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A Universal Set of Test Data for Computer Implementations of Elementary Mathematical Functions

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Mathematical Analysis Division
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U.S. DEPARTMENT OF COMMERCE, Juanita M. Kreps, *Secretary*
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Jordan J. Baruch, *Assistant Secretary for Science and Technology*
NATIONAL BUREAU OF STANDARDS, Ernest Ambler, *Director*

1. Introduction

Computer implementations of algorithms for the evaluation of mathematical functions have been an essential feature of computer systems since the very first days of the stored-program computer. This does not mean, however, that these implementations have always been done correctly. Because of the fundamental importance of accurate function evaluation, the programmer needs to have the utmost confidence in the quality of computed function values. Comparing computed function values with published tables in an effort to gain this confidence is often unsatisfactory because either the precision or argument range of the tabulated function is inadequate. Published tables were often intended as an aid to hand computation. The density of data in such tables is highest in regions of presumed highest usage, whereas most possible inputs to a computer subroutine lie outside these regions. Thus there is a need for test data that is specifically created for the purpose of testing computer subroutines.

The ideal approach would be to create a different set of test data for each particular computer implementation of a function. Then every feature of the algorithm, the computer arithmetic, or the actual coded implementation could be tested. But this ideal would require a means of generating function values correct to the precision of the subroutine being tested. The tester may lack such a capability or the will to use it. The approach taken here is much more modest. The objective is to provide a small amount of data chosen to sample the function over a large range of arguments and in regions where special difficulties might be expected. The serious limitations of such an approach will become apparent later in this note.

The data presented here was developed about four years ago in response to needs at the NBS. No general method of data selection was used, and other data could undoubtedly be found that would serve the purpose as well or better. More recently, a project has been started to develop a general theory of test data selection for mathematical functions that we hope will lead to a sound and unified basis for mathematical function testing in the future.

2. Interpreting Test Results

The reader will note that the test data is presented in decimal form with five digit test arguments. Although many pocket calculators use decimal arithmetic, most computers use binary (or hexadecimal). The data may be used to test decimal or non-decimal implementations of functions, but the interpretation of the results is slightly different in these two cases. Let us first consider the decimal case. We may as well assume the computing device is a calculator. In this case there is no conversion between incommensurate internal and external number bases. The test arguments are exactly representable, provided at least five digits can be accommodated by the calculator, so any error in the function value is entirely attributable to the implementation.

This section is mainly concerned with real functions of one real variable, although test data is given in this report for other functions. The other functions have simple definitions in terms of the real functions for which test data is presented. It is assumed that computer implementations will make use of this fact. Therefore, data for the other functions is intended only to test these definitions.

The calculator very likely uses one or more guard figures. These are extra decimal places that are used for calculations but are not ordinarily displayed. Their purpose is to make all of the displayed figures correct for most calculations.

Let us consider the effect of guard digits on the trigonometric functions. These functions make the heaviest demand for guard figures, of the functions considered in this note. Let $t(x)$ be one of the trigonometric functions, and let p be the period of $t(x)$. We first consider x restricted to the interval

$$0 < x \leq p .$$

Then, depending on the particular trigonometric function, $t(x)$ has 0, 1, or 2 zeros and 0, 1, or 2 poles in this interval.

(Note that we are excluding the left end point.) In a neighborhood of a zero x_0 or a pole x_∞, $t(x)$ is asymptotic to $\pm(x - x_0)$ or $\pm(x - x_\infty)^{-1}$. Let x^* denote either a zero or a pole. In order to compute $t(x)$ in a neighborhood of x^*, the difference $x - x^*$ must be formed. Suppose the normally displayed precision of the calculator is d digits, and suppose x is exactly representable. As x approaches x^*, there is an increase in the number of leading digits that are the same. Therefore, there is a progressive loss of precision through cancellation that occurs in forming $x - x^*$, as x gets closer to x^*. Guard digits mitigate this loss; let us see how.

Suppose g guard digits are used, and that the period p is known to $d + g$ digits. Then x^* is known essentially to $d + g$ digits also, because the zeros and poles of the trigonometric function are rational fractions of p. Since, for x in radians, x^* is irrational, its $(d + 1)$st digit is unlikely to be zero. Thus not more than d digits are likely to match between x and x^*. Put another way, at least g digits will usually survive in the difference $x - x^*$. This suggests that nothing much is to be gained by taking more than d guard digits.

We remark that $g = d$ permits the calculation of $t(x)$ to full relative precision even in the neighborhoods of zeros and poles. This advantage is illusory if x is not stored exactly, as will be the usual case when x is generated by prior calculation. In this case the digits of x would not be known beyond the d-th, and there would be no basis for forming $x - x^*$ beyond the d-th digit. It is common, therefore, to advocate the use of no guard digits† in neighborhoods of zeros of functions. This corresponds to using the absolute error criterion rather than the relative error criterion in the calculation of $t(x)$.

Because only one irrational number is needed to specify all zeros and poles of the trigonometric functions, and because the other elementary functions commonly found in computer libraries require not more than d guard digits, an unofficial standard is adhered to by many of today's digital computers: function values are computed to full relative accuracy through the use of up to d guard digits, as necessary. In calculators, however, we typically find $g < d$. Thus in calculators, if x^* is approached too closely, then $t(x)$ will have reduced relative precision. The better calculators display only the digits that are likely to be correct.

So far we have considered $t(x)$ only for x in a single period interval. Now suppose x satisfies

$$np < x \leq np + p$$

for some integer n. Then the usual way of proceeding is to determine the number pf such that $0 < f \leq 1$ and

$$x = pn + pf .$$

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† At least one guard digit is essential to obtain desirable rounding error behavior in floating-point arithmetic. See Olver [4] or Sterbenz [5] for a full account.

Then the relation

$$t(x) = t(pf)$$

may be employed. Determination of pf is done, in effect, by dividing x by p, taking the integer part of the result, subtracting this from x/p and finally multiplying by p. The subtraction step causes a loss of precision in f, which will be propagated to pf. Clearly, a measure of the number of digits lost is the number of digits in the integer part of x/p, which is given by

$$\text{int}\,\{\log_{10} |\tfrac{x}{p}|\} \quad .$$

Here, $\text{int}\{\cdot\}$ is the integer part of the quantity in braces. If g guard digits are used, then the number of significant digits s in pf is estimated by

$$s = d + g - \text{int}\,\{\log_{10} |\tfrac{x}{p}|\} \ .$$

If $g = d$, then $s \geq d$ as long as x satisfies $x \leq 10^d p$. Recalling our earlier remarks, we see that if d guard digits are used, then full relative precision can be maintained uniformly, not only for x in the fundamental period interval, but for very large values of x as well. A new feature of the present analysis is that a fixed number of guard digits does not suffice for all x; computer implementations of $t(x)$ must have an error path for inputs that are too large. We remark, as we did before, that $g = d$ can in no way be considered an irrefutable requirement. The advantage, as before, is only of value for exactly representable arguments.

Before turning to a consideration of non-decimal implementations, let us briefly consider how to interpret the results of tests based on comparing the computed function values against the data. It is clear that successful comparison to the full displayed precision of the calculator does not guarantee the function is accurate at other arguments. An attempt has been made, however, to select the data so that this would be a good indication that the accuracy of the function is nearly constant throughout the tested range. Obviously, the loophole here is that the data cannot test the algorithmic details of the computer implementation. There may be an interval of arguments, perhaps very short but with reasonable probability of use under normal circumstances, for which an approximation to the function is incorrectly coded or improperly chosen. It is highly unlikely that such an interval would be tested by any universal set of test data.

On the other hand, if some of the computed function values do not compare well against the data, it does not necessarily mean the function is unacceptable. The preceding discussion of trigonometric functions indicates there are limits to what one can reasonably require. It is reasonable for a subroutine designer to specify a function so that loss of accuracy occurs under certain stated conditions. One purpose of testing is to verify that a subroutine conforms to the specification of its designer. The tester, however, may not know what specification the designer had in mind. But he can infer quite a lot about the function from the test results, such as whether the internal base of calculation is two or ten (as we shall see) and what the number of guard figures is. In this way he can construct a plausible model of the algorithm that will explain the test results. Without such a model, he has no basis for predicting the accuracy of the function at other arguments.

