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# HIGHER ORDER MODES IN RECTANGULAR COAXIAL LINE WITH INFINITELY THIN INNER CONDUCTOR

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## ABSTRACT

The singular integral equation approach is used to derive the secular equations for both TE and TM waves in a rectangular coaxial line with zero thickness inner conductor. Approximations for the secular equations are found that reduce to simple expressions in terms of well-known special functions (elliptic integrals). When the strip width is exceedingly small or nearly equal to the width of the outer conductor, closed form expressions for the cut-off frequencies can be found by replacing the elliptic integrals by their asymptotic forms for modulus either near zero or one.

Key words: Higher order modes; rectangular coaxial line; striplines.

## I. INTRODUCTION

In many applications involving rectangular coaxial waveguiding structures an understanding of the propagation characteristics of higher order modes is just as important as that of the fundamental TEM mode. One such application is that of the so-called "TEM cell" which is used in some EMI measurement systems [1]. This device consists of a section of rectangular coaxial line which is used as a transducer for coupling EM energy from a device under test into the TEM mode of the transmission line. The useful frequency range of the TEM cell is limited by the cut-off frequencies of the higher order modes. Thus, in order to design a cell for use over a specified frequency range, these cut-off frequencies must be known.

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The device analyzed in this paper consists of a zero thickness metal plate located inside a rectangular waveguide. The inner conductor may be offset vertically from its central position but is located symmetrically about the y-axis. Both the inner and outer conductor are perfectly conducting and the medium separating the two conductors is a homogeneous dielectric. A cross-sectional view of the rectangular line is shown in Fig. 1.

Although the transmission line properties of the TEM mode in these structures have been extensively studied, comparitively little work has been done in analyzing the higher order mode structure. One can find in the literature, however, this problem [2-4] as well as related problems, such as shielded strip lines [5-7], various coupling configurations [8-9], rectangular lines with thick inner conductors [10-13], and ridge waveguides [14-16] analyzed using a variety of techniques. These include functional equation techniques, finite difference techniques, mode matching techniques, integral equation techniques, and methods based on transmission line theory.

The purpose of this paper is to obtain a relatively simple, closed form expression for the characteristic equation for both TE and TM modes which is valid for arbitrary strip widths. Thus, very little computer programming is necessary to calculate the cut-off frequencies. In addition, approximate solutions for either small or large strip widths can be obtained without resorting to any numerical analysis. This is achieved by using the singular integral equation technique similar to that used in waveguide diaphragm problems [17]. This method has the advantage of handling the edge condition exactly and eliminates the problems encountered in any numerical solution associated with the discontinuities of the fields near the sharp edges of the inner conductor. More specifically, the problem is formulated using an integral equation--Green's function type of approach.

The singular part of the kernel of the resulting integral equation is extracted (as is done, e.g., in [18-20]) and the nonsingular part of the kernel that remains is expanding in terms of Chebyshev polynomials as suggested by [19].

## II. FORMULATION

The cut-off frequencies of the higher-order modes of a rectangular coaxial line are just the eigenvalues K of the reduced wave equation

$$(\nabla^2 + K^2)\begin{pmatrix} H_z \\ E_z \end{pmatrix} = 0 \qquad (1)$$

where

$$\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$$

and the unknowns satisfy the following boundary conditions on the metal walls

$$\partial_n H_z = 0$$

and

$$E_z = 0.$$

In the first case, $H_z$ represents the z-component of the magnetic field of a TE mode; and in the second case, $E_z$ represents the z-component of the electric field of a TM mode.

In order to solve this eigenvalue problem, we will convert (1) into an integral equation. This is accomplished by expanding $\begin{pmatrix} H_z \\ E_z \end{pmatrix}$ into a complete set of basis functions appropriate to each of the subregions above and below the strip (j=1,2, respectively), as

$$\begin{pmatrix} H_z^{(j)} \\ E_z^{(j)} \end{pmatrix} = \sum_{m,n} \begin{pmatrix} \bar{A}_{mn}^{(j)} & 0 \\ 0 & \bar{B}_{mn}^{(j)} \end{pmatrix} \begin{pmatrix} \psi_{mn}^{(j)} \\ \phi_{mn}^{(j)} \end{pmatrix} \qquad (j = 1,2)$$

where the $\overline{A}_{mn}^{(j)}$'s and $\overline{B}_{mn}^{(j)}$'s are as yet undetermined coefficients.

For our geometry, $\psi_{mn}^{(j)}$ and $\phi_{mn}^{(j)}$ are just the TE and TM basis functions of a rectangular waveguide of height $b_j$, i.e.,

$$\psi_{mn}^{(j)}(x,y) = \left(\frac{2}{ab_j}\right)^{\frac{1}{2}} \cos\left(\frac{m\pi}{2a}(x+a)\right)\cos\left(\frac{n\pi y}{b_j}\right)$$

and

$$\phi_{mn}^{(j)}(x,y) = \left(\frac{2}{ab_j}\right)^{\frac{1}{2}} \sin\left(\frac{m\pi}{2a}(x+a)\right)\sin\left(\frac{n\pi y}{b_j}\right) .$$

These basis functions also satisfy (1) but with known eigenvalues

$$K_{mn}^{(j)} = \left(\left(\frac{m\pi}{2a}\right)^2 + \left(\frac{n\pi}{b_j}\right)^2\right)^{\frac{1}{2}} .$$

They do not, except in the rather trivial case of an unperturbed rectangular waveguide mode, satisfy the same continuity conditions as the unknowns $\begin{pmatrix} H_z \\ E_z \end{pmatrix}$ in the gap regions. We can still expand $\begin{pmatrix} H_z \\ E_z \end{pmatrix}$ in terms of these basis functions, however, since $\partial_n H_z$ or $E_z$ need not be uniformly convergent at $y = 0$. In this case the limit $y \to 0$ cannot be taken inside the appropriate expansion. See [21] for a discussion of this phenomenon.

Using the orthogonality properties of the basis functions to solve for the unknown expansion coefficients, applying Green's theorem, and matching boundary conditions in the gap regions, results in the following integral equations for the TE and TM modes, respectively.

$$\int_w^a E_x(x',0)G^{TE}(x,x')dx' = 0 \tag{2}$$

and

$$P\int_w^a \partial_{x'}E_z(x',0)G^{TM}(x,x')dx' = 0 . \tag{3}$$

where P denotes that the integral is to be interpreted in the principal value sense, $E_x(x,y) \propto \partial_y H_z(x,y)$, and the kernels $G^{TE}$ and $G^{TM}$ which contain the unknown eigenvalues $K_{TE}$ and $K_{TM}$ are given by

$$G^{TE}(x,x') = \sum_{j=1}^{2} \sum_{m,n} \Delta_m \Delta_n \frac{\psi_{mn}^{(j)}(x,0)\psi_{mn}^{(j)}(x',0)}{[K_{TE}^2 - K_{mn}^{(j)2}]} \tag{4}$$

and

$$G^{TM}(x,x') = \sum_{j=1}^{2} \frac{\partial}{\partial y} \left\{ \sum_{m,n} \frac{K_{on}^{(j)}}{K_{mo}^{(j)}} \frac{\phi_{mn}^{(j)}(x,y)\psi_{mn}^{(j)}(x',0)}{[K_{TM}^{2} - K_{mn}^{(j)2}]} \right\} \Bigg|_{y=0} \tag{5}$$

where

$$\Delta_i = \begin{cases} \frac{1}{2} & i = 0 \\ 1 & i > 0 \end{cases} .$$

The symmetry about the y-axis allows the integrations appearing in (2) and (3) to be written only over the one gap from x = w to a and permits a decomposition of the TE and TM polarizations into even and odd modes. This is manifest in the summations on "m" in (4) and (5). For TE modes with odd symmetry, i.e., $H_z(x,y) = -H_z(-x,y)$, (4) is summed over odd values, while for TM modes with odd symmetry, i.e., $E_z(x,y) = -E_z(-x,y)$, (5) is summed over even values. The reverse is true for modes with even symmetry.

