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Mathematical Methods of Site Selection for Electronic Message Systems (EMS)

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U.S. DEPARTMENT OF COMMERCE, Rogers C.B. Morton, ***Secretary***
James A. Baker, III, *Under Secretary*
Dr. Betsy Ancker-Johnson, *Assistant Secretary for Science and Technology*

NATIONAL BUREAU OF STANDARDS, Ernest Ambler, ***Acting Director***

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ABSTRACT

The concept of electronic message (mail) transmission has been the subject of several feasibility studies during the past decade. It requires the installation of electronic message handling facilities at selected locations. If transmission is to be via communications satellite, then any such facility can transmit to and receive from any other one. In this report, the mathematical aspects of choosing the number and locations of these facilities are examined. An inventory of solution methods is presented, along with recommendations as to which among them should be employed or developed further.

Key words: communication, cost-benefit, deployment, electronic transmission, facility location, mail, mathematical programming, message network synthesis, network optimization, satellite, service improvement.

INTRODUCTION

The work reported here has its source in earlier studies by postal personnel on planning an electronic message system with the aid of postal-volume data. This previous work is described in detail in references [1] and [2] and more briefly below.

Both of these references are concerned with designing an electronic message system (EMS) made up of individual electronic message facilities (EMFs) so as to maximize the volume carried within the EMS. Both obtain average daily volumes from a large postal data base called ODIS (Origin Destination Information System [3]). The ODIS data are obtained from 20-30 million individual mail pieces which are selected each fiscal year by a nationwide sampling system.

The concept of the message service considered here is to transmit the information of the mail piece electronically (e.g., via satellite) rather than transporting the material mail piece. Thus the electronic message system requires an electronic transmission subsystem. It also requires an input subsystem to convert mail information to electronic form and an output subsystem to convert the transmitted information to the form in which is to be delivered to addressee.

This concept dates back within the Postal Service to the late 1950s, when an experimental system called "Speed Mail" was operated between Washington, D.C., Detroit, Michigan, and Battle Creek, Michigan. However, in 1961 the Post Office discontinued the Speed Mail program and, until recently, there have been no significant postal programs on electronic mail.

The final configuration for a network of EMFs will depend upon customer satisfaction with and use of electronic message services and on postal policies concerning the nature and extent of these services. However, for initial planning purposes, data on mail flows and volumes may be used to identify market areas, to estimate network volume, and to identify network configurations likely to yield the highest revenue, give the greatest improvement in service, or provide the greatest profit (or smallest loss).

The studies of references [1] and [2] were initial planning efforts which developed methodology as a by-product during their performance. Numerical results were obtained manually. By contrast, the present study has an explicitly methodological purpose: to determine what mathematical methods are "useful" for solving the large problems posed in references [1] and [2], where "useful" means that the method is practical to code on a computer and that excessive computer time will not be required for solving systems involving 100 or more potential EMF sites. An additional concern is to explore alternate problem formulations related to those in [1] and [2].

The body of the paper is divided into four parts. The first of these outlines three different formulations of the problem of synthesizing networks: optimal deployment of a given number of EMFs, cost-benefit optimization of the size of the EMS, and, finally, generation of an optimal "nested" sequence of networks. Solution methods for each of these three problem areas are discussed in the three subsequent parts, respectively. Finally, our recommendations as to preferred problem formulations and solution methods are summarized.

A collection and description of pertinent methodologies has also been given as part of a General Dynamics Study [12]. It mentions some network flow techniques which are also referred to in our report. The two studies, however, differ radically in their approaches and formulations.

The exposition of the material is predominantly mathematical. That is because this report is intended as a technical basepoint and reference for use by mathematical analysts pursuing the recommended courses of action. A less detailed account is given in a letter report (Witzgall [13]) previously submitted in completion of our contractual documentation obligations.

The author gratefully acknowledges the lively interest and partipation by other NBS mathematicians in this project, as well as the advice, guidance, and general helpfulness of Mr. Emile Sherrard of the U.S. Postal Service.

1. PROBLEM FORMULATION

In this Part we will consider essentially three ways of formulating EMS network site selection as an optimization problem. It is understood that all such optimization formulations are more or less idealizations in that they are based on estimates, involve concept-simplifying assumptions and ignore some side conditions of a politico-societal nature which are present in real world planning. These limitations do not detract from the usefulness of solving idealized optimization problems in order to assess the overall utility of a given system, with the option of then adjusting for special conditions as they are recognized. It does, however, warn against solution techniques which expend excessive computational efforts in order to find a "true" optimum.

1.1 Prescribed Number of EMFs

The following two sets will recur in our discussion:

(1.1.1) N = set of all <u>potential</u> sites which are considered,

K = set of all <u>actual</u> EMF sites (to be chosen).

Clearly, K is a subset of N, symbolically $K \subseteq N$, and

$$0 \leq k \leq n ,$$

where

(1.1.2) $n = |N|$ = number of potential sites,

$k = |K|$ = number of EMF sites.

Roughly speaking, the problem is to optimize the selection of actual sites from a multitude of potential sites, in terms of some measure of goodness.

This statement has to be clarified. A main problem, clearly, is to decide how many EMFs should be built in the first place. Since the facilities will obviously not be installed all at once, one might also ask in which sequence the facilities are to be introduced.

With an eye on these higher level problem formulations, we initially pose the somewhat simpler question of how to select a

given number k

of installations in an optimal fashion. Plotting the values of these optimal solutions for different values of k may then yield (Sherrard [2]) a criterion for deciding on the number of facilities to be installed.