Here again we see how the power of testing with a universal set of data is limited. Without the possibility of generating additional test data, there is no way the tester can refine his model of the algorithm.

When the implementation is non-decimal, the test arguments are not necessarily exactly representable in the computer. Thus the subroutine may receive not x, but an approximation $\tilde{x}$. Let β be the base and d the number of base-β digits in the computer arithmetic. Then it is known that, with proper base conversion procedures, the relative error ε in $\tilde{x}$ satisfies a bound at least as good as

$$|\varepsilon| \leq \beta^{1-d} .$$

Thus the error in representing x, if indeed the error is non-zero, is dependent only on parameters of the floating-point representation. The error in $f(x)$ resulting from an error in x, however, satisfies no bound independent of x

and f. The size of the error in $f(x)$ is predicted by standard analytical (calculus) methods and a catalog of cases will not be given here. Three handbooks of particular value are given in the references [1,2,3].

By what we have just stated, the test data is divided into two classes: that for which x is exactly representable, and that for which this is not true. Interpretation of test results for data of the first class is identical to what we stated for the case of decimal implementations. For data of the second class, the tester must take into account the effect that an input error of the indicated size will have on the function value. Any loss of accuracy that is predicted by this effect is intrinsic and cannot be reduced by any such device as the use of guard digits. Thus the tester may observe dramatic differences in the sizes of the errors in function values at neighboring arguments, just because one argument is exactly representable and another is not.

3. The Test Data

The functions for which test data is given are listed in Table 1. There are 13 real functions, five complex functions, the complex modulus, and the special arctangent function of two variables that is useful for converting from rectangular coordinates to polar coordinates. In the table, x is a real variable and z is a complex variable. All angles are expressed in radians.

Table 1. Functions for which test data is given.

e^x	$\sin x$	e^z
$\sinh x$	$\cos x$	$\sqrt{z}$
$\cosh x$	$\tan x$	$\ln z$
$\tanh x$	$\arcsin x$	$\sin z$
$\sqrt{x}$	$\arccos x$	$\cos z$
$\ln x$	$\arctan x$	$\|z\|$
$\log_{10} x$		$\arctan (x_1, x_2)$

The test data is presented in Tables 2, 3 and 4. There is one sample point per line in each table, and each line has 80 positions. A function identification code and seouence number appear in positions 74-80 of each line. Table 2 presents the data for real functions $g(x)$. The line format is

Starting Position	Width	Content
1	10	x, 5D
12	12	g(x), 7D
25	16	g(x), 11D
42	21	g(x), 16D

Table 3 presents the data for complex functions $g(z)$. The line format is

Starting Position	Width	Content
1	10	Re(z), 5D
12	10	Im(z), 5D
23	12	$\mathrm{Re}\{g(z)\}$, 7D
36	12	$\mathrm{Im}\{g(z)\}$, 7D

Table 4 presents the data for $g(x_1,x_2) = |x_1 + ix_2|$ and $g(x_1,x_2) = \arctan(x_1,x_2)$. The line format is

Starting Position	Width	Content
1	10	x_1, 5D
12	10	x_2, 5D
23	12	$g(x_1,x_2)$, 7D
36	16	$g(x_1,x_2)$, 11D
53	21	$g(x_1,x_2)$, 16D

The notation 5D, 7D, etc. indicates the number of decimal places to which a number is given. All function values are believed to be in error by not more than one unit in the last place.

x	g(x)			Ident.	
-.87000+02	.1645811-37	.16458114311-37	.1645811431082274-37	EXP	1
-.80590+02	.1000478-34	.10004783692-34	.1000478369173656-34	EXP	2
-.50000+02	.1928750-21	.19287498480-21	.1928749847963918-21	EXP	3
-.36841+02	.1000362-15	.10003615532-15	.1000361553249357-15	EXP	4
-.18421+02	.9996808-08	.99968079491-08	.9996807949091546-08	EXP	5
-.16000+02	.1125352-06	.11253517472-06	.1125351747192591-06	EXP	6
-.92103+01	.1000040-03	.10000403728-03	.1000040372791142-03	EXP	7
-.80000+01	.3354626-03	.33546262790-03	.3354626279025118-03	EXP	8
-.46052+01	.9999702-02	.99997018643-02	.9999701864325246-02	EXP	9
-.40000+01	.1831564-01	.18315638889-01	.1831563888873418-01	EXP	10
-.23979+01	.9090866-01	.90908661165-01	.9090866116450398-01	EXP	11
-.20000+01	.1353353+00	.13533528324+00	.1353352832366127+00	EXP	12
-.16094+01	.2000076+00	.20000758263+00	.2000075826305572+00	EXP	13
-.10986+01	.3333374+00	.33333742958+00	.3333374295812052+00	EXP	14
-.33333+00	.7165337+00	.71653369902+00	.7165336990154719+00	EXP	15
.00000	.1000000+01	.10000000000+01	.1000000000000000+01	EXP	16
.25000+00	.1284025+01	.12840254167+01	.1284025416687741+01	EXP	17
.50000+00	.1648721+01	.16487212707+01	.1648721270700128+01	EXP	18
.69315+00	.2000006+01	.20000056389+01	.2000005638888059+01	EXP	19
.10000+01	.2718282+01	.27182818285+01	.2718281828459045+01	EXP	20
.13863+01	.4000023+01	.40000225556+01	.4000022555584032+01	EXP	21
.20794+01	.7999668+01	.79996676735+01	.7999667673464062+01	EXP	22
.23026+01	.1000015+02	.10000149071+02	.1000014907117064+02	EXP	23
.27726+01	.1600018+02	.16000180445+02	.1600018044518101+02	EXP	24
.50000+01	.1484132+03	.14841315910+03	.1484131591025766+03	EXP	25
.69078+01	.1000045+04	.10000447220+04	.1000044722017863+04	EXP	26
.11000+02	.5987414+05	.59874141715+05	.5987414171519782+05	EXP	27
.11513+02	.1000075+06	.10000745378+06	.1000074537807576+06	EXP	28
.25000+02	.7200490+11	.72004899337+11	.7200489933738587+11	EXP	29
.25328+02	.9995641+11	.99956407211+11	.9995640721096829+11	EXP	30
.46052+02	.1000298+21	.10002981846+21	.1000298184567269+21	EXP	31
.75000+02	.3733242+33	.37332419968+33	.3733241996799002+33	EXP	32
.85000+02	.8223013+37	.82230127146+37	.8223012714622913+37	EXP	33
-.85000+02	-.4111506+37	-.41115063573+37	-.4111506357311457+37	SINH	1
-.50000+02	-.2592353+22	-.25923527643+22	-.2592352764293536+22	SINH	2
-.16000+02	-.4443055+07	-.44430552603+07	-.4443055260253880+07	SINH	3
-.80000+01	-.1490479+04	-.14904788258+04	-.1490478825789550+04	SINH	4
-.40000+01	-.2728992+02	-.27289917197+02	-.2728991719712775+02	SINH	5
-.20000+01	-.3626860+01	-.36268604078+01	-.3626860407847019+01	SINH	6
-.80000+00	-.8881060+00	-.88810598219+00	-.8881059821876230+00	SINH	7
-.41421+00	-.4261564+00	-.42615635230+00	-.4261563523008389+00	SINH	8
-.30000+00	-.3045203+00	-.30452029345+00	-.3045202934471426+00	SINH	9
-.19891+00	-.2002242+00	-.20022424916+00	-.2002242491609640+00	SINH	10
-.10000+00	-.1001668+00	-.10016675002+00	-.1001667500198440+00	SINH	11
-.10000-02	-.1000000-02	-.10000001667-02	-.1000000166666675-02	SINH	12
-.10000-04	-.1000000-04	-.10000000000-04	-.1000000000016667-04	SINH	13
-.10000-10	-.1000000-10	-.10000000000-10	-.1000000000000000-10	SINH	14
-.10000-24	-.1000000-24	-.10000000000-24	-.1000000000000000-24	SINH	15
.00000	.0000000	.00000000000	.0000000000000000	SINH	16
.10000-34	.1000000-34	.10000000000-34	.1000000000000000-34	SINH	17
.10000-15	.1000000-15	.10000000000-15	.1000000000000000-15	SINH	18
.10000-07	.1000000-07	.10000000000-07	.1000000000000000-07	SINH	19
.10000-03	.1000000-03	.10000000017-03	.1000000001666667-03	SINH	20
.10000-01	.1000017-01	.10000166668-01	.1000016666750000-01	SINH	21
.98491-01	.9865031-01	.98650312199-01	.9865031219861936-01	SINH	22