## III. EXTRACTION OF THE SINGULAR PARTS OF THE KERNEL FUNCTIONS

The summations on "n" in (4) and (5) can both be summed exactly using the residue series technique [22]. The derivative in (5) can then be taken and when evaluated at y = 0 one finds

$$G^{TE}(x,x') = \sum_{j=1}^{2} \frac{b_j^2}{2\pi} \sum_m \frac{\Delta m \cot \pi\gamma_m^{(j)}}{\gamma_m^{(j)}} \psi_{mn}^{(j)}(x,0)\psi_{mn}^{(j)}(x',0) \tag{6}$$

and

$$G^{TM}(x,x') = -\frac{\pi}{2} \sum_{j=1}^{2} \sum_m \frac{\Gamma_m^{(j)} \cot \pi\Gamma_m^{(j)}}{K_{mo}^{(j)2}} \partial_x \psi_{mn}^{(j)}(x,0)\psi_{mn}^{(j)}(x',0) \tag{7}$$

where

$$\gamma_m^{(j)} = \frac{b_j}{\pi} [K_{TE}^2 - K_{mo}^{(j)2}]^{\frac{1}{2}} \tag{8}$$

and

$$\Gamma_m^{(j)} = \frac{b_j}{\pi} [K_{TM}^2 - K_{mo}^{(j)2}]^{\frac{1}{2}} . \tag{9}$$

Although the summations on "m" in (6) and (7) cannot be done in closed form, we can extract the singular part in each sum by replacing the coefficients in front of the basis functions by their asymptotic form for large "m", i.e.,

$$\frac{\cot \pi\gamma_m^{(j)}}{\gamma_m^{(j)}} \sim -\left(\frac{2a}{b_j}\right) \frac{1}{m}$$

and

$$\Gamma_m^{(j)} \cot \pi\Gamma_m^{(j)} \sim \left(\frac{b_j}{2a}\right) m \quad .$$

The kernels will then consist of two parts, a singular part $\tilde{G}$ which can be summed in closed form (see, e.g., [22]) and a correction series $\hat{G} = G-\tilde{G}$. Recognizing that the summations on "m" are either over even or odd indices, four cases need to be considered. These are not independent, however, since it is easily verified that

$$\tilde{G}^{TM}(x,x') = \partial_x \tilde{G}^{TE}(x,x') \quad . \tag{10}$$

Thus, only the kernels for the TE polarization are needed, i.e.,

$$\tilde{G}^{TE}_{odd} = \frac{1}{\pi} \ln[\tan \frac{\pi}{4a}|t+t'| \tan \frac{\pi}{4a}|t-t'||] \tag{11}$$

and

$$\tilde{G}^{TE}_{even} = \frac{1}{\pi} \ln [4 \sin \frac{\pi}{2a}|t+t'| \sin \frac{\pi}{2a}|t-t'||] \tag{12}$$

where the following change of variables was introduced

$$t = a-x \tag{13}$$

and

$$t' = a-x'. \tag{14}$$

The singular kernels for the TM polarization can then be found using (10).

Although the correction series $\hat{G}$ cannot be summed in closed form, they are rapidly convergent and are given as

$$\hat{G}^{TE}(t,t') = \sum_m A_m \cos\left(\frac{m\pi t}{2a}\right) \cos\left(\frac{m\pi t'}{2a}\right) \tag{15}$$

and

$$\hat{G}^{TM}(t,t') = \sum_m B_m \sin\left(\frac{m\pi t}{2a}\right) \cos\left(\frac{m\pi t'}{2a}\right) \tag{16}$$

where

$$A_m = \sum_{j=1}^{2} \frac{b_j}{a\pi} \left(\frac{\Delta m \cot \pi\gamma_m^{(j)}}{\gamma_m^{(j)}} + \left(\frac{2a}{b_j}\right) \frac{(1-\delta_{mo})}{m}\right) \tag{17}$$

$$B_m = \sum_{j=1}^{2} \frac{-2}{mb_j} \left(\Gamma_m^{(j)} \cot\pi\Gamma_m^{(j)} - \left(\frac{b_j}{2a}\right) m\right) \tag{18}$$

and $\delta_{mo}$ is the Kroneker delta defined as

$$\delta_{mo} = \begin{cases} 1 & m = 0 \\ 0 & m \neq 0 \end{cases} .$$

## IV. SOLUTION OF THE SINGULAR INTEGRAL EQUATIONS

Having extracted the singular parts of both kernels we can move the nonsingular correction series to the right-hand sides of both integral equations and treat these terms as forcing terms. Since the singular parts of the kernels of (2) and (3) are related according to (10), it is convenient to differentiate (2) with respect to x. Both equations will then be of the same canonical form. In terms of the new variables defined in (13) and (14), (2) and (3) can be rewritten as follows

$$P \int_0^g U(t')\tilde{G}^{TM}(t,t')dt' = \int_0^g U(t')\partial_t \hat{G}^{TE}(t,t')dt' \tag{19}$$

and

$$P \int_0^g V(t')\tilde{G}^{TM}(t,t')dt' = -\int_0^g V(t')\hat{G}^{TM}(t,t')dt' \tag{20}$$

where

$$U(t') \equiv E_x(x',0)$$

and

$$V(t') \equiv \partial_{x'} E_z(x',0) .$$

Equations (19) and (20) are both singular integral equations. The standard form of the singular integral equation, however, has integration limits -1 and +1 respectively. These limits can be achieved through the use of the Schwinger transformation [17]

$$\cos\left(\frac{\pi t'}{a}\right) = \alpha+\beta u \tag{21}$$

and

$$\cos\left(\frac{\pi t}{a}\right) = \alpha+\beta v \tag{22}$$

where

$$\alpha = \frac{1}{2}\,[\cos\left(\frac{\pi g}{a}\right) + 1]$$

and

$$\beta = \frac{1}{2}\,[\cos\left(\frac{\pi g}{a}\right) - 1] \quad .$$

Defining

$$T_v[F(u)] \equiv \frac{1}{\pi}P\int_{-1}^{1}\frac{F(u)du}{u-v}$$

and transforming to the new variables u and v defined in (21) and (22) we find that (19) and (20) can be rewritten in standard form as