Further specification is needed with respect to the measure of optimality to be employed. The alternatives of foremost interest appear to be

revenue, service improvement, cost-benefit.

Use of these measures requires a capability to predict transmitted volumes. ODIS [3] data will provide

(1.1.3) transmissible volumes,

i.e. the presently observed amounts of mail suitable for EMS-transmission which move between potential sites. A first guess is to assume that the transmitted volume would be proportional to this transmissible volume, and that revenue in turn is proportional to transmitted volume. These two assumptions are somewhat contradictory. The latter of them seems to envision a flat rate. Under a flat rate, however, the portion of electronically transmissible mail which will actually be transmitted must be expected to be larger for greater distances because of the greater time-savings provided and of the consequently increased attractiveness to the customer.

The improvement in the quality of service between two sites is roughly proportional to their distance. This improvement should be weighed by the number of customers who are able to take advantage of it. Thus it is plausible to assume that the measure of service improvement is proportional to the product of distance with transmissible volume.

Cost-benefit would be given by the difference of revenues realized by different modes of transmission and adjusted by the corresponding difference in costs.

Obviously there are many alternative measures of optimality. The rate structure will enter into most of them, and almost all measures of benefit will be strongly dependent on distance. Exploration of these alternatives should certainly be an important part of further EMS planning.

Regardless of which particular benefit criterion has been selected, we will use the notation

(1.1.4) $$v_{ij} = \left\{ \begin{array}{l} \text{predicted benefit of EMS service} \\ \text{from site } i \text{ to site } j, \end{array} \right\} = \underline{\text{link benefit}},$$

- realized whenever both sites i and j are chosen for EMFs - and make the

(1.1.5) Assumption: The total predicted benefit of the system is the sum of all realized link benefits.

In other words:

(1.1.6) $$\text{Total benefit} = V = \Sigma_{i\varepsilon K}\Sigma_{j\varepsilon K, j\neq i} v_{ij} .$$

The assumption (1.1.5) is not automatically satisfied. For example, the benefit derived from one EMF may conceivably be influenced by the proximity of a competing facility. There may be other effects which influence the total benefit in a nonadditive way. One will have to assess the extent of such effects in view of the overall accuracy required.

We are now able to state the

(1.1.7) Network Synthesis Problem for Prescribed Number of Installations:

$$\underline{\text{Maximize}} \quad \Sigma_{i\varepsilon K}\Sigma_{j\varepsilon K, j\neq i} v_{ij} \quad \underline{\text{subject to}}:$$

$$\text{total number of EMFs} = |K| = k = \underline{\text{given}}.$$

Suppose all potential sites are numbered sequentially:

$$i = 1, \ldots , n .$$

An equivalent expression of the above problem - a formulation more common in Operations Research literature - is as follows:

$$\text{(1.1.8)} \qquad \underline{\text{Max}} \quad \sum_{i=1}^{n}\sum_{\substack{j=1 \\ j\neq i}}^{n} v_{ij}x_i x_j \quad \underline{\text{subject to}}$$

$$\sum_{i=1}^{n} x_i = k ,$$

$$x_i = \begin{cases} 0 \\ 1 \end{cases} \quad \underline{\text{for}} \quad 1 \leq i \leq n .$$

Here the variables x_i are capable of only two values, namely 1 and 0, depending on whether or not i is the site of an EMF. Note that the total benefit is formally stated as a summation over <u>all</u> potential sites; but since the product $x_i x_j$ will be zero unless both values x_i and x_j equal 1, only those coefficients v_{ij} for which both sites i and j carry EMFs will enter into the summation. Thus the two formulations are indeed equivalent.

There are simple operations, useful for later purposes, which can be applied to the matrix of link benefits v_{ij} without thereby changing the optimal installation pattern. <u>Adding a common constant</u> to all v_{ij} is such an operation, since it changes the total benefit function only by an additive constant. One may therefore assume without loss of generality that all link benefits v_{ij} are positive. This relies on the fact that the number k of installations is fixed. One wants to keep careful track of such modifications in order to permit proper adjustments if the resulting optimal benefits are to be compared for different values of k.

Each pair of counter-directed link benefits v_{ij} and v_{ji} may be combined, since

$$\sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} v_{ij} x_i x_j = \sum_{i=2}^{n} \sum_{j=1}^{i-1} (v_{ij}+v_{ji}) x_i x_j .$$

In other words, the total benefit depends only on the sums $v_{ij}+v_{ji}$. One can therefore

(1.1.9) <u>symmetrize</u>

the problem by replacing the v_{ij}'s with the modified benefits

$$\bar{v}_{ij} = \frac{1}{2}(v_{ij}+v_{ji}) = \bar{v}_{ji} .$$

This symmetrization does not require the number of installations to be fixed, nor does it affect the total benefit in any way.

1.2 The Cost-Benefit Approach

Synthesizing an EMS-network with a prescribed number of installations may be of direct interest in some planning situations. More likely, however, it will be employed as a subanalysis in the process of gaining some idea of how many installations should be built.

