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ELECTROMAGNETIC REMOTE SENSING OF INHOMOGENEOUS MEDIA

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CONTENTS

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This report deals with the electromagnetic response of inhomogeneous dielectrics; i.e., media whose permittivity is a function of depth. The resulting boundary value problem is solved for a large number of permittivity functions which can model almost any medium of interest.

Since those permittivity profiles are characterized by only a few parameters, they are particularly useful for the inverse problem; i.e., the retrieval of profiles from the measured electromagnetic response.

It is shown how the non-uniformity of the permittivity changes the response and how the change is related to the profile characteristics.

Key words: Inhomogeneous dielectrics; profile inversion.

1. INTRODUCTION

With the advent of more accurate electromagnetic measurement techniques, especially over large frequency bands, the interest in non-destructive testing methods has considerably increased. A host of literature is available on theory [1,2]* as well as on applications [3,4,5] which range from atmospheric sounding to geologic explorations.

The problem of obtaining information about an object using electromagnetic means is one in the theory of inverse scattering. Unlike a conventional radar which determines the position and shape of a conducting object from the return signal, the kind of remote probing technique we are interested in concerns the reconstruction of the electrical profile of vertically-stratified dielectrics from information obtained by illuminating them with electromagnetic waves.

Once the scatter-field is successfully "inverted," the electrical properties of the medium which caused it become known. More specifically, we are interested in the determination of the moisture profile of grain as stored in bins, trucks, etc. Since the moisture changes the grain's effective permittivity and conductivity, we therefore attempt to measure the electrical properties of grain using microwave signals. For the particular application we have in mind, the pile of grain is generally assumed to be several meters on each side, and the operating frequency is typically in the L- and S-band. This means that the unknown medium is basically a dielectric material at these frequencies, with a loss tangent in the neighborhood of 0.1. Hence, the remote-sensing method for this problem is substantially different from the magneto-telluric techniques where only the conductivity profile is considered [4]. Indeed, in this report we have only treated the case of dissipationless dielectrics. The extension to lossy dielectrics for which the ratio of the conductivity profile to the permittivity profile is independent of depth will be dealt with in a later report.

*Figures in brackets indicate the literature reference at the end of this paper.

A number of theories dealing with this kind of remote measurement have been developed. For example:

1. the multilayer approach [6] in which one views the medium as a system of uniform transmission lines, connected in series;
2. the Parameter Optimization Method [5] where one approximates permittivity profiles by polynomials whose coefficients are obtained from an optimization process involving the numerical integration of the wave-equation; and
3. the Synthesis approach [7] which is based on the assumption that the input admittance is a rational function of frequency.

For details on these and other methods see the collection [1] which also contains an extensive bibliography on the subject. The implementation of these theories, however, usually requires extensive computer facilities. Yet, in many remote sensing problems it is desired to process all measurement data in situ in order to obtain the required information immediately -- thus imposing severe limitations on the available machinery. Our objective is then to develop an optimized microwave sensing method which can describe the profile of an unknown medium with sufficient accuracy without employing any elaborate numerical computation schemes. In this report we consider mainly the "forward" problem of determining the reflection coefficient and the surface admittance of a plane wave as a result of electromagnetic interference with a vertically stratified dielectric. A new exact solution for a large class of inhomogeneous media is presented and compared with previously obtained results. While we shall discuss a conceptually attractive approach to the remote-sensing problem, the actual inversion based upon this analysis will be given in our next report.

2. THE REFLECTION OF WAVES BY INHOMOGENEOUS MEDIA

Generally speaking, the determination of the reflection of plane waves by media whose permittivity is a known function of depth is directly related to the problem of finding a solution to the general linear second order differential equation with two boundary values which incidentally is also encountered in the theory of non-uniform transmission lines [8], quantum mechanics [9], and other disciplines of physics.

In the special case of a normally incident linearly polarized plane wave, the electric field inside the medium must satisfy the well-known wave equation in its time-harmonic form [3]:

$$\left(\frac{d^2}{dx^2} + k^2\varepsilon_r(x)\right)E_y = 0, \qquad 0 < x < 1, \tag{1}$$

where x is the normalized depth, $\varepsilon_r(x)$ is the relative permittivity of the medium, $k = k_oD$ (with D being the total depth) is the normalized wave number, and $k_o = \omega\sqrt{\mu_o\varepsilon_o}$ is the free-space propagation constant. The time dependence factor $\exp(j\omega t)$ is suppressed; ω being the operating angular frequency. The geometry of the problem is depicted in figure 1. A number of exact solutions have been derived when $\varepsilon_r(x)$ is of a special form. For example, when the permittivity obeys an

$$\text{inverse square law:} \quad \varepsilon_r(x) = a^2 - \frac{b^2}{(x+x_o)^2} , \tag{2}$$

$$\text{exponential law:} \quad \varepsilon_r(x) = (a^2-b^2)e^{-\alpha x} + b^2, \tag{3}$$

or a

$$\text{power law:} \quad \varepsilon_r(x) = a(1 + \frac{x}{b})^{-\alpha} . \tag{4}$$

Where a, b, and α are constants, the resulting equation (1) may be transformed into Bessel's differential equation. Similarly, if the permittivity varies linearly, (1) transforms into Stokes' differential equation. A detailed discussion of these results may be found in [6]. We note, however, that these profiles are monotonic with depth, hence are not useful to model for instance a medium containing a denser slab between to less dense layers.