$$T_v[F_{ij}(u)] = H_{ij}(v) \quad (i,j = 1,2) \tag{23}$$

where

$$F_{11}(u) = U(t')\cos\left(\frac{\pi t'}{2a}\right)/\sin\left(\frac{\pi t'}{a}\right)$$

$$F_{12}(u) = U(t')/\sin\left(\frac{\pi t'}{a}\right)$$

$$F_{21}(u) = V(t')\cos\left(\frac{\pi t'}{2a}\right)/\sin\left(\frac{\pi t'}{a}\right)$$

$$F_{22}(u) = V(t')/\sin\left(\frac{\pi t'}{a}\right)$$

$$H_{11}(v) = \int_0^g U(t')\partial_t\hat{G}^{TE}(t,t')dt'/2\sin\left(\frac{\pi t}{2a}\right)$$

$$H_{12}(v) = \int_0^g U(t')\partial_t\hat{G}^{TE}(t,t')dt'/\sin\left(\frac{\pi t}{a}\right)$$

$$H_{21}(v) = -\int_0^g V(t')\hat{G}^{TM}(t,t')dt'/2\sin\left(\frac{\pi t}{2a}\right)$$

and

$$H_{22}(v) = -\int_0^g V(t')\hat{G}^{TM}(t,t')dt'/\sin\left(\frac{\pi t}{a}\right) \quad .$$

In (23) the subscript "i" refers to the type of mode, TE or TM respectively, and the subscript "j" refers to the type of symmetry, odd or even respectively.

As shown in [17] (23) can be inverted exactly for the unknown $F_{ij}(v)$ as

$$F_{ij}(v) = \frac{1}{\sqrt{1-v^2}} \{C_{ij} - T_v[\sqrt{1-u^2}\, H_{ij}(u)]\} \tag{24}$$

where the $C_{ij}$'s are constants to be determined.

In order to make use of (24) we need to express $H_{ij}$ in terms of u. The most convenient expansion to use is one in terms of Chebyshev polynomials of the second kind $U_n$. In terms of these polynomials $H_{ij}$ can be written as

$$H_{ij}(u) = \sum_{n=0}^{\infty} P_{ij,n}\, U_n(u) \tag{25}$$

where the $P_{ij,n}$'s are expansion coefficients to be determined.

To express $F_{ij}$ as a function of v we can use the following identity [23]

$$T_v[\sqrt{1-u^2}\, U_n(u)] = -T_{n+1}(v) \tag{26}$$

where $T_n$ is the Chebyshev polynomial of the first kind.

Thus, $F_{ij}$ is given as

$$F_{ij}(v) = \frac{1}{\sqrt{1-v^2}} \{C_{ij} + \sum_{n=0}^{\infty} P_{ij,n} T_{n+1}(v)\} . \tag{27}$$

## V. DERIVATION OF THE SECULAR EQUATIONS

In order to complete the solution given in (27) we need to evaluate the unknown constants $C_{ij}$ and $P_{ij,n}$. We begin by first expressing $C_{ij}$ in terms of the $P_{ij,n}$'s. For the TM polarization this is accomplished by requiring

$$\int_w^a \partial_{x'} E_z(x',0)dx' = E(a,0) - E(w,0) = 0. \tag{28}$$

For the TE polarization, we substitute the solution given in (27) back into the undifferentiated form of the integral equation, i.e., eq. (2). As shown in Appendix 1, one then finds for $C_{ij}$ the following

$$C_{ij} = -\frac{1}{R_{ij,0}} \sum_{m=0}^{\infty} R_{ij,m+1} P_{ij,m} \tag{29}$$

where

$$R_{11,m} = I_{11,m} - \frac{\pi}{4\Delta_m} \sum_{n=m}^{\infty} A_{2n+1} P_{mn}$$

$$R_{12,m} = I_{12,m} + \frac{\pi}{2\Delta_m} \sum_{n=m}^{\infty} A_{2n} q_{mn}$$

$$R_{21,m} = I_{21,m}$$

and

$$R_{22,m} = 0 \quad (m > 0) \quad .$$

$I_{11}$, $I_{12}$, and $I_{21}$ are three canonical integrals, the recursion formulae for which, are derived in Appendix II. For modes with odd symmetry, the integrals $I_{11}$ and $I_{12}$ can be expressed in terms of complete elliptic integrals of modulus $\sqrt{\alpha}$ for the TE polarization and $\sqrt{-\beta}$ for the TM polarization, while for modes with even symmetry, $I_{12}$ contains no special functions.

$p_{mn}$ and $q_{mn}$ are the coefficients of the following expansions

$$\frac{\cos(2n+1)\left(\frac{\pi t'}{2a}\right)}{\cos\left(\frac{\pi t'}{2a}\right)} = \sum_{m=0}^{n} p_{mn} \, T_m(u)$$

and

$$\cos\left(\frac{n\pi t'}{a}\right) = \sum_{m=0}^{n} q_{mn} \, T_m(u) \quad .$$

We are now in a position to substitute our solution for $F_{ij}$ (expressed solely in terms of the unknown $P_{ij,n}$'s), back into the definition of the $H_{ij}$'s defined in (23). Upon matching coefficients of $U_n(v)$ we then obtain an infinite set of equations for the unknown $P_{ij,n}$'s. Setting the determinant of the coefficients to zero we obtain the secular equation from which K can be determined. The details of this procedure are outlined in Appendix III. Four infinite sets of equations are derived of which the following for the odd TM polarization is representative.

$$P_{22,m} = \frac{a\beta}{2} \sum_{n=m}^{\infty} \sum_{k=0}^{n} B_{2n+2} \, s_{mn} \, q_{k+1,n+1} \, P_{22,k} \tag{30}$$

where $s_{mn}$ is the coefficient in the following expansion

$$\frac{\sin[(n+1)\frac{\pi t}{a}]}{\sin\left(\frac{\pi t}{a}\right)} = \sum_{m=0}^{n} s_{mn} \, U_m(v) \quad .$$

Equation (30) is the simplest of the four equations given in Appendix III since $C_{22} = 0$ for this case only. Fortunately, however, these equations can be greatly simplified. In most cases, only a few of the $A_m$'s or $B_m$'s, given respectively in (15) and (16), are significant, and thus the infinite matrix equations given in Appendix III reduce to equations of very small order. The number of $A_m$'s or $B_m$'s that we must keep depends upon the number of higher order modes that we allow to propagate. We can see from (8) or (9) that once "m" becomes large enough so that $K_{mo} > K_{TE}$ or $K_{TM}$, respectively, then $\gamma_m^{(j)}$ or $\Gamma_m^{(j)}$ becomes imaginary. The cotangents then in (15) or (16) can be replaced by hyperbolic cotangents and thus the succeeding $A_m$'s or $B_m$'s become negligibly small. Thus, we can truncate our infinite matrix equation to a finite one at the sacrifice of being able to calculate all of the higher order modes. We will only find those modes for which $K_{TE}$ or $K_{TM} \lesssim K_{Mo}$ where we have truncated the matrix to one of the finite order M.