Table 2. Test data for the real functions.

x	g(x)			Ident.
.20000+00	.2013360+00	.20133600254+00	.201336002541094+00	SINH 23
.26795+00	.2711679+00	.27116787342+00	.271167873422011+00	SINH 24
.50000+00	.5210953+00	.52109530549+00	.521095305493747+00	SINH 25
.10000+01	.1175201+01	.11752011936+01	.117520119364380+01	SINH 26
.30000+01	.1001787+02	.10017874927+02	.100178749274099+02	SINH 27
.50000+01	.7420321+02	.74203210578+02	.742032105777888+02	SINH 28
.11000+02	.2993707+05	.29937070849+05	.299370708492481+05	SINH 29
.25000+02	.3600245+11	.36002449669+11	.360024496686929+11	SINH 30
.75000+02	.1866621+33	.18666209984+33	.186662099839950+33	SINH 31
-.85000+02	.4111506+37	.41115063573+37	.411150635731146+37	COSH 1
-.50000+02	.2592353+22	.25923527643+22	.259235276429354+22	COSH 2
-.16000+02	.4443055+07	.44430552603+07	.444305526025399+07	COSH 3
-.80000+01	.1490479+04	.14904791613+04	.149047916125218+04	COSH 4
-.40000+01	.2730823+02	.27308232836+02	.273082328360165+02	COSH 5
-.20000+01	.3762196+01	.37621956911+01	.376219569108363+01	COSH 6
-.80000+00	.1337435+01	.13374349463+01	.133743494630485+01	COSH 7
-.41421+00	.1087019+01	.10870185079+01	.108701850794104+01	COSH 8
-.30000+00	.1045339+01	.10453385141+01	.104533851412886+01	COSH 9
-.19891+00	.1019848+01	.10198479053+01	.101984790530357+01	COSH 10
-.10000+00	.1005004+01	.10050041681+01	.100500416805580+01	COSH 11
-.10000-02	.1000001+01	.10000005000+01	.100000050000004+01	COSH 12
-.10000-04	.1000000+01	.10000000000+01	.100000000005000+01	COSH 13
-.10000-10	.1000000+01	.10000000000+01	.100000000000000+01	COSH 14
-.10000-24	.1000000+01	.10000000000+01	.100000000000000+01	COSH 15
.00000	.1000000+01	.10000000000+01	.100000000000000+01	COSH 16
.10000-34	.1000000+01	.10000000000+01	.100000000000000+01	COSH 17
.10000-15	.1000000+01	.10000000000+01	.100000000000000+01	COSH 18
.10000-07	.1000000+01	.10000000000+01	.100000000000000+01	COSH 19
.10000-03	.1000000+01	.10000000050+01	.100000000500000+01	COSH 20
.10000-01	.1000050+01	.10000500004+01	.100005000041667+01	COSH 21
.98491-01	.1004854+01	.10048541606+01	.100485416061082+01	COSH 22
.20000+00	.1020067+01	.10200667556+01	.102006675561908+01	COSH 23
.26795+00	.1036114+01	.10361139009+01	.103611390087008+01	COSH 24
.50000+00	.1127626+01	.11276259652+01	.112762596520638+01	COSH 25
.10000+01	.1543081+01	.15430806348+01	.154308063481524+01	COSH 26
.30000+01	.1006766+02	.10067661996+02	.100676619957778+02	COSH 27
.50000+01	.7420995+02	.74209948525+02	.742099485247878+02	COSH 28
.11000+02	.2993707+05	.29937070866+05	.299370708659498+05	COSH 29
.25000+02	.3600245+11	.36002449669+11	.360024496686929+11	COSH 30
.75000+02	.1866621+33	.18666209984+33	.186662099839950+33	COSH 31
-.10000+26	-.1000000+01	-.10000000000+01	-.100000000000000+01	TANH 1
-.10000+03	-.1000000+01	-.10000000000+01	-.100000000000000+01	TANH 2
-.85000+02	-.1000000+01	-.10000000000+01	-.100000000000000+01	TANH 3
-.50000+02	-.1000000+01	-.10000000000+01	-.100000000000000+01	TANH 4
-.16000+02	-.1000000+01	-.10000000000+01	-.999999999999747+00	TANH 5
-.80000+01	-.9999998+00	-.99999977493+00	-.999999774929676+00	TANH 6
-.40000+01	-.9993293+00	-.99932929974+00	-.999329299739067+00	TANH 7
-.20000+01	-.9640276+00	-.96402758008+00	-.964027580075817+00	TANH 8
-.80000+00	-.6640368+00	-.66403677027+00	-.664036770267849+00	TANH 9
-.41421+00	-.3920415+00	-.39204148705+00	-.392041487047022+00	TANH 10
-.30000+00	-.2913126+00	-.29131261245+00	-.291312612451591+00	TANH 11
-.19891+00	-.1963276+00	-.19632755837+00	-.196327558373879+00	TANH 12
-.10000+00	-.9966799-01	-.99667994625-01	-.996679946249558-01	TANH 13
-.10000-02	-.9999997-03	-.99999966667-03	-.999999666666800-03	TANH 14
-.10000-04	-.1000000-04	-.99999999997-05	-.999999999666667-05	TANH 15

Table 2 (cont.). Test data for the real functions.