If estimates of costs for amortizing, operating and maintaining EMFs are available, then one could introduce them into the model as factors limiting the proliferation of facilities and hopefully producing a most cost-efficient EMS-network. Note that <u>direct</u> operating costs, i.e. the costs per volume processed, can and should be spread over the links by adjusting the link benefits v_{ij}. The

(1.2.1) c_i = <u>indirect operating costs</u>,

which are independent of volume processed, are the costs to be associated with sites. The quantity now to be maximized is the difference between total benefit accruing on the links - adjusted for direct operating costs - and the total (indirect operating) cost

(1.2.2) $$C = \Sigma_{i\varepsilon K} c_i$$

incurred at the facilities. This gives rise to the following

(1.2.3) <u>Network Synthesis Problem with Cost-Benefit Optimization</u>:

$$\underline{\text{Max}} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} v_{ij} x_i x_j - \sum_{i=1}^{n} c_i x_i \quad \underline{\text{subject to}}$$

$$x_i = \begin{cases} 0 \\ 1 \end{cases} \quad \underline{\text{for}} \qquad 1 \leq i \leq n .$$

This approach has been explored - for a more general setting - in an important paper by J.M.W. Rhys [4], who was able to show that the problem is dual to a network-flow problem and is therefore particularly easy to solve. The paper of Rhys is discussed in greater detail in Part 3. It was brought to our attention by D.R. Shier, Post-doctoral Research Fellow, NBS Applied Mathematics Division. The problem is also treated in a recent report by Picard and Ratliff [5].

The main difficulty of the cost-benefit approach is that it requires cost and revenue figures, which may not be available or may need to await further decisions on rate structure, etc. It also requires demand estimates that may be hard to come by. In this situation the model should be used

(1.2.4) <u>parametrically</u>,

i.e. one should explore the performance of the system for a variety of possible values of the unknown but critical factors like rates, demand responses, operating costs, etc.

1.3 Nested Solutions

Taking into account the fact that EMFs are not built all at once, and that the total number to be built is unknown, one is led to formulate a somewhat different class of optimization problems in which interest centers on some kind of optimal sequence in which to install EMFs.

A first cut at this problem is to require that the network solutions be

(1.3.1) nested,

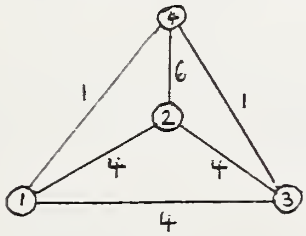
that is, the network generated for k-1 EMFs is a subnetwork of the one generated for k EMFs. Methods which successively generate nested networks will be called

(1.3.2) sequential méthods,

and two prototypes will be discussed in Sections 2.1 and 2.2.

The problem then presents itself as finding optimal nested sequences of networks, where one of the tasks is again to suitably define optimality.

Note that the optimal solutions to the Network Synthesis Problem (1.1.7) for a prescribed number of EMFs are not necessarily nested. The following example involving 4 potential sites is a case in point:



The numbers attached to the links describe the combined transmission and reception benefit. The optimal solution for 2 facilities is {2,4}, whereas the optimal solution for 3 facilities is {1,2,3}.

2. PRESCRIBED NUMBER OF INSTALLATIONS

This Part discusses various methods and approaches directed at optimal Network Synthesis for a given number of EMFs. The first two methods represent what one might call "sequential" methods in that they generate nested sequences of optimal solutions. They have been employed previously to a limited extent for synthesizing networks based on ODIS data. While these methods will in general not find a true optimum, they will come close enough for practical purposes. Furthermore, they are attractive because they mimic the sequential process of actual construction. Subsequently, various possible integer programming formulations are described and some heuristic approaches are discussed. Application of Dynamic Programming has not been considered.

2.1 The Ranking Method.

The following method for arriving at a first cut at EMS-network synthesis was proposed by R. Ruckman [1]: The potential sites are ranked in linear order by the estimated amount of mail volume generated for EMS transmission, including the amount of comparable local business. The sites are then selected in this order, namely the highest ranking site first, to be followed by the second ranking site etc.

The rationale behind this method is as follows: Suppose revenue - and therefore volume - is to be maximized, and one makes the

(2.1.1) Assumption: The transmitted volume at each site will distribute among the other sites proportionally to the respective totals of transmission volumes of the receiving sites.

Let then

$$v_i, \; 1 \leq i \leq n ,$$

be the total "transmissible" mail at site i:

$$v_i = \sum_{j=1}^{n} v_{ij} ,$$

where

$$v_{ii}$$

is the estimated amount of local business of the transmissible kind. Assumption (2.1.1) then means that there are factors

$$\rho_i , i = 1, \dots , n ,$$

such that

$$v_{ij} = \rho_i v_j .$$

Consequently

$$v_i = \sum_{j=1}^{n} v_{ij} = \sum_{j=1}^{n} \rho_i v_j = \rho_i \sum_{j=1}^{n} v_j = \rho_i V ,$$

where

$$V = \sum_{j=1}^{n} v_j$$

is the grand total transmissible mail volume. Thus

$$\rho_i = \frac{v_i}{V}$$

and

$$v_{ij} = \frac{v_i v_j}{V} . \tag{2.1.2}$$

It now follows that

$$\sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} v_{ij} x_i x_j = \frac{1}{V} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} (v_i x_i)(v_j x_j) ,$$

and it is clear that this expression is maximized by selecting the k nodes of highest ranking v_i .

The above argument shows the <u>limitation</u> of the ranking method: If distance enters the benefit assessment - as it does if one seeks to maximize service improvement or cost benefit rather than volume - then assumption (2.1.1) is clearly not valid. Even for volume maximization, the assumed proportionality of the volumes is problematical. This is particularly true if there is a large discrepancy between transmitted and received mail volumes such as exists, for instance, for Washington, D.C., as pointed out by Sherrard [2]. Ranking by combined transmitted and received volumes, however, would avoid the latter difficulty.

