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Mathematical Models for Selecting Catalogs of Standard-Sized Items

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ABSTRACT

This report identifies and discusses various mathematical models for selecting a "best" catalog of standard sizes. A survey of existing models for continuous and discrete versions of the catalog selection problem is presented. The advantages and disadvantages of such models are assessed with regard to both range of applicability and computational feasibility. This evaluation shows that a frequently-advocated iterative procedure may produce erroneous results and identifies another approach as the most promising. Various refinements and extensions are then proposed for this latter (discrete) model and its associated solution technique (dynamic programming). In particular, a multidimensional version of the catalog selection problem is formulated and analyzed. Areas for further investigation, and unresolved issues, are also discussed.

Key Words: Catalog; dynamic programming; iterative procedure; models; optimization; stability; standards.

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1. INTRODUCTION

This report concentrates on the formulation and analysis of mathematical models for selecting a "best" catalog (or ensemble) of standard-sized items. This situation can arise when a relatively small subset of standard sizes for some design element or component must be used to meet demands for a much larger number of potential sizes. The restriction to this smaller subset of sizes might reasonably be dictated by limitations of storage and economies of scale in manufacturing. The problem, then, is to find which standard-sized items should be produced (or stocked) in order to meet forecasted demand at minimum cost.

One instance of this catalog selection problem arises in the stocking of structural steel beams. On the basis of design calculations, the professional engineer may find that a beam possessing a certain section modulus is required. Of course, it is unlikely that this precise section modulus will be available from an existing catalog of beam section moduli, and so a steel beam from the catalog with a somewhat larger section modulus than that required will be used. This larger section modulus is assumed more costly (without conferring needed benefit) than the smaller required section modulus. The problem in this context is thus to select a catalog of beams with specified section moduli that will adequately meet demand, while minimizing the relevant costs of stocking such a catalog.

More generally, this framework appears appropriate whenever only some of the possible sizes for an industrial product are to be stocked, and demands for unstocked sizes are met by using a larger stocked size. Often, existing catalogs for such products (e.g., windows, lumber, glass) reflect the historical evolution of standard industrial products and not a deliberately rationalized selection of standard sizes. In such cases, it would be useful to have an analytical method for selecting a catalog of standard sizes that is in some sense "best". Quite likely, such a method would suggest a streamlining in existing, and sometimes unnecessarily large, collections of standard-sized products.

A precise statement of the catalog selection problem is provided in Section 2, where a survey of existing models and alternative approaches to the problem is presented. In Section 3, the advantages and disadvantages of the various approaches are assessed with regard both to range of applicability and to computational feasibility. As a result of such assessments, one particular approach is shown to be treacherous and inadvisable for use, while another approach is recommended for actual implementation. Section 4 details the basic model associated with this recommended approach and shows how the basic model can be modified to allow for a more flexible and realistic formulation. In particular, a multidimensional version of the catalog selection problem is formulated and analyzed. Computational aspects and input requirements to the model are also discussed in this section. Areas for further investigation, and unresolved issues, are addressed in Section 5.

2. SURVEY OF CATALOG SELECTION PROBLEMS

This section discusses a variety of different approaches taken in the formulation and solution of catalog selection problems. As will soon become apparent, this type of problem has in fact been studied in a number of different guises by a number of different authors. The catalog selection problem has been studied at times as an "assortment problem" [11], at other times as a "dynamic economic lot size problem" [14], and also as a "vehicle dispatching problem" [8]. Thus, it is not surprising that authors studying one manifestation of the catalog selection problem were often quite unaware of what authors studying a completely equivalent manifestation had already discovered.

We shall first describe in some generality the important defining characteristics of a catalog selection problem. While a number of somewhat different versions of the catalog selection problem have been examined in the literature, virtually all such versions share certain basic features:

1. A catalog of "supply sizes" $x_1, x_2, \ldots, x_n$ is to be chosen from some set S of allowable sizes.
2. The underlying demand for each existing (or potential) size x from the set S is known.
3. The demand for any size x can be met by any supply size x_i which is at least as large as x. All demands must be met.
4. There is no limit to the demand that can be met by using a given supply size.

5. Various costs are associated with the selection of a catalog:

- the cost of failing to meet demand exactly,
- the cost of stocking the specified catalog of sizes.

Once a model incorporating the above features has been formulated, two distinct questions arise. First, what is the overall cost of a specified catalog and, second, what should be the composition of a catalog in order to minimize the sum of all relevant costs? It is therefore important for a model to permit the <u>evaluation</u> of a given catalog as well as the <u>construction</u> of an optimal one.

Previous investigations considered two rather distinct versions of the catalog selection problem, differing in whether the set S postulated above represents a continuous interval (e.g., all sizes between some lower and upper limits) or a discrete (finite) set of sizes. The first case presents a <u>continuous</u> problem. Here, the sizes selected for the catalog as well as the sizes which could conceivably be demanded come from an infinite set, and the demand function (defined over the set S) is assumed to be a continuous function over this set. In the second case, we have a <u>discrete</u> problem, where selected sizes and demanded sizes arise from a finite set. The demand function here is accordingly a discrete function defined over this finite set.

Because the continuous and discrete versions of the catalog selection problem are formulated in somewhat different terms (and also because different analytical techniques are used to analyze the associated models), we shall discuss them separately here. Accordingly, Section 2.1 is devoted to a survey of models for the continuous problem, while Section 2.2 focuses on models for the discrete problem.

2.1 Models for the Continuous Problem

Several models have been considered for the continuous version of the catalog selection problem. The earliest such model was proposed by Hanssmann [4] and it served as the basis for continuous models subsequently proposed.

The problem is to select a fixed number n (specified in advance) of sizes $x_1, x_2, \ldots, x_n$ which are to be chosen from the continuous interval $[a,b]$. Sizes x which could potentially be demanded are also located within this interval, and the known demand function, denoted by $f(x)$, is assumed to be a continuous function throughout the interval $[a,b]$. A typical such function is depicted in Figure 1, where an arbitrary catalog $\{x_1, x_2, \ldots, x_n\}$ has also been indicated. The interpretation of the demand function $f(x)$ is that the number of items demanded with sizes between x and $x + \Delta x$ is approximately $f(x)\Delta x$, for Δx small.

For simplicity, it is assumed that the catalog sizes are arranged so that

$$a \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq b .$$

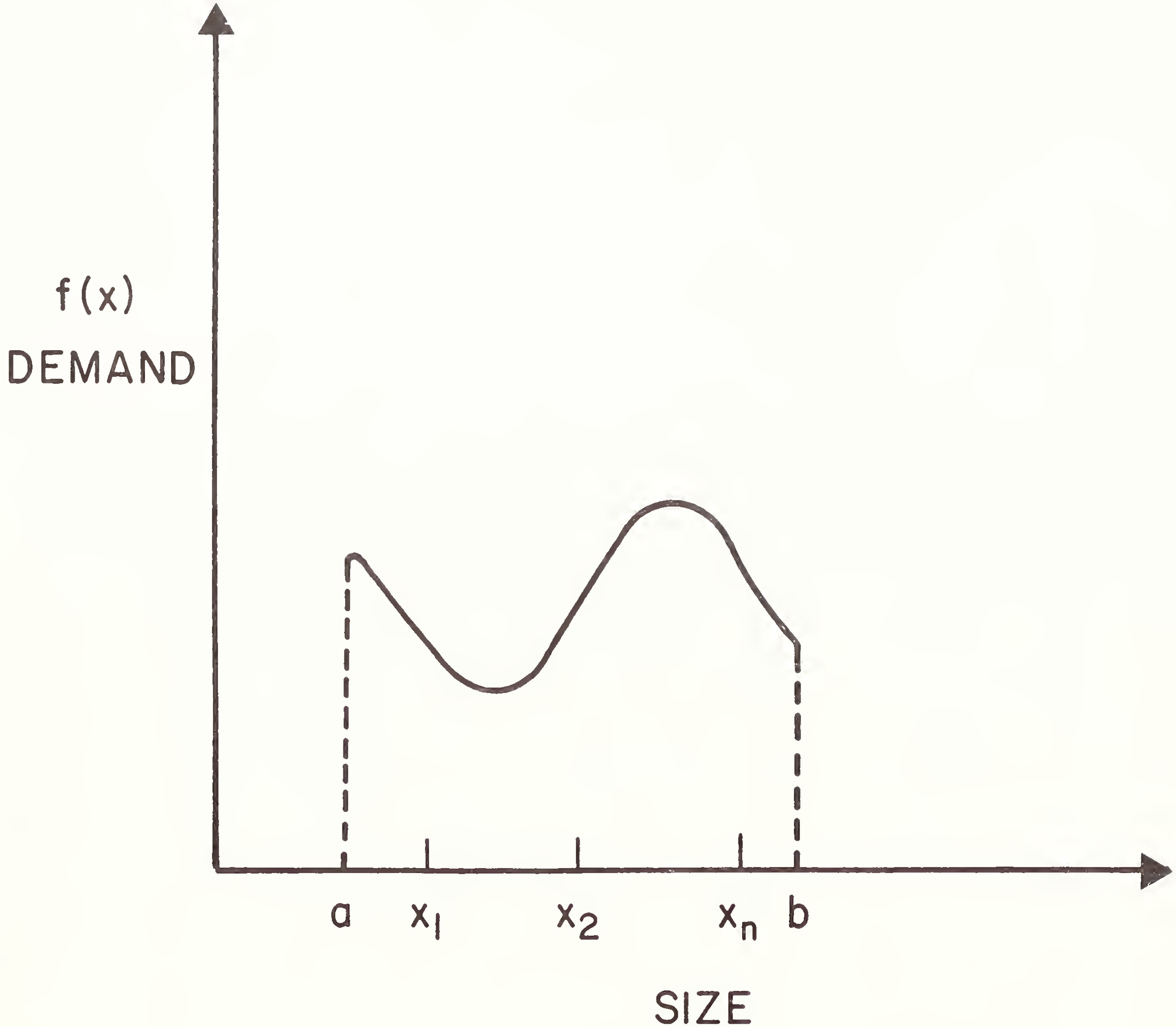

Figure 1. Demand Function for the Continuous Model

Since demand for any size x in the interval [a,b] must be met (using a larger size if necessary), it follows that the largest size in a catalog needs to be located at the right-hand endpoint of the interval, whence $x_n = b$.

In the model proposed by Hanssmann, the costs of stocking a given catalog are not explicitly considered. This omission is not invariably serious. Indeed, if the cost K of stocking a particular size x_i does not depend on the size x_i itself, then the total stocking cost for a catalog with precisely n supply sizes is just nK. Since this stocking cost will be the same for any catalog of n sizes, such a term can be ignored when comparing different catalogs with n sizes. In this instance, then, the Hanssmann model is not unreasonable; extensions of the model to deal with the situation of more general stocking costs are discussed later.

The present model thus concentrates on the costs of failing to meet demand exactly. For example, if a size x is demanded but the next largest size available from the catalog is x_i, then an amount of excess material x_i-x could be considered as "wasted". The objective of the Hanssmann model is to minimize the total "wastage" resulting from the use of a catalog. More precisely, the total cost of a catalog $\{x_1, x_2 \ldots, x_n\}$ is given by

$$C(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} [\int_{x_{i-1}}^{x_i} (x_i - x)\, f(x)dx], \qquad (2.1)$$

where for convenience we define $x_0 = a$ and set $x_n = b$. In the above expression, each bracketed term of the sum represents the total amount of excess material that results from using catalog size x_i to satisfy the demands for sizes x, where $x_{i-1} < x \leq x_i$.

By using (2.1), we can therefore evaluate the cost of any specific catalog $\{x_1, x_2, \ldots, x_n\}$. An optimal catalog (i.e., one that minimizes this overall cost) can also be found within the framework of this model. In fact, since it is assumed that there is no limit to the demand that can be satisfied by a size in the catalog, the techniques of differential calculus can be employed to find (at least in theory) an optimal catalog having certain sizes $x_1^*, x_2^*, \ldots, x_n^*$. Particular solution techniques for this type of model are discussed and analyzed in Section 3.1.

It should be emphasized that the above model addresses the problem of finding a catalog with exactly n sizes. As mentioned by Hanssmann, one can repeatedly solve the continuous problem for various values of n. Using information on the cost of stocking catalogs with different numbers of items, one could then determine a best number n* of sizes to be stocked, as well as the particular sizes to be stocked.

An alternative approach found in the literature takes such stocking costs into explicit account. Sadowski [11] formulates a model identical to Hanssmann's, except that a stocking cost $c(x_i)$ is associated with using size x_i in the catalog. Equation (2.1) then becomes

$$C(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} [\int_{x_{i-1}}^{x_i} (x_i - x) f(x)dx + c(x_i)]. \tag{2.2}$$

By using this expression, the costs for a number of proposed alternative catalogs can be evaluated. As before, differential calculus techniques can be employed to characterize an optimal catalog with n sizes. Sadowski also proposed solving for an optimal catalog by using a technique known as (continuous) dynamic programming [2]. The relative merits of these solution methods are discussed in Section 3.1. By finding an optimal catalog for each fixed value of n, an overall optimal value n* can then be identified. That is, a value of n and a catalog $\{x_1, x_2, \ldots, x_n\}$ can be found that minimizes the cost (2.2) over all possible catalogs with any number of sizes.