x	g(x)			Ident.
-.10000-10	-.1000000-10	-.10000000000-10	-.1000000000000000-10	TANH 16
-.10000-24	-.1000000-24	-.10000000000-24	-.1000000000000000-24	TANH 17
.00000	.0000000	.00000000000	.0000000000000000	TANH 18
.10000-34	.1000000-34	.10000000000-34	.1000000000000000-34	TANH 19
.10000-15	.1000000-15	.10000000000-15	.1000000000000000-15	TANH 20
.10000-07	.1000000-07	.10000000000-07	.1000000000000000-07	TANH 21
.10000-03	.1000000-03	.99999999667-04	.9999999966666667-04	TANH 22
.10000-01	.9999667-02	.99996666800-02	.9999666679999460-02	TANH 23
.98491-01	.9817376-01	.98173760995-01	.9817376099498100-01	TANH 24
.20000+00	.1973753+00	.19737532022+00	.1973753202249040+00	TANH 25
.26795+00	.2617163+00	.26171627771+00	.2617162777126113+00	TANH 26
.50000+00	.4621172+00	.46211715726+00	.4621171572600098+00	TANH 27
.10000+01	.7615942+00	.76159415596+00	.7615941559557649+00	TANH 28
.30000+01	.9950548+00	.99505475369+00	.9950547536867305+00	TANH 29
.50000+01	.9999092+00	.99990920426+00	.9999092042625951+00	TANH 30
.11000+02	.1000000+01	.99999999944+00	.9999999994421064+00	TANH 31
.25000+02	.1000000+01	.10000000000+01	.1000000000000000+01	TANH 32
.75000+02	.1000000+01	.10000000000+01	.1000000000000000+01	TANH 33
.10000+11	.1000000+01	.10000000000+01	.1000000000000000+01	TANH 34
.10000+36	.1000000+01	.10000000000+01	.1000000000000000+01	TANH 35
.00000	.0000000	.00000000000	.0000000000000000	SQRT 1
.10000-34	.3162278-17	.31622776602-17	.3162277660168379-17	SQRT 2
.10000-15	.1000000-07	.10000000000-07	.1000000000000000-07	SQRT 3
.10000-07	.1000000-03	.10000000000-03	.1000000000000000-03	SQRT 4
.10000-03	.1000000-01	.10000000000-01	.1000000000000000-01	SQRT 5
.10000-01	.1000000+00	.10000000000+00	.1000000000000000+00	SQRT 6
.98491-01	.3138328+00	.31383275801+00	.3138327580097400+00	SQRT 7
.20000+00	.4472136+00	.44721359550+00	.4472135954999579+00	SQRT 8
.26795+00	.5176389+00	.51763887026+00	.5176388702560889+00	SQRT 9
.50000+00	.7071068+00	.70710678119+00	.7071067811865475+00	SQRT 10
.10000+01	.1000000+01	.10000000000+01	.1000000000000000+01	SQRT 11
.30000+01	.1732051+01	.17320508076+01	.1732050807568877+01	SQRT 12
.50000+01	.2236068+01	.22360679775+01	.2236067977499790+01	SQRT 13
.11000+02	.3316625+01	.33166247904+01	.3316624790355400+01	SQRT 14
.25000+02	.5000000+01	.50000000000+01	.5000000000000000+01	SQRT 15
.75000+02	.8660254+01	.86602540378+01	.8660254037844386+01	SQRT 16
.10000+11	.1000000+06	.10000000000+06	.1000000000000000+06	SQRT 17
.10000+36	.3162278+18	.31622776602+18	.3162277660168379+18	SQRT 18
.10000-19	-.4605170+02	-.46051701860+02	-.4605170185988091+02	LOGE 1
.10000-07	-.1842068+02	-.18420680744+02	-.1842068074395237+02	LOGE 2
.11254-06	-.1599996+02	-.15999957123+02	-.1599995712294128+02	LOGE 3
.10000-04	-.1151293+02	-.11512925465+02	-.1151292546497023+02	LOGE 4
.33546-03	-.8000008+01	-.80000078337+01	-.8000007833697665+01	LOGE 5
.10000-01	-.4605170+01	-.46051701860+01	-.4605170185988091+01	LOGE 6
.18316-01	-.3999980+01	-.39999802842+01	-.3999980284187288+01	LOGE 7
.31250-01	-.3465736+01	-.34657359028+01	-.3465735902799727+01	LOGE 8
.90909-01	-.2397896+01	-.23978962728+01	-.2397896272798871+01	LOGE 9
.13534+00	-.1999965+01	-.19999651482+01	-.1999965148178058+01	LOGE 10
.20000+00	-.1609438+01	-.16094379124+01	-.1609437912434100+01	LOGE 11
.33333+00	-.1098622+01	-.10986222887+01	-.1098622288718110+01	LOGE 12
.50000+00	-.6931472+00	-.69314718056+00	-.6931471805599453+00	LOGE 13
.75000+00	-.2876821+00	-.28768207245+00	-.2876820724517809+00	LOGE 14
.90000+00	-.1053605+00	-.10536051566+00	-.1053605156578263+00	LOGE 15
.99000+00	-.1005034-01	-.10050335854-01	-.1005033585350144-01	LOGE 16
.99900+00	-.1000500-02	-.10005003336-02	-.1000500333583534-02	LOGE 17

Table 2 (cont.). Test data for the real functions.

x	g(x)			Ident.
.99990+00	-.1000050-03	-.10000500033-03	-.1000050003333583-03	LOGE 18
.99999+00	-.1000005-04	-.10000050000-04	-.1000005000033334-04	LOGE 19
.10000+01	.0000000	.00000000000	.0000000000000000	LOGE 20
.10001+01	.9999500-04	.99995000333-04	.9999500033330834-04	LOGE 21
.10010+01	.9995003-03	.99950033308-03	.9995003330835332-03	LOGE 22
.10100+01	.9950331-02	.99503308532-02	.9950330853168083-02	LOGE 23
.11000+01	.9531018-01	.95310179804-01	.9531017980432486-01	LOGE 24
.15000+01	.4054651+00	.40546510811+00	.4054651081081644+00	LOGE 25
.20000+01	.6931472+00	.69314718056+00	.6931471805599453+00	LOGE 26
.27183+01	.1000007+01	.10000066849+01	.1000006684913988+01	LOGE 27
.40000+01	.1386294+01	.13862943611+01	.1386294361119891+01	LOGE 28
.80000+01	.2079442+01	.20794415417+01	.2079441541679836+01	LOGE 29
.10000+02	.2302585+01	.23025850930+01	.2302585092994046+01	LOGE 30
.16000+02	.2772589+01	.27725887222+01	.2772588722239781+01	LOGE 31
.20086+02	.3000023+01	.30000230550+01	.3000023054971147+01	LOGE 32
.64000+02	.4158883+01	.41588830834+01	.4158883083359672+01	LOGE 33
.14841+03	.4999979+01	.49999787139+01	.4999978713907727+01	LOGE 34
.10000+04	.6907755+01	.69077552790+01	.6907755278982137+01	LOGE 35
.10000+05	.9210340+01	.92103403720+01	.9210340371976183+01	LOGE 36
.59874+05	.1100000+02	.10999997633+02	.1099999763311237+02	LOGE 37
.10000+11	.2302585+02	.23025850930+02	.2302585092994046+02	LOGE 38
.10000+36	.8059048+02	.80590478255+02	.8059047825479160+02	LOGE 39
.10000-19	-.2000000+02	-.20000000000+02	-.2000000000000000+02	LG10 1
.10000-07	-.8000000+01	-.80000000000+01	-.8000000000000000+01	LG10 2
.11254-06	-.6948693+01	-.69486930892+01	-.6948693089182026+01	LG10 3
.10000-04	-.5000000+01	-.50000000000+01	-.5000000000000000+01	LG10 4
.33546-03	-.3474359+01	-.34743592574+01	-.3474359257357683+01	LG10 5
.10000-01	-.2000000+01	-.20000000000+01	-.2000000000000000+01	LG10 6
.18316-01	-.1737169+01	-.17371693651+01	-.1737169365144340+01	LG10 7
.31250-01	-.1505150+01	-.15051499783+01	-.1505149978319906+01	LG10 8
.90909-01	-.1041393+01	-.10413931195+01	-.1041393119452924+01	LG10 9
.13534+00	-.8685738+00	-.86857382785+00	-.8685738278525498+00	LG10 10
.20000+00	-.6989700+00	-.69897000434+00	-.6989700043360188+00	LG10 11
.33333+00	-.4771256+00	-.47712559769+00	-.4771255976861963+00	LG10 12
.50000+00	-.3010300+00	-.30102999566+00	-.3010299956639812+00	LG10 13
.75000+00	-.1249387+00	-.12493873661+00	-.1249387366083000+00	LG10 14
.90000+00	-.4575749-01	-.45757490561-01	-.4575749056067513-01	LG10 15
.99000+00	-.4364805-02	-.43648054025-02	-.4364805402450085-02	LG10 16
.99900+00	-.4345118-03	-.43451177402-03	-.4345117740176913-03	LG10 17
.99990+00	-.4343162-04	-.43431619808-04	-.4343161980751038-04	LG10 18
.99999+00	-.4342967-05	-.43429665339-05	-.4342966533901379-05	LG10 19
.10000+01	.0000000	.00000000000	.0000000000000000	LG10 20
.10001+01	.4342728-04	.43427276863-04	.4342727686266964-04	LG10 21
.10010+01	.4340775-03	.43407747932-03	.4340774793186407-03	LG10 22
.10100+01	.4321374-02	.43213737826-02	.4321373782642574-02	LG10 23
.11000+01	.4139269-01	.41392685158-01	.4139268515822504-01	LG10 24
.15000+01	.1760913+00	.17609125906+00	.1760912590556812+00	LG10 25
.20000+01	.3010300+00	.30102999566+00	.3010299956639812+00	LG10 26
.27183+01	.4342974+00	.43429738512+00	.4342973851245086+00	LG10 27
.40000+01	.6020600+00	.60205999133+00	.6020599913279624+00	LG10 28
.80000+01	.9030900+00	.90308998699+00	.9030899869919436+00	LG10 29
.10000+02	.1000000+01	.10000000000+01	.1000000000000000+01	LG10 30
.16000+02	.1204120+01	.12041199827+01	.1204119982655925+01	LG10 31
.20086+02	.1302893+01	.13028934584+01	.1302893458356505+01	LG10 32
.64000+02	.1806180+01	.18061799740+01	.1806179973983887+01	LG10 33

Table 2 (cont.). Test data for the real functions.