It is interesting to see that in order to infer the relationship (2.1.2) from assumption (2.1.1), it was necessary to include the "local" volume v_{ii} . Without this inclusion the representation of v_{ij} will be more complicated and not even symmetric (see (4.2.7)). However, we will see in Part 4, that even then the ranking method produces nested solutions that are truly optimal.

<u>2.2 The Maximum Increment Method</u>

The best solution for two installations (i.e. k=2) is to find the highest symmetrized (1.1.9) link benefit - combining transmission and reception - and build installations at the two ends of this link.

The idea of the maximum increment method is to simply add to a best solution S for k facilities by building at a new site i (not in S) which maximizes the incremental benefit

$$\Sigma_{j\varepsilon S}(v_{ij}+v_{ji})$$

achieved by the enlarged capacity of the system. The networks thus synthesized will be nested but not necessarily optimal. Nevertheless they are sufficiently close to optimal to provide a workable model and in general are superior to the solutions obtained by ranking.

Obvious variations are to add two or three sites at once, maximizing their combined contribution. Adding more than three facilities at once will, however, be too expensive computationally.

One might also suggest scanning each solution as to whether any of its installations could profitably be shifted to another site. The sequence of solutions thus generated may, however, no longer be nested.

The maximum increment method has been employed by Sherrard [2] in manual synthesis of some volume maximizing networks. It is excellently suited for automation on a computer.

2.3 Quadratic Integer Programming

Problem (1.1.8) can be transformed into a quadratic integer programming problem of the kind solvable by an algorithm of Witzgall [6]. The transformation is to replace the products $x_i x_j$ by the expressions

$$\frac{x_i+x_j-(x_i-x_j)^2}{2} \quad . \quad *$$

* The writer does not remember from what source he learned this "trick" in 1971.

Indeed, these expressions have the same value as $x_i x_j$ for all 4 combinations of 0,1 values which x_i and x_j can assume. The quadratic programming algorithm further requires integer coefficients in the function to be maximized. This means that the link benefits v_{ij} should be nonnegative integers divisible by two. This can always be achieved by adding a suitable constant to insure nonnegativity and by suitably scaling so that the rounding to an even integer does not introduce an appreciable error.

The method will produce an exact solution. It is, however, not recommended for problems with more than 20 potential sites, since running time will be a problem.

2.4 Linear Integer Programming

The Problem (1.1.7) can in various ways be formulated as a Linear Integer Programming Problem. We will present three such formulations, two of them closely related to the reformulations of general polynomial 0,1-programs developed by Glover and Woolsey [7]. In all three cases one replaces the product $x_i x_j$ by a new variable y_{ij}:

(2.4.1) $$y_{ij} = x_i x_j \quad \text{for all} \quad 1 \leq i,j \leq n \ , \ i \neq j \ .$$

Linear side conditions will have to be introduced to enforce this nonlinear relationship.

The first formulation is due to A.J. Goldman (NBS Applied Mathematics Division):

$$(2.4.2) \qquad \underline{\text{Max}} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}} v_{ij} y_{ij} \qquad \underline{\text{subject to}}$$

$$\sum_{\substack{j=1 \\ j\neq i}} y_{ij} = (k-1)x_i \qquad \underline{\text{for}} \quad 1 \leq i \leq n ,$$

$$y_{ij} = y_{ji} \qquad \underline{\text{for}} \quad 1 \leq i,j \leq n ,$$

$$\sum_{i=1}^{n} x_i = k ,$$

$$x_i = \begin{cases} 0 \\ 1 \end{cases} \qquad \underline{\text{for}} \quad 1 \leq i \leq n ,$$

$$0 \leq y_{ij} \leq 1 \qquad \underline{\text{for}} \quad 1 \leq i,j \leq n .$$

The first set of equations expresses the fact that for each site i with value $x_i=1$ precisely $k-1$ of the products $x_i x_j$, $i\neq j$, have value 1 , whereas $x_i x_j$ is always zero if $x_i=0$. These equations are therefore necessary. They are also sufficient. Indeed, $\Sigma x_i=k$ implies that there are precisely k nonzero values x_i . For an i with $x_i=0$, all values y_{ij} must vanish. For an i with $x_i=1$, there can be at most $k-1$ nonzero values y_{ij} since $y_{ji}=y_{ij}>0$ requires $x_j=1$; since the values y_{ij} for fixed i add up to $k-1$, all y_{ij} must assume their maximum value, namely 1 . It follows that $y_{ij}=1$ whenever $x_i=1$ and $x_j=1$, and $y_{ij}=0$ whenever $x_i=0$ or $x_j=0$. In other words, $y_{ij}=x_i x_j$, which was to be shown.

Clearly only half of the variables y_{ij} are needed, as $y_{ij}=y_{ji}$. However, the formulation becomes pesky for expository purposes if carried out in terms of only those variables y_{ij} for which, say, $i < j$.

Note that the integrality of the variables x_i enforces integrality of the variables y_{ij} so that the latter need not be explicitly required. The problem is therefore a <u>mixed integer program</u> (see Garfinkel and Nemhauser [8]) i.e., some but not all variables are required to be integral. There are general purpose computer programs available for such mixed integer programs.

They are expensive to run and not recommended for $n > 30$. "Branch and Bound" methods can be devised for solving problem (2.4.2) which should be superior to the use of a general purpose package, but they will still be expensive inasmuch as they will require the solution of a large continuous linear program (see Gass [9]) as a much repeated subprogram.