Epstein and others [10,11] have transformed (1) into the hypergeometric differential equation and obtained field solutions for a much larger class of profiles; namely, the so-called Epstein profiles:

$$\varepsilon_r(x) = a + \frac{e^{\xi}}{(e^{\xi}+1)} [(\varepsilon_2-\varepsilon_1)(e^{\xi}+1) + \varepsilon_3]; \quad \xi = kx \tag{5}$$

where a, ε_1, ε_2, and ε_3 are adjustable parameters. However, a careful examination shows that except in trivial cases these parameters are not independent of frequency. Consequently, they are not suitable to model inhomogeneous media in any remote-sensing method which is based upon frequency scanning. Therefore, it is the purpose of this report to develop exact solutions of the wave-equation within the framework of Epstein's, but with profiles characterized by genuine constants; i.e., do not vary with the operating frequency.

3. TRANSFORMATION OF THE HYPERGEOMETRIC DIFFERENTIAL EQUATION INTO THE WAVE EQUATION

Starting from the hypergeometric differential equation (henceforth abbreviated as hg. DE)

$$\theta(1-\theta) \frac{d^2\Pi}{d\theta^2} + \{c-(a+b+1)\theta\} \frac{d\Pi}{d\theta} - ab\Pi = 0 , \tag{6}$$

one obtains upon transforming the independent variable $\theta = f(x)$, where f is any meromorphic function, an alternative differential equation of the form

$$\frac{d^2\Pi}{dx^2} + A \frac{d\Pi}{dx} - B\Pi = 0 \tag{7}$$

with

$$A = \frac{(c-1)-(a+b)f}{f(1-f)} f' + \left(\frac{f'}{f} - \frac{f''}{f'}\right) \text{ and}$$

$$B = ab \frac{f'^2}{f(1-f)} \tag{8}$$

where from now on " ' " denotes the derivative with respect to x.

Now let $\Pi = EG$, where E will play the role of the electric field E_y in our model and G is a function of x yet to be determined. Then (7) transforms into

$$E'' + \left(2\frac{G'}{G} + A\right)E' + \left(\frac{G''}{G} + \frac{G'}{G}A - B\right)E = 0 \ . \tag{9}$$

Forcing the linear term to vanish, one obtains the equation

$$\frac{G'}{G} = -\frac{A}{2} = -\frac{1}{2}\left[\frac{(c-1)-(a+b)f}{f(1-f)} f' + \left(\frac{f'}{f} - \frac{f''}{f'}\right)\right] \tag{10}$$

which determines G(x) in terms of f(x) modulo an integration constant. Also,

$$\frac{G''}{G} = \frac{A^2}{4} - \frac{A'}{2}; \quad A' = \frac{f(1-f)f'' - f'^2(1-2f)}{f^2(1-f)^2}\{(c-1)-(a+b)f\} -$$

$$- \frac{f'^2}{f(1-f)}(a+b) + \left(\frac{f'}{f} - \frac{f''}{f'}\right)' \ . \tag{11}$$

If now one defines

$$\delta = \frac{G''}{G} + \frac{G'}{G}A = -\left(\frac{A^2}{4} + \frac{A'}{2}\right) \tag{12}$$

or equivalently,

$$-4\delta = A^2+2A'$$

$$= \left(\frac{(c-1)-(a+b)f}{f(1-f)} f'\right)^2 + 2\,\frac{(c-1)-(a+b)f}{f(1-f)} f' \left(\frac{f'}{f} - \frac{f''}{f'}\right) + \left(\frac{f'}{f} - \frac{f''}{f'}\right)^2 +$$

$$+ 2\,\frac{f(1-f)f'' - f'^2(1-2f)}{f^2(1-f)^2}\{(c-1)-(a+b)f\} -$$

$$- 2(a+b)\frac{f'^2}{f(1-f)} + 2\left(\frac{f'}{f} - \frac{f''}{f'}\right)' \tag{13}$$

Then the function $E(x) = \Pi(\theta(x))/G(x)$ satisfies the differential equation

$$\left(\frac{d^2}{dx^2} + \Delta(x)\right)E = 0 \tag{14}$$

with

$$\Delta(x) = \delta(x) - B(x). \tag{15}$$

Now (14) must be identified with the wave-equation (1). When the inhomogeneous medium is lossless, ε_r is only a function of depth, but not of frequency. On the other hand, k_o is proportional to frequency, and since (14) is intended to be used for many different frequencies, the following conditions on $\Delta(x)$ must be imposed:

(I) $\Delta(x)$ must have a factor which can be identified with k_o^2

(II) $\dfrac{\Delta(x)}{k_o^2}$ must be a function of x only

(III) $\Delta(x) > 0$ for $0 < x < 1$.

It is easily seen that (I) and (II) are satisfied if and only if

$$c-1 = (k_oD)\hat{c}, \quad a = (k_oD)\hat{a}, \quad b = (k_oD)\hat{b} \tag{16}$$

where D is the total depth of the medium, and $\hat{c}$, $\hat{a}$, $\hat{b}$ are independent of frequency and therefore may be extracted from data at different frequencies containing them; i.e., may be "sensed" from surface impedance data at frequencies f_o, f_1, ... f_N.

Furthermore, to satisfy (II), all terms not containing k_o^2 must be eliminated by imposing conditions on f(x), which clearly must not contain k_o^2. This can be achieved by setting the terms in (13) which are linear in (k_oD) and the terms which are constant with respect to k_oD to zero; i.e.,

$$2\{(c-1)-(a+b)f\} \frac{1}{f^2(1-f)^2} [f'^2(1-f) - f''f(1-f) + ff''(1-f) - f'^2(1-2f)] -$$

$$- 2(a+b) \frac{f'^2}{f(1-f)} + \left(\frac{f'}{f} - \frac{f''}{f'}\right)^2 + 2\left(\frac{f'}{f} - \frac{f''}{f'}\right)' = 0 \,. \tag{17}$$