x	g(x)			Ident.
.14841+03	.2171463+01	.21714631651+01	.2171463165083844+01	LG10 34
.10000+04	.3000000+01	.30000000000+01	.3000000000000000+01	LG10 35
.10000+05	.4000000+01	.40000000000+01	.4000000000000000+01	LG10 36
.59874+05	.4777238+01	.47772382730+01	.4777238273009532+01	LG10 37
.10000+11	.1000000+02	.10000000000+02	.1000000000000000+02	LG10 38
.10000+36	.3500000+02	.35000000000+02	.3500000000000000+02	LG10 39
-.10000+05	.3056144+00	.30561438889+00	.3056143888882521+00	SIN 1
-.10000+03	.5063656+00	.50636564111+00	.5063656411097588+00	SIN 2
-.50000+02	.2623749+00	.26237485370+00	.2623748537039288+00	SIN 3
-.16000+02	.2879033+00	.28790331667+00	.2879033166650653+00	SIN 4
-.80000+01	-.9893582+00	-.98935824662+00	-.9893582466233818+00	SIN 5
-.40000+01	.7568025+00	.75680249531+00	.7568024953079283+00	SIN 6
-.20000+01	-.9092974+00	-.90929742683+00	-.9092974268256817+00	SIN 7
-.50000+00	-.4794255+00	-.47942553860+00	-.4794255386042030+00	SIN 8
-.10000+00	-.9983342-01	-.99833416647-01	-.9983341664682815-01	SIN 9
-.10000-02	-.9999998-03	-.99999983333-03	-.9999998333333417-03	SIN 10
-.10000-04	-.1000000-04	-.99999999998-05	-.9999999999833333-05	SIN 11
-.10000-10	-.1000000-10	-.10000000000-10	-.1000000000000000-10	SIN 12
-.10000-24	-.1000000-24	-.10000000000-24	-.1000000000000000-24	SIN 13
.00000	.0000000	.00000000000	.0000000000000000	SIN 14
.10000-34	.1000000-34	.10000000000-34	.1000000000000000-34	SIN 15
.10000-15	.1000000-15	.10000000000-15	.1000000000000000-15	SIN 16
.10000-07	.1000000-07	.10000000000-07	.1000000000000000-07	SIN 17
.10000-03	.1000000-03	.99999999833-04	.9999999983333333-04	SIN 18
.10000-01	.9999833-02	.99998333342-02	.9999833334166665-02	SIN 19
.25000+00	.2474040+00	.24740395925+00	.2474039592545229+00	SIN 20
.26179+00	.2588100+00	.25880997717+00	.2588099771734655+00	SIN 21
.52360+00	.5000011+00	.50001060036+00	.5000010603626028+00	SIN 22
.75000+00	.6816388+00	.68163876002+00	.6816387600233342+00	SIN 23
.78539+00	.7071010+00	.70710100877+00	.7071010087692932+00	SIN 24
.10000+01	.8414710+00	.84147098481+00	.8414709848078965+00	SIN 25
.10472+01	.8660266+00	.86602662818+00	.8660266281835432+00	SIN 26
.13089+01	.9659007+00	.96590073209+00	.9659007320923110+00	SIN 27
.15708+01	.1000000+01	.99999999999+00	.9999999999932538+00	SIN 28
.18325+01	.9659506+00	.96595059462+00	.9659505946243037+00	SIN 29
.20944+01	.8660230+00	.86602295497+00	.8660229549706499+00	SIN 30
.23561+01	.7071736+00	.70717359269+00	.7071735926855460+00	SIN 31
.26180+01	.4999947+00	.49999469818+00	.4999946981757422+00	SIN 32
.28797+01	.2589091+00	.25890913181+00	.2589091318126125+00	SIN 33
.30000+01	.1411200+00	.14112000806+00	.1411200080598672+00	SIN 34
.31416+01	-.7346410-05	-.73464102067-05	-.7346410206695457-05	SIN 35
.34033+01	-.2587301+00	-.25873013885+00	-.2587301388516711+00	SIN 36
.36652+01	-.5000074+00	-.50000742252+00	-.5000074225224785+00	SIN 37
.39269+01	-.7070426+00	-.70704256096+00	-.7070425609630992+00	SIN 38
.41888+01	-.8660303+00	-.86603030135+00	-.8660303013496971+00	SIN 39
.44505+01	-.9659026+00	-.96590263414+00	-.9659026341449996+00	SIN 40
.47124+01	-.1000000+01	-.99999999994+00	-.9999999999392840+00	SIN 41
.49741+01	-.9659487+00	-.96594869389+00	-.9659486938865708+00	SIN 42
.50000+01	-.9589243+00	-.95892427466+00	-.9589242746631385+00	SIN 43
.52360+01	-.8660193+00	-.86601928171+00	-.8660192817110177+00	SIN 44
.54977+01	-.7071684+00	-.70716839846+00	-.7071683984608595+00	SIN 45
.57596+01	-.4999883+00	-.49998833596+00	-.4999883359618970+00	SIN 46
.60213+01	-.2589020+00	-.25890203590+00	-.2589020358956423+00	SIN 47
.62832+01	.1469282-04	.14692820413-04	.1469282041299443-04	SIN 48
.11000+02	-.9999902+00	-.99999020655+00	-.9999902065507035+00	SIN 49

Table 2 (cont.). Test data for the real functions.

x	g(x)			Ident.	
.25000+02	-.1323518+00	-.13235175010+00	-.1323517500977730+00	SIN	50
.75000+02	-.3877816+00	-.38778163541+00	-.3877816354094304+00	SIN	51
.10000+04	.8268795+00	.82687954053+00	.8268795405320026+00	SIN	52
.10000+06	.3574880-01	.35748797972-01	.3574879797201651-01	SIN	53
-.10000+05	-.9521554+00	-.95215536826+00	-.9521553682590149+00	COS	1
-.10000+03	.8623189+00	.86231887229+00	.8623188722876839+00	COS	2
-.50000+02	.9649660+00	.96496602849+00	.9649660284921133+00	COS	3
-.16000+02	-.9576595+00	-.95765948032+00	-.9576594803233846+00	COS	4
-.80000+01	-.1455000+00	-.14550003381+00	-.1455000338086135+00	COS	5
-.40000+01	-.6536436+00	-.65364362086+00	-.6536436208636119+00	COS	6
-.20000+01	-.4161468+00	-.41614683655+00	-.4161468365471424+00	COS	7
-.50000+00	.8775826+00	.87758256189+00	.8775825618903727+00	COS	8
-.10000+00	.9950042+00	.99500416528+00	.9950041652780258+00	COS	9
-.10000-02	.9999995+00	.99999950000+00	.9999995000000417+00	COS	10
-.10000-04	.1000000+01	.99999999995+00	.9999999999500000+00	COS	11
-.10000-10	.1000000+01	.10000000000+01	.1000000000000000+01	COS	12
-.10000-24	.1000000+01	.10000000000+01	.1000000000000000+01	COS	13
.00000	.1000000+01	.10000000000+01	.1000000000000000+01	COS	14
.10000-34	.1000000+01	.10000000000+01	.1000000000000000+01	COS	15
.10000-15	.1000000+01	.10000000000+01	.1000000000000000+01	COS	16
.10000-07	.1000000+01	.10000000000+01	.1000000000000000+01	COS	17
.10000-03	.1000000+01	.99999999500+00	.9999999950000000+00	COS	18
.10000-01	.9999500+00	.99995000042+00	.9999500004166653+00	COS	19
.25000+00	.9689124+00	.96891242171+00	.9689124217106448+00	COS	20
.26179+00	.9659283+00	.96592825599+00	.9659282559877158+00	COS	21
.52360+00	.8660248+00	.86602479158+00	.8660247915829389+00	COS	22
.75000+00	.7316889+00	.73168886887+00	.7316888688738209+00	COS	23
.78539+00	.7071126+00	.70711255356+00	.7071125535566795+00	COS	24
.10000+01	.5403023+00	.54030230587+00	.5403023058681397+00	COS	25
.10472+01	.4999979+00	.49999787927+00	.4999978792725456+00	COS	26
.13089+01	.2589127+00	.25891267977+00	.2589126797658577+00	COS	27
.15708+01	-.3673205-05	-.36732051034-05	-.3673205103372509-05	COS	28
.18325+01	-.2587266+00	-.25872659072+00	-.2587265907187627+00	COS	29
.20944+01	-.5000042+00	-.50000424145+00	-.5000042414459138+00	COS	30
.23561+01	-.7070400+00	-.70703996337+00	-.7070399633742194+00	COS	31
.26180+01	-.8660285+00	-.86602846477+00	-.8660284647724626+00	COS	32
.28797+01	-.9659017+00	-.96590168313+00	-.9659016831251715+00	COS	33
.30000+01	-.9899925+00	-.98999249660+00	-.9899924966004455+00	COS	34
.31416+01	-.1000000+01	-.99999999997+00	-.9999999999730151+00	COS	35
.34033+01	-.9659496+00	-.96594964426+00	-.9659496442619537+00	COS	36
.36652+01	-.8660211+00	-.86602111835+00	-.8660211183466762+00	COS	37
.39269+01	-.7071710+00	-.70717099558+00	-.7071709955779735+00	COS	38
.41888+01	-.4999915+00	-.49999151707+00	-.4999915170721927+00	COS	39
.44505+01	-.2589056+00	-.25890558386+00	-.2589055838558741+00	COS	40
.47124+01	.1101962-04	.11019615310-04	.1101961530991928-04	COS	41
.49741+01	.2587337+00	.25873368698+00	.2587336869810885+00	COS	42
.50000+01	.2836622+00	.28366218546+00	.2836621854632263+00	COS	43
.52360+01	.5000106+00	.50001060359+00	.5000106035922969+00	COS	44
.54977+01	.7070452+00	.70704515854+00	.7070451585424392+00	COS	45
.57596+01	.8660321+00	.86603213792+00	.8660321379152469+00	COS	46
.60213+01	.9659036+00	.96590358515+00	.9659035851517953+00	COS	47
.62832+01	.1000000+01	.99999999989+00	.9999999998920605+00	COS	48
.11000+02	.4425698-02	.44256979881-02	.4425697988050786-02	COS	49
.25000+02	.9912028+00	.99120281186+00	.9912028118634736+00	COS	50
.75000+02	.9217513+00	.92175126972+00	.9217512697247493+00	COS	51

