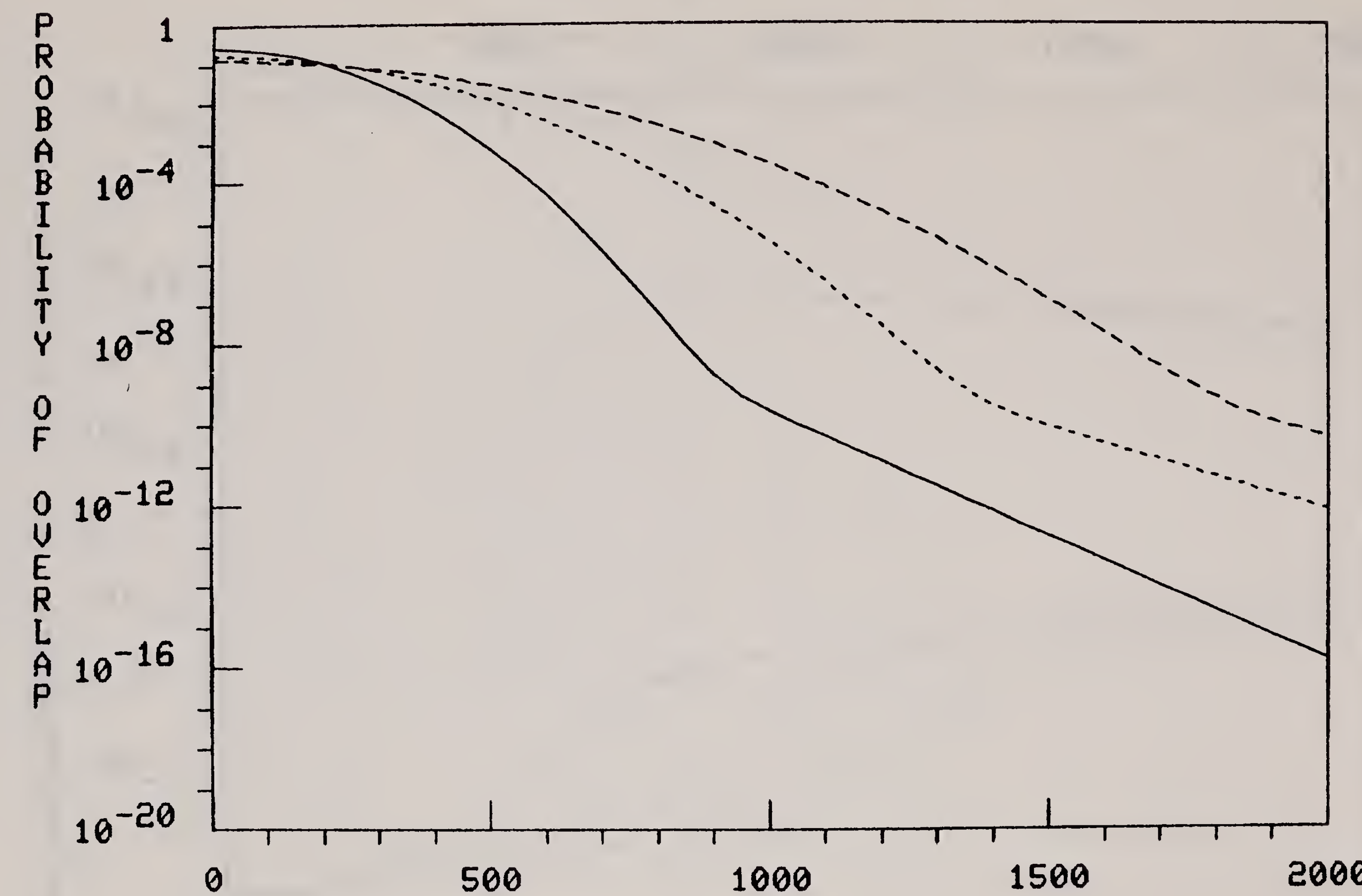
FIGURE 25



FAA VERTICAL SEPARATION PROJECT
PROBABILITY OF OVERLAP
1
10-4
10-8
10-12
10-16
10-20
0
500
1000
1500
2000
SEPARATION STANDARDS (FT)
NORMAL/DOUBLE EXPONENTIAL, H=30,GAMMA=.99,RHO=7
SOLID!SIGMA=100,DOT!SIGMA=150,DASH!SIGMA=200 FT

FIGURE 26

FAA VERTICAL SEPARATION PROJECT



SEPARATION STANDARDS (FT)
NORMAL/DOUBLE EXPONENTIAL, H=50,GAMMA=.99,RHO=1
SOLID:SIGMA=100,DOT:SIGMA=150,DASH:SIGMA=200 FT