After some manipulations one arrives finally at the following condition for f(x):

$$\frac{f'^2}{f(1-f)^2} \{(c-1)-(a+b)\} + \frac{1}{2}\left(\frac{f'}{f} - \frac{f''}{f'}\right)^2 + \left(\frac{f'}{f} - \frac{f''}{f'}\right)' = 0 \,. \tag{18}$$

Since f(x) must be independent of k_o, (18) is equivalent to the condition

$$(c-1)-(a+b) = 0 \text{ or } \underline{\hat{c} - \hat{a} - \hat{b} = 0} \tag{19}$$

plus the condition

$$\frac{1}{2}\left(\frac{f'}{f} - \frac{f''}{f'}\right)^2 + \left(\frac{f'}{f} - \frac{f''}{f'}\right)' = 0. \tag{20}$$

In order to solve the differential equation (20) for f(x), let $g(x) = \left(\frac{f'}{f} - \frac{f''}{f'}\right)$. Then

$$\frac{g^2}{2} = -g' \,. \tag{21}$$

So

$$g(x) = \frac{2}{x-r} = \frac{f'(x)}{f(x)} - \frac{f''(x)}{f'(x)} \tag{22}$$

where r is an integration constant. Integrating twice yields the most general transformations of the hg. DE into the wave-equation; namely,

$$\underline{\left\{ f(x) = q \exp\left(\frac{p}{x-r}\right) : \quad p,q,r \right\}} \tag{23}$$

where p,q,r are arbitrary constants. From (15) it then follows that the class of ε_r-profiles this method can handle, henceforth called "permissible profiles," is defined by the family

$$\begin{aligned} \varepsilon_r(x;\hat{a},\hat{b};p,q,r) &= -\left(\frac{f'}{f}\right)^2 \left(\frac{\hat{a}+\hat{b}}{2}\right)^2 - \frac{f'^2}{f(1-f)} \hat{a}\hat{b} \\ &= \frac{p^2}{(x-r)^4} \left(\hat{a}\hat{b} \frac{q}{q-\exp(p/x-r)} - \left(\frac{\hat{a}+\hat{b}}{2}\right)^2\right) . \end{aligned} \tag{24}$$

To illustrate the versatility of the class of permissible profiles $\varepsilon_r(x)$ a few of its members are shown in figure 2. The profiles depicted there are special cases of (24); namely,

$$\hat{a} = -\hat{b}$$

$$\varepsilon_r(x) = \frac{y^4}{p^2} \frac{\hat{a}^2 q}{\exp(y)-q}. \qquad \text{(four parameter family)}; \; y = \frac{p}{x - r} . \tag{25}$$

For the sake of simplicity we will limit ourselves in this report to profiles of the form (25).

We further note that if they are rewritten as

$$\varepsilon_r(x) = y^4(\hat{a}/p)^2\theta(1-\theta)^{-1} \tag{26}$$

where

$$\theta = q \exp(-y), \quad y = \frac{p}{x - r} \tag{27}$$

it follows that the range of the hypergeometric differential equation extends only from $\theta = 0$ to 1, if the relative permittivity is to remain positive.

The solution to the wave equation (1) with permittivity profiles (24), therefore, is given by

$$E_y = \Pi(a,b,c;\theta(x))G^{-1}(x) \tag{28}$$

where Π is the solution to the hg. DE (6) and G is calculated from (10) and (23):

$$G(x) = \frac{1}{x - r} . \tag{29}$$

4. EXPRESSIONS FOR THE SURFACE ADMITTANCE AND THE REFLECTION COEFFICIENT

With the geometry chosen as in figure 1, the transverse admittance is given by

$$Y(x) = \frac{H_z}{E_y} = \frac{-1}{j\omega\mu_o D}\,\frac{1}{E_y}\,\frac{dE_y}{dx} \text{ mhos} \tag{30}$$

where D is the total depth of the medium, and the reflection coefficient at the air-dielectric interface becomes

$$R = \frac{1 - Y_o}{1 + Y_o} \tag{31}$$

where Y_o is now the surface admittance evaluated at $x = 0$ and normalized to free-space admittance; i.e., $Y_o = Y(0)\eta_o$ with $\eta_o \doteq 120\ \pi$ ohms.

Furthermore, we have shown that the profiles obtained by setting $\hat{a} = -\hat{b}$ in equation (24) can model many media of interest; and for those profiles the hypergeometric function Π satisfies a much simpler differential equation containing only one parameter; namely,

$$\theta(1-\theta)\,\frac{d^2\Pi}{d\theta^2} + (1-\theta)\,\frac{d\Pi}{d\theta} + a^2\Pi = 0, \quad 0 < \theta < 1, \quad a = \hat{a}(k_o D). \tag{32}$$

The solution of this equation is given in terms of Weierstrass sums as [13]

$$\Pi = AF\begin{pmatrix} a,-a; \\ 1; \end{pmatrix}\theta + B(1-\theta)F\begin{pmatrix} 1-a,1+a; \\ 2; \end{pmatrix}1-\theta \tag{33}$$

where A,B are the two arbitrary constants to be determined from the boundary condition at $x = 1$. For instance, one of the cases of interest is when the pile of grain is supported at the bottom by a perfectly conducting plane. In that event, the boundary condition of a vanishing tangential electric field now yields the condition that

$$A/B = -(1-\theta)F\begin{pmatrix} 1-a,1+a; \\ 2; \end{pmatrix}\theta \Big/ F\begin{pmatrix} a,-a; \\ 1; \end{pmatrix}\theta$$

for $x = 1$.