In many applications (e.g., [24]) one is only interested in the first few higher order modes. This method is especially applicable for finding these modes since we can usually truncate the matrices to matrices of order one. In this case, (30) reduces to

$$1 = \frac{a\beta^2}{2} B_2 \tag{31}$$

since $s_{00} = 1$ and $q_{11} = \beta$. Substituting for $B_2$ from (16) we then find the following secular equation for the odd TM modes

$$\sum_{j=1}^{2} \frac{a\Gamma_2^{(j)}}{2b_j} \cot\pi\Gamma_2^{(j)} = 1 - \frac{1}{\beta^2} \tag{32}$$

where $\Gamma_2^{(j)}$ was defined in (8). Similarly, for the even TM modes we find

$$\sum_{j=1}^{2} \frac{a\Gamma_1^{(j)}}{b_j} \cot\pi\Gamma_1^{(j)} = \frac{2E-\alpha K}{2E-(1+\alpha)K} \qquad (\mathrm{mod}\ \sqrt{-\beta}\ ) \tag{33}$$

## VI. CONCLUSIONS

In addition to the higher-order modes whose cut-off wavelengths are plotted in Figs. 2-4, there are also some higher-order modes which are unperturbed by the presence of the inner conductor. If $b_1 = b_2 = b$, these modes are just the $TE_{m,2n}$ and $TM_{m,2n}$ modes of the full guide of cross-section 2a x 2b. Alternatively, we can view this system as two rectangular sub-waveguides of cross-section 2a x b which are coupled through the gap. The sub-waveguide modes can then be combined to form a system mode of the entire structure which is either symmetric with respect to the inner conductor (i.e., $E_y(x,y) = E_y(x,-y)$ or antisymmetric. As found also in [27] only the symmetric combination is unperturbed by the presence of the inner conductor. If the inner conductor is offset then there exists fewer unperturbed modes.

Since the perturbed modes reduce to rectangular waveguide modes in the limit of either zero strip width ($\beta \to -1$) or zero gap ($\beta \to 0$) we can obtain approximate solutions for the cut-off wavelengths by replacing the cotangents and elliptic integrals in (32) to (34) by their asymptotic forms for beta either near 0 or -1. One can then find that the cut-off frequencies are perturbed as follows. For small strip widths

$$\delta^2_{TM_{11}} \propto 1/\ln\left(\frac{8a}{\pi w}\right)$$

$$\delta^2_{TM_{21}} \propto \left(\frac{w}{a}\right)^2$$

and

$$\delta^2_{TE_{11}} \propto \left(\frac{w}{a}\right)^2$$

while for small gaps one finds

$$\delta^2_{TM_{11}} \propto \left(\frac{g}{a}\right)^2$$

$$\delta^2_{TM_{21}} \propto \left(\frac{g}{a}\right)^4$$

and

$$\delta^2_{TE_{11}} \propto 1/\ln \left(\frac{8a}{\pi g}\right)$$

where

$$K^2 - K^2_{\text{unperturbed}} \equiv \pm \left(\frac{\delta}{b}\right)^2 \quad .$$

The significance of the logarithmic perturbation of the cut-off wavelength of the $TE_{11}$ mode for small gap and the $TM_{11}$ mode for small strip width is apparent in Figs. 3 and 4. For any finite gap width or strip width, respectively, the perturbation is quite significant. Thus, in contrast to the other cases for which the perturbation is algebraic, the cut-off frequencies are not accurately predicted by their unperturbed values. This same type of logarithmic or algebraic behavior was found by [8] in an analysis of a related problem of two rectangular waveguides coupled through a longitudinal slot.

In reference [11] are plotted similar curves for the case of a thick center conductor. The curves show the same qualitative behavior as a function of strip width. In particular, the cut-off wavelength of the $TE_{11}$ mode for a small gap tends to be that of a "coaxial $TE_{10}$ mode" in each sub-waveguide.

Although we have only presented the numerical results for a one-term approximation to the secular equations, higher-order solutions can be easily found, since the canonical integrals needed in the computation of the matrix elements are given recursively. For example, if we keep two of the $B_m$'s in the expression for the secular equation for the odd TM mode, (30) reduces to

$$\left(1 - \frac{a\beta^2}{2} B_2\right) \left(1 - \frac{a\beta^4}{2} B_4\right) = 0 \quad . \tag{36}$$

The first term in (36) gives the same roots as we found earlier in our one-term approximation as given in (31). In addition, however, we have an extra term which gives rise to additional higher-order modes. One can see how continuing this process would ultimately recover all of the higher-order modes.

## VII. ACKNOWLEDGMENTS

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## APPENDIX 1. EXPRESSION FOR $C_{ij}$ IN TERMS OF $P_{ij,n}$

We begin with the odd TM Polarization and transform (28) into an equation in terms of u as follows

$$\int_{w}^{a} \partial_{x'} E_z(x',0)dx' = \int_{0}^{g} V(t')dt' = \int_{-1}^{1} F_{22}(u)du = 0 \qquad (1.1)$$

If we now substitute for $F_{22}(u)$ from (27), we obtain

$$C_{22} \int_{-1}^{1} \frac{du}{\sqrt{1-u^2}} = - \sum_{n=o}^{\infty} P_{22,n} \int_{-1}^{1} \frac{T_{n+1}(u)du}{\sqrt{1-u^2}} \quad . \qquad (1.2)$$

The right-hand side of (1.2) vanishes because of the orthogonality property of the Chebyshev polynomials, i.e., [25]

$$\int_{-1}^{1} \frac{T_m(u)\ T_n(u)du}{\sqrt{1-u^2}} = \frac{\pi}{2\Delta_m} \delta_{mn} \qquad (1.3)$$

where

$$\Delta_m = \begin{cases} \frac{1}{2} & m=0 \\ 1 & m>0 \end{cases} \quad .$$

Thus,

$$C_{22} = 0 \ . \qquad (1.4)$$

For the even TM Polarization, we transform (1.1) into an equation in terms of a new variable $\chi$ defined as

$$\chi = \cos\left(\frac{\pi t'}{2a}\right) \qquad (1.5)$$

to obtain

$$\int_w^a \partial_{x'} E_z(x',0)dx' = 2\int_0^g F_{21}(u)\sin\left(\frac{\pi t'}{2a}\right)dt' = \frac{4a}{\pi}\int_{\sqrt{\alpha}}^1 F_{21}\left(\frac{2\chi^2-1-\alpha}{\beta}\right)d\chi=0.$$