The second formulation is due to D. Shier (NBS). Here it is convenient to introduce only the variables y_{ij} with $i<j$. As a consequence we will consider the symmetrized (1.1.9) benefits

$$\bar{v}_{ij} = 1/2(v_{ij} + v_{ji}), \qquad 1 \leq i < j \leq n .$$

Shier's formulation will require that

$$\bar{v}_{ij} > 0 \quad \underline{\text{for}} \quad 1 \leq i < j \leq n .$$

As was shown in Section 1.1, this can be achieved by adding a sufficiently large constant to all link benefits v_{ij}. As the number k of installations is fixed, this will not influence the optimality pattern.

(2.4.3) $$\text{Max} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \bar{v}_{ij} y_{ij} \qquad \underline{\text{subject to}}$$

$$y_{ij} \leq x_i , \; y_{ij} \leq x_j \qquad \underline{\text{for}} \quad 1 \leq i < j \leq n ,$$

$$\sum_{i=1}^{n} x_i = k ,$$

$$x_i = \begin{cases} 0 \\ 1 \end{cases} \qquad \underline{\text{for}} \quad 1 \leq i \leq n .$$

Clearly $y_{ij} \leq x_i x_j$. Since we are maximizing a linear function with positive coefficients, the variables y_{ij} will assume their largest possible values, whence $y_{ij} = x_i x_j$.

This is a <u>mixed integer program</u> (see for instance Garfinkel and Nemhauser [8]). It has many more constraints than the program (2.4.2), and it is consequently less suitable for the application of a general purpose mixed integer package. It is, however, better suited than (2.4.2) for a Branch and Bound method, since its continuous linear program with $0 \leq x_i \leq 1$ has a particularly tractable structure: If it were not for the presence of equation $\Sigma x_i = k$, the continuous linear program would be the dual of a <u>network</u> problem (a proof of this statement will be given in Section 3.1). Such problems are particularly easy to solve, even in the presence of one (or several) side-conditions.

A formulation suggested by J. Edmonds (NBS Applied Mathematics Division) works only with variables y_{ij} , $i < j$, and does not consider variables x_i . Note that the y-variables are in 1-1 correspondence with all possible links between sites. These links define a

(2.4.4) <u>graph</u> G .

The actual installations and the links between them define the

(2.4.5) <u>subgraph</u> S .

For each of the many other subgraphs S , including $S = G$, we define

(2.4.6) $$y(S) = \Sigma_{\ell \epsilon S} y_\ell$$

where ℓ denotes links and y_ℓ denotes the corresponding y-variables. Finally if $(P,\bar{P})$ denotes a

(2.4.7) <u>partition</u>

of the sites into two classes, then the

(2.4.8) <u>cutset</u> $C(P,\bar{P})$

is defined as the set of all links with one end in P and the other end in $\bar{P}$. In these terms, Edmonds suggests the formulation ($|P|$, $|\bar{P}|$ = cardinalities of P, $\bar{P}$):

$$\text{Max} \quad \Sigma_{\ell \epsilon G}\, y_\ell v_\ell \qquad \underline{\text{subject to}}$$

$$y(G) = \frac{k(k-1)}{2}$$

$$y_\ell = \begin{cases} 0 \\ 1 \end{cases} \qquad \underline{\text{for}}\ \ell \epsilon G\ .$$

<u>for partitions</u> $(P,\bar{P})$:

$$y(C(P,\bar{P})) \leq \max\{(k-h)h \mid \max\{0,k-|\bar{P}|\} \leq h \leq \min\{k,|P|\}\}$$

The above conditions are necessary. Indeed, if h denotes the number of EMFs in P, then there are $k-h$ EMFs in $\bar{P}$, and the cut set $C(P,\bar{P})$ consequently contains $(k-h)h$ links between EMFs. As $h \leq |P|$ and $k-h \leq |\bar{P}|$, h can range as follows

$$\max\{0,k-|\bar{P}|\} \leq h \leq \min\{k,|P|\}\ ,$$

and the maximum of $(k-h)h$ achieved for integers in this range is clearly an upper bound on $C(P,\bar{P})$.

It is conjectured that the inequalities also suffice to make this a correct problem formulation. It is not suitable for a general purpose integer programming package because now all variables have to be integer and because of the huge number of constraints. It might lend itself, however, to an efficient stepwise computational method based on a linear program which at each stage imposes only a subset of the constraints, adding one or more of those violated by the "solution" at the preceding stage. Development of such a method would be an advanced research project.

2.5. Concave Minimization Program

For the sake of completeness, and without discussing possible solution methods, we note that the problem (1.1.7) of selecting a fixed number of EMFs can also be formulated in terms of minimizing a concave function subject to linear equations and inequality conditions.

The reader verifies immediately that

$$\min_{\substack{\Sigma x_i = k \\ 0 \le x_i \le 1}} \ \max_{\substack{y_{ij} \le x_i \\ y_{ij} \le x_j}} \ \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} y_{ij} = \min_{\substack{\Sigma x_i = k \\ 0 \le x_i \le 1}} \ \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \min\{x_i, x_j\} \ .$$

We now claim that

(2.5.1) <u>the expression</u>

$$Y(x) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \min\{x_i, x_j\} \ ,$$

<u>where</u>

$$\sum_{i=1}^{n} x_i = k \ , \ 0 \le x_i \le 1 \quad \underline{\text{for}} \quad 1 \le i \le n \ ,$$

<u>is minimized for 0,1-values of</u> x_i , <u>satisfying the above conditions</u>.