Substitution of (28) into (30) finally provides a formal expression for the transverse admittance as follows:

$$Y(x) = \frac{j}{\eta_o \varepsilon_o D}\,(x-r)^{-2}\left\{\frac{p}{\Pi}\,\frac{d\Pi}{d\theta} - (x-r)\right\} \text{ mhos.} \tag{34}$$

As mentioned earlier, the grain pile is typically several meters deep, while the operating frequencies lie above several hundred megahertz. This means that a is usually in the order of 10 or larger. Consequently, an approximate expression for the transverse admittance can be derived from the asymptotic evaluation of the hypergeometric functions. In Appendix A we have shown that as $|a| \to \infty$ the asymptotic expansion of Π, accurate to the order of a^{-2}, is given as

$$\Pi(\theta) = A^{+}h(\theta)\exp\left[-ja\alpha(\theta) - \frac{j}{a}\beta(\theta)\right] + A^{-}h(\theta)\exp\left[ja\alpha(\theta) + \frac{j}{a}\beta(\theta)\right] \tag{35}$$

where

$$h(\theta) = \theta^{-1/4}(1-\theta)^{1/4} ,$$

$$\alpha(\theta) = 2 \arccos(\theta^{1/2}) ,$$

$$\beta(\theta) = -\frac{1}{16}\theta^{-1/2}(1-\theta)^{-1/2}(2\theta+1) ,$$

and A^{+}, A^{-} are some linear combination of A and B in (33). It is of particular interest to note that the first term in (35) actually represents a wave propagating in the forward x-direction, while the second term represents a wave reflected from the boundary at x = 1. In particular, when a perfectly conducting plate is located at x = 1, the boundary conditions require that $A^{-} = -A^{+}$, and the transverse admittance at x = 0 takes the following form:

$$Y_o = \frac{j}{\eta_o}\hat{a}\left\{\frac{A}{a} - \left(B + \frac{\beta'}{a^2}\right)\cot\left[\left(a\alpha(\theta) + \frac{1}{a}\beta(\theta)\right)_{\theta=\theta_0}^{\theta_1}\right]\right\} \text{ mhos} \tag{36}$$

where

$$A = -\left(.25\, pr^{-2}(1-\theta_0)^{-1} + r^{-1}\right) ,$$

$$B = pr^{-2}\theta_0^{1/2}(1-\theta_0)^{-1/2} ,$$

$$\theta_{0,1} = \theta(x=0,1) ,$$

and β' is the derivative of β with respect to x.

The constants in Y_o as written in (36), which is a function of frequency alone, relate the profile characteristics quite explicitly to the measurement characteristics if one transforms θ back to the normalized depth variable x:

$$Y_o = \frac{j}{\eta_o}\left\{\frac{1}{k_oD}\varepsilon_r^{1/4}(\varepsilon_r^{-1/4})' - \left[\varepsilon_r^{1/2} + .5(k_oD)^{-2}\varepsilon_r^{-1/4}(\varepsilon_r^{-1/4})''\right]\cot\left[k_oDL + \frac{1}{2k_oD}F_1\right]\right\} \text{ mhos} \tag{37}$$

where the profile-function ε_r is assumed to be evaluated at x = 0, and L and F_1 are its optical depth and average degree of concavity, respectively; i.e.,

$$L = \int_0^1 \varepsilon_r^{1/2}(x)dx$$

$$F_1 = \int_0^1 \varepsilon_r^{-1/4}(\varepsilon_r^{-1/4})''\, dx . \tag{38}$$

Expression (37) resembles the input admittance for a uniform medium terminated in a perfect conductor; namely,

$$Y_{in} = \frac{-j}{\eta_o} \varepsilon_r^{1/2} \cot(k_o D \varepsilon_r^{1/2}) \text{ mhos.} \tag{39}$$

In fact, this is the limit that Y_o attains when the profile tends to a constant. For uniform media are indeed limiting cases of the permissible profiles (25) and are obtained as follows: Let $\hat{a} = r^2$ and take

$$\lim_{r\to\infty} \varepsilon_r(x) = \frac{p^2 q}{1 - q} = \text{constant.}$$

Thus it appears that the electromagnetic response of an inhomogeneous medium depends on its permittivity profile, specifically on the profile's initial value at the surface as well as on its overall optical length, i.e., the integral of $\varepsilon_r^{1/2}(x)$, and its concavity, i.e., $\varepsilon_r''(x)$.

This demonstrates how the non-uniformity of the profile enters analytically into the electromagnetic response. In the next section we will investigate the numerical aspect of these conclusions and discuss some experimental implications.

5. NUMERICAL RESULTS AND DISCUSSION

In 1973 Kuester and Chang [12] developed a computer program to calculate the transverse impedance to analyze propagating modes in dielectric slab guides with arbitrary permittivity profiles. Based on the invariant embedding principle, it integrates a first order Riccati differential equation for the impedance numerically across the slab using a fourth-order Runge-Kutta method with error estimation. This program is not only highly accurate, but also very versatile in that it can handle all possible profiles without limitations on frequency. This approach, while possibly too complex for inversion purposes, will be used as a guideline and standard in our investigations.

In table 1 we compare the normalized input admittance for a lossless inhomogeneous medium as calculated by Kuester and Chang with the asymptotic expression (37). Specifically, the relative permittivity used for this table is profile (e) in figure 3. Since the medium considered is lossless, the conductive part of the admittance is of course zero in both cases. As expected, the agreement between our asymptotic result and the numerically "exact" result is excellent everywhere except close to the sharp resonances. Even then, the frequency where any given resonance occurs, differs only very insignificantly in all our studies.