If we now substitute for $F_{21}$ from (27) we obtain

$$\int_{\sqrt{\alpha}}^1 \frac{C_{21}d\chi}{\sqrt{(1-\chi^2)(\chi^2-\alpha)}} = -\sum_{n=0}^{\infty} P_{21,n}\int_{\sqrt{\alpha}}^1 \frac{T_{n+1}\left(\frac{2\chi^2-1-\alpha}{\beta}\right)d\chi}{\sqrt{(1-\chi^2)(\chi^2-\alpha)}}$$

since

$$\sqrt{1-u^2} = -\frac{2}{\beta}\sqrt{(1-\chi^2)(\chi^2-\alpha)} \quad .$$

Defining $I_{21,n}$ as

$$I_{21,n} = \int_{\sqrt{\alpha}}^1 T_n\left(\frac{2\chi^2-1-\alpha}{\beta}\right)\frac{d\chi}{\sqrt{(1-\chi^2)(\chi^2-\alpha)}}$$

we find for $C_{21}$ the following

$$C_{21} = -\frac{1}{I_{21,0}}\sum_{n=0}^{\infty} P_{21,n}\, I_{21,n+1} \tag{1.6}$$

Now for the odd TE polarization we transform (2) into an equation in terms of t and t', and without loss of generality, evaluate the resulting equation at t = 0 to obtain

$$\int_0^g U(t')\tilde{G}^{TE}(0,t')dt' = -\int_0^g U(t')\hat{G}^{TE}(0,t')dt' \quad . \tag{1.7}$$

The left-hand side (L.H.S.) of (1.7) can be transformed into an equation in terms of the variable $\chi$ defined in (1.5) as follows

$$\tilde{G}(0,t') = \frac{1}{\pi}\ln\tan^2\frac{\pi t'}{4a} = \frac{1}{\pi}\ln\left(\frac{1-\chi}{1+\chi}\right) \quad .$$

Thus,

$$\text{L.H.S.} = \frac{2}{\pi}\int_0^g F_{11}(u)\,\ell n\left(\frac{1-\chi}{1+\chi}\right)\sin\left(\frac{\pi t'}{2a}\right)dt' = \frac{4a}{\pi^2}\int_{\sqrt{\alpha}}^1 F_{11}\left(\frac{2\chi^2-1-\alpha}{\beta}\right)\ell n\left(\frac{1-\chi}{1+\chi}\right)d\chi$$

If we now substitute for $F_{11}$ from (27) we obtain

$$\text{L.H.S.} = \frac{2a}{\pi^2}\int_{\sqrt{\alpha}}^1 \frac{\ell n\left(\frac{1+\chi}{1-\chi}\right)d\chi}{\sqrt{(1-\chi^2)(\chi^2-\alpha)}}\left(C_{11} + \sum_{n=0}^{\infty} P_{11,n}\, T_{n+1}\left(\frac{2\chi^2-1-\alpha}{\beta}\right)\right) . \tag{1.8}$$

Defining $I_{11,n}$ as

$$I_{11,n} = \frac{1}{\pi}\int_{\sqrt{\alpha}}^1 T_n\left(\frac{2\chi^2-1-\alpha}{\beta}\right)\ell n\left(\frac{1+\chi}{1-\chi}\right)\frac{d\chi}{\sqrt{(1-\chi^2)(\chi^2-\alpha)}}$$

we see that (1.8) reduces to

$$\text{L.H.S.} = \frac{2a\beta}{\pi}\left[C_{11}\, I_{11,0} + \sum_{n=0}^{\infty} P_{11,n} I_{11,n+1}\right] . \tag{1.9}$$

Now the right-hand side (R.H.S.) of (1.7) can be written in terms of u as follows

$$\text{R.H.S.} = -\int_0^g F_{11}(u)\,\frac{\sin\left(\frac{\pi t'}{a}\right)}{\cos\left(\frac{\pi t'}{2a}\right)}\sum_{n=0}^{\infty} A_{2n+1}\cos\left[(2n+1)\frac{\pi t'}{2a}\right]dt' =$$

$$= \frac{a\beta}{\pi}\sum_{n=0}^{\infty} A_{2n+1}\int_{-1}^{1} F_{11}(u)\,\frac{\cos\left[(2n+1)\frac{\pi t'}{2a}\right]}{\cos\left(\frac{\pi t'}{2a}\right)}\,du.$$

If we now make use of the following expansion in terms of Chebyshev polynomials of the first kind $T_m$

$$\frac{\cos[(2n+1)\frac{\pi t'}{2a}]}{\cos\left(\frac{\pi t'}{2a}\right)} = \sum_{m=0}^{n} p_{mn}\, T_m(u) \tag{1.10}$$

and substitute for $F_{11}(u)$ from (27) we find the following

$$\text{R.H.S.} = \frac{a\beta}{\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{n} A_{2n+1}\, p_{mn} \int_{-1}^{1} \frac{T_m(u)\,du}{\sqrt{1-u^2}} [C_{11} + \sum_{k=0}^{\infty} P_{11,k}\, T_{k+1}(u)] . \tag{1.11}$$

The integrations in (1.11) can be evaluated by using the orthogonality relation given in (1.3) and thus (1.11) reduces to

$$\text{R.H.S.} = a\beta \sum_{n=0}^{\infty} A_{2n+1} [p_{on} C_{11} + \frac{(1-\delta_{no})}{2} \sum_{k=1}^{n} P_{11,k-1}\, p_{kn}] \tag{1.12}$$

where $\delta_{no}$ is the Kroneker delta. Finally (1.9) and (1.12) when equated and solved for $C_{11}$ gives

$$C_{11} = \frac{-\left(\sum_{n=0}^{\infty} P_{11,n}\, I_{11,n+1} - \frac{\pi}{4} \sum_{n=0}^{\infty} A_{2n+1}(1-\delta_{no}) \sum_{k=1}^{n} P_{11,k-1}\, p_{kn}\right)}{\left(I_{11,0} - \frac{\pi}{2} \sum_{n=0}^{\infty} A_{2n+1}\, p_{on}\right)} . \tag{1.13}$$

For the even TE polarization we again start with (1.7), but we transform the left-hand side of that equation into an equation in terms of u as follows

$$\tilde{G}^{TE}(0,t') = \frac{1}{\pi} \ln[(-2\beta)(1+u)]$$

Thus,

$$\text{L.H.S.} = -\frac{a\beta}{\pi^2} \int_{-1}^{1} F_{12}(u) \ln[(-2\beta)(1+u)]du .$$

If we now substitute for $F_{12}(u)$ from (27) we obtain

$$\text{L.H.S.} = -\frac{a\beta}{\pi^2}\int_{-1}^{1}\frac{\ln[(-2\beta)(1+u)]du}{\sqrt{1-u^2}}\,[C_{12} + \sum_{n=0}^{\infty} P_{12,n}\,T_{n+1}(u)]\,. \quad (1.14)$$

Defining $I_{12,n}$ as

$$I_{12,n} = \frac{1}{\pi}\int_{-1}^{1}\frac{T_n(u)\ln(1+u)}{\sqrt{1-u^2}}\,du + \delta_{no}\,\ln(-2\beta)$$

we see that (1.14) reduces to

$$\text{L.H.S.} = -\frac{a\beta}{\pi}\,[C_{12}\,I_{12,0} + \sum_{n=0}^{\infty} P_{12,n}\,I_{12,n+1}]\,. \quad (1.15)$$