A further generalization of the Sadowski model has recently been advanced by Lind [7]. Instead of considering the "wastage" to be simply $x_i - x$, a more general expression for wastage $g(x_i) - g(x)$ is used. Here, $g(x)$ refers to the specific per unit cost of producing an item of size x. Since $g(x)$ would realistically be an increasing function of the size x, then $g(x_i) - g(x)$ represents the additional cost incurred by using a (possibly larger) stock size x_i instead of the demanded size x. In this context, equation (2.2) becomes

$$C(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} [\int_{x_{i-1}}^{x_i} \{g(x_i) - g(x)\} f(x)dx + c(x_i)]. \tag{2.3}$$

This expression can be used either for evaluating or for optimizing catalogs with n sizes. Again, differential calculus has been proposed as the basis of a solution technique. In order to find a best value n^* for n, Lind gives an approximate expression for the optimum number of items in the catalog when all the stocking costs $c(x_i) = K$, a constant independent of x_i. Then the search for the overall optimum can be confined to the vicinity of this approximate value.

The three models discussed so far were formulated with the example of selecting an optimal catalog of (industrial) items in mind. For this reason, the problem has often been referred to as the <u>assortment problem</u> [11]: that is, find a best assortment of sizes to meet demand at minimum cost. Quite independently of investigations into the assortment problem, authors in the area of transportation were studying a seemingly different problem known as the <u>vehicle dispatching problem</u> [8]. In this latter problem, a given number n of vehicles are to be dispatched from a central terminal over a given period of time. Moreover, the vehicles are to be dispatched in order to minimize the total waiting time of passengers (arriving at the terminal according to some predictable pattern).

As a matter of fact, these two problems are in all essential respects the same. The basic identification that illustrates this equivalence is the identification of sizes, chosen from some interval [a,b], with departure times, also chosen from an interval [a,b]. The demand function $f(x)$ for sizes x could equally represent the arrival rate of passengers at time x. Furthermore, just as a demanded size x can be met by a (possibly) larger size x_i, the arrival time x of a customer cannot exceed the departure time x_i of the vehicle he will board. Finally, the cost of failing to meet demand for a size exactly ("wastage") corresponds to the waiting time of a customer, and the stocking cost of an item corresponds to the cost of operating a vehicle at a particular time of day.

Historically, these two related lines of investigation remained virtually separate from one another. Accordingly, it is not surprising to find that certain results obtained by authors working on one version of the problem had actually been obtained earlier by investigators of a different version. We shall therefore briefly indicate the contributions of transportation analysts to an understanding of the catalog selection problem.

In particular, vom Saal [13] studied the case when there are a fixed number of departures (sizes) and operating (stocking) costs are not explicitly considered. The total cost for the departure schedule $\{x_1, x_2, \ldots, x_n\}$ is therefore given by equation (2.1). A further generalization of the cost structure of this model was also considered. Namely, the cost associated with a delay (wastage) $x_i - x$ is allowed to be a nonlinear function $g(x_i - x)$ of delay. Accordingly, the total cost for a schedule $\{x_1, x_2, \ldots, x_n\}$ is given by

$$C(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} [\int_{x_{i-1}}^{x_i} g(x_i-x)\, f(x)dx]. \tag{2.4}$$

As in the other models of this section, calculus was proposed for use in characterizing an optimal schedule with n departures. Generalizations were considered when capacity constraints are imposed on the number of passengers that can be transported by a vehicle, and when maximum/minimum "headways" are imposed between vehicle departures. Such generalizations correspond in the assortment problem to relaxing Assumption 4 (p. 3), and to insisting upon certain "spacing" requirements on the sizes selected for a catalog.

Newell [8] also proposed models for scheduling a fixed number of vehicle departures. In his basic model, the total cost of scheduling n departures is given by an expression which is mathematically identical to equation (2.1). In addition, Newell considered a model that took explicit account of the vehicle costs $c(x_i)$, assumed to be independent of x_i. For the latter model, an approximation to the optimal number of dispatches n^* is derived. This approximation represents a somewhat more refined estimate of n^* than that given later (and independently) by Lind [7]. Newell also considered an extension of these models to the case where capacity restrictions are imposed on the vehicles.

In all the models discussed here (whether for the assortment problem or the vehicle dispatching problem), the quantities x_i are continuous variables, freely ranging over some fixed interval of the real line. Accordingly, the techniques of calculus can in theory be used to find those sizes defining an optimal catalog. To calculate such sizes in practice is, however, another matter. We postpone a discussion of numerical solution methods until Section 3.1, and proceed to a description of discrete models.

2.2 Models for the Discrete Problem

In this section, various models for versions of the discrete catalog selection problem are surveyed. It is useful to consider first those models predicated on a fixed number n of sizes in the catalog. Subsequently, those models are considered which allow for the simultaneous determination of both the optimal number of sizes and the optimal sizes themselves.

2.2.1. Fixed Number of Sizes

The model proposed by Wolfson [16], and later reformulated by Cohen [3], deals with selecting a fixed number (n) of sizes which are to be chosen from a finite set S. Sizes that could potentially be demanded also arise from this finite set. For simplicity, these potentially demanded sizes (m in number) are denoted by $S = \{s_1, s_2, \ldots, s_m\}$, where $s_1 < s_2 < \ldots < s_m$. The demand for such sizes is specified by the quantities $d_1, d_2, \ldots, d_m$; namely, the demand (over the period of interest) for items of the i-th size s_i is equal to d_i, as illustrated in Figure 2.

Since the demand for any size has to be met, it again follows that the largest size s_m must be part of any optimal catalog. More generally, the demand for size s_i can be met by a (possibly) larger size s_j, with $s_j \geq s_i$. The problem of interest is that of selecting a subset of n

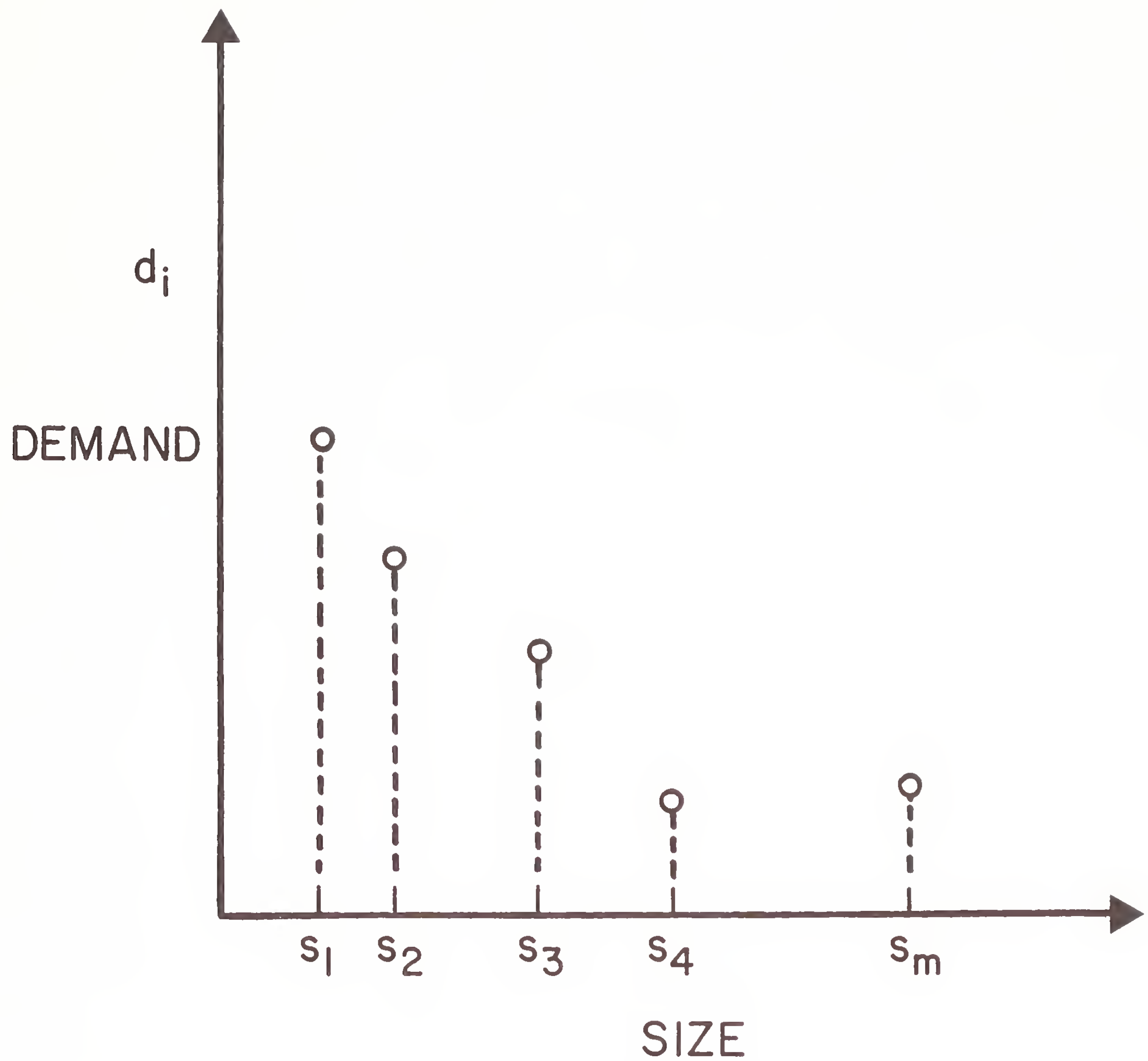

Figure 2. Demand Function for the Discrete Model

items (a catalog) from the set S of m items (where $m \geq n$). There is assumed to be no limit to the demand that can be fulfilled by using a size stocked in the catalog.

Under these assumptions, the cost of instituting a catalog with sizes $s_{i_1}, s_{i_2}, \ldots, s_{i_n}$ can be expressed as

$$C(i_1, i_2, \ldots, i_n) = \sum_{j=1}^{n} [\sum_{i_{j-1}<k<i_j} (s_{i_j} - s_k)d_k], \qquad (2.5)$$

where for convenience we define $i_0 = 0$ and note that $i_n = m$. Each of the above terms $(s_{i_j} - s_k)$ can be interpreted as the (per unit) "excess material" resulting from using the j-th catalog size s_{i_j} instead of the demanded size s_k. Thus, the quantity within brackets in equation (2.5) represents the total excess material resulting from using the j-th catalog size to satisfy all demands met by that size. The total cost for the entire catalog is then simply the sum of such costs for all sizes in the catalog, as indicated by equation (2.5). Note that for this case (fixed n) only the costs of failing to meet demand exactly are considered. Stocking costs for the catalog are not explicitly included in this model.

By using equation (2.5) one can directly evaluate the cost of any specified catalog. An optimal catalog of n sizes (i.e., one that minimizes the cost expression given above) can be determined by a technique known as (discrete) dynamic programming [2]. The essential idea of this technique is to solve a given problem by solving instead a series of smaller subproblems. Details of performing such recursive calculations are deferred to Section 3.2. Wolfson [16] presented such a procedure for finding an optimal catalog with n sizes, and Cohen [3] subsequently simplified the mechanics of Wolfson's method.

In order to find the best number of items in a catalog, Wolfson suggests using a dynamic programming technique to solve the problem for various values of n. As a matter of fact, dynamic programming is quite amenable to these calculations. Indeed, the technique does not need to solve from scratch each catalog problem with a different value of n, but can effectively use the information obtained when solving for n sizes in order to solve for n+1 sizes. Once the catalog problems have been solved for various values of n, the cost (2.5) can be balanced against the production and stocking costs to define a best n* and a best catalog of sizes.

The work by Wolfson and Cohen was subsequently extended by Jackson and Zerbe [6] in two notable respects. First, the excess material cost $s_{i_j} - s_k$ in equation (2.5) was replaced by the more general cost difference $g(s_{i_j}) - g(s_k)$, where g(s) can be viewed as the manufacturing cost for size s. Second, the possibility was allowed for certain sizes to be "custom-made"; these represent catalog sizes that meet demand for only one particular size. The appropriate modification of equation (2.5) to reflect these changes is straightforward, but notationally cumbersome, and therefore is not shown here. Again, the solution technique chosen for determining optimal catalogs was dynamic programming.

2.2.2 Optimal Number of Sizes

In all the models previously discussed, the method for finding the overall best catalog of sizes to stock has (explicitly or implicitly) been a two-step process. First, find an optimal catalog for a fixed number of sizes n, and then vary the value of n parametrically to find the best value n*. As a result, one obtains the optimal number of sizes to stock and the optimal catalog having this number of sizes. The catalog thus obtained will minimize the relevant costs over all possible choices of catalogs with any number of sizes.

In the models presented in this section, the optimal value n* as well as the (overall) optimal catalog can be found <u>directly</u>, without solving for each value of n separately. Moreover, the work involved in solving this seemingly more difficult problem turns out to be less than that needed to solve any particular "fixed n" problem.

The first model of this type was proposed by Wagner and Whitin [14]. As in the case for a fixed number of sizes (Section 2.2.1), demands $d_1, d_2, \ldots, d_m$ are given for sizes in some finite set $S = \{s_1, s_2, \ldots, s_m\}$, with $s_1 < s_2 < \ldots < s_m$. The problem here is to select a best subset of sizes to form an optimal catalog. This optimal catalog is one which minimizes the sum of all costs associated with the catalog: namely, the costs of failing to meet demand exactly as well as the costs of stocking the catalog.