We note that due to the very nature of the asymptotic formulations of the admittance, in particular the expansion (A-1), one should expect increasingly better agreement between the two results with increasing frequency. However, at higher frequencies the perturbation terms in (27) of order $(k_o D)^{-1}$ decrease. Hence, a non-uniform medium responds more and more like a uniform one with dielectric constant equal to the square of the optical

Length of the non-uniform medium. From the viewpoint of microwave remote-sensing this means that the constants in Y_o containing information about the profile, i.e., F_1, $\varepsilon_r^{1/4}(\varepsilon_r^{-1/4})'$, $\varepsilon_r^{-1/4}(\varepsilon_r^{-1/4})''$, being dependent on $(k_oD)^{-1}$ are more difficult to extract from measurements made beyond a certain frequency. We also note that the constants in Y_o as expressed by (37) comprise all the information about the profile one can measure, no matter how many additional frequencies one is determined to employ. (All neglected terms of Y_o are of order $(k_oD)^{-n}$ for $n \geq 3$ which is expected to lie below the noise level of the experiment.) Consequently, the maximum number of equations containing the unknown profile parameters is five. This means that any profile inversion-technique using high-frequency measurements is limited to profiles characterized by less than five parameters, especially since at least two of the equations are non-linear.

Next, we will present and discuss the results for the input admittance for a number of profiles. The permittivity profiles used are shown in figure 4.

First, we note that the optical length L can easily be determined from the average separation between adjacent resonant frequencies. Since we have arbitrarily set the total optical length equal to 1.499 for all profiles considered in the following, the separation in resonance frequencies is approximately the same; namely, about 100 kHz.

Now in order to distinguish between different profiles, we have tabulated the shift of each resonance and anti-resonance frequency (for which the surface admittance is infinite or zero, respectively, for a lossless dielectric) as well as the resonance and anti-resonance frequencies for a uniform medium with the same optical length.

The shifts of resonance and anti-resonance frequency as compared to those of a uniform medium (table 2) may now be interpreted from (37) as being due to the profile's average concavity (F_1) and other functionals of the relative permittivity profile $\varepsilon_r(x)$.

From the above observations it now appears that the proper procedure for the remote measurement of a permittivity profile could be divided into the following steps:

(1) From swept frequency measurement, one can determine three consecutive resonances (i.e., $Y_o \to \infty$) at f_1, f_2, and f_3, say. Then the following three equations in L and F_1 are obtained.

$$k_iL + \frac{1}{2k_i} F_1 = N_i\pi \tag{40}$$

where N is a positive integer and $N_i = N + i - 1$ $(i = 1,2,3)$, $k_i = 2\pi f_iD(\mu_o\varepsilon_o)^{1/2}$. Solving for L and F_1 yields:

$$F_1 = 2\pi(k_1-2k_2+k_3)/C \quad \text{where } C = (k_2/k_1-k_1/k_2) + (k_1/k_3-k_3/k_1) + (k_3/k_2-k_2/k_3) \tag{41}$$

$$L = (2/k_2 - 1/k_3 - 1/k_1)\pi/C. \tag{42}$$

(2) Since the argument of the cotangent function in (37) is now known, linear equations in $\varepsilon_r(0)$ and $\varepsilon_r^{-1/4}(0)(\varepsilon_r^{-1/4}(0))''$ may be determined from the measured value of Y_o at other non-resonant frequencies.

Consequently, all constants in Y_o are measurable within at least 5 frequencies. More measurements will only yield consistency results, within the accuracy of the experiment. In our next report, we will show that the profile-constants $\hat{a}$, p, q, and r can be determined from two independent equations containing the "measurable" quantities and a single unknown.

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APPENDIX A

ASYMPTOTIC SOLUTION OF THE HYPERGEOMETRIC DIFFERENTIAL EQUATION

The asymptotic form of the hypergeometric function Π with a large parameter may be obtained as follows.

First, one assumes the general form

$$\Pi(\theta) = A^{+}h_o(\theta)\exp[-ja\alpha_o(\theta) - \frac{j}{a}\beta_o(\theta)] + A^{+}h_1(\theta)\exp[ja\alpha_1(\theta) + \frac{j}{a}\beta_1(\theta)] \tag{A-1}$$

and then attempts to satisfy the differential equation (32), repeated below for convenience, with the unknown functions.

$$\theta(1-\theta)\Pi'' + (1-\theta)\Pi' + a^2\Pi = 0 \ . \tag{32}$$

Equating terms containing like powers of the parameter a shows that $\alpha_o = \alpha_1$, $\beta_o = \beta_1$, and $h_o = h_1$. We therefore may drop the subscripts and obtain α, β, and h as the solution of the following differential equations:

$$\begin{aligned} &\alpha' = \theta^{-1}(1-\theta)^{-1} \\ &2\theta\alpha'h' + (\theta\alpha''+\alpha')h = 0 \\ &2\theta\alpha'h\beta' = h' + \theta h'' \ . \end{aligned} \tag{A-2}$$

The asymptotic form for Π in the range $0 < \theta < 1$ and accurate to the order a^{-2} is then given by (A-1), with

$$\begin{aligned} &\alpha(\theta) = 2 \arccos \theta^{1/2} \\ &\beta(\theta) = -\frac{1}{16}\ \theta^{-1/2}\ (1-\theta)^{-1/2}\ (2\theta+1) \\ &h(\theta) = \theta^{-1/4}\ (1-\theta)^{1/4} \end{aligned} \tag{A-3}$$

with the limitation given by the solution of the remaining differential equation, i.e. the a^{-2} term in the expansion:

$$a^{-2}(1+\theta)(1-\theta)^{-2}\theta^{-1}/16 \ll 1 \ . \tag{A-4}$$

Consequently, the asymptotic form (A-1) need not carry this correction term as long as the condition (A-4) is satisfied.

To first order, the result (A-2) agrees with [14] where it is confirmed by a steepest descent integration.

Table 1. Comparison of the input susceptance as calculated by Kuester and Chang (B_1) with expression (37), (B_2).