Now the right-hand side of (1.7) can be written in terms of u as follows

$$\text{R.H.S.} = -\int_0^g F_{12}(u)\,\sin\left(\frac{\pi t'}{a}\right)\sum_{n=0}^{\infty} A_{2n}\,\cos\left(\frac{n\pi t'}{a}\right)dt' =$$

$$= \frac{a\beta}{\pi}\sum_{n=0}^{\infty} A_{2n}\int_{-1}^{1} F_{12}(u)\,\cos\left(\frac{n\pi t'}{a}\right)du.$$

If we now make use of the following expansion

$$\cos\left(\frac{n\pi t'}{a}\right) = \sum_{m=0}^{n} q_{mn}\,T_m(u)$$

and substitute for $F_{12}(u)$ from (27) we find the following

$$\text{R.H.S.} = \frac{a\beta}{\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{n} A_{2n}\,q_{mn}\int_{-1}^{1}\frac{T_m(u)du}{\sqrt{1-u^2}}\,[C_{12} + \sum_{k=0}^{\infty} P_{12,k}\,T_{k+1}(u)]\,. \quad (1.16)$$

Again, the integrations in (1.16) can be evaluated using the orthogonality relation given in (1.3) to obtain

$$\text{R.H.S.} = a\beta \sum_{n=0}^{\infty} A_{2n} \left[q_{on} C_{12} + \frac{(1-\delta_{no})}{2} \sum_{k=1}^{n} P_{12,k-1}\, q_{kn}\right] \qquad (1.17)$$

Finally (1.15) and (1.17) when equated and solved for $C_{12}$ gives

$$C_{12} = - \frac{\left\{\sum_{n=0}^{\infty} P_{12,n} I_{12,n+1} + \frac{\pi}{2} \sum_{n=0}^{\infty} A_{2n} (1-\delta_{no}) \sum_{k=1}^{n} P_{12,k-1}\, q_{kn}\right\}}{\left\{I_{12,0} + \pi \sum_{n=0}^{\infty} A_{2n}\, q_{on}\right\}} \quad . (1.18)$$

Equations (1.4), (1.6), (1.13) and (1.18) can all be expressed in one equation as

$$C_{ij} = - \frac{1}{R_{ij,0}} \sum_{m=0}^{\infty} R_{ij,m+1}\, P_{ij,m}$$

where

$$R_{11,m} = I_{11,m} - \frac{\pi}{4\Delta_m} \sum_{n=m}^{\infty} A_{2n+1}\, P_{mn}$$

$$R_{12,m} = I_{12,m} + \frac{\pi}{2\Delta_m} \sum_{n=m}^{\infty} A_{2n} q_{mn}$$

$$R_{21,m} = I_{21,m}$$

and

$$R_{22,m} = 0 \quad (m > 0) \quad .$$

## APPENDIX 2. EVALUATION OF THE CANONICAL INTEGRALS

The canonical integrals appearing in (29) are defined by

$$I_{11,n} = \frac{1}{\pi} \int_{\sqrt{\alpha}}^{1} T_n\left(\frac{2\chi^2-1-\alpha}{B}\right) \ell n\left(\frac{1+\chi}{1-\chi}\right) \frac{d\chi}{\sqrt{(1-\chi^2)(\chi^2-\alpha)}}$$

$$I_{12,n} = \frac{1}{\pi} \int_{-1}^{1} \frac{T_n(u)\ell n(1+u)}{\sqrt{1-u^2}} du + \delta_{no} \ell n(-2\beta)$$

and

$$I_{21,n} = \int_{\sqrt{\alpha}}^{1} T_n\left(\frac{2\chi^2-1-\alpha}{\beta}\right) \frac{d\chi}{\sqrt{(1-\chi^2)(\chi^2-\alpha)}}$$

The evaluation of $I_{12}$ can be found in [19] and is given by

$$I_{12,n} = \begin{cases} \ell n(-\beta) & n = 0 \\ \dfrac{(-1)^{n+1}}{n} & n > 0 \end{cases} .$$

The remaining two integrals can both be evaluated by recognizing that [25]

$$T_n(u)-T_{n+2}(u) = 2(1-u^2)U_n(u)$$

where

$$\sqrt{1-u^2} = \frac{-2}{\beta} \sqrt{(1-\chi^2)(\chi^2-\alpha)}$$

and

$$u = \frac{2\chi^2-1-\alpha}{\beta} .$$

Thus, if we take $I_{i1,n} - I_{i1,n+2}$ we can convert the integral into one for which the square root is now in the numerator. This integral can then be integrated by parts to obtain the following recursion relation

$$I_{i1,n+2} = \frac{2\left(\frac{1+\alpha}{1-\alpha}\right)(2n+2)\, I_{i1,n+1} - (2n+1)\, I_{i1,n} + J_{i1,n}}{(2n+3)} \qquad (i = 1,2) \qquad (2.1)$$

where

$$J_{11,n} = \frac{16}{\pi\beta^2} \int_{\sqrt{\alpha}}^{1} U_n\left(\frac{2\chi^2-1-\alpha}{\beta}\right) \left(\frac{\chi^2-\alpha}{1-\chi^2}\right)^{\frac{1}{2}} \chi \, d\chi$$

and

$$J_{21,n} = 0 .$$

$J_{21}$ is zero because the integrand of $I_{21}$ does not contain a logarithmic function as does $I_{11}$. The evaluation of $J_{11}$ is easily accomplished by substituting

$$\chi^2 = \cos^2\theta + \alpha\sin^2\theta$$

to give

$$J_{11,n} = \frac{4}{\beta}\,(-1)^{n+1} \quad .$$

The starting values for the recurrence relations given in (2.1) are given as [26]

$$I_{11,0} = K\,(\sqrt{\alpha})$$

$$I_{11,1} = \frac{1}{\beta}\left\{2[1+K(\sqrt{\alpha})-E(\sqrt{\alpha})] - (1+\alpha)\, K(\sqrt{\alpha})\right\}$$

$$I_{21,0} = K(\sqrt{-\beta})$$

and

$$I_{21,1} = \frac{1}{\beta}\left(2E(\sqrt{-\beta}) - (1+\alpha)\ K\ (\sqrt{-\beta})\right)$$

where K and E are complete elliptic integrals of the first and second kind, respectively.