Table 2 (cont.). Test data for the real functions.

x	g(x)			Ident.	
.10000+04	.5623791+00	.56237907629+00	.5623790762907030+00	COS	52
.10000+06	-.9993608+00	-.99936080744+00	-.9993608074382125+00	COS	53
-.10000+05	-.3209711+00	-.32097113462+00	-.3209711346238147+00	TAN	1
-.10000+03	.5872139+00	.58721391516+00	.5872139151569291+00	TAN	2
-.50000+02	.2719006+00	.27190061200+00	.2719006119976307+00	TAN	3
-.16000+02	-.3006322+00	-.30063224202+00	-.3006322420239034+00	TAN	4
-.80000+01	.6799711+01	.67997114552+01	.6799711455220379+01	TAN	5
-.40000+01	-.1157821+01	-.11578212823+01	-.1157821282349578+01	TAN	6
-.20000+01	.2185040+01	.21850398633+01	.2185039863261519+01	TAN	7
-.50000+00	-.5463025+00	-.54630248984+00	-.5463024898437905+00	TAN	8
-.10000+00	-.1003347+00	-.10033467209+00	-.1003346720854505+00	TAN	9
-.10000-02	-.1000000-02	-.10000003333-02	-.1000000333333467-02	TAN	10
-.10000-04	-.1000000-04	-.10000000000-04	-.1000000000033333-04	TAN	11
-.10000-10	-.1000000-10	-.10000000000-10	-.1000000000000000-10	TAN	12
-.10000-24	-.1000000-24	-.10000000000-24	-.1000000000000000-24	TAN	13
.00000	.0000000	.00000000000	.0000000000000000	TAN	14
.10000-34	.1000000-34	.10000000000-34	.1000000000000000-34	TAN	15
.10000-15	.1000000-15	.10000000000-15	.1000000000000000-15	TAN	16
.10000-07	.1000000-07	.10000000000-07	.1000000000000000-07	TAN	17
.10000-03	.1000000-03	.10000000033-03	.1000000003333333-03	TAN	18
.10000-01	.1000033-01	.10000333347-01	.1000033334666721-01	TAN	19
.25000+00	.2553419+00	.25534192122+00	.2553419212210363+00	TAN	20
.26179+00	.2679391+00	.26793913064+00	.2679391306436292+00	TAN	21
.52360+00	.5773519+00	.57735190173+00	.5773519017263813+00	TAN	22
.75000+00	.9315965+00	.93159645994+00	.9315964599440725+00	TAN	23
.78539+00	.9999837+00	.99998367334+00	.9999836733383840+00	TAN	24
.10000+01	.1557408+01	.15574077247+01	.1557407724654902+01	TAN	25
.10472+01	.1732061+01	.17320606028+01	.1732060602824032+01	TAN	26
.13089+01	.3730604+01	.37306042059+01	.3730604205888268+01	TAN	27
.15708+01	-.2722418+06	-.27224180841+06	-.2722418084073541+06	TAN	28
.18325+01	-.3733480+01	-.37334801651+01	-.3733480165068527+01	TAN	29
.20944+01	-.1732031+01	-.17320312173+01	-.1732031217307841+01	TAN	30
.23561+01	-.1000189+01	-.10001889982+01	-.1000188998243733+01	TAN	31
.26180+01	-.5773421+00	-.57734210654+00	-.5773421065404694+00	TAN	32
.28797+01	-.2680492+00	-.26804915690+00	-.2680491569027118+00	TAN	33
.30000+01	-.1425465+00	-.14254654307+00	-.1425465430742778+00	TAN	34
.31416+01	.7346410-05	.73464102069-05	.7346410206893699-05	TAN	35
.34033+01	.2678505+00	.26785054520+00	.2678505452003734+00	TAN	36
.36652+01	.5773617+00	.57736169700+00	.5773616969953855+00	TAN	37
.39269+01	.9998184+00	.99981838252+00	.9998183825189700+00	TAN	38
.41888+01	.1732090+01	.17320899891+01	.1732089989088061+01	TAN	39
.44505+01	.3730714+01	.37307137983+01	.3730713798288306+01	TAN	40
.47124+01	-.9074727+05	-.90747269466+05	-.9074726946585296+05	TAN	41
.49741+01	-.3733370+01	-.37333704210+01	-.3733370420981070+01	TAN	42
.50000+01	-.3380515+01	-.33805150062+01	-.3380515006246586+01	TAN	43
.52360+01	-.1732002+01	-.17320018325+01	-.1732001832539456+01	TAN	44
.54977+01	-.1000174+01	-.10001743028+01	-.1000174302754119+01	TAN	45
.57596+01	-.5773323+00	-.57733231144+00	-.5773323114376475+00	TAN	46
.60213+01	-.2680413+00	-.26804128267+00	-.2680412826658625+00	TAN	47
.62832+01	.1469282-04	.14692820415-04	.1469282041458037-04	TAN	48
.11000+02	-.2259508+03	-.22595084645+03	-.2259508464541951+03	TAN	49
.25000+02	-.1335264+00	-.13352640702+00	-.1335264070215359+00	TAN	50
.75000+02	-.4207010+00	-.42070095062+00	-.4207009506211244+00	TAN	51
.10000+04	.1470324+01	.14703241557+01	.1470324155702718+01	TAN	52
.10000+06	-.3577166-01	-.35771662953-01	-.3577166295289877-01	TAN	53