Note that no weights $\bar{v}_{ij}$ are considered in the above minimization. All feasible integer solutions are therefore optimal with $Y(x) = k(k-1)/2$ the minimum value.

Proof: Consider any non-integral solution, and assume for simplicity of notation that

(2.5.2) $$x_1 \le x_2 \le \cdots \le x_n ,$$

so that $\min\{x_i,x_j\} = x_i$ for $i < j$ and therefore

$$Y(x) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} x_i = \sum_{i=1}^{n-1} (n-i)x_i = \sum_{i=1}^{n} (n-i)x_i = nk - \sum_{i=1}^{n} ix_i .$$

Let s be the smallest index such that

$$x_s > 0 ,$$

and t the largest index such that

$$x_t < 1 .$$

We **claim** that $Y(x)$ is not minimal as long as there exists a non-integer x_i ,

$$0 < x_i < 1 ,$$

in other words, as long as $s \le t$. Note that $s = t$ would imply that just one of the values x_i was not integer, contradicting the fact that all x_i add up to integer k. Thus $s \le t$ implies

$$s < t .$$

In the latter case, let

$$\theta = \min\{x_s,\ 1-x_t\} > 0 ,$$

and modify the values x_i as follows:

$$\bar{x}_i = \begin{cases} x_s - \theta & \text{if } i = s \\ x_t + \theta & \text{if } i = t \\ x_i & \text{otherwise.} \end{cases}$$

Clearly, $\Sigma \bar{x}_i = k$ and $0 \leq \bar{x}_i \leq 1$ for $1 \leq i \leq n$. Furthermore, the order-relationships (2.5.2) remain unchanged. Thus

$$Y(\bar{x}) = nk - \sum_{i=1}^{n} i\bar{x}_i$$

and

$$Y(\bar{x}) - Y(x) = - \sum_{i=1}^{n} i(\bar{x}_i - x_i) = s\theta - t\theta < 0 ,$$

which shows that integrality is necessary for $Y(x)$ to be minimized. It is clearly sufficient, as all 0,1 values x_i with $\Sigma x_i = k$ yield $Y(x) = k(k-1)/2$, which thus is the actual minimum value.

We now denote by

$$M$$

a very big number, and claim that

(2.5.3) <u>the solutions</u> $(x_1,\ldots,x_n)$ <u>of the minimization problem</u>

$$\underline{\text{Min}} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (M - \bar{v}_{ij}) \min\{x_i, x_j\} \quad \underline{\text{subject to}}$$

$$\sum_{i=1}^{n} x_i = k ,$$

$$0 \leq x_i \leq 1 \quad \underline{\text{for}} \quad 1 \leq i \leq n$$

<u>are also the solutions of the problem</u> (1.1.8).

<u>Proof</u>: For sufficiently large M, the above minimization will behave like the minimization of $Y(x)$ in (2.5.1), and therefore admit only integer values x_i as minimum solutions. For such solutions

$$\min\{x_i, x_j\} = x_i x_j$$

and therefore

$$\mathrm{Min}\Sigma\Sigma(M-\bar{v}_{ij})\min\{x_i, x_j\} = -\ \mathrm{Max}\Sigma\Sigma\bar{v}_{ij} x_i x_j + \text{const.}$$

3. THE RHYS APPROACH

In this part we will elaborate on the cost-benefit approach to network synthesis. We follow the lines established by J.M.W. Rhys [4] for more general system design problems.

3.1 Network formulation

We are dealing with the Network Synthesis Problem with Cost-Benefit Optimization as spelled out in (1.2.3). We linearize the problem using the method described in (2.4.3) for the case of positive link benefits $\bar{v}_{ij}$:

$$\text{(3.1.1)} \qquad \underline{\text{Max}} \quad \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \bar{v}_{ij} y_{ij} - \sum_{i=1}^{n} c_i x_i \qquad \underline{\text{subject to}}$$

$$y_{ij} \leq x_i \ , \ y_{ij} \leq x_j \qquad \underline{\text{for}} \ \ 1 \leq i < j \leq n \ ,$$

$$x_i = \begin{cases} 0 \\ 1 \end{cases} \quad \underline{\text{for}} \quad 1 \leq i \leq n \ .$$

The difference between formulations (2.4.3) and (3.1.1) is that the constraint $\Sigma x_i = k$ has been removed and instead of it a penalty term $\Sigma c_i x_i$ has been added to the objective function.

Now, the important observation is that in (3.1.1) the condition

$$x_i = \begin{cases} 0 \\ 1 \end{cases}$$

can be replaced by

$$0 \leq x_i \leq 1$$

without changing the problem. In other words, problem (3.1.1) is in reality not an integer programming problem but a normal continuous linear program:

$$(3.1.2) \qquad \underline{\text{Max}} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \bar{v}_{ij} y_{ij} - \sum_{i=1}^{n} c_i x_i \qquad \underline{\text{subject to}}$$

$$y_{ij} \leq x_i \ , \ y_{ij} \leq x_j \quad \underline{\text{for}} \quad 1 \leq i < j \leq n \ ,$$

$$0 \leq x_i \leq 1 \quad \underline{\text{for}} \quad 1 \leq i \leq n \ .$$

The equivalence of formulations (3.1.1) and (3.1.2) is due to the fact - to be shown below - that problem (3.1.2) is the dual of a network flow problem of the kind described for instance in Ford and Fulkerson [10]. We have mentioned this fact already in Section 2.4. Such network problems and their duals automatically yield integer values for variables with integer bounds. Furthermore, problems of this kind are computationally much easier to handle than continuous linear programs and certainly much easier than general integer linear programs.