## APPENDIX 3. DERIVATION OF THE INFINITE MATRIX EQUATIONS

The procedure for deriving the infinite matrix equations for the four cases, even and odd TE and TM, is basically the same. Therefore, we will derive only the odd TE case and quote the results for the remaining cases. We begin with the definition at $H_{11}$ defined in (23), i.e.,

$$H_{11}(v) = \int_0^g U(t')\partial_t \hat{G}^{TE}(t,t')\, dt' \,/2\sin\left(\frac{\pi t}{2a}\right) \tag{3.1}$$

From the definition of $\hat{G}^{TE}$ given in (15) we can derive the following result

$$\frac{\partial_t \hat{G}^{TE}}{2\sin\left(\frac{\pi t}{2a}\right)} = \frac{\pi}{4a} \sum_{n=0}^{\infty} (2n+1)A_{2n+1} \cos\left(\frac{\pi t'}{2a}\right) \left\{\frac{\sin[(2n+1)\frac{\pi t}{2a}]}{\sin\left(\frac{\pi t}{2a}\right)} \quad \frac{\cos\,[(2n+1)\,\frac{\pi t'}{2a}]}{\cos\left(\frac{\pi t'}{2a}\right)}\right\}. \tag{3.2}$$

If we use the following expansion in (3.2),

$$\frac{\sin\,[(2n+1)\frac{\pi t}{2a}]}{\sin\left(\frac{\pi t}{2a}\right)} = \sum_{m=0}^{n} r_{mn}\, U_m(v)$$

insert (3.2) into (3.1), and use the expression for $H_{11}$ given in (25) we obtain the following

$$\sum_{n=0}^{\infty} P_{11,n}\, U_n(v) = \frac{\pi}{4a} \sum_{n=0}^{\infty} \sum_{m=0}^{n} (2n+1)A_{2n+1}\, r_{mn} \int_0^g U(t')\cos\left(\frac{\pi t'}{2a}\right)\left\{\frac{\cos[(2n+1)\,\frac{\pi t'}{2a}]}{\cos\,\frac{\pi t'}{2a}}\right\} dt'\, U_m(v)\,. \tag{3.3}$$

We can now equate coefficients of $U_n(v)$ to obtain

$$P_{11,m} = \frac{\pi}{4a} \sum_{n=m}^{\infty} (2n+1)A_{2n+1}\, r_{mn} \int_0^g V(t')\, \cos\left(\frac{\pi t'}{2a}\right) \left\{\frac{\cos[(2n+1)\frac{\pi t'}{2a}]}{\cos\left(\frac{\pi t'}{2a}\right)}\right\} dt' \tag{3.4}$$

If we now transform the integration in (3.4) from t' to u via (21) and use the expansion given in (1.10), we obtain the following

$$P_{11,m} = \frac{\beta}{4} \sum_{n=m}^{\infty} \sum_{k=0}^{n} (2n+1) A_{2n+1}\, r_{mn}\, p_{kn} \int_{-1}^{1} F_{11}(u)\, T_k(u)\, du\,. \tag{3.5}$$

Next, we insert the expression for $F_{11}(u)$ given in (27) and note that the integration in (3.5) can be performed using the orthogonality relation given in (1.3). Equation (3.5) then reduces to the following

$$P_{11,m} = \frac{\pi\beta}{8} \sum_{n=m}^{\infty} \sum_{k=0}^{n} \frac{(2n+1)\, A_{2n+1}\, r_{mn}\, p_{kn}\, P_{11,k-1}}{\Delta_k}$$

where we have defined $P_{11,-1} \equiv C_{11}$ .

The final step involves substituting for $C_{11}$ from (29) with the result that

$$P_{11,m} = \frac{\pi\beta}{8} \sum_{n=m}^{\infty} (2n+1)\, A_{2n+1}\, r_{mn} \left\{ (1-\delta_{no}) \sum_{k=0}^{n-1} p_{k+1,n}\, P_{11,k} - \frac{2p_{on}}{R_{11,0}} \sum_{k=0}^{\infty} R_{11,k+1}\, P_{11,k} \right\} . \tag{3.6}$$

Equation (3.6) is thus the infinite matrix equation desired. The following equations for the other three cases can be derived in an analogous manner.

$$P_{12,m} = \frac{\pi\beta}{2} \sum_{n=m}^{\infty} (n+1)\, A_{2n+2}\, s_{mn} \left\{ \sum_{k=0}^{n} q_{k+1,n+1}\, P_{12,k} - \frac{2q_{o,n+1}}{R_{12,0}} \sum_{k=0}^{\infty} R_{12,k+1}\, P_{12,k} \right\}$$

$$P_{21,m} = \frac{a\beta}{4} \sum_{n=m}^{\infty} B_{2n+1}\, r_{mn} \left\{ (1-\delta_{no}) \sum_{k=0}^{n-1} p_{k+1,n}\, P_{21,k} - \frac{2p_{on}}{R_{21,0}} \sum_{k=0}^{\infty} R_{21,k+1}\, P_{21,k} \right\}$$

and

$$P_{22,m} = \frac{a\beta}{2} \sum_{n=m}^{\infty} \sum_{k=0}^{n} B_{2n+2}\, s_{mn}\, q_{k+1,n+1}\, P_{22,k}$$

where we have made use of the following expansion

$$\frac{\sin\left((n+1)\frac{\pi t}{a}\right)}{\sin\left(\frac{\pi t}{a}\right)} = \sum_{m=0}^{n} s_{mn}\, U_m(v) \quad .$$

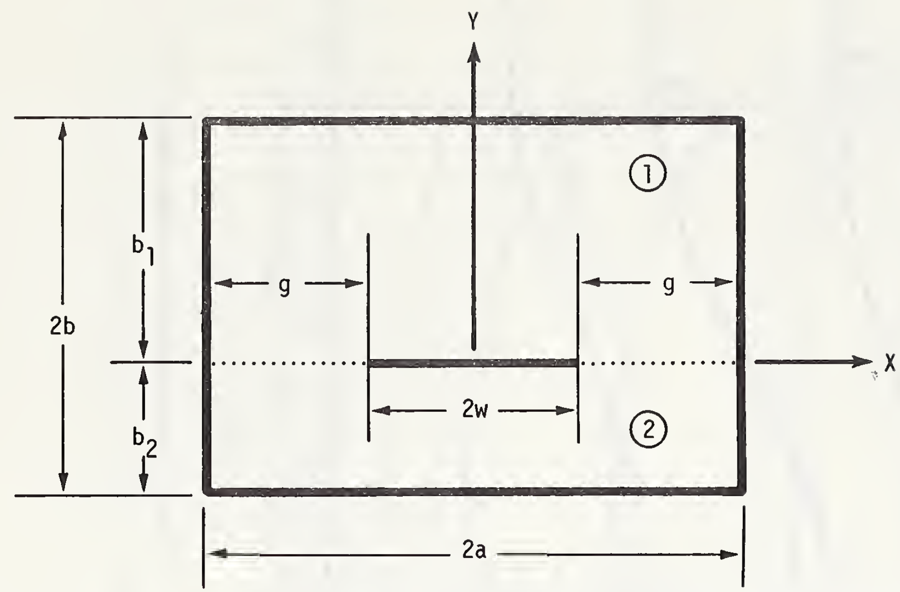


Fig. 1. Cross Section of a rectangular coaxial line.