x	g(x)			Ident.
-.10000+01	-.1570796+01	-.15707963268+01	-.1570796326794897+01	ASIN 1
-.99990+00	-.1556654+01	-.15566540733+01	-.1556654073317384+01	ASIN 2
-.99000+00	-.1429257+01	-.14292568535+01	-.1429256853470469+01	ASIN 3
-.80000+00	-.9272952+00	-.92729521800+00	-.9272952180016122+00	ASIN 4
-.60000+00	-.6435011+00	-.64350110879+00	-.6435011087932844+00	ASIN 5
-.40000+00	-.4115168+00	-.41151684607+00	-.4115168460674880+00	ASIN 6
-.20000+00	-.2013579+00	-.20135792079+00	-.2013579207903308+00	ASIN 7
-.10000-01	-.1000017-01	-.10000166674-01	-.1000016667416711-01	ASIN 8
-.10000-03	-.1000000-03	-.10000000017-03	-.1000000001666667-03	ASIN 9
-.10000-07	-.1000000-07	-.10000000000-07	-.1000000000000000-07	ASIN 10
-.10000-15	-.1000000-15	-.10000000000-15	-.1000000000000000-15	ASIN 11
-.10000-34	-.1000000-34	-.10000000000-34	-.1000000000000000-34	ASIN 12
.00000	.0000000	.00000000000	.0000000000000000	ASIN 13
.10000-24	.1000000-24	.10000000000-24	.1000000000000000-24	ASIN 14
.10000-10	.1000000-10	.10000000000-10	.1000000000000000-10	ASIN 15
.10000-04	.1000000-04	.10000000000-04	.1000000000016667-04	ASIN 16
.10000-02	.1000000-02	.10000001667-02	.1000000166666742-02	ASIN 17
.10000+00	.1001674+00	.10016742116+00	.1001674211615598+00	ASIN 18
.30000+00	.3046927+00	.30469265402+00	.3046926540153975+00	ASIN 19
.50000+00	.5235988+00	.52359877560+00	.5235987755982989+00	ASIN 20
.70000+00	.7753975+00	.77539749661+00	.7753974966107531+00	ASIN 21
.90000+00	.1119770+01	.11197695150+01	.1119769514998634+01	ASIN 22
.99900+00	.1526071+01	.15260712396+01	.1526071239626163+01	ASIN 23
.99999+00	.1566324+01	.15663241871+01	.1566324187113109+01	ASIN 24
-.10000+01	.3141593+01	.31415926536+01	.3141592653589793+01	ACOS 1
-.99990+00	.3127450+01	.31274504001+01	.3127450400112280+01	ACOS 2
-.99000+00	.3000053+01	.30000531803+01	.3000053180265366+01	ACOS 3
-.80000+00	.2498092+01	.24980915448+01	.2498091544796509+01	ACOS 4
-.60000+00	.2214297+01	.22142974356+01	.2214297435588181+01	ACOS 5
-.40000+00	.1982313+01	.19823131729+01	.1982313172862385+01	ACOS 6
-.20000+00	.1772154+01	.17721542476+01	.1772154247585227+01	ACOS 7
-.10000-01	.1580796+01	.15807964935+01	.1580796493469064+01	ACOS 8
-.10000-03	.1570896+01	.15708963268+01	.1570896326795063+01	ACOS 9
-.10000-07	.1570796+01	.15707963368+01	.1570796336794897+01	ACOS 10
-.10000-15	.1570796+01	.15707963268+01	.1570796326794897+01	ACOS 11
-.10000-34	.1570796+01	.15707963268+01	.1570796326794897+01	ACOS 12
.00000	.1570796+01	.15707963268+01	.1570796326794897+01	ACOS 13
.10000-24	.1570796+01	.15707963268+01	.1570796326794897+01	ACOS 14
.10000-10	.1570796+01	.15707963268+01	.1570796326784897+01	ACOS 15
.10000-04	.1570786+01	.15707863268+01	.1570786326794896+01	ACOS 16
.10000-02	.1569796+01	.15697963266+01	.1569796326628230+01	ACOS 17
.10000+00	.1470629+01	.14706289056+01	.1470628905633337+01	ACOS 18
.30000+00	.1266104+01	.12661036728+01	.1266103672779499+01	ACOS 19
.50000+00	.1047198+01	.10471975512+01	.1047197551196598+01	ACOS 20
.70000+00	.7953988+00	.79539883018+00	.7953988301841436+00	ACOS 21
.90000+00	.4510268+00	.45102681180+00	.4510268117962624+00	ACOS 22
.99900+00	.4472509-01	.44725087169-01	.4472508716873343-01	ACOS 23
.99999+00	.4472140-02	.44721396818-02	.4472139681787927-02	ACOS 24
-.10000+26	-.1570796+01	-.15707963268+01	-.1570796326794897+01	ATAN 1
-.10000+03	-.1560797+01	-.15607966601+01	-.1560796660108231+01	ATAN 2
-.85000+02	-.1559032+01	-.15590321636+01	-.1559032163645138+01	ATAN 3
-.50000+02	-.1550799+01	-.15507989928+01	-.1550798992821746+01	ATAN 4
-.16000+02	-.1508378+01	-.15083775168+01	-.1508377516798939+01	ATAN 5
-.80000+01	-.1446441+01	-.14464413322+01	-.1446441332248135+01	ATAN 6
-.40000+01	-.1325818+01	-.13258176637+01	-.1325817663668032+01	ATAN 7

x	g(x)			Ident.
-.20000+01	-.1107149+01	-.11071487178+01	-.1107148717794091+01	ATAN 8
-.80000+00	-.6747409+00	-.67474094222+00	-.6747409422235527+00	ATAN 9
-.41421+00	-.3926960+00	-.39269604102+00	-.3926960410192606+00	ATAN 10
-.30000+00	-.2914568+00	-.29145679448+00	-.2914567944778671+00	ATAN 11
-.19891+00	-.1963473+00	-.19634726357+00	-.1963472635716957+00	ATAN 12
-.10000+00	-.9966865-01	-.99668652491-01	-.9966865249116203-01	ATAN 13
-.10000-02	-.9999997-03	-.99999966667-03	-.9999996666668667-03	ATAN 14
-.10000-04	-.1000000-04	-.99999999997-05	-.9999999999666667-05	ATAN 15
-.10000-10	-.1000000-10	-.10000000000-10	-.1000000000000000-10	ATAN 16
-.10000-24	-.1000000-24	-.10000000000-24	-.1000000000000000-24	ATAN 17
.00000	.0000000	.00000000000	.0000000000000000	ATAN 18
.10000-34	.1000000-34	.10000000000-34	.1000000000000000-34	ATAN 19
.10000-15	.1000000-15	.10000000000-15	.1000000000000000-15	ATAN 20
.10000-07	.1000000-07	.10000000000-07	.1000000000000000-07	ATAN 21
.10000-03	.1000000-03	.99999999667-04	.9999999966666667-04	ATAN 22
.10000-01	.9999667-02	.99996666867-02	.9999666686665238-02	ATAN 23
.98491-01	.9817437-01	.98174370943-01	.9817437094269847-01	ATAN 24
.20000+00	.1973956+00	.19739555985+00	.1973955598498808+00	ATAN 25
.26795+00	.2618001+00	.26180014127+00	.2618001412710175+00	ATAN 26
.50000+00	.4636476+00	.46364760900+00	.4636476090008061+00	ATAN 27
.10000+01	.7853982+00	.78539816340+00	.7853981633974483+00	ATAN 28
.30000+01	.1249046+01	.12490457724+01	.1249045772398254+01	ATAN 29
.50000+01	.1373401+01	.13734007669+01	.1373400766945016+01	ATAN 30
.11000+02	.1480136+01	.14801364396+01	.1480136439594152+01	ATAN 31
.25000+02	.1530818+01	.15308176397+01	.1530817639671607+01	ATAN 32
.75000+02	.1557464+01	.15574637835+01	.1557463783500751+01	ATAN 33
.10000+11	.1570796+01	.15707963267+01	.1570796326694897+01	ATAN 34
.10000+36	.1570796+01	.15707963268+01	.1570796326794897+01	ATAN 35

Table 2 (cont.). Test data for the real functions.