More specifically, the total cost of a catalog $\{s_{i_1}, s_{i_2}, \ldots, s_{i_n}\}$ under the Wagner-Whitin model is given by

$$C(i_1, i_2, \ldots, i_n) = \sum_{j=1}^{n} [k_{i_j} + \sum_{i_{j-1} < r \leq i_j} c_{r,i_j} d_r], \qquad (2.6)$$

where it is assumed $i_0 = 0$ and $i_n = m$. In the above formulation, k_t represents the cost of stocking size s_t, and c_{rt} represents the per unit substitution cost in using size s_t to meet demand for size s_r ($r \leq t$). The classification into these two types of costs is made so as to reflect the fact that certain costs (substitution) depend on the number of units demanded, while other costs (stocking) are fixed costs, independent of demand. Accordingly, a (possibly nonzero) term $c_{tt}d_t$ is included in (2.6) to allow for certain quantity-sensitive costs in producing or inventorying size s_t.

Equation (2.6) thus allows evaluation of the costs of alternative catalogs that may be proposed. In addition, an optimal catalog (one that minimizes this cost expression over all possible catalogs) can be found, again by using the technique of (discrete) dynamic programming. The details of such calculations are deferred to Section 3.2.

Equation (2.6) reveals that an optimal catalog will be one that manages to balance the stocking costs against the substitution costs. The former costs increase with increasing n, while the latter costs decrease with increasing n. It is reasonable, then, that for some intermediate value of n (not too large and not too small) the total contribution of such costs will be minimized.

It should be noted that the problem actually considered by Wagner and Whitin was not one of finding a best collection of sizes (the assortment problem). Rather, these authors were concerned with another problem, the <u>dynamic economic lot size problem</u>, which is in all important respects equivalent to the assortment problem. In brief, this problem refers to the selection of ordering points in time for a particular item. There are m discrete time periods at which orders could be placed; during each time period there is a known demand for the item and this demand draws down the existing inventory. In order that demands can always be filled from existing inventory, the inventory will have to be replenished at certain points in time. The problem is thus to determine such ordering points (as well as the amounts ordered) in order to balance the inventory costs of holding quantities of the item in storage against the (fixed) cost of placing an order. Furthermore, both the inventory and ordering costs can vary from period to period.

The equivalence of this problem to the assortment problem can be seen by identifying catalog sizes with the ordering times. Also, the stocking and substitution costs correspond, respectively, to the ordering and inventory costs of the economic lot size problem. Furthermore, the assumption that demand for a given size can be met by a larger size corresponds to the notion that inventory ordered in one period can be used to meet demand in subsequent periods.

Several generalizations of the Wagner-Whitin model have been subsequently proposed. Most notably, Zangwill [17] considered the possibility of allowing the stocking and substitution costs to be more general nonlinear functions. In addition, the possibility of "backlogging" was considered - that of allowing the current inventory to drop below zero. This corresponds in the assortment problem to allowing a <u>smaller</u> size to be used to fulfill demand for a given size, but at an additional cost.

In parallel with such developments of the economic lot size problem, similar models were being proposed for discrete versions of the assortment problem, where n is not required to be fixed. Independently of Wagner and Whitin, and at just about the same time, Sadowski [11] considered a model similar to that described by (2.6). Sadowski's model was proposed in order to allow for capacity restrictions on the demand that could be met by a given item in the catalog. (As will be discussed in Section 3.1, such capacity restrictions cannot be easily incorporated into solution techniques for continuous models.) The work of Sadowski was subsequently generalized by Pentico [9, 10], who considered more general structures for the stocking and substitution costs.

Pentico also considered a variant of the assortment problem where the demands are not known in advance, but vary according to known (and independent) probability distributions.

In addition, the same basic models have been studied (again, independently) by authors in connection with vehicle dispatching problems. In particular, Ward [15] considered a series of models for this problem, including models with constant stocking costs, and (a) linear substitution costs, (b) nonlinear substitution costs, (c) differing capacities, and (d) probabilistic demand. In these, as well as the other models considered in this section, the indicated method of solution was by dynamic programming.

To summarize, the models of Section 2 have been classified into those dealing with continuous catalog selection problems and those dealing with discrete catalog selection problems. For models in the first class, the techniques of differential calculus are appropriate in determining optimal catalogs. By contrast, for models of the second type, dynamic programming emerges as the appropriate computational tool. Accordingly, the following section will address the relative merits of alternative models by studying the properties of their associated solution methods.

3. ASSESSMENT OF MODELS

In this section we will study the properties of solution methods for both the continuous and discrete versions of the catalog selection problem. This study will provide valuable information about the range of applicability and computational feasibility of the associated models. First, solution methods for the continuous catalog selection problem will be investigated, followed by a discussion of solution methods for the discrete problem.

3.1 Solution Methods for the Continuous Problem

We choose the model described by equation (2.2) to illustrate how the techniques of differential calculus have been used to study optimal catalogs for the continuous problem. Thus, it is required to find a catalog (with a fixed number of sizes n) that minimizes (2.2):

$$C(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} \left[\int_{x_{i-1}}^{x_i} (x_i - x)\, f(x)dx + c(x_i) \right].$$

Because the catalog sizes $x_1, x_2, \ldots, x_n$ are subject to no essential constraints, the minimizing values $x_1^*, x_2^*, \ldots, x_n^*$ can be found by differentiating the above cost expression with respect to each x_i and setting the result equal to zero. This procedure yields the following set of equations that must be satisfied at $x_1^*, x_2^*, \ldots, x_n^*$ for i = 1, 2, ..., n-1:

$$c'(x_i) + \int_{x_{i-1}}^{x_i} f(x)dx - (x_{i+1} - x_i)\, f(x_i) = 0, \qquad (3.1)$$

where $x_0 = a$ and $x_n = b$. In the above expression, $c'(x_i)$ denotes the derivative of $c(x)$, evaluated at $x = x_i$.

Equation (3.1) can also be rewritten, assuming $f(x_i) \neq 0$, in the following form:

$$x_{i+1} = x_i + \frac{1}{f(x_i)} \{c'(x_i) + \int_{x_{i-1}}^{x_i} f(x)dx\}. \tag{3.2}$$

In the case that $f(x) > 0$ and $c'(x) \geq 0$, the above relation can be used to show that $x_{i+1} > x_i$, as desired. These two stipulations mean that demand is always positive and the stocking cost is a nondecreasing function of size stocked.

Several authors [4,7,8,11,13] have used (3.2) as the basis of an iterative scheme to solve the nonlinear equations (3.1) for x_1^*, x_2^*, ..., x_n^*. Namely, one guesses a trial value for x_1 and then uses (3.2) with $i = 1$ and $x_0 = a$ in order to obtain a value for x_2. This value of x_2, together with the assumed value for x_1, is then used in (3.2) with $i = 2$ to produce a value for x_3. The process is repeated until x_4, x_5, ..., x_{n-1}, x_n have been successively calculated. Of course, an optimal catalog must have $x_n = b$; however, the value obtained for x_n by repeatedly applying (3.2) can certainly differ from b. It has therefore been suggested [4,7] that if x_n exceeds b, the trial value x_1 should be reduced and the process should be started again with this new trial value. On the other hand, if x_n is less than b, the process is begun with an increased value of x_1. Through successive iterations of this procedure, the discrepancy between x_n and b is expected to be reduced. Accordingly, the iterative process will be terminated once $|x_n - b|$ is small enough.

For example, Lind [7] suggested the following procedure for obtaining an optimal catalog.

LIND'S ALGORITHM

L1. Select x_1.

L2. Use (3.2) to calculate successively $x_2, x_3, \ldots, x_n$.

L3. If $|x_n - b| < \varepsilon$, where ε is a small pre-specified "tolerance", STOP. Otherwise, calculate $z = x_1 - \frac{1}{n}(x_n - b)$ and using $x_1 = z$ return to Step L2.

In addition to being suitable for computer calculation, an iterative algorithm such as the above can be implemented [7,8] in the form of a graphical algorithm, suitable for hand calculation.

Let us now summarize and evaluate the efficacy of the above solution procedures. First, it must be emphasized that equation (3.1) is only a first-order condition defining the optimal $x_1^*, x_2^*, \ldots, x_n^*$. Namely, this equation will indeed be satisfied by any $x_1, x_2, \ldots, x_n$ that minimize $C(x_1, x_2, \ldots, x_n)$, but it will also be satisfied by any values that maximize or define some other "critical point" of this function[1]. In other words, even should we be able to solve equation (3.1), it is not at all clear that the $x_1, x_2, \ldots, x_n$ so determined will actually minimize the catalog cost. However, if all solutions of (3.1) could be found, then one could search through these to find one that minimizes $C(x_1, x_2, \ldots, x_n)$.

The difficulty, of course, is solving equation (3.1). Because it represents a highly nonlinear system of equations, it can be quite arduous to determine a single solution, let alone the set of all solutions. For this reason, several authors have reformulated (3.1) as (3.2) and then employed an iterative solution method (e.g., Lind's algorithm) on the latter system of equations.

However, as noted by Sadowski [11], vom Saal [13] and Lind [7], there still remain difficulties in identifying an optimal catalog. Indeed, the solution produced by using such an iterative solution procedure may in fact represent a local minimum, but not a global minimum of the cost function. That is, it may be better than other catalogs similar to it in composition, but it may still fail to be the best catalog available. This possibility is illustrated in Figure 3, which shows a hypothetical cost function $C(x_1, x_2, \ldots, x_n)$ having

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[1] Recall that when a function of a single variable has its first derivative set equal to zero, the sign of the second derivative must be checked to see whether a (relative) minimum or maximum has been found. The situation is even more complicated when n variables $x_1, \ldots, x_n$ are involved [1, p. 152].

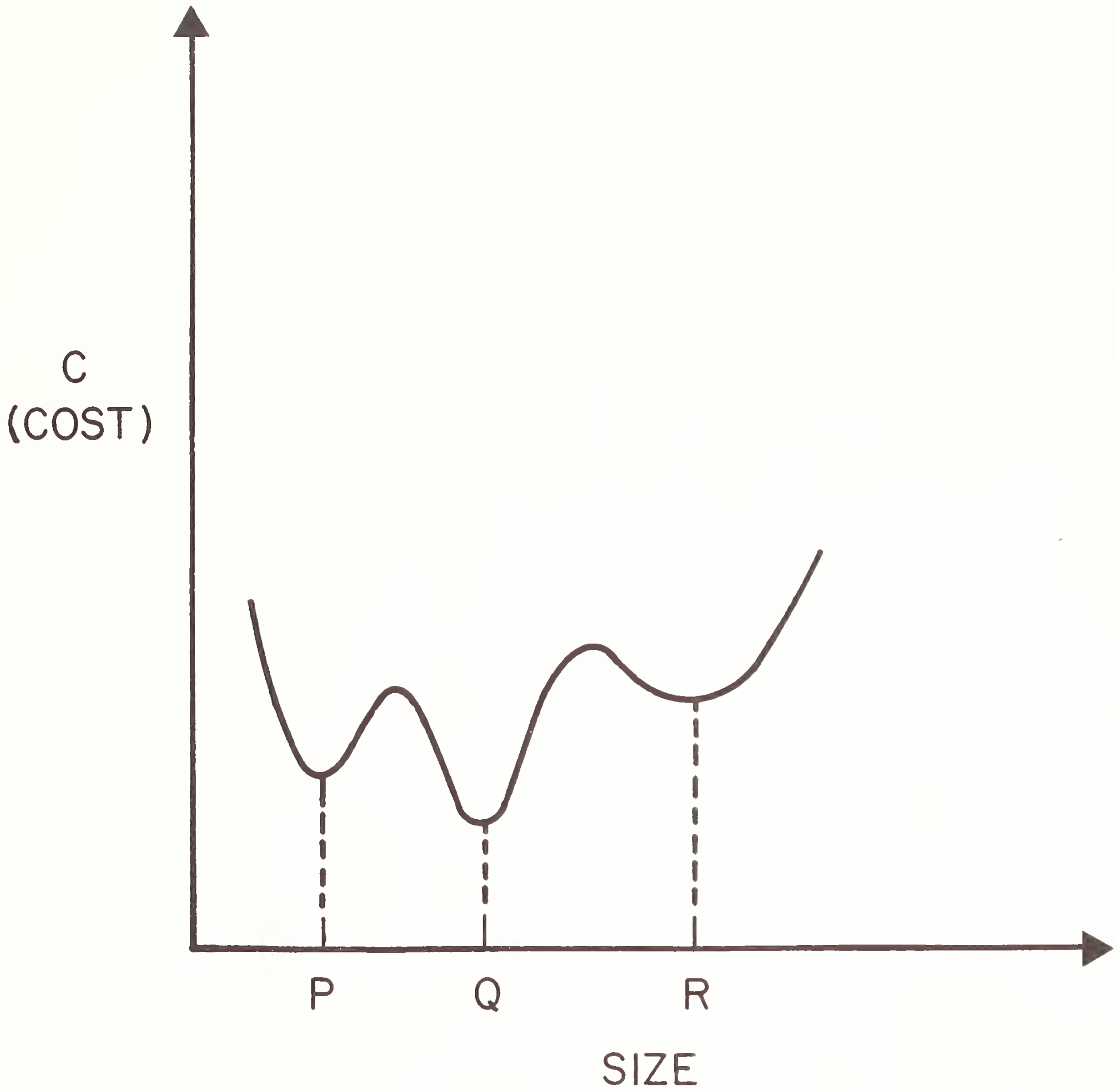

Figure 3. A Catalog Cost Function with Several Local Minima

several local minima (at P, Q and R), but only a single global minimum (at Q). In [13], vom Saal suggested that this difficulty could be overcome by repeatedly using the iterative procedure to find all such local minima, and then choosing the best from among these. Lind [7] attempted to avoid such issues by investigating conditions under which the iterative procedure generates a unique solution. In practice, however, such conditions are not only extremely difficult to check, but are also unlikely to be satisfied.