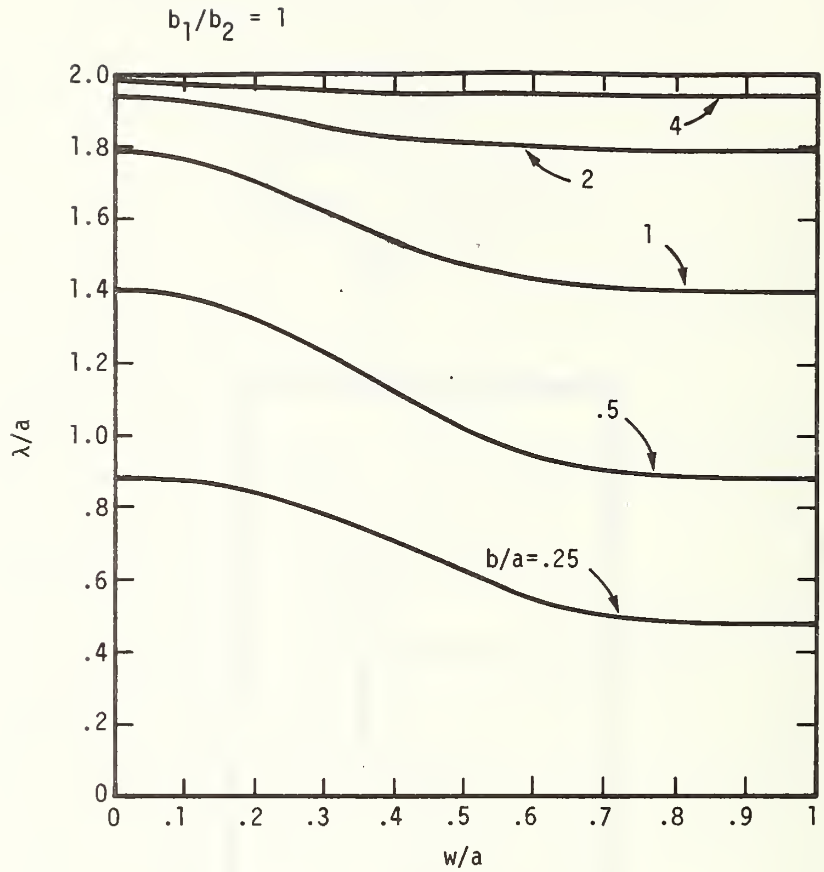


Fig. 2. Cutoff wavelength vs. strip width of the $TM_{21}$ mode.

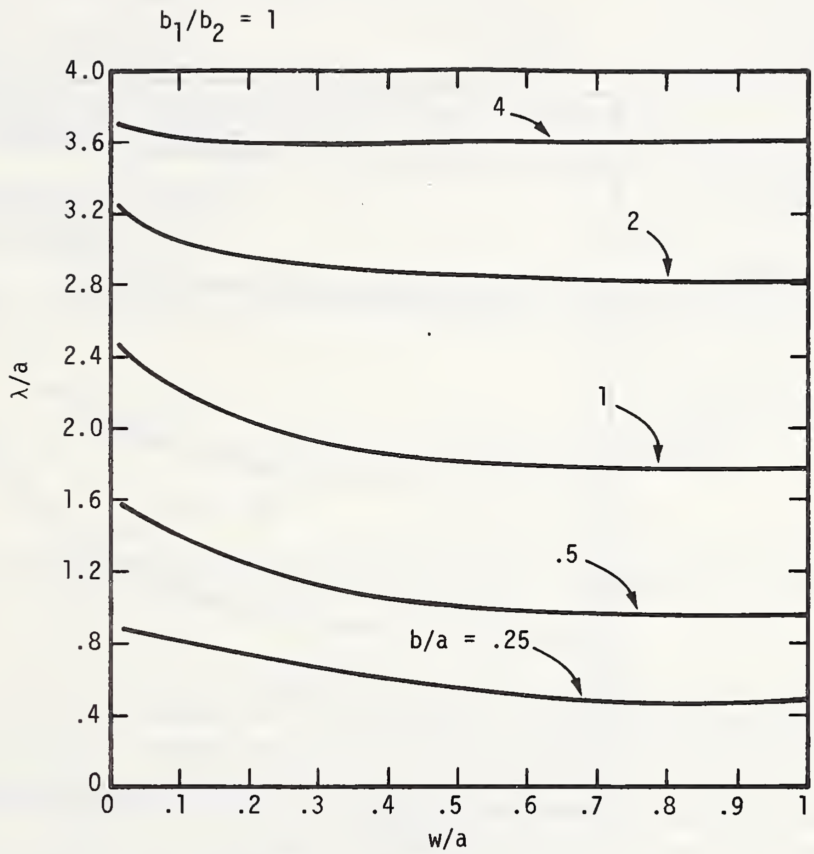


Fig. 3. Cutoff wavelength vs. strip width of the $TM_{11}$ mode.

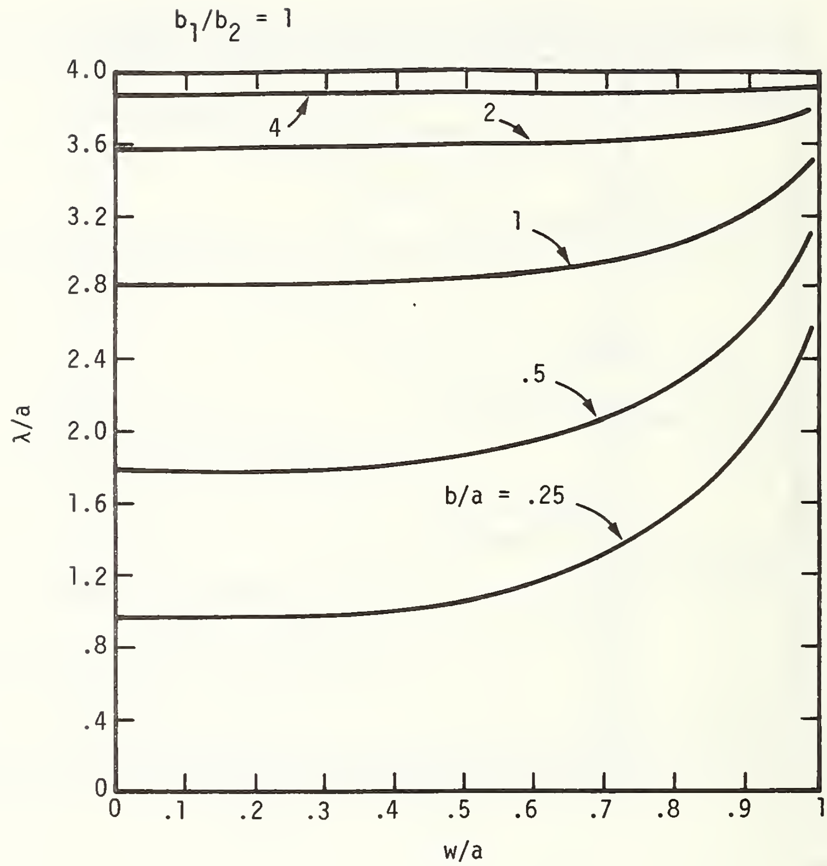


Fig. 4. Cutoff wavelength vs. strip width of the $TE_{11}$ mode.

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The singular integral equation approach is used to derive the secular equations for both TE and TM waves in a rectangular coaxial line with zero thickness inner conductor. Approximations for the secular equations are found that reduce to simple expressions in terms of well-known special functions (elliptic integrals). When the strip width is exceedingly small or nearly equal to the width of the outer conductor, closed form expressions for the cut-off frequencies can be found by replacing the elliptic integrals by their asymptotic forms for modulus either near zero or one.

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