x_1	x_2	$g(x_1 + ix_2)$		Ident.
.00000	.00000	.1000000+01	.0000000	CEXP 1
.10000-34	.00000	.1000000+01	.0000000	CEXP 2
.00000	.10000-34	.1000000+01	.1000000-34	CEXP 3
-.10000-34	.10000-34	.1000000+01	.1000000-34	CEXP 4
-.10000+01	.10000-34	.3678794+00	.3678794-35	CEXP 5
-.10000+01	-.10000+01	.1987661+00	-.3095599+00	CEXP 6
-.10000-34	-.10000+01	.5403023+00	-.8414710+00	CEXP 7
.85000+02	-.10000+01	.4442913+37	-.6919427+37	CEXP 8
.85000+02	.85000+02	-.8094542+37	-.1447872+37	CEXP 9
.10000+01	.85000+02	-.2675813+01	-.4786232+00	CEXP 10
.00000	.00000	.0000000	.0000000	CSQR 1
.10000+01	.00000	.1000000+01	.0000000	CSQR 2
.10000+01	.10000-34	.1000000+01	.5000000-35	CSQR 3
.10000+01	.10000-04	.1000000+01	.5000000-05	CSQR 4
.10000+01	.50000+00	.1029086+01	.2429341+00	CSQR 5
.10000+01	.10000+01	.1098684+01	.4550899+00	CSQR 6
.10000+01	.20000+01	.1272020+01	.7861514+00	CSQR 7
.10000+01	.10000+06	.2236079+03	.2236057+03	CSQR 8
.10000+01	.10000+36	.2236068+18	.2236068+18	CSQR 9
.10000-34	.10000+36	.2236068+18	.2236068+18	CSQR 10
.00000	.10000+01	.7071068+00	.7071068+00	CSQR 11
-.10000+01	.10000+01	.4550899+00	.1098684+01	CSQR 12
-.10000+01	.00000	.0000000	.1000000+01	CSQR 13
-.10000+01	-.10000+01	.4550899+00	-.1098684+01	CSQR 14
.00000	-.10000+01	.7071068+00	-.7071068+00	CSQR 15
.10000+01	-.10000+01	.1098684+01	-.4550899+00	CSQR 16
.10000+01	.00000	.0000000	.0000000	CLOG 1
.10000+01	.10000-04	.5000000-10	.1000000-04	CLOG 2
.10000+01	.50000+00	.1115718+00	.4636476+00	CLOG 3
.10000+01	.10000+01	.3465736+00	.7853982+00	CLOG 4
.10000+01	.20000+01	.8047190+00	.1107149+01	CLOG 5
.10000+01	.10000+06	.1151293+02	.1570786+01	CLOG 6
.10000+01	.10000+36	.8059048+02	.1570796+01	CLOG 7
.10000-34	.10000+36	.8059048+02	.1570796+01	CLOG 8
.00000	.10000+01	.0000000	.1570796+01	CLOG 9
-.10000+01	.10000+01	.3465736+00	.2356194+01	CLOG 10
-.10000+01	.00000	.0000000	.3141593+01	CLOG 11
-.10000+01	-.10000+01	.3465736+00	-.2356194+01	CLOG 12
.00000	-.10000+01	.0000000	-.1570796+01	CLOG 13
.10000+01	-.10000+01	.3465736+00	-.7853982+00	CLOG 14
.00000	.00000	.0000000	.0000000	CSIN 1
.10000-34	.00000	.1000000-34	.0000000	CSIN 2
.00000	.10000-34	.0000000	.1000000-34	CSIN 3
-.10000-34	.10000-34	-.1000000-34	.1000000-34	CSIN 4
-.10000+01	.10000-34	-.8414710+00	.5403023-35	CSIN 5
-.10000+01	-.10000+01	-.1298458+01	-.6349639+00	CSIN 6
-.10000-34	-.10000+01	-.1543081-34	-.1175201+01	CSIN 7
.85000+02	-.10000+01	-.2716989+00	.1156841+01	CSIN 8
.85000+02	.85000+02	-.7239360+36	-.4047271+37	CSIN 9
.10000+01	.85000+02	.3459713+37	.2221456+37	CSIN 10

50:

Table 3. Test data for the complex functions.

x_1	x_2	$g(x_1 + ix_2)$		Ident.
.00000	.00000	.1000000+01	.0000000	CCOS 1
.10000-34	.00000	.1000000+01	.0000000	CCOS 2
.00000	.10000-34	.1000000+01	.0000000	CCOS 3
-.10000+01	.10000-34	.5403023+00	.8414710-35	CCOS 4
-.10000+01	-.10000+01	.8337300+00	-.9888977+00	CCOS 5
-.10000-34	-.10000+01	.1543081+01	-.1175201-34	CCOS 6
.85000+02	-.10000+01	-.1518973+01	-.2069243+00	CCOS 7
.85000+02	.85000+02	-.4047271+37	.7239360+36	CCOS 8
.10000+01	.85000+02	.2221456+37	-.3459713+37	CCOS 9

59:

Table 3 (cont.). Test data for the complex functions.

x_1	x_2	$g(x_1,x_2)$			Ident.	
.00000	.00000	.0000000	.00000000000	.0000000000000000	CABS	1
.10000+01	.00000	.1000000+01	.10000000000+01	.1000000000000000+01	CABS	2
.10000+01	.10000-34	.1000000+01	.10000000000+01	.1000000000000000+01	CABS	3
.10000+01	.10000-04	.1000000+01	.10000000000+01	.1000000000050000+01	CABS	4
.10000+01	.50000+00	.1118034+01	.11180339887+01	.1118033988749895+01	CABS	5
.10000+01	.10000+01	.1414214+01	.14142135624+01	.1414213562373095+01	CABS	6
.10000+01	.20000+01	.2236068+01	.22360679775+01	.2236067977499790+01	CABS	7
.10000+01	.10000+06	.1000000+06	.10000000000+06	.1000000000050000+06	CABS	8
.10000+01	.10000+36	.1000000+36	.10000000000+36	.1000000000000000+36	CABS	9
.10000-34	.10000+36	.1000000+36	.10000000000+36	.1000000000000000+36	CABS	10
.00000	.10000+01	.1000000+01	.10000000000+01	.1000000000000000+01	CABS	11
-.10000+01	.10000+01	.1414214+01	.14142135624+01	.1414213562373095+01	CABS	12
-.10000+01	.00000	.1000000+01	.10000000000+01	.1000000000000000+01	CABS	13
-.10000+01	-.10000+01	.1414214+01	.14142135624+01	.1414213562373095+01	CABS	14
.00000	-.10000+01	.1000000+01	.10000000000+01	.1000000000000000+01	CABS	15
.10000+01	-.10000+01	.1414214+01	.14142135624+01	.1414213562373095+01	CABS	16
.10000+01	.00000	.1570796+01	.15707963268+01	.1570796326794897+01	ATN2	1
.10000+01	.10000-34	.1570796+01	.15707963268+01	.1570796326794897+01	ATN2	2
.10000+01	.10000-04	.1570786+01	.15707863268+01	.1570786326794897+01	ATN2	3
.10000+01	.50000+00	.1107149+01	.11071487178+01	.1107148717794091+01	ATN2	4
.10000+01	.10000+01	.7853982+00	.78539816340+00	.7853981633974483+00	ATN2	5
.10000+01	.20000+01	.4636476+00	.46364760900+00	.4636476090008061+00	ATN2	6
.10000+01	.10000+06	.1000000-04	.99999999997-05	.9999999999666667-05	ATN2	7
.10000+01	.10000+36	.1000000-34	.10000000000-34	.1000000000000000-34	ATN2	8
.00000	.10000+01	.0000000	.00000000000	.0000000000000000	ATN2	9
-.10000+01	.10000+01	-.7853982+00	-.78539816340+00	-.7853981633974483+00	ATN2	10
-.10000+01	.00000	-.1570796+01	-.15707963268+01	-.1570796326794897+01	ATN2	11
-.10000+01	-.10000+01	-.2356194+01	-.23561944902+01	-.2356194490192345+01	ATN2	12
.00000	-.10000+01	.3141593+01	.31415926536+01	.3141592653589793+01	ATN2	13
.10000+01	-.10000+01	.2356194+01	.23561944902+01	.2356194490192345+01	ATN2	14

30:

Table 4. Test data for $g(x_1,x_2) = |x_1 + ix_2|$ and $g(x_1,x_2) = \arctan (x_1,x_2)$.

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16. ABSTRACT *(A 200-word or less factual summary of most significant information. If document includes a significant bibliography or literature survey, mention it here.)*

A short table of values of 20 mathematical functions commonly found in computer libraries is given. The data was chosen to sample the functions throughout the range of a typical floating-point arithmetic. Computer implementations can be tested by comparing computed function values against the tabular values. A discussion on how to interpret the test results is included, for both decimal and non-decimal implementations.

17. KEY WORDS *(six to twelve entries; alphabetical order; capitalize only the first letter of the first key word unless a proper name; separated by semicolons)*
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