FIGURE 27

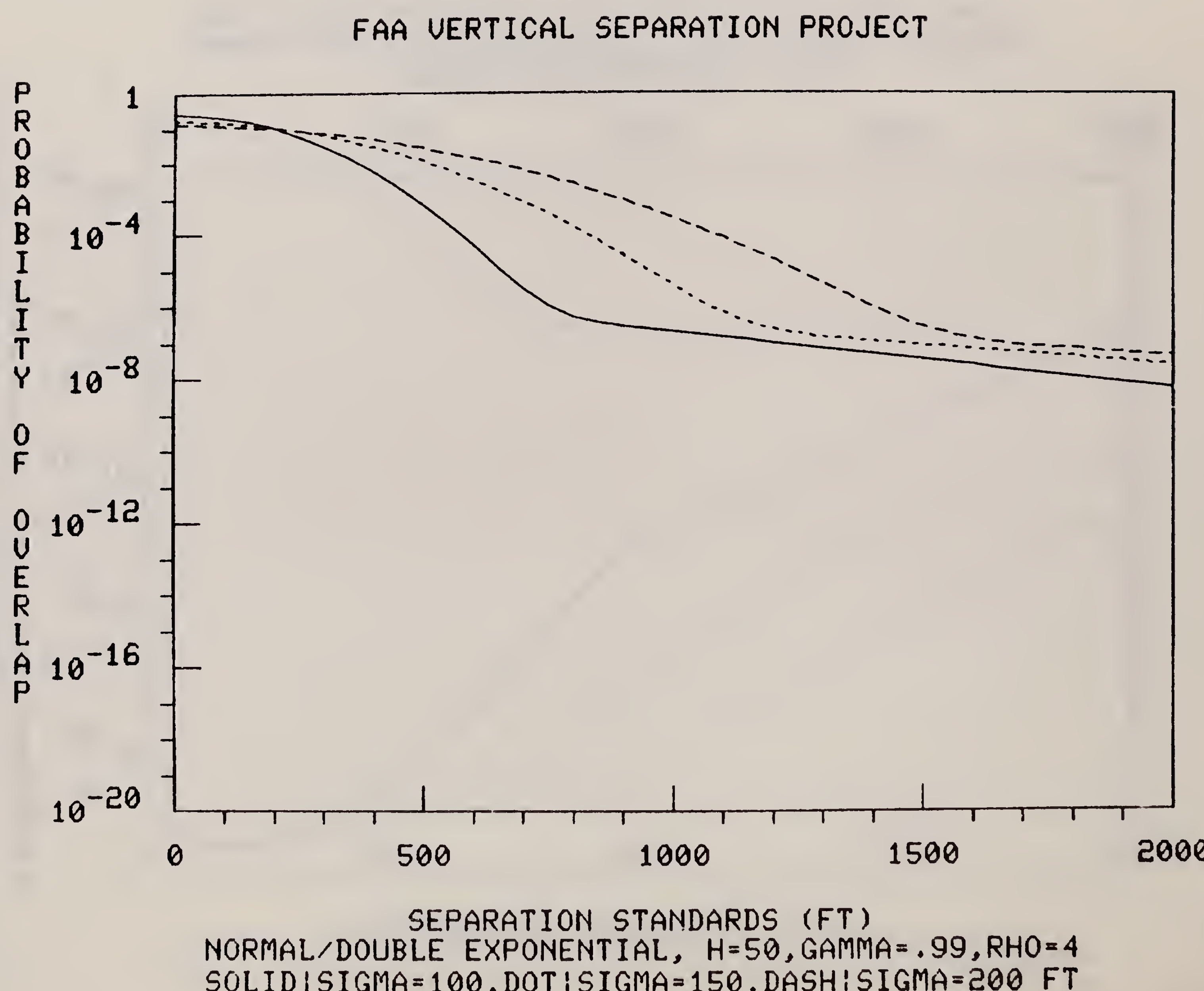

FAA VERTICAL SEPARATION PROJECT
PROBABILITY OF OVERLAP
1
10-4
10-8
10-12
10-16
10-20
0
500
1000
1500
2000
SEPARATION STANDARDS (FT)
NORMAL/DOUBLE EXPONENTIAL, H=50,GAMMA=.99,RHO=4
SOLID:SIGMA=100,DOT:SIGMA=150,DASH:SIGMA=200 FT