(3.1.3) Theorem: Problem (3.1.2) is the dual of a minimum cost flow network problem.

Proof: In order to dualize (3.1.2), we introduce multipliers corresponding to its constraints (see for instance Gass [9]):

$$r_{ij} \geq 0 \quad \text{for} \quad 1 \leq i < j \leq n \quad (y_{ij} - x_i \leq 0) \ ,$$

$$s_{ij} \geq 0 \quad \text{for} \quad 1 \leq i < j \leq n \quad (y_{ij} - x_j \leq 0) \ ,$$

$$t_i \geq 0 \quad \text{for} \quad 1 \leq i \leq n \quad (x_i \leq 1) \ ,$$

$$w_i \geq 0 \quad \text{for} \quad 1 \leq i \leq n \quad (-x_i \leq 0) \ .$$

$$\underline{\text{Min}} \sum_{i=1}^{n} t_i \quad \underline{\text{subject to}}$$

$$r_{ij} + s_{ij} = \bar{v}_{ij} \quad \underline{\text{for}} \quad 1 \leq i < j \leq n$$

$$- \sum_{j=i+1}^{n} r_{ij} - \sum_{j=1}^{i-1} s_{ji} + t_i - w_i = - c_i \quad \text{for} \quad 1 \leq i \leq n \ ,$$

$$r_{ij}, s_{ij}, t_i, w_i \geq 0 \ .$$

Substituting

$$s_{ij} = \bar{v}_{ij} - r_{ij} \, , \quad c_i^* = -c_i + \sum_{j=1}^{i-1} \bar{v}_{ji} \, ,$$

and adding an irrelevant constraint gives

(3.1.4) $$\underline{\text{Min}} \quad \sum_{i=1}^{n} t_i \quad \underline{\text{subject to}}$$

$$- \sum_{i=1}^{n} t_i + \sum_{i=1}^{n} w_i = - \sum_{i=1}^{n} c_i^* \, ,$$

$$- \sum_{j=i+1}^{n} r_{ij} + \sum_{j=1}^{i-1} r_{ji} + t_i - w_i = c_i^* \quad \underline{\text{for}} \quad 1 \leq i \leq n \, ,$$

$$0 \leq r_{ij} \leq \bar{v}_{ij} \quad \underline{\text{for}} \quad 1 \leq i < j \leq n \, ,$$

$$t_i \, , \, w_i \geq 0 \quad \underline{\text{for}} \quad 1 \leq i \leq n \, .$$

In order to see that the first constraint equation is indeed superfluous, we note that the sum of all constraint equations is identically zero, as each coefficient r_{ij} occurs exactly twice with opposite signs.

Now construct a network as follows: nodes are the numbers $0,1,\ldots,n$. The links adjacent to node 0 are one to and one from each node $i \neq 0$, with flows corresponding to variables t_i and w_i, respectively. The remaining arcs point from each index $i > 0$ to all indices $j > i$, with capacities $\bar{v}_{ij}$ and flows corresponding to the variables r_{ij}. The constraint equations are "Kirchhoff conditions", stating for each node the balance equation:

$$-(\text{flow out}) + (\text{flow in}) = (\text{sink strength}) \, .$$

This proves Theorem (3.1.3).

The integrality of x_i follows from the fact that x_i is the shadow-price for the constraint equation corresponding to node i , and these shadow-prices are obtained as additive combinations of the cost-coefficients, which are in our case of value 1 and 0.

3.2 Penalty Parametrization

Suppose that all site-related costs are equal:

$$c_i = C \quad \underline{\text{for}} \quad 1 \le i \le n \text{ .}$$

Then the objective function of (3.1.1) becomes

$$\text{(3.2.1)} \qquad \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \bar{v}_{ij} y_{ij} - C \sum_{i=1}^{n} x_i \text{ .}$$

We can now treat C as a

(3.2.2) <u>parameter</u>

of the problem, i.e. we can ask for the solutions of (3.1.1) in <u>dependence</u> on the value of C. A small change in C will in general not change the optimal solution (the optimal deployment of facilities). The C-axis is thus divided into a number of intervals ("regimes") in each of which a certain network is optimal. Parametric Programming (see Gass [9]) is a computational technique for efficiently determining in succession the "breakpoints" which are the boundaries of the various regimes.

Since $\bar{v}_{ij} > 0$, the optimal solution for $C=0$ is to put an EMF at every site. As C increases, a network of fewer sites is encountered, and so on, until finally the penalty C becomes so big that the optimal strategy is to build no facilities at all. There will thus be a sequence of sizes

$$\text{(3.2.3)} \qquad n=k_o > k_1 > k_2 > \ldots > 0$$

of optimal solutions. Clearly, each of these networks will be optimal when compared to others of the same size.

If the link-benefits are indeed revenues expressed in monetary units, then the penalties C can be interpreted as average installation costs. However, if the link-benefits represent quantities other than revenues as, for instance, mail volumes or service improvements, then C becomes just an abstract

(3.2.4) penalty.