Apart from the question of nonuniqueness, the possibility has also been raised [11] that the iterative procedure might not even converge. That is to say, is it assured that the sequence of values x_n generated by the procedure get closer and closer to b (see Step L3 of the Lind algorithm)? Is such a procedure always guaranteed to terminate?

We now show, by means of simple examples, that the iterative procedure may <u>not</u> in fact converge. Moreover, it will also be demonstrated that even when the procedure does converge, it may fail to reach the true (global) solution. These examples are constructed for the simplest case when n = 2 and when there are no stocking costs: $c(x_i) = 0$ for i = 1,2. Clearly, the situation can only get worse when $n \geq 2$ and when more general cost structures are assumed.

<u>Example 1</u>. Consider the problem of determining an optimal catalog with n = 2 sizes, where the demand function is given by $f(x) = e^{-x}$ over

the interval $[0,\ 1+e^2] = [0,8.389]$. The demand function shown in Figure 4 reflects the (not unrealistic) situation when larger sizes are less in demand than smaller sizes of the same product. Because the largest size in the catalog must be at the right-hand end of the interval, $x_2^* = b = 8.389$. The problem then is to determine what value of x_1 will minimize (2.1) with $x_2 = b$. It is straightforward to verify that

$$C = C(x_1) = x_1 + (b-x_1)e^{-x_1} + K,$$

where K is a constant independent of x_1. Thus,

$$C'(x_1) = 1 - e^{-x_1}(1 + b - x_1)$$

and $C'(2) = 0$. It can easily be shown that the only solution of $C'(x_1) = 0$ is $x_1 = 2$. Since $C''(2) > 0$, it follows that $x_1 = 2$ is the unique minimum of $C(x_1)$ and so $x_1^* = 2$, $x_2^* = b$ represents the optimal catalog for $n = 2$.

When the Lind algorithm is applied to this problem, using a starting value of $x_1 = 1.5$, the results shown in Figure 5 are obtained. In this figure, the dotted line represents the true value $x_1^* = 2$; the successive values of x_1 obtained by Lind's algorithm are plotted for each iteration (up to 25 iterations). It is clear that, even though the starting value was close to $x_1^* = 2$, the procedure yields wildly oscillating values of x_1, which do not seem to be settling down to the true value $x_1^* = 2$.

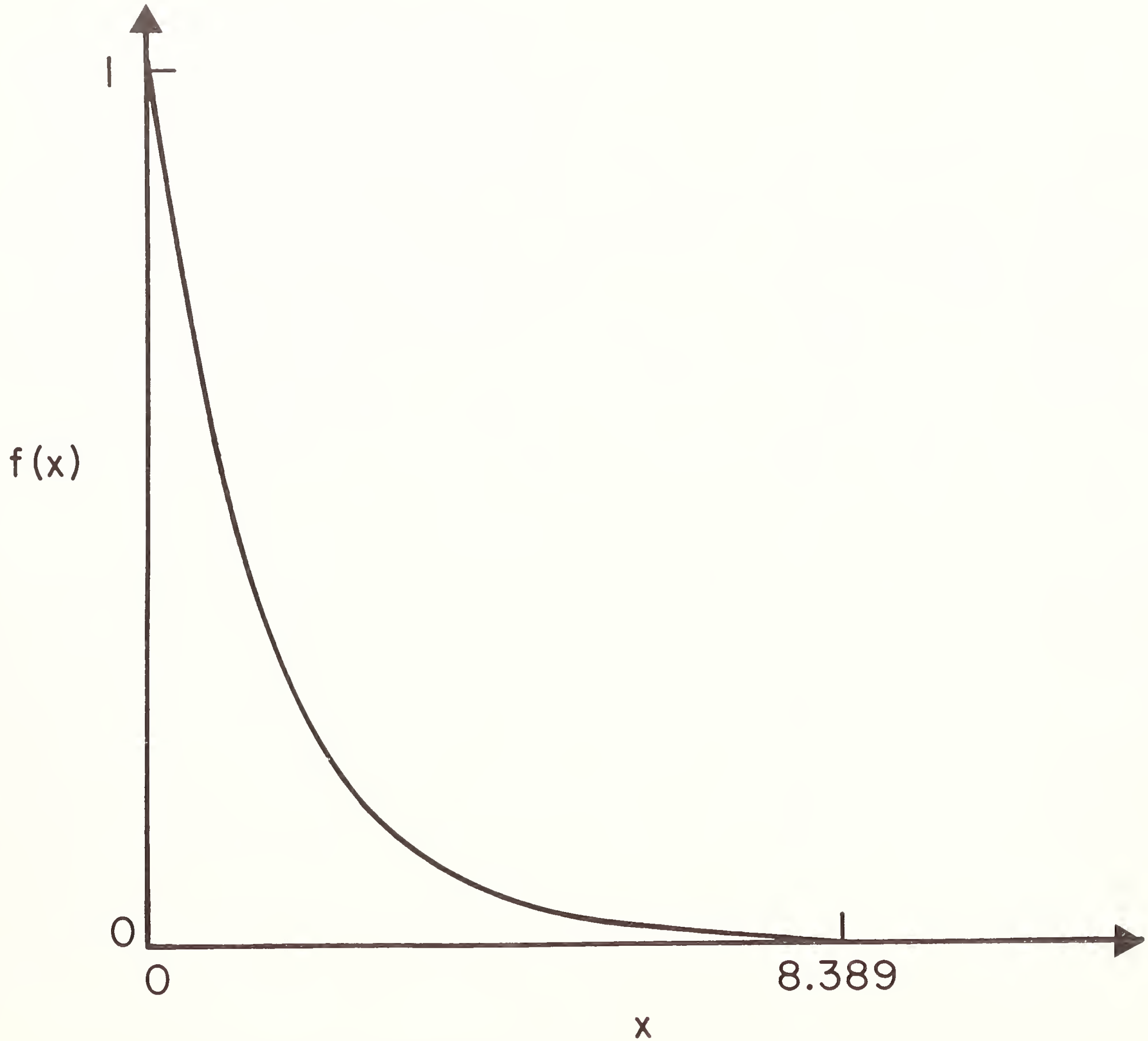

Figure 4. Demand Function for Example 1

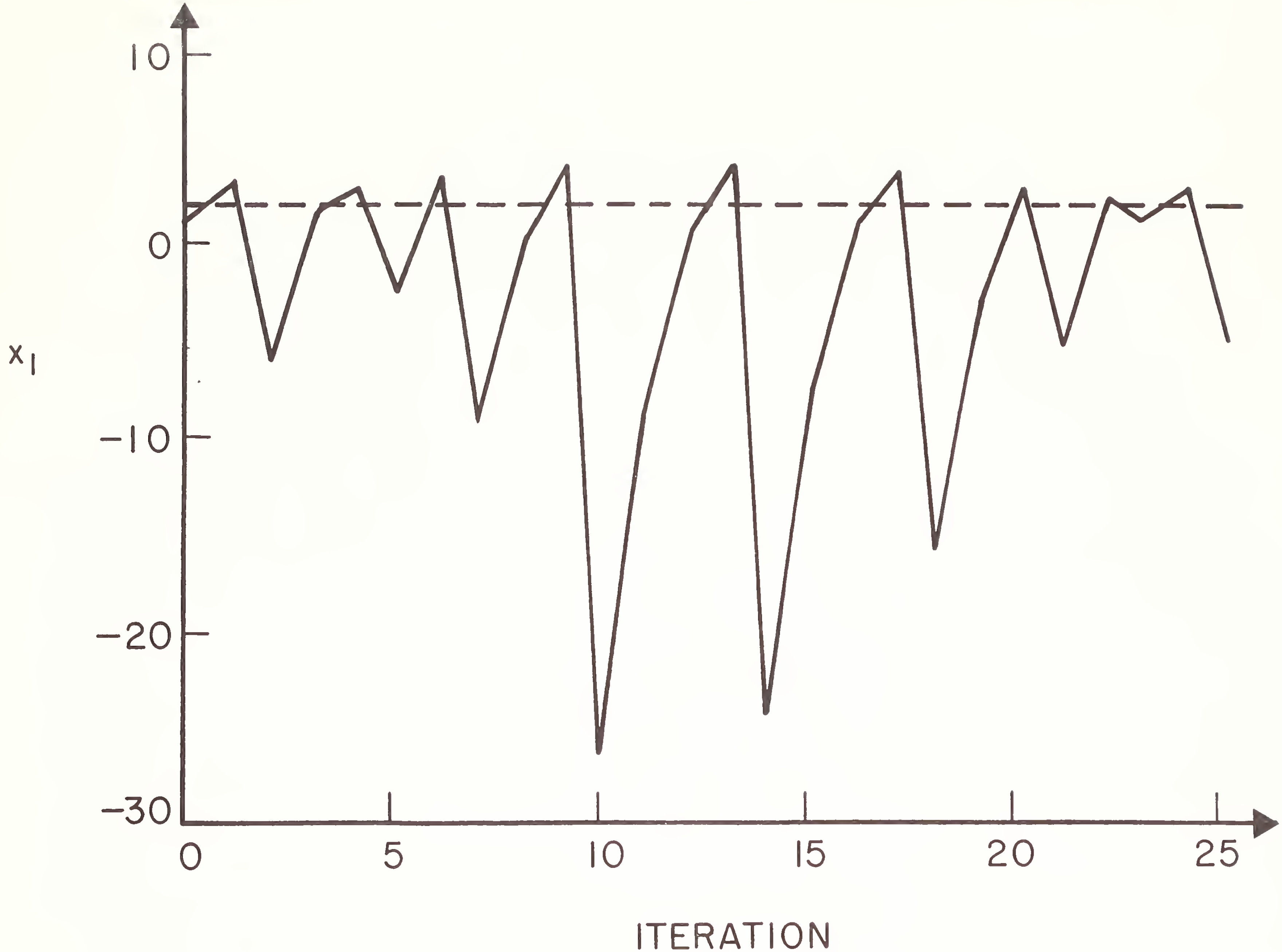

Figure 5. Application of Lind's Algorithm to Example 1, with Starting Value x_1 = 1.5

It is shown in Appendix A that for this example the Lind procedure will <u>never</u> converge. The difficulty here is that, while $x_1^* = 2$ represents the unique solution of (3.1), the iterative procedure cannot reach it. More generally, if the procedure is started using <u>any</u> trial value in $[0,b]$ other than $x_1 = 2$, the same type of divergent behavior will occur. The reason is that the value $x_1^* = 2$, the solution to (3.1), is <u>unstable</u> with respect to the iterative scheme based on (3.2). That is, whenever a given iteration value gets sufficiently close to $x_1^* = 2$, the next iteration value produced will be <u>further</u> from (rather than closer to) $x_1^* = 2$. The only instance for which the Lind procedure will converge is when the trial value chosen happens to be precisely the true solution $x_1 = 2$, an extremely unlikely occurrence.

To summarize, we have illustrated by way of a concrete example that the Lind procedure will not always achieve convergence. Indeed, any iterative procedure based on (3.2) will also suffer from this rather serious defect. Even when convergence is assured, such iterative procedures can fail to converge to the <u>correct</u> solution, as the following example illustrates.