GHz	B_1	B_2	GHz	B_1	B_2
.40	-189.682	-179.298	1.00	-448.563	-443.582
.41	-6.521	-6.523	1.01	-6.586	-6.586
.42	-1.590	-1.597	1.02	-1.578	-1.579
.43	1.587	1.584	1.03	1.573	1.573
.44	6.938	6.951	1.04	6.764	6.765
.45	-210.695	-201.193	1.05	-470.469	-465.666
.46	-6.529	-6.530	1.06	-6.589	-6.589
.47	-1.588	-1.593	1.07	-1.578	-1.578
.48	1.583	1.581	1.08	1.573	1.573
.49	6.904	6.914	1.09	6.759	6.759
.50	-231.888	-223.137	1.10	-492.414	-487.752
.51	-6.537	-6.538	1.11	-6.592	-6.592
.52	-1.586	-1.590	1.12	-1.578	-1.578
.53	1.581	1.579	1.13	1.573	1.573
.54	6.878	6.885	1.14	6.754	6.754
.55	-253.222	-245.114	1.15	-514.360	-509.839
.56	-6.545	-6.544	1.16	-6.595	-6.594
.57	-1.585	-1.587	1.17	-1.577	-1.578
.58	1.579	1.577	1.18	1.573	1.572
.59	6.857	6.862	1.19	6.750	6.750
.60	-274.667	-267.118	1.20	-536.334	-531.928

Table 2. Resonance and anti-resonance frequencies for a uniform medium and profiles (i) - (j) in figure 4.

Anti-resonance Frequencies
(in MHz)

Profile						
uniform	550.0	650.0	750.0	850.0	950.0	1,050.0
(i)	551.4	651.2	751.0	850.8	950.7	1,050.7
(j)	550.5	650.4	750.3	850.3	950.3	1.050.2
(k)	552.2	651.8	751.6	851.4	951.3	1.051.1
(l)	550.3	650.2	750.2	850.2	950.1	1,050.1
(m)	551.9	651.7	751.5	851.4	951.2	1.051.1

Resonance Frequencies
(in MHz)

Profile						
uniform	500.0	600.0	700.0	800.0	900.0	1,000.0
(i)	502.1	601.8	701.5	801.3	901.2	1,001.1
(j)	499.1	599.3	699.4	799.5	899.5	999.6
(k)	501.2	601.0	700.8	800.8	900.7	1,000.6
(l)	499.9	600.0	700.0	800.0	900.0	1,000.0
(m)	497.5	597.9	698.2	798.4	898.6	998.8

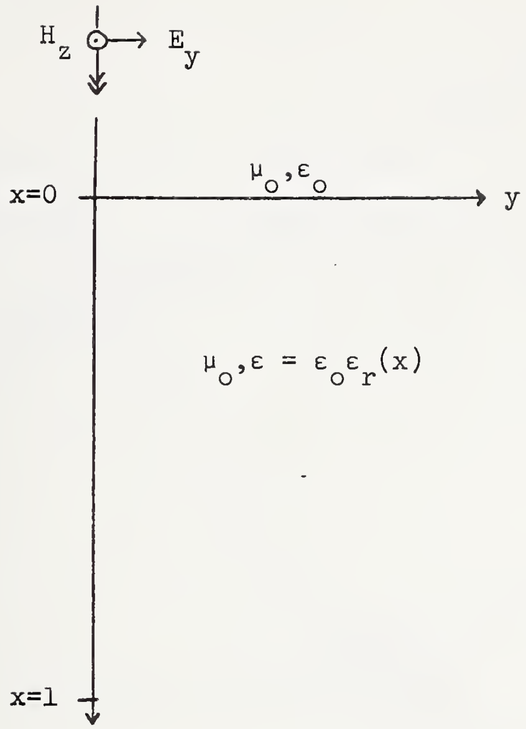

Figure 1. Geometry of the problem.