FIGURE 28

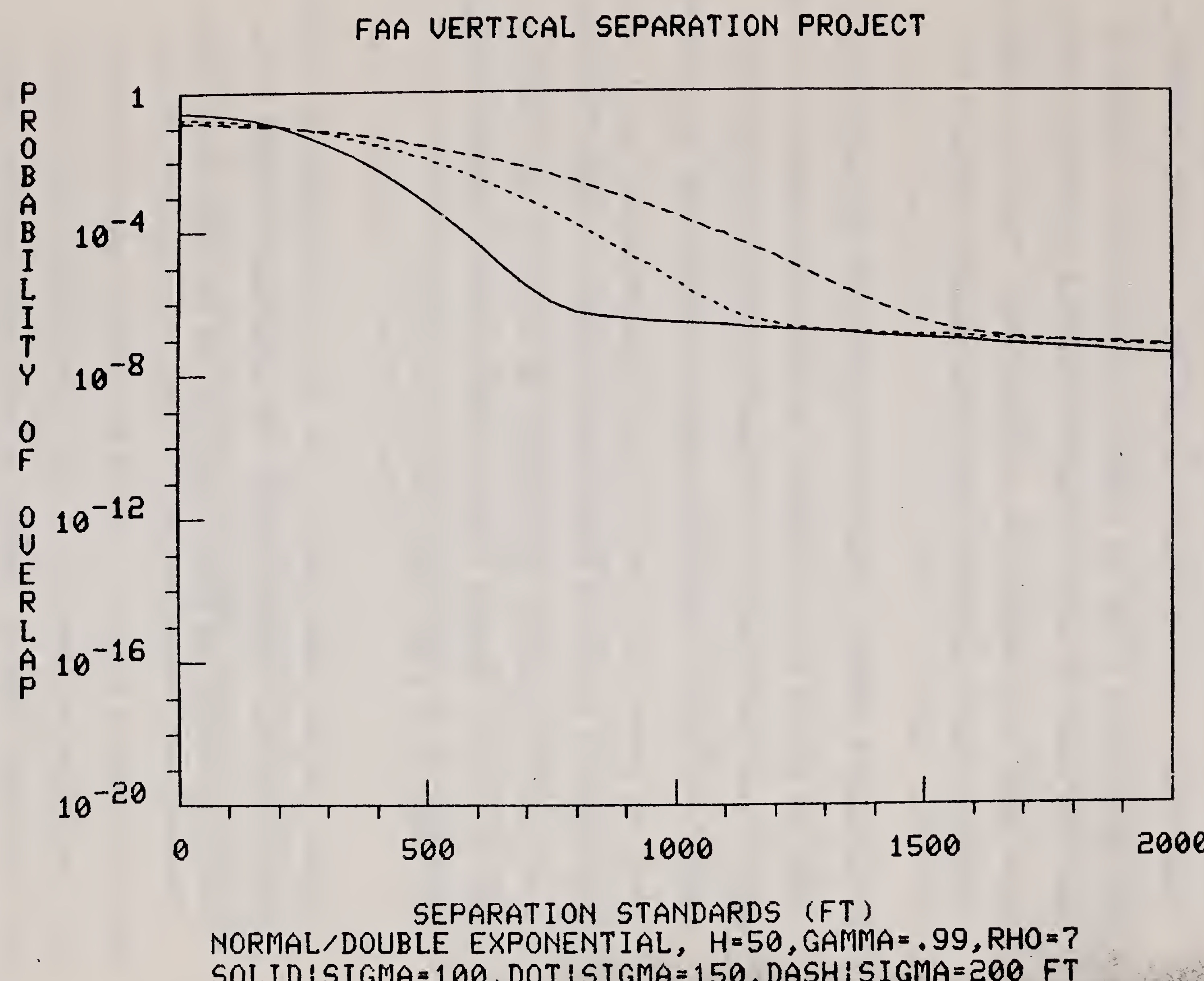

FAA VERTICAL SEPARATION PROJECT
PROBABILITY OF OVERLAP
1
10^-4
10^-8
10^-12
10^-16
10^-20
0
500
1000
1500
2000
SEPARATION STANDARDS (FT)
NORMAL/DOUBLE EXPONENTIAL, H=50,GAMMA=.99,RHO=7
SOLID:SIGMA=100,DOT:SIGMA=150,DASH:SIGMA=200 FT

FIGURE 29

6.0 Discussion and Conclusions

The phenomenon of decreasing probabilities of vertical overlap with increasing separation standards, appears intuitively reasonable. Probabilities of vertical overlap between two aircraft arise from the overlapping region of the two probability distributions. Increasing separation standards results in pulling these two probability distributions apart, thus reducing the overlapped area. Therefore, probabilities of vertical overlap decrease as separation standards increase.

Of the three families of probability distributions that we have investigated, for a fixed value of standard deviation the probabilities of vertical overlap under the assumption of a normal distribution decrease most rapidly when vertical separation standards increase. For the t distribution, the probability of vertical overlap decreases more rapidly for a large number of degrees of freedom than for smaller numbers. For the normal/double-exponential distribution, the probabilities of vertical overlap initially decrease rapidly as the separation standard increases, then change to an almost steady rate of decrease.