<u>Example 2</u>. Again we consider the problem of finding an optimal catalog with $n = 2$ sizes, where the demand function is now given by $f(x) = 3x^2-2.04x + 0.415$ over the interval $[0,1]$. Figure 6 illustrates this demand function. Since the right-hand end of the interval must be the largest size in the catalog ($x_2^* = 1$), it is required then to find a value of x_1 that minimizes (2.1) with $x_2 = 1$. It is readily verified that

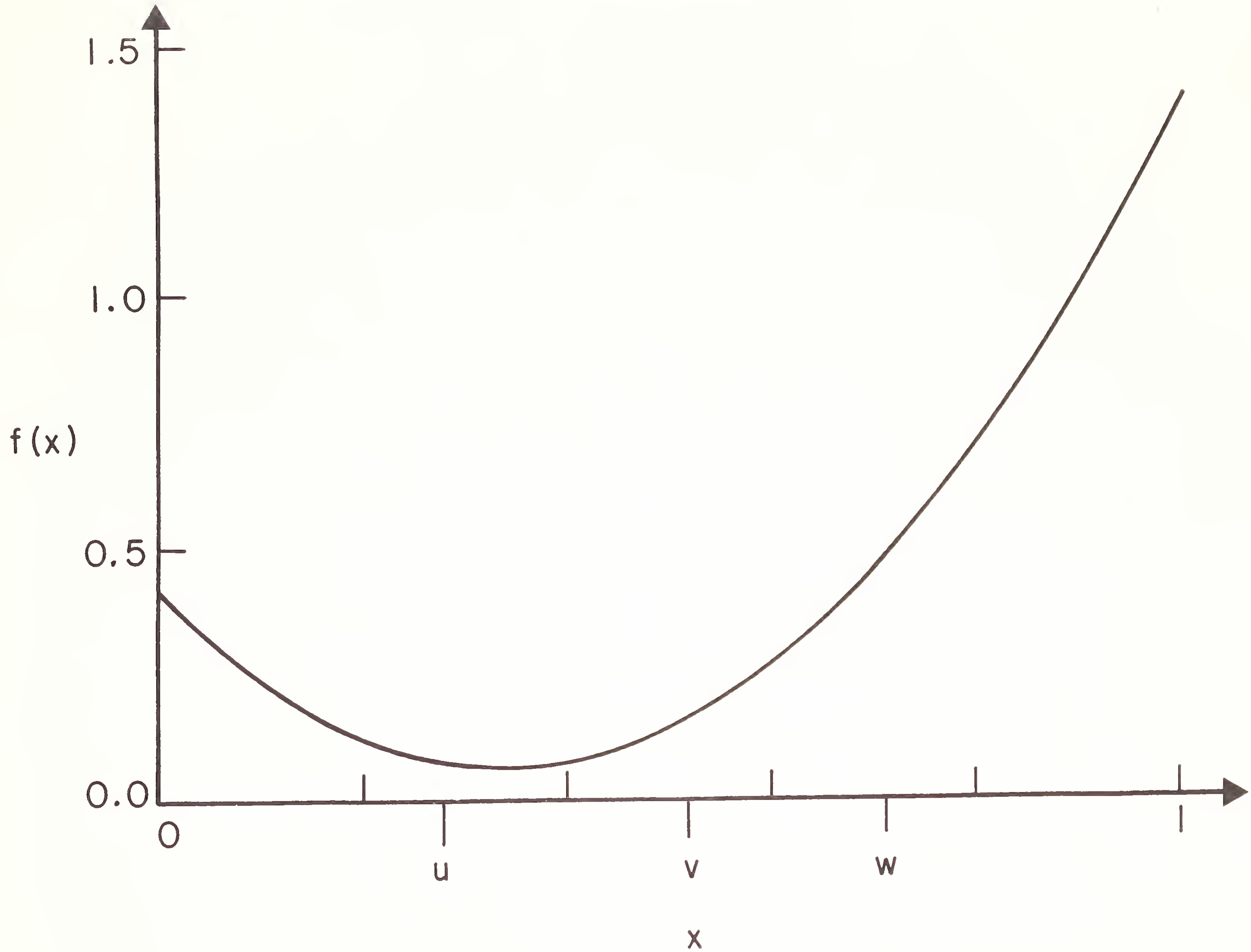

Figure 6. Demand Function for Example 2

$$C(x_1) = -x_1(1-x_1)(x_1^2-1.02x_1+0.415) + K,$$

where K is a constant independent of x_1. Upon differentiation one obtains

$$C'(x_1) = 4x_1^3 - 6.06x_1^2 + 2.87x_1 - 0.415,$$

from which it can be determined that u = 0.2769, v = 0.5265 and w = 0.7116 are the three solutions of $C'(x_1) = 0$. Thus, equation (3.1) possesses multiple solutions in this case. By examining second-order conditions on the sign of $C''(x_1)$, it can be inferred that u and w are local minima of $C(x_1)$, while v is a local maximum; the global minimum is located at $x_1 = u$. Accordingly, the optimal catalog for this problem consists of $x_1^* = u$, $x_2^* = 1$.

However, when Lind's algorithm is applied to this problem using a starting value $x_1 = 0.6$, the sequence of x_1- values generated converges to $x_1 = w$. While this resultant value for x_1 corresponds to a solution of (3.1), it represents a local but not the global solution -- a relative cost minimum but not the absolute minimum. If one attempts to vary the starting value x_1 (beginning, say, with any value $x_1 > v$), then the Lind procedure will <u>always</u> converge to $x_1 = w$. If, on the other hand, the procedure is begun with a starting value x_1 (not equal to u) with $x_1 < v$, then the resulting sequence of x_1-values does <u>not</u> converge. Thus, no matter how hard one tries (by varying the starting value), the Lind procedure will invariably fail to converge to the true solution $x_1^* = u$.

Again, the reason for the failure of the Lind algorithm is related to stability properties of the solutions to (3.1), as discussed in Appendix A. It is shown there that $x_1 = u$ is in fact an <u>unstable</u> minimum, while $x_1 = w$ is a <u>stable</u> minimum; $x_1 = v$ corresponds to an <u>unstable</u> maximum. Thus, it is not surprising that when the Lind algorithm is started with an x_1-value near $x_1 = w$, the process converges to $x_1 = w$. However, when the procedure is begun with an x_1-value near $x_1 = u$, the process will not even converge.

To summarize, this second example illustrates the fact that the Lind iterative procedure may not converge for certain starting values. Even when it does converge, the convergence need not be to the true cost minimum. This treacherous behavior is a serious drawback to the use of the Lind procedure, and it applies as well to other proposed iterative procedures based on (3.2).

With the above analysis in hand, it is now possible to assess the advantages and disadvantages of models for the continuous problem. First, equations (3.1) represent only first-order conditions on the required solution, and do not in general uniquely characterize the optimal catalog. These equations can be highly nonlinear in nature and difficult to solve. To find <u>all</u> solutions of (3.1) and then choose the best represents an extremely burdensome (as well as uncertain) computational task. Second, if one instead devises an iterative procedure based on (3.2) to determine an optimal catalog, convergence is not necessarily guaranteed, nor can one be confident (when convergence does occur) that the answer obtained is a correct one.

Even if such technical difficulties could somehow be resolved, there are other distinct disadvantages to using models formulated for the continuous problem:

1. Realistically, the given demands are discrete and not continuous. In order to obtain an analytic representation (an algebraically defined continuous function) for $f(x)$, some additional work is necessary to do curve-fitting with known analytic forms. Not only will error be introduced by such a procedure, but any irregularities in the underlying discrete demand will require the use of a rather complex analytic representation.

2. While the use of a well-established and easily understood technique (calculus) is a real advantage of the approach, this technique is no longer applicable when additional constraints (e.g., on capacity) are allowed.

3. In order to find the optimum n^*, a number of problems with n fixed have to be successively solved. There is no effective way to solve for the optimum n^* directly.

In summary, the disadvantages of existing models for the continuous problem seem to outweigh decidedly their advantages. While certain difficulties in convergence and correctness can be overcome by a continuous dynamic programming approach [2], the use of a discrete dynamic programming approach is not only computationally simpler but also avoids the three disadvantages listed above. It is therefore to a discussion of dynamic programming approaches, in the context of discrete models, that the discussion now turns.

3.2 Solution Methods for the Discrete Problem

All the solution methods proposed for solving the discrete catalog selection problem are based, in one way or another, on the technique of dynamic programming [2]. In order to illustrate both the philosophy and computational aspects of this technique, the Wagner-Whitin model will first be studied. For this model it is required to find a catalog of n sizes (n is to be determined) to minimize (2.6):

$$C(i_1, i_2, \ldots, i_n) = \sum_{j=1}^{n} [k_{i_j} + \sum_{i_{j-1} < r \leq i_j} c_{r,i_j} d_r].$$

The philosophy of dynamic programming is to attack the solution of the original problem - selecting an optimal catalog from $S=\{s_1, s_2, \ldots, s_m\}$ - through solution of a sequence of simpler subproblems. Specifically, consider the subproblem defined by the subset S_i of S, where $S_i = \{s_i, s_{i+1}, \ldots, s_m\}$. In this subproblem there are demands $d_i, d_{i+1}, \ldots, d_m$ for the potentially occurring sizes in S_i, and one is required to find an optimal catalog of sizes chosen from S_i. Let G(i) denote the cost of an optimal catalog for S_i; thus,

G(i) = the minimum cost achievable using sizes chosen from S_i to meet demands over S_i.

In this context, the dynamic programming approach produces the following recursive relation satisfied by the G(i)'s, where i = 1, ..., m:

$$G(i) = \min_{i \leq j \leq m} \{k_j + \sum_{t=i}^{j} c_{tj} d_t + G(j+1)\}. \quad (3.3)$$

The interpretation of (3.3) is that, assuming we have an optimal catalog for S_i and the smallest catalog size in S_i is s_j $(i \leq j \leq m)$, then the remainder of the catalog should be optimal for S_{j+1}. The bracketed expression in (3.3) thus represents the cost for a catalog having smallest size s_j: the first two terms represent the stocking cost of instituting size s_j in the catalog and the substitution costs associated with that catalog size, while the last term $G(j+1)$ represents the minimum cost for a catalog to meet demands from S_{j+1}. Finally, the minimum cost over all possible values of j (corresponding to the smallest catalog size in S_i) should be selected in forming $G(i)$.

Having obtained the recursive relation (3.3) it is now direct to obtain an optimal catalog for S. First, it is convenient to define

$$G(m+1) = 0.$$

Thus, using (3.3) with $i = m$ produces

$$G(m) = k_m + c_{mm} d_m,$$

indicating that in order to meet demands in $S_m = \{s_m\}$, the optimal catalog has the single size s_m. Next, using (3.3) with $i = m-1$, one can calculate $G(m-1)$, also noting which index j^* in (3.3) yields the minimum value of the expression in brackets. The interpretation of this j^* is as the smallest catalog size in an optimal catalog for $\{s_{m-1}, s_m\}$. Continuing in this manner, one can successively calculate $G(m-2)$, ..., $G(2)$, $G(1)$. However $G(1)$, the minimum cost achievable for a catalog over $S_1 = S$, is precisely what is required.

Moreover, by keeping track of the minimizing j^* at each step of the process, it is straightforward to construct an optimal catalog corresponding to the minimum cost $G(1)$. In fact, by keeping track of all minimizing j^* at each step, it is also possible to reconstruct <u>all</u> optimal catalogs.

While the above discussion has focused on the Wagner-Whitin model for determining a best catalog (and optimal n^*), it is not difficult to modify the basic recursion (3.3) to find the best catalog with a fixed value for n. In this case, define the quantity $G_r(i)$ by

$G_r(i)$ = the minimum cost achievable using r sizes chosen from S_i to meet demands over S_i.

Then the appropriate recursion governing these quantities is

$$G_r(i) = \min_{i \le j \le m-r+1} \{k_j + \sum_{t=i}^{j} c_{tj} d_t + G_{r-1}(j+1)\}. \tag{3.4}$$

Again, this relation has a direct interpretation. The first two terms within brackets represent the costs associated with the smallest size s_j in the catalog, while $G_{r-1}(j+1)$ represents the minimum cost achievable using r-1 remaining catalog sizes to meet demands arising from S_{j+1}.

To calculate an optimal catalog of n sizes for S, it is first convenient to define

$$G_1(i) = k_m + c_{mm}d_m, \text{ for } i = 1, \ldots, m.$$

Then, by using (3.4), one can determine in succession

$$G_2(i),\ i = m-1, \ldots, 1;$$
$$G_3(i),\ i = m-2, \ldots, 1;$$
$$\vdots$$
$$G_n(i),\ i = m-n+1, \ldots, 1.$$

The value $G_n(1)$ finally obtained thus represents the minimum cost achievable for a catalog of n sizes chosen from S. By keeping track of the various minimizing indices j*, <u>all</u> optimal catalogs with n sizes can be found.

It should be pointed out that in determining an optimal catalog for a fixed number of sizes n, it has been necessary to solve subproblems with one fixed size, two fixed sizes, ..., and n-1 fixed sizes. In particular, when solving the catalog selection problem for a fixed number of sizes we use information previously generated for one fewer number of fixed sizes. Recall that when determining (for the continuous catalog selection problem) the optimal number of sizes n*, it is necessary to solve a series of unrelated subproblems. In the present case of discrete catalog problems with n fixed, it is also necessary to solve a series of subproblems. But here the information generated in one subproblem is extremely useful in solving the next subproblem. Suppose, for example, that an optimal catalog of n=10 sizes has been calculated using the dynamic programming approach (3.4). If an optimal catalog of n=11 sizes is subsequently required, the calculation procedure does not have to begin again from scratch. Indeed, all the required "carryover" information is already captured in the calculated quantities $G_{10}(i)$ currently available.

To summarize, the technique of dynamic programming has been illustrated in relation to the "optimal n*" and "fixed n" versions of the discrete catalog selection problem. In contrast to the case for continuous problems (where the former version requires more computation than the latter one), the "optimal n*" version is easier to solve here than the "fixed n" version.

Certain advantages of a discrete formulation of the catalog selection problem can be noted:

1. Demand, which occurs naturally in discrete form, is handled in discrete form. Analytic (closed form) representation of demand is not required.

2. The technique of dynamic programming always terminates and is guaranteed to produce globally optimal solutions.

3. All optimal catalogs can be found through appropriate implementation of the dynamic programming calculations.

4. It is not difficult to incorporate additional constraints on the problem variables. In fact, since imposition of further constraints narrows the region to be searched for the minimizations in (3.3) or (3.4), it makes solution of the problem easier, not harder.

On the other hand, it has been noted that solution of the "fixed n" version of the discrete problem requires solution of a number of subproblems. (Yet, these subproblems do share common information and need not be restarted from scratch.) A more substantive drawback in using models for the discrete problem is that demand estimates may be available only for certain "historically determined" sizes and not for certain "new" sizes. A procedure for estimating demands over the entire set S, given certain historical demands, can however be devised to remedy this situation (see Section 4.3).