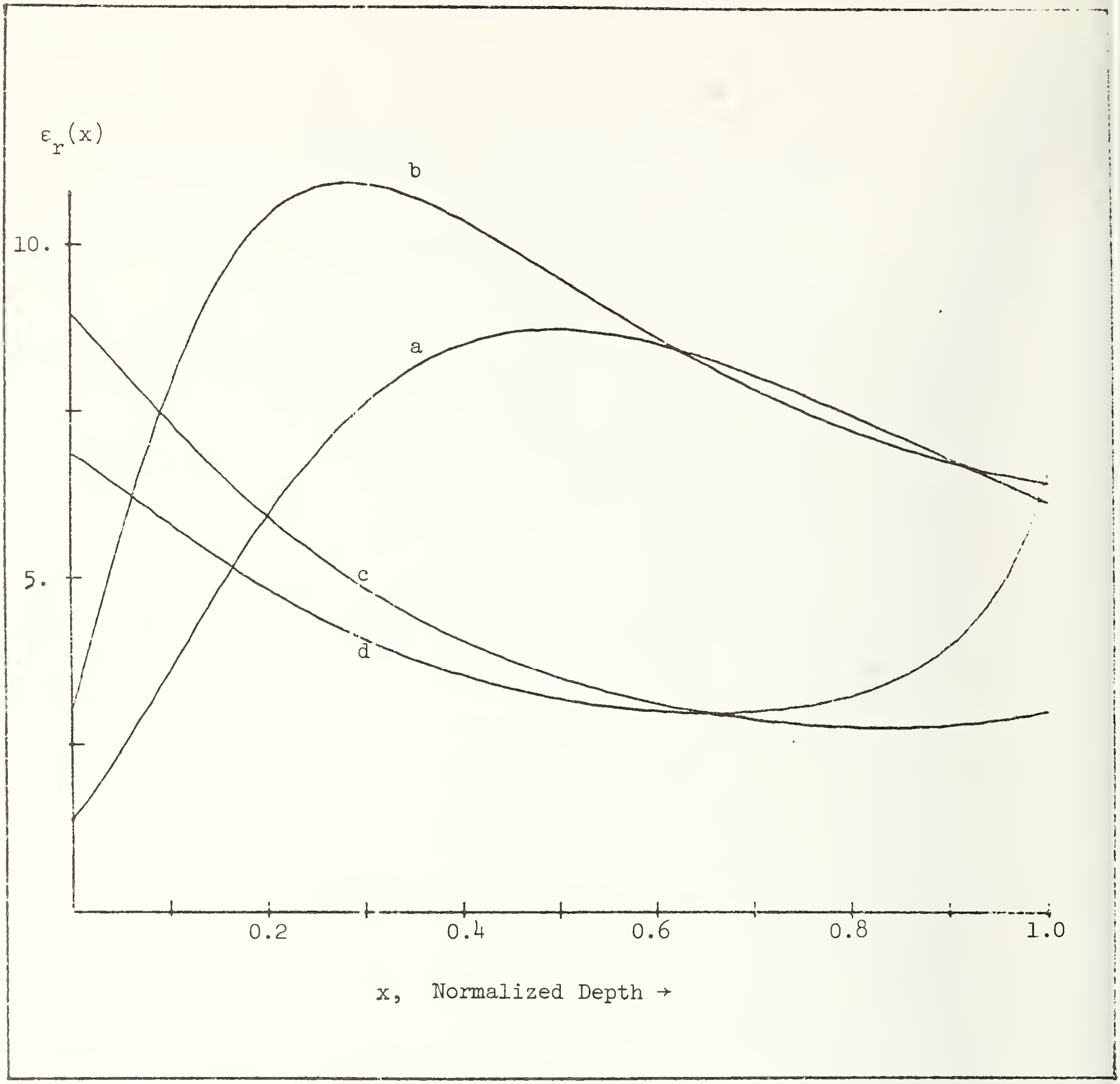

Figure 2. Permissible profiles (a)-(d) with parameters as given below:

	$\hat{a}$	p	q	r
(a)	13.0	3.5	.135	-.38
(b)	1.8	2.0	2.719	-.25
(c)	1.2	1.2	2.0	-.40
(d)	1.0	1.25	2.3	-.40

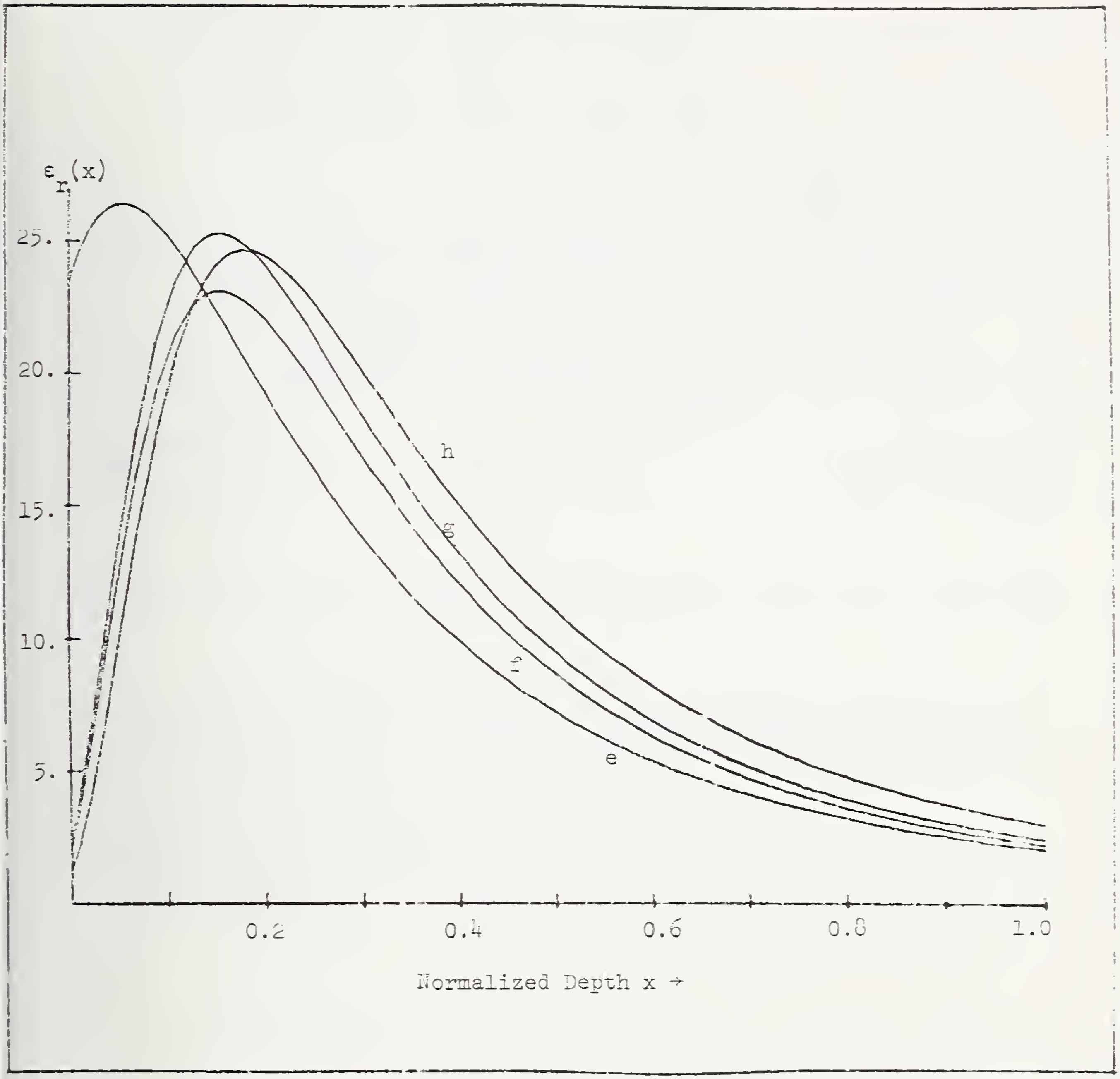

Figure 3. Permissible profiles (e)-(h) with parameters as given below:

	$\hat{a}$	p	q	r	
(e)	2.35	1.0	1.0	-0.2	(used in table 1)
(f)	2.2	1.0	1.0	-0.1	
(g)	2.3	1.0	1.0	-0.1	
(h)	2.5	1.1	1.0	-0.1	

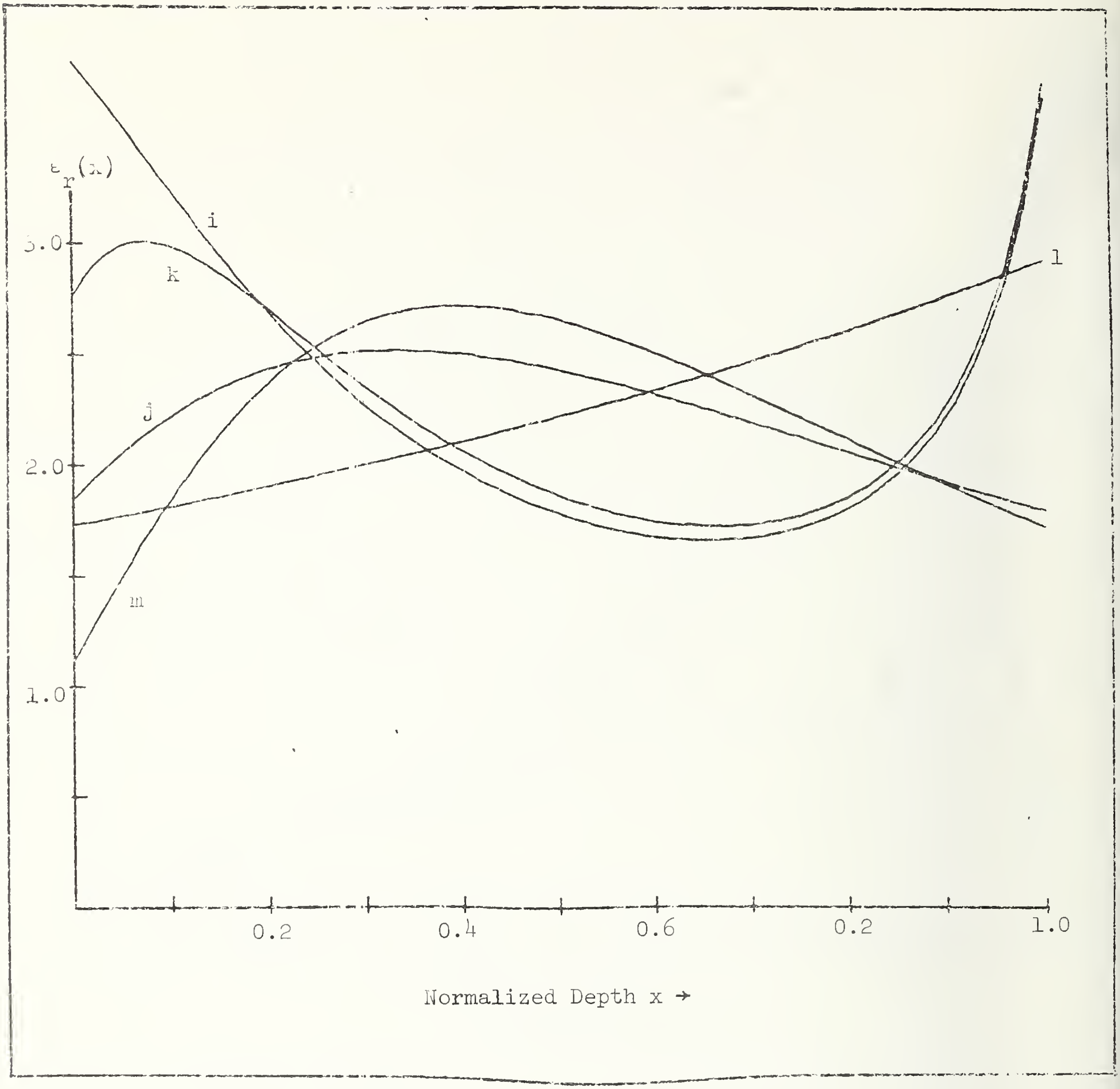

Figure 4. Permissible profiles (i)-(m) with optical length L = 1.499 used in table 2.

	$\hat{a}$	p	q	r
(i)	0.747	1.25	2.3	-0.4
(j)	2.255	4.0	1.64	-0.7
(k)	0.661	1.4	2.718	-0.3
(l)	20.150	-1.0	1.0	6.0
(m)	7.257	3.5	.135	-0.48

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16. ABSTRACT *(A 200-word or less factual summary of most significant information. If document includes a significant bibliography or literature survey, mention it here.)*

This report deals with the electromagnetic response of inhomogeneous dielectrics, i.e., media whose permittivity is a function of depth. The resulting boundary value problem is solved for a large number of permittivity functions which can model almost any medium of interest.

Since those permittivity profiles are characterized by only a few parameters, they are particularly useful for the inverse problem; i.e., the retrieval of profiles from the measured electromagnetic response.

It is shown how the non-uniformity of the permittivity changes the response and how the change is related to the profile characteristics.

17. KEY WORDS *(six to twelve entries; alphabetical order; capitalize only the first letter of the first key word unless a proper name; separated by semicolons)*

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