Vertical overlaps usually occur in the tail of the probability distribution of the total vertical errors. As can be seen in Figure 2, the normal distribution has small short tails and the Cauchy distribution has large long tails. The tails of the normal/double exponential distribution and the t distribution lie in between these two extremes. The shapes of the tails of these probability distributions in fact determine the rate of decrease of the probability of vertical overlap as the separation standard increases. The above observations indicate the importance of properly estimating the tail of the probability distribution of the total vertical error. Therefore, it is necessary in the data collection plan that sufficient amounts of data be collected to present a clear picture of the tail portion of the empirical probability distribution of the total vertical errors. Attempts should be made to fit a theoretical probability distribution to these data. Another advantage of dealing with a theoretical probability distribution is that the probability of vertical overlap can be calculated with relative ease.

From the point of view of air traffic safety, the desirable probability distributions for total vertical error of an aircraft would be normal distributions with small standard deviations. The t distributions and the normal/double exponential distributions have long tails which cause relatively large probability of vertical overlap.

It is observed in Figure 9 that the curves of probabilities of vertical overlap for normal distributions with standard deviations of 100 feet and 200 feet cross each other when the two aircraft are separated somewhere between 150 feet and 200 feet. A check by numerical methods supports these computational results. An intuitive explanation of the curves' crossing can be given as follows: For small (near-zero) values of the separation S, overlap is associated mainly with the bodies of the two aircrafts' error distributions, so that larger probabilities of overlap are associated with "large body" situations and thus with smaller standard deviations. For larger values of the separation standard, the overlap phenomenon is associated with the distribution-tails, and thus the liklihood of overlap is greater for larger tails, hence for larger standard deviations. Machol [8] also observed this; he observed that under some circumstances, the better the navigation, the greater the probability of collision, and he called this phenomenon "navigational paradox."

The effect of the vertical dimension of aircraft on the probability of vertical overlap is visible using all three probability distributions for the total vertical errors. In our limited computation of varying separation standards up to 2,000 feet, we observed that the minimum ratio of probabilities of vertical overlap is approximately the ratio of the respective aircraft vertical dimensions which was 50/30, and this happened at the larger separation values (see Table 2 and Table 5). However, for those distributions where the slope of the tail of the distribution is much greater than zero, such as normal distribution, or for lesser separation values, the ratio of probabilities of vertical overlap is generally higher than the ratio of the respective aircraft vertical dimensions. It should be mentioned that the sensitivity studies presented in this report are limited to the probability of vertical overlap defined in Section 2. The effects of aircraft phugoid and aircraft wake vortices were not considered.

In this report, we have examined the effects of the functional form of vertical error probability distribution, the parameters of the distribution, the aircraft vertical dimension, and the vertical separation standard on the probability of vertical overlap between two aircraft. In Sections 2 and 3, the probability of vertical overlap was defined and the factors affecting it discussed. A brief outline of computer programs developed to compute the probability of vertical overlap was presented in Section 4, and computational results along with their plots were presented in Section 5. Of the four factors that we have examined, the aircraft vertical dimension is the least influential.

The functional form and the parameter such as the standard deviation of the probability distribution of the total vertical error are critical to the probability of vertical overlap. Once these two factors are known, the vertical separation standard can be set at a level which assures that the probability of vertical overlap between two aircraft is sufficiently small to be acceptable to the air traffic community.

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15. SUPPLEMENTARY NOTES

☐ Document describes a computer program; SF-185, FIPS Software Summary, is attached.

16. ABSTRACT (A 200-word or less factual summary of most significant information. If document includes a significant bibliography or literature survey, mention it here.)

Because of the potential increase in traffic at FL 290 and above, both current and alternative vertical separation standards are being reviewed. A plan to collect data on the vertical navigational performance of aircraft is also contemplated. This report documents a sensitivity analysis carried out to assess how different assumptions about the probability distribution of "total vertical error" affect the probability of vertical overlap. The four factors affecting the probability of vertical overlap which are examined in this study are: the functional form of the vertical-error distribution function; the standard deviation of this probability distribution; the vertical dimensions of the aircraft; and the vertical separation standard. Probabilities of vertical overlap were computed over a range of possibilities for each of these four factors in order to discern the effect of each factor. A final section discusses the findings of this study and draws some conclusions.

17. KEY WORDS (six to twelve entries; alphabetical order; capitalize only the first letter of the first key word unless a proper name; separated by semicolons)

Air safety; collision probability; probability distribution; sensitivity analysis; separation standard; vertical error; vertical overlap

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