On balance, therefore, models for the discrete problem appear rather attractive, especially when compared to analogous models for the continuous problem. These discrete models are guaranteed to produce reliable answers with a minimum of computation. Moreover, such models can be easily modified and extended to incorporate more realistic constraints. The following section discusses a number of such modifications and extensions.

4. REFINEMENTS OF MODELS FOR THE DISCRETE PROBLEM

In light of the assessments of alternative models given in Sections 3.1 and 3.2, attention is subsequently confined to models for the discrete version of the catalog selection problem. In particular, the basic model chosen here for further study is the Wagner-Whitin model, which is described in Section 2.2 and whose associated solution technique is outlined in Section 3.2. The present section develops various modifications and extensions of this basic model.

4.1 Computational Observations

The basic Wagner-Whitin model involves evaluating and selecting catalogs from a finite set S, where the cost of a given catalog $\{s_{i_1}, s_{i_2}, \ldots, s_{i_n}\}$ is provided by the expression (2.6). There is certainly no computational difficulty in evaluating (2.6) for different proposed catalogs. In order to find an optimal catalog, the dynamic programming approach of Section 3.2 is recommended. The basic recursion governing the dynamic programming approach is given by (3.3) for the optimal n* case. The amount of computation required to calculate optimal catalogs using (3.3) can be estimated directly. Assuming that S contains m potential sizes, the calculation of G(1), the minimum cost of a catalog, necessitates approximately $m^3/6$ additions and $m^2/2$ comparisons. Finding an optimal catalog is therefore computationally feasible for catalogs of certainly several hundred sizes. In addition, only the m values G(1), G(2), ..., G(m) need to be stored, together with appropriate minimizing values of

j in (3.3). Thus, the storage requirements are rather minimal for this solution method.

We now show how this computational effort, already modest, can be further reduced by modifying the recursion (3.3). Namely, let $j(i)$ be any minimizing value of j for $G(i)$ in (3.3). That is,

$$G(i) = k_{j(i)} + \sum_{t=i}^{j(i)} c_{t,j(i)} d_t + G(j(i) + 1).$$

Similarly, define $j(i+1)$ to be any minimizing value of j for $G(i+1)$. Then it is certainly true that

$$i \leq j(i) \leq m.$$

In fact, under an extremely mild restriction, one can demonstrate (see Appendix B) that

$$i \leq j(i) \leq j(i+1) \leq m. \tag{4.1}$$

Relation (4.1) holds as long as $d_i > 0$ and the substitution costs c_{ij} satisfy

$$c_{ik} < c_{i\ell} \text{ for } k < \ell. \tag{4.2}$$

Relation (4.2) states a rather natural requirement: namely, that the cost of substituting a catalog size for a given size increases as the difference between those two sizes increases. It is inconceivable that such a condition would not be satisfied in any realistic situation.

The import of (4.1) on the calculation of optimal catalogs results from the fact that the minimization in (3.3) does not therefore need to be carried out for all j with $i \leq j \leq m$. Rather, it only needs to be performed for $i \leq j \leq j(i+1)$, where $j(i+1)$ is some minimizing value for $G(i+1)$. By choosing $J(i+1)$ to be the smallest value of j that minimizes $G(i+1)$, one obtains the recursion

$$G(i) = \min_{i \leq j \leq J(i+1)} \{k_j + \sum_{t=i}^{j} c_{tj} d_t + G(j+1)\}. \qquad (4.3)$$

This reformulation of (3.3) can reduce substantially the computation involved in the determination of optimal catalogs. A similar sort of reduction in computational effort can be achieved in the case of models for fixed n, through an analogous reformulation of (3.4).

4.2 Extension to a Two-Dimensional Model

In all the models heretofore considered, the problem addressed has been one of selecting a catalog of items characterized by a single dimension of size. For example, structural steel beams can be characterized by section modulus, plate glass by thickness, and so forth. There may, however, be situations in which a single scalar quantity would not be considered adequate to characterize an element in potential demand. In this case, a multidimensional version of the catalog selection problem might be of interest, where several quantities (e.g., section modulus and unbraced length) are used to characterize a catalog of standard items. A two-dimensional version of the (discrete) catalgo selection problem will be developed in this section.

First, the essential features of this two-dimensional model are briefly summarized:

1. Each potentially demanded element e is characterized by two scalar quantities, s(e) and t(e). The finite sets S and T indicate the possible range of values for these two characteristics.

2. The underlying demand for each possible element is known over the set $S \times T$, defined below.

3. The demand for any element u can be met by an element v whenever $s(v) \geq s(u)$ and $t(v) \geq t(u)$. All demands must be met.

4. A catalog of supply elements $\{e_1, e_2, \ldots, e_n\}$ is to be chosen from $S \times T$, with $s(e_1) < s(e_2) < \ldots < s(e_n)$ and $t(e_1) < t(e_2) < \ldots < t(e_n)$.[2]

5. There is no limit to the demand that can be met by using a given supply element.

6. Various costs are associated with the selection of a catalog:
   - o the cost of failing to meet demand exactly,
   - o the cost of stocking the specified catalog of elements.

In the above, suppose that the sets $S = \{s_1, s_2, \ldots, s_p\}$ and $T = \{t_1, t_2, \ldots, t_q\}$ represent possible discrete sizes in each of the two "size dimensions". The set $S \times T$ denotes the Cartesian product set, consisting of all (s,t) with s from S and t from T. An example of this product set is shown by the rectangle in Figure 7. Clearly, not every possible size s_i and size t_j are feasible in combination. Indeed, it appears likely that large values of s_i will be associated with large values of t_j, and small values of s_i will be associated with small values of t_j. In such cases, the nonexistent elements are simply assigned a corresponding demand $d(i,j) = 0$ and a very large stocking cost. More generally, the demand for an element with first size characteristic s_i and second size characteristic t_j is given by $d(i,j) \geq 0$. A typical pattern of zero-nonzero demands is shown in Figure 7, with the positive $d(i,j)$ indicated by solid dots.

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[2]The imposition of strict inequalities here is not an essential requirement, but rather is used to simplify the subsequent exposition.

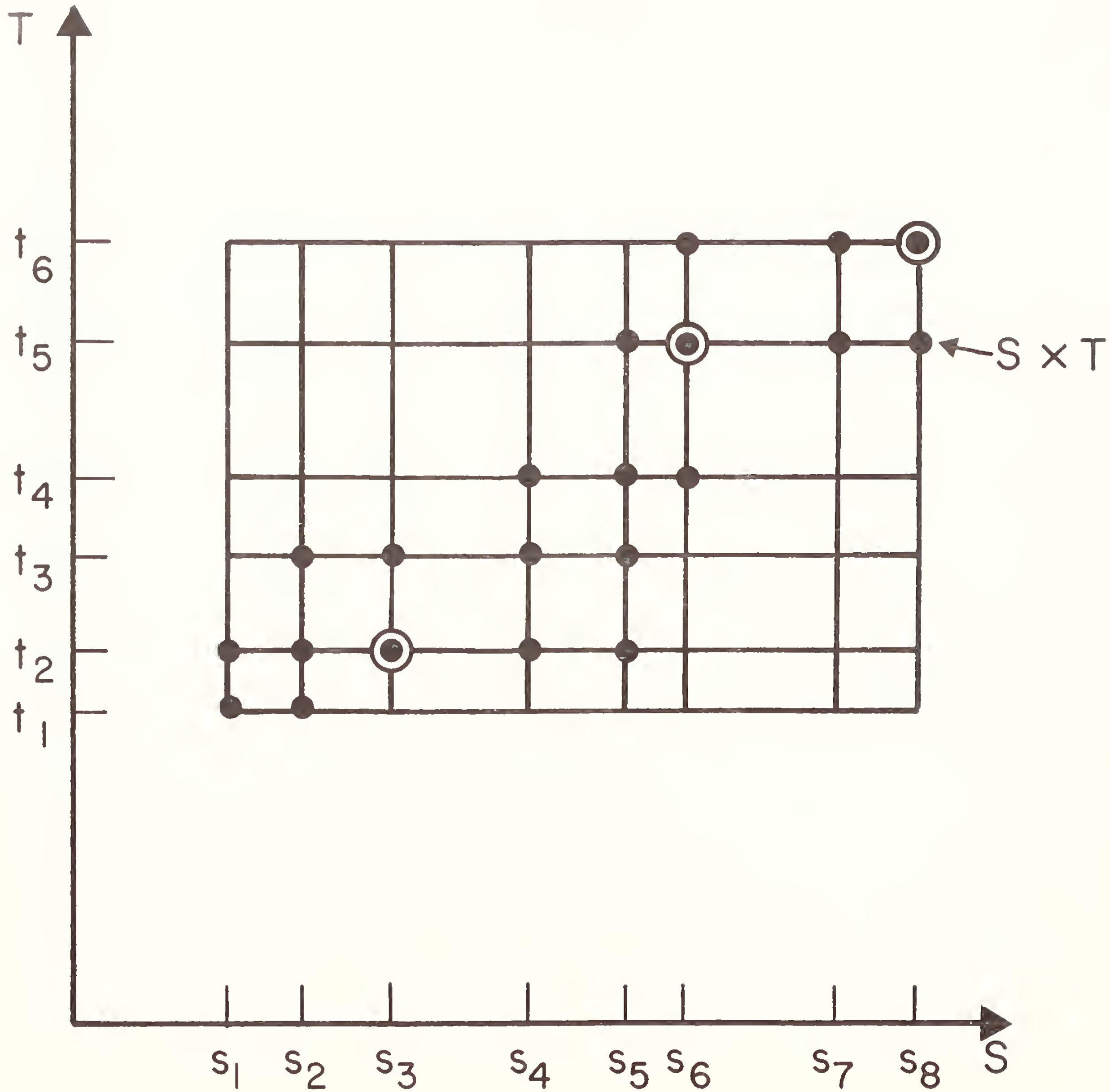

Figure 7. The Sets S, T and S × T for the Two-Dimensional Problem

For this formulation of the problem, Assumption 3 above means that if a demanded element is not available, the catalog element which is supplied must be at least as large in each dimension as the requested element. Note that since all demands <u>must</u> be met, the element corresponding to the upper right-hand corner of the rectangle $S \times T$ always needs to be in the catalog. Assumption 4 indicates that only certain collections of elements are possible candidates for a catalog. That is, the catalog elements must be "nested": in other words, it should be possible to arrange the catalog elements in sequence so that each element is greater in both its size characteristics than the preceding element in the sequence. An example of such a nested catalog is given by the circled dots in Figure 7.

The assumption of a nested sequence of catalog elements is not at all unreasonable, inasmuch as actual elements can often be characterized by two quantitative features which tend to increase (or decrease) together. Thus, it is believed that actual problems do exist where such an assumption is realistic. More importantly, this assumption enables efficient solution of the multidimensional catalog selection problem. Non-nested problems (which also exist) are excluded from study here, but it appears that insights into techniques for their solution can be gained from investigating nested catalogs.

As before, there are two costs associated with any given catalog. Let $k(a,b)$ denote the stocking cost for the element with size dimensions s_a and t_b, and let $c[u,v; a,b]$ denote the substitution cost in using a catalog element (s_a, t_b) instead of a demanded element (s_u, t_v). Then the cost of any specified catalog $\{(s_{i_1}, t_{j_1}), (s_{i_2}, t_{j_2}), \ldots, (s_{i_n}, t_{j_n})\}$ is given by the following expression:

$$C(i_1, \ldots, i_n; j_1, \ldots, j_n) = \sum_{\ell=1}^{n} [k(i_\ell, j_\ell) + \Sigma^\ell\, c[u,v;\, i_\ell, j_\ell]\, d(u,v)], \quad (4.4)$$

where Σ^ℓ indicates a summation taken over all pairs (u,v) with

<u>either</u> $\quad i_{\ell-1} < u \leq i_\ell$ and $v \leq j_\ell$

<u>or</u> $\quad j_{\ell-1} < v \leq j_\ell$ and $u \leq i_\ell$.

The set of (u,v) defined in the summation Σ^ℓ is illustrated by the reversed L-shaped region in Figure 8. For convenience, we specify $i_0 = j_0 = 0$ as well as $i_n = p$, $j_n = q$. It is noted that the collection of Σ^ℓ, $\ell = 1, \ldots, n$, partitions the rectangle of Figure 7 into disjoint subsets.

Using equation (4.4), one can evaluate the cost of any specified catalog, and thus compare (on a cost basis) a number of proposed alternative configurations. In addition, it is possible to solve for an optimal catalog using an appropriate dynamic programming formulation. To this end, let us define $W(i,j)$ to be the set of all elements (s_u, t_v) where either $i \leq u \leq p$ <u>or</u> $j \leq v \leq q$. Also, define $Z(i,j)$ to be the set of all elements (s_u, t_v) where both $i \leq u \leq p$ <u>and</u> $j \leq v \leq q$. Then the important quantity to be calculated is

$G(i,j)$ = the minimum cost achievable using elements chosen from $Z(i,j)$ to meet demands over $W(i,j)$.

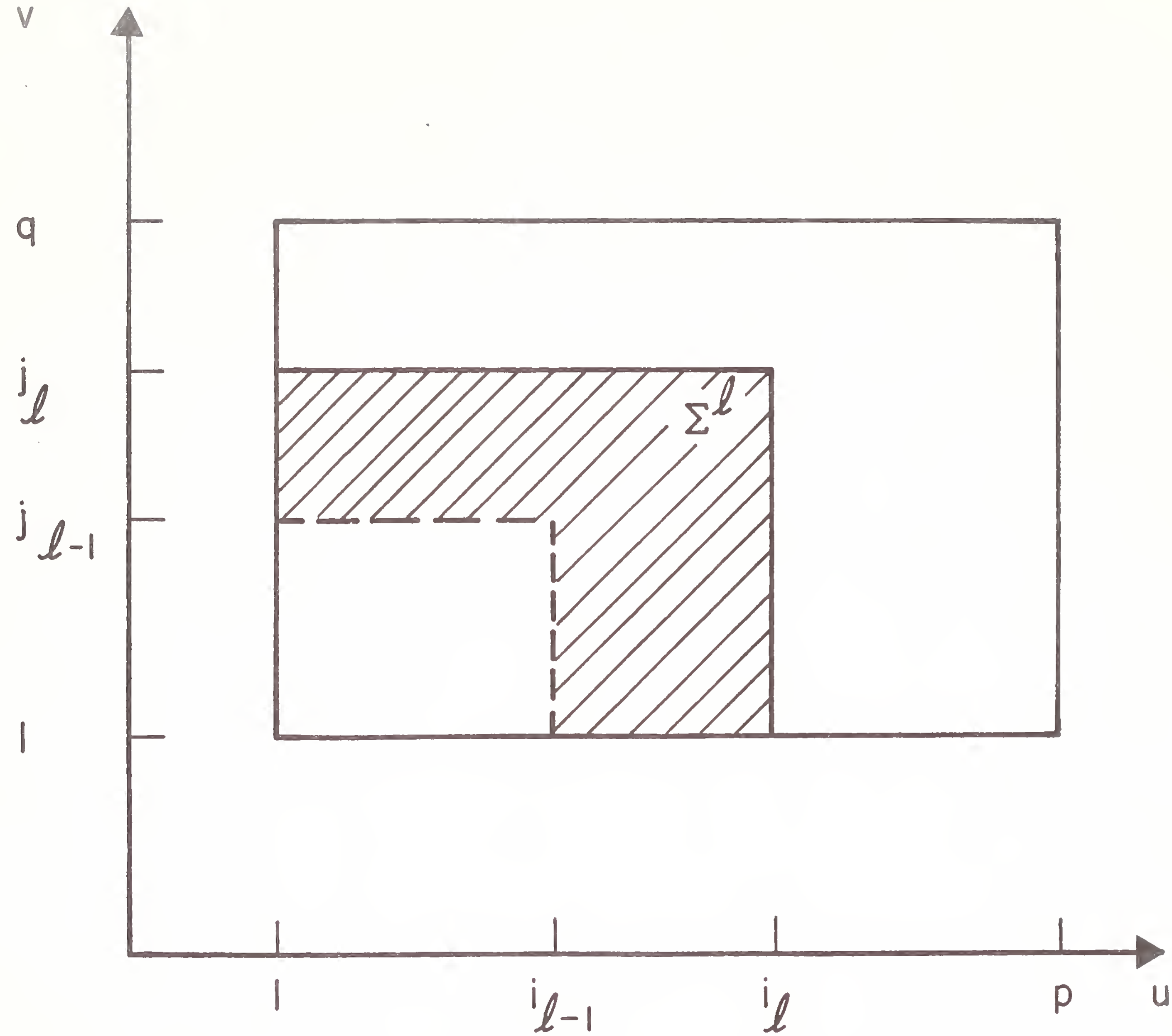

Figure 8. The Index Set Σ^ℓ

If the quantity $G(1,1)$ can be determined, then it represents the minimum cost of a (nested) catalog chosen from $Z(1,1) = S \times T$ to meet demands over $W(1,1) = S \times T$.

As a matter of fact, a recursive relation can be found linking the values $G(i,j)$. Namely, for $i < p$ and $j < q$

$$G(i,j) = \min_{(s_a,t_b) \varepsilon Z^*(i,j)} \{k(a,b) + \Sigma_{ab}^{ij} c[u,v;\ a,b]d(u,v) + G(a+1,b+1)\}, \quad (4.5)$$

where Σ_{ab}^{ij} indicates a summation over all u,v with $u \leq a$, $v \leq b$ and either $u \geq i$ or $v \geq j$. In addition, $Z^*(i,j)$ is derived from the set $Z(i,j)$ by removing all $(s_u,\ t_v)$ with $u = p$ or $v = q$, but not both. Thus, $Z^*(i,j)$ consists of all elements $(s_u,\ t_v)$ with $i \leq u < p$ and $j \leq v < q$, together with $(s_p,\ t_q)$. Because the catalog must always contain (s_p, t_q) and because strict inequalities hold in Assumption 4, the use of $Z^*(i,j)$ instead of $Z(i,j)$ in (4.5) is permissible.

In addition, define for $i = p$ or $j = q$ (or both)

$$G(i,j) = k(p,q) + \Sigma_{pq}^{ij} c[u,v;\ p,q]d(u,v), \quad (4.6)$$

and set

$$G(p+1,\ q+1) = 0. \quad (4.7)$$

Then using (4.5) - (4.7), one can calculate in succession all $G(i,j)$ for $1 \leq i \leq p$, $1 \leq j \leq q$. For example, all $G(i,j)$ with $i = p$ and/or $j = q$ can first be calculated, followed by all $G(i,j)$ with $i = p-1$ and/or $j = q-1$, and so forth until all the values $G(i,j)$ have been calculated. Eventually, $G(1,1)$ will have been determined, and this quantity represents the minimum cost of any two-dimensional (nested) catalog over $S \times T$. By keeping track of the minimizing a^* and b^* in (4.5), an optimal catalog (or all optimal catalogs) can be found.

Certainly, the two-dimensional version of the problem does not lend itself to as transparent a dynamic programming formulation as the basic Wagner-Whitin model. The computations, while certainly increased over those for the single-attribute model, are nonetheless straightforward. It should also be clear that the model and its associated solution strategy can be generalized appropriately for catalog selection problems involving more than two size dimensions.

4.3 Estimation of Demands

In the basic model of Wagner and Whitin, it is assumed that demands are known for all potential sizes in the set $S = \{s_1, s_2, \ldots, s_m\}$. Therefore, this model is quite appropriate when an existing catalog with many sizes needs to be reduced to a smaller number of standard sizes. There is also the possibility that new sizes, not in the current catalog, should be considered for introduction. Thus, there are known historical demands for certain sizes, but these do not make up the entire set S of potential sizes in a new catalog. As mentioned at the end of Section 3.2, it is important in this case to have demand estimates for <u>all</u> sizes potentially in demand.

In this section, a simplified procedure for obtaining such demand estimates is developed. Suppose, then, that historical demands are known for some subset E of S, where $E = \{e_1, e_2, \ldots, e_r\}$ and $e_1 < e_2 < \ldots < e_r$. It appears reasonable that the current (historical) demand for size e_1 represents the cumulative demand for all possible sizes $s \leq e_1$, that the demand for e_2 represents the cumulative demand for sizes s with $e_1 < s \leq e_2$, and so forth. This is just another way of saying that the underlying demand for various sizes is already reflected through the existing sizes e_i.

Therefore, the cumulative demand for sizes $e_1, e_2, \ldots, e_r$ can be plotted in the form of a (cumulative) distribution curve, as illustrated in Figure 9. The ordinate value corresponding to e_1 represents the existing demand for e_1, the ordinate value for e_2 represents the existing demand for e_1 and e_2, and so forth. A continuous curve which approximates this cumulative demand curve can be efficiently obtained - for example, through the use of spline functions [12]. Note that it is not necessary (cf. Section 3.1) to obtain in this case an analytical representation for the approximating curve $h(x)$ over $[e_1, e_r]$.

Using this curve, it is then direct to estimate the resulting demand for sizes $s_1, s_2, \ldots, s_m$. Thus, assuming that $s_1 < s_2 < \ldots < s_m$ and that every s_i lies in the range $[e_1, e_r]$, the demand for various sizes s_i can be found by differencing. Namely, the demand for size s_1 can be estimated as $h(s_1)$, the demand for size s_2 as $h(s_2) - h(s_1)$, and so forth. If the sizes s_i do not all lie in the range from e_1 to e_r, then determining the maximum and minimum sizes for which any demand could exist will be useful in extrapolating the curve $h(x)$.

The above procedure represents a simple method of obtaining demands for sizes not already in an existing catalog. It should be possible to refine such a procedure, if needed. In any event, the sensitivity of optimal catalogs to changes in the demand profile should definitely be analyzed. Specifically, the dynamic programming approach should be exercised for a variety of demand scenarios, in order to see how sensitive (or how robust) selected catalogs are to the demand distribution assumed.

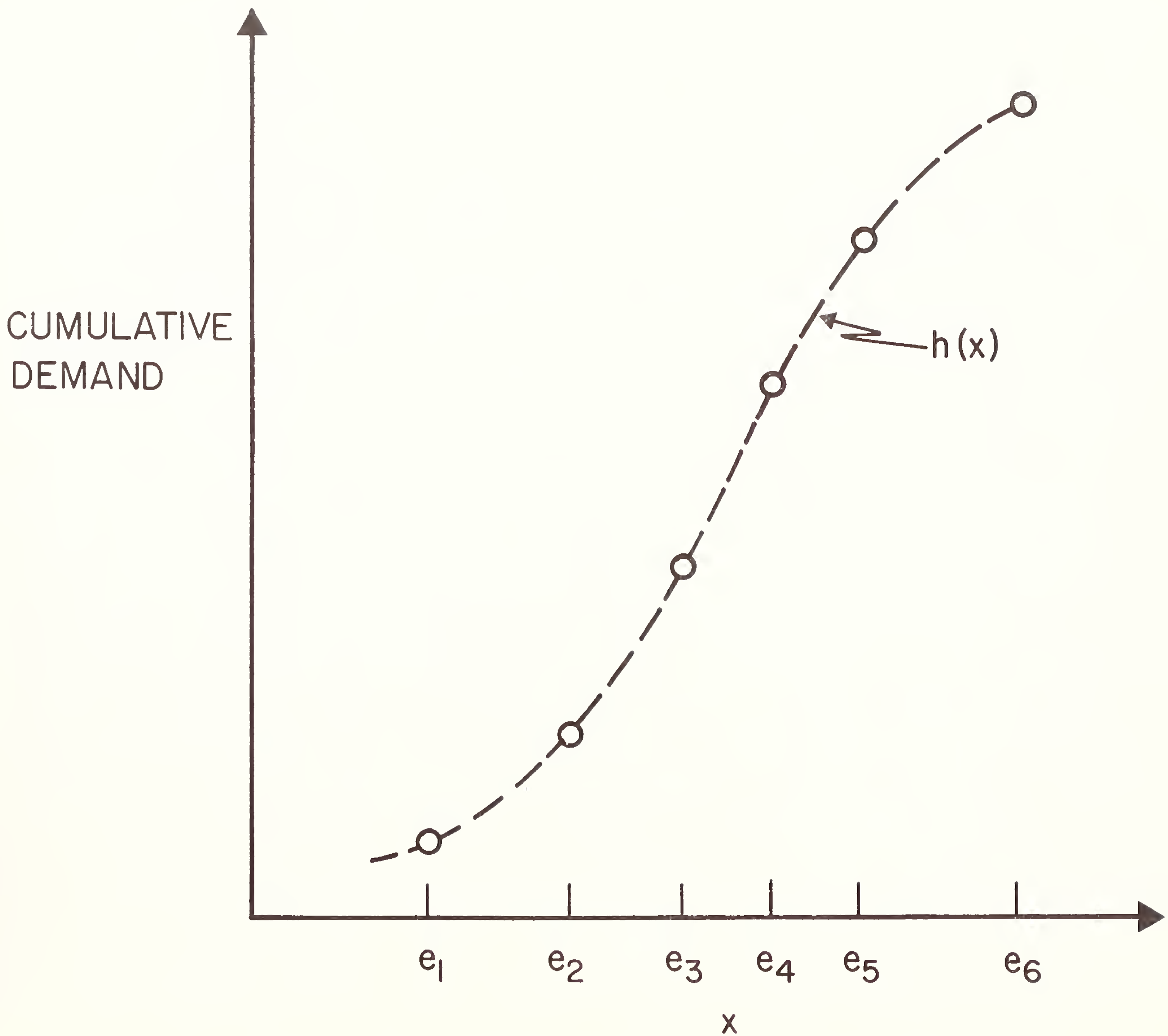

Figure 9. Approximating Demand Curve for Existing Sizes e_i

5. AREAS FOR FURTHER INVESTIGATION

In summary, this report has presented and analyzed various models for solving different versions (continuous and discrete) of the catalog selection problem. After a careful assessment of both the models and their solution procedures, one model was singled out for further examination. Certain refinements and extensions were then proposed for this discrete model and its associated dynamic programming calculations. This model emerges as not only the most promising among existing models, but also as one flexible enough to allow further modification.

It has been stressed throughout that such a model can be used in two distinct ways. First, the model allows one to evaluate, on a cost basis, any number of proposed alternative catalogs. In this sense, the model allows the user to weigh the cost aspects of proposed configurations against any other desirable characteristics of the configurations. Second, the model permits calculation of an optimal (and indeed every optimal) catalog relative to the stated cost expression. This information can be especially valuable as a <u>baseline</u> representation of what the minimum possible cost would be for any catalog. Therefore, it allows a trade-off analysis to be performed between the additional cost of a proposed catalog and other desirable properties of the catalog (e.g., inclusion of a preferred number series [5]). Moreover, the pattern of sizes indicated by an optimal catalog, or the collection of all optimal catalogs, can be valuable when attempting to structure catalogs in order to satisfy several criteria (and not just that of minimum cost). As the above discussion indicates, the process of catalog selection can usefully be viewed as an interactive procedure, involving several iterations between the user and the model.

A computer program in FORTRAN for finding optimal catalogs under the basic model (3.5) has already been developed. Preliminary testing of the programmed version of the model has also been undertaken. Before any intensive testing and evaluation of this automated version of the model are performed, several other issues need to be explored and resolved:

1. Further investigation of methods for obtaining a realistic demand profile should be undertaken. Implementation of a computerized scheme for obtaining such estimates (through use of appropriate curve-fitting techniques) can then be pursued.

2. Realistic estimates for the various cost parameters need to be obtained, preferably through consultation with and cooperation of interested industrial groups.

3. Decisions about the advisability of further modifications of the model, such as the incorporation of capacity restraints[3] or inclusion of preferred number series,[4] should also be made. Again,

[3] In the case of capacity constraints, certain alternative models may become attractive. They appear in the literature in contexts where possible catalog sizes correspond to possible warehouse sites, and demanded sizes correspond to "customer sites" to be supplied from the warehouses. Such models, which are solved by "mixed integer programming" techniques, are not discussed in this report but warrant further investigation.

[4] The inclusion of preferred number series poses an especially interesting challenge for modeling and analysis, and this topic certainly merits additional research effort.

these decisions should be based on consultation with potential (industrial) users.

4. Once demand and cost estimates are available, and after appropriate modifications of the model have been agreed upon, an automated version of the model should be employed within the framework of a realistic setting.

5. Parametric analyses should be performed in order to assess the sensitivity of optimal catalogs to demand and cost estimates.

Thus, while the present report has identified and developed a reasonable approach to catalog selection problems, actual implementation will require further steps. The joint efforts of modelers and users are essential in order for that implementation to be effective and acceptable.

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APPENDIX A

Stability Aspects of Iterative Procedures

In this appendix, we examine certain stability properties of iterative procedures for solving the catalog selection problem. For concreteness, attention will be focused on the behavior of the Lind procedure when n = 2 and all $c(x_i) = 0$. However, the concepts and results presented here can be extended to other iterative schemes, to $n \geq 2$, and to more general stocking costs.

Suppose that an optimal catalog of n = 2 sizes $\{x_1^*, x_2^*\}$ is to be determined. Then the basic iterative relation (3.2) becomes

$$x_2 = x_1 + \frac{1}{f(x_1)} \int_a^{x_1} f(x)dx \ . \tag{A.1}$$

Recall that it is required to find a value x_1 in [a, b] such that the x_2 calculated as above equals b. The Lind algorithm reduces in this instance to the following:

L1. Select x_1.

L2. Calculate x_2 from (A.1).

L3. If $|x_2-b| < \varepsilon$, STOP.

Otherwise, calculate $z = x_1 - \frac{1}{2}(x_2-b)$ and using $x_1 = z$ return to Step L2.

Suppose that x_1^0 is a solution of (A.1) with $x_2 = b$. Then x_1^0 satisfies the first-order conditions for a relative extremum over [a,b]. Intuitively, we consider that x_1^0 is a "stable" solution of (A.1) under the Lind scheme if, after a small perturbation is applied to x_1^0, the scheme insures convergence back to the value x_1^0. More precisely, $y = x_1^0$ is termed <u>stable</u> for our purposes if

$$b = y + \frac{1}{f(y)} \int_a^y f(x)dx \tag{A.2}$$

and if for all δ sufficiently small, the Lind scheme started with the trial value $y+\delta$ yields at the next iteration a value $x_1 = \bar{y}$ with $|\bar{y} - y| < |\delta|$.

Note that in Step L3 of the Lind algorithm

$$\bar{y} = x_1 - h(x_1), \tag{A.3}$$

where

$$h(x_1) = \frac{1}{2}[x_1 + \frac{1}{f(x_1)} \int_a^{x_1} f(x)dx - b]. \tag{A.4}$$

Since $h(y) = 0$, for the starting value $x_1 = y + \delta$, we have

$$\begin{aligned} \bar{y} &= y + \delta - h(y + \delta) \\ &= y + \delta - [h(y+\delta) - h(y)], \end{aligned}$$

or equivalently

$$(\bar{y}-y)/\delta = 1 - [h(y+\delta)-h(y)]/\delta \ .$$

Using the definition of the derivative,

$$h'(y) = \lim_{\delta\to 0}[h(y+\delta)-h(y)]/\delta,$$

we see that y is stable if and only if

$$|1-h'(y)| < 1. \tag{A.5}$$

Using (A.4), h'(y) can be evaluated as

$$h'(y) = \frac{1}{2}[1 + \frac{1}{f^2(y)}\{f(y)f(y) - f'(y)\int_a^y f(x)dx\}]$$

$$= \frac{1}{2}[2 - \frac{f'(y)}{f^2(y)}\int_a^y f(x)dx].$$

Condition (A.5) then becomes

$$\left|\frac{f'(y)}{f^2(y)}\int_a^y f(x)dx\right| < 2$$

or, using (A.2),

$$|Q(y)| = \left|\frac{(b-y)f'(y)}{f(y)}\right| < 2. \tag{A.6}$$

The indicated quotient Q(y) in (A.6) can be readily computed and used to characterize solutions y of (A.2). Namely, a solution y of the first-order condition (A.2) is <u>stable</u> for the Lind algorithm if (A.6) holds at this value y. On the other hand, if (A.6) is not satisfied at y then we have an <u>unstable</u> solution. For example, it can be directly shown that any relative maximum of the catalog cost function

$$C(x_1) = \int_a^{x_1} (x_1 - x)\, f(x)dx + \int_{x_1}^{b} (b-x)\, f(x)dx \tag{A.7}$$

is always unstable. Indeed,

$$\begin{aligned} C(x_1) &= x_1 \int_a^{x_1} f(x)dx + b \int_{x_1}^{b} f(x)dx - \int_a^b xf(x)dx \\ &= (x_1 - b) \int_a^{x_1} f(x)dx + b \int_a^b f(x)dx - \int_a^b xf(x)dx \\ &= (x_1 - b) \int_a^{x_1} f(x)dx + K, \end{aligned}$$

where K is a constant independent of x_1. Therefore,

$$C'(x_1) = (x_1 - b)\, f(x_1) + \int_a^{x_1} f(x)dx,$$

$$C''(x_1) = (x_1 - b)\, f'(x_1) + 2f(x_1).$$

If x_1 corresponds to a relative maximum of C(x) then $C''(x_1) \leq 0$ or equivalently (since $f(x) > 0$)

$$\frac{(b-x_1)\, f'(x_1)}{f(x_1)} \geq 2.$$

Relation (A.6) is thus not satisfied at $y = x_1$ and so x_1 is an unstable solution.

For Example 1 of Section 3.1, $f(x) = e^{-x}$ over $[0,b]$, with $b = 1 + e^2$. It can be shown that $y = 2$ is the only solution to (A.2). Moreover, the quotient in (A.6) is

$$Q(y) = \frac{-(b-y)\, e^{-y}}{e^{-y}} = y-b,$$

whence

$$Q(2) = 2 - (1 + e^2) = 1 - e^2 = -6.389 < -2,$$

and so $y = 2$ is <u>unstable</u> with respect to the Lind procedure. As a result the Lind procedure, begun with any starting value $x_1 \neq 2$, will <u>never</u> converge to the required solution $y = 2$.

For Example 2 of Section 3.1, $f(x) = 3x^2 - 2.04x + 0.415$ over $[0,1]$. The quotient in (A.6) thus becomes

$$Q(y) = \frac{(1-y)(6y-2.04)}{3y^2-2.04y + 0.415} .$$

When evaluated at the three solutions of (A.2): $u = 0.2769$, $v = 0.5265$ and $w = 0.7116$, this quotient yields

$$Q(u) = -3.42,$$

$$Q(v) = 3.07,$$

$$Q(w) = 1.33.$$

It follows that u represents an unstable (minimum) solution, v represents an unstable (maximum) solution and w represents a stable (minimum) solution, with respect to the Lind scheme.

APPENDIX B

Computational Refinement for the Wagner-Whitin Model

This appendix considers the basic Wagner-Whitin model, described by the dynamic programming formulation (3.3):

$$G(i) = \min_{i \leq j \leq m} \{k_j + \sum_{t=i}^{j} c_{tj} d_t + G(j+1)\}.$$

Here $G(i)$ represents the minimum cost achievable using sizes chosen from $S_i = \{s_i, s_{i+1}, \ldots, s_m\}$ to meet demands over S_i. Let $j(i)$ be any minimizing value of j in the above equation; similarly $j(i+1)$ refers to any minimizing value of j in the corresponding expression for $G(i+1)$.

We demonstrate in this appendix the validity of the relation

$$j(i) \leq j(i+1), \tag{B.1}$$

under the assumption that $d_i > 0$ and

$$c_{ik} < c_{i\ell} \text{ for } k < \ell. \tag{B.2}$$

First, note that if $j(i) = i$, then $j(i) = i < i + 1 \leq j(i+1)$, and so the result (B.1) holds automatically. Accordingly, it is assumed in what follows that $j(i) > i$.

Since $j(i)$ and $j(i+1)$ are minimizing values of $G(i)$ and $G(i+1)$, respectively,

$$G(i) = k_{j(i)} + \sum_{t=i}^{j(i)} c_{t,j(i)} d_t + G(j(i) + 1) \tag{B.3}$$

and

$$G(i+1) = k_{j(i+1)} + \sum_{t=i+1}^{j(i+1)} c_{t,j(i+1)} d_t + G(j(i+1) + 1). \tag{B.4}$$

Also, because $j(i)$ is a minimizing value of j for $G(i)$ with $j(i) \geq i$ and because $j(i+1) \geq i+1$,

$$G(i) \leq k_{j(i+1)} + \sum_{t=i}^{j(i+1)} c_{t,j(i+1)} d_t + G(j(i+1) + 1)$$

$$= G(i+1) + c_{i,j(i+1)} d_i,$$

using (B.4). Furthermore, because $j(i+1)$ is a minimizing value of j for $G(i+1)$ with $j(i+1) \geq i+1$, and because $j(i) > i$,

$$G(i+1) \leq k_{j(i)} + \sum_{t=i+1}^{j(i)} c_{t,j(i)} d_t + G(j(i) + 1)$$

$$= G(i) - c_{i,j(i)} d_i,$$

using (B.3). Therefore,

$$G(i+1) \leq G(i) - c_{i,j(i)} d_i \leq G(i) \leq G(i+1) + c_{i,j(i+1)} d_i$$

and so

$$c_{i,j(i)} d_i \leq c_{i,j(i+1)} d_i .$$

In the case where $d_i > 0$, then

$$c_{i,j(i)} \leq c_{i,j(i+1)} . \tag{B.5}$$

From this relation it follows that $j(i) \leq j(i+1)$, since otherwise $j(i) > j(i+1)$ and so by (B.2)

$$c_{i,j(i)} > c_{i,j(i+1)} ,$$

contradicting (B.5).

In the case when $d_i = 0$, the expressions for $G(i)$ and $G(i+1)$ become

$$G(i) = \min_{i \leq j \leq m} \{k_j + \sum_{t=i+1}^{j} c_{tj} d_t + G(j+1)\} ,$$

$$G(i+1) = \min_{i+1 \leq j \leq m} \{k_j + \sum_{t=i+1}^{j} c_{tj} d_t + G(j+1)\}.$$

If the first minimum occurs for $j=i$, then i is a $j(i)$ for which (B.1) holds; if it does not occur for $j=i$, then $j(i+1)$ is a $j(i)$ for which (B.1) holds. As a consequence,

$$J(i) \leq J(i+1),$$

where J(i) represents the smallest value of j minimizing G(i), and similarly for J(i+1). As a result, even when $d_i = 0$, an optimal catalog can be found by using the formulation (4.3). However, it may not be possible to retrieve <u>all</u> optimal catalogs in this case.

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16. ABSTRACT *(A 200-word or less factual summary of most significant information. If document includes a significant bibliography or literature survey, mention it here.)*

This report identifies and discusses various mathematical models for selecting a "best" catalog of standard sizes. A survey of existing models for continuous and discrete versions of the catalog selection problem is presented. The advantages and disadvantages of such models are assessed with regard to both range of applicability and computational feasibility. This evaluation shows that a frequently-advocated iterative procedure may produce erroneous results and identifies another approach as the most promising. Various refinements and extensions are then proposed for this latter (discrete) model and its associated solution technique (dynamic programming). In particular, a multidimensional version of the catalog selection problem is formulated and analyzed. Areas for further investigation, and unresolved issues, are also discussed.

17. KEY WORDS *(six to twelve entries; alphabetical order; capitalize only the first letter of the first key word unless a proper name; separated by semicolons)*

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