No meaning attaches to its values as such. Its purpose is to generate a sequence of networks of sizes (3.2.3). These networks will then be solutions to the Network Synthesis Problem with Prescribed Number of Installations (1.1.7) for the k-values in (3.2.3). The sequence (3.2.3) may not contain all numbers between 0 and n, but it is expected that it contains a majority of them. The penalty interval associated with each optimal network, i.e. the range over which the penalty can vary so that the same network will be optimal, yields a measure of preference for the network: if the penalty interval is big, then the associated optimal network is a preferred one. The above penalty parametrization method will therefore produce a set of preferred network sizes for arbitrary measures of benefit.

3.3 Cost-Benefit Parametrization

For the purposes of a direct cost-benefit analysis, let

r_{ij} = electronic mail revenue - conventional mail revenue

c_{ij} = direct electronic operating cost - direct conventional operating cost.

Then

$$p_{ij} = r_{ij} - c_{ij} = \text{operating revenue (benefit)}$$

for transmissions from site i to site j.

In order to demonstrate one possible parametric approach - not necessarily a recommended one - we assume that there are flat rates for both electronic and conventional transmission per unit of mail; that costs are distance independent and proportional to volume; and that demand volume is a uniform fraction of

v_{ij} = ODIS transmissible volumes.

This uniform fraction is denoted by

$$D = \text{demand level} .$$

If

$$R = \text{electronic rate} - \text{conventional rate},$$
$$B = \text{electronic unit cost} - \text{conventional unit cost} ,$$

then

$$r_{ij} = RDv_{ij} , \quad c_{ij} = BDv_{ij} ,$$

and

$$p_{ij} = r_{ij} - c_{ij} = D(R-B)v_{ij} = Tv_{ij}$$

where

$$T = D(R-B) .$$

With

$$c_i = \text{indirect operating costs at site } i \text{ ,}$$

we then have the parametric problem

$$\underline{\text{Max}} \quad T \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} v_{ij} y_{ij} - \sum_{i=1}^{n} c_i x_i \qquad \underline{\text{subject to}}$$

$$y_{ij} \leq x_i \text{ , } y_{ij} \leq x_j \text{ ,}$$

$$0 \leq x_i \leq 1 \text{ .}$$

This is a parametric program in its simplest form: linear in the single parameter T. More sophisticated formulations would be nonlinear in a single parameter or linear in two parameters.

Finding "best nested" solutions (1.3.1) can be defined and treated as finding the best approximation of the given problem by problems which are solvable by ranking.

4.1 Potentials

We recall the observation of Section 2.1 that if the link benefits v_{ij} have the special form

(4.1.1) $$v_{ij} = \frac{v_i v_j}{V} , \; i \neq j ,$$

where v_i , $v_j > 0$ will be called

(4.1.2) <u>potentials</u>

associated with sites, then ranking by potentials will yield an optimal solution for each prescribed number of installations.

Note that if potentials exist satisfying (4.1.1), then there also exist potentials v_i satisfying

(4.1.3) $$v_{ij} = v_i v_j .$$

Indeed, all one has to do is replace v_i by $v_i/\sqrt{V}$.

For arbitrary link benefits v_{ij}, potentials will not in general exist such that (4.1.3) is satisfied. However, one may ask the question what is the "best" approximation - say, in the sense of least squares - of the given link benefits v_{ij} by numbers of the form $v_i v_j$. Ranking by the potentials v_i so determined, can be considered to define a best sequence of nested solutions.

If a least squares approach is adopted directly, then the following minimization problem results:

$$(4.1.4) \qquad \underline{\text{Min}} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (\bar{v}_{ij} - v_i v_j)^2 \quad \underline{\text{subject to}}$$

$$v_i \geq 0 \quad \underline{\text{for}} \quad 1 \leq i \leq n .$$

Note that we have passed from the v_{ij} to the symmetrized quantities $\bar{v}_{ij}$ (1.1.9), since we seek to approximate by symmetric quantities $(v_i v_j = v_j v_i)$. This is a nonlinear programming problem - the objective function is a polynomial of fourth degree - whose solution does not appear easy enough for this approach to be attractive.

A more tractable approach can be obtained by passing to logarithms:

$$\bar{u}_{ij} = \log \bar{v}_{ij} , \quad u_i = \log v_i .$$

Noting that the desired exact relation $\bar{v}_{ij} = v_i v_j$ is equivalent to $\bar{u}_{ij} = u_i + u_j$, we are led to replace (4.1.4) by

$$(4.1.5) \qquad \text{Min} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (\bar{u}_{ij} - u_i - u_j)^2 ,$$

This problem admits a simple closed-form solution (A.J. Goldman, NBS). Indeed, equating the i-th partial derivative of the objective function (4.1.5) to zero yields

$$\sum_{\substack{j=1 \\ j \neq i}}^{n} (\bar{u}_{ij} - u_i - u_j) = 0 ,$$

or equivalently

$$(n-1)u_i + \sum_{\substack{j=1 \\ j \neq i}}^{n} u_j = \sum_{\substack{j=1 \\ j \neq i}}^{n} \bar{u}_{ij} ,$$

which can be rewritten

$$(4.1.6) \qquad (n-2)u_i + \sum_{j=1}^{n} u_j = \sum_{\substack{j=1 \\ j\neq i}}^{n} \bar{u}_{ij} .$$

Summing (4.1.6) over $i=1,2,\ldots,n$ yields

$$(n-2) \sum_{i=1}^{n} u_i + n \sum_{j=1}^{n} u_j = \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} \bar{u}_{ij} ,$$

or equivalently

$$(2n-2) \sum_{j=1}^{n} u_j = 2 \sum_{i=1}^{n} \sum_{j=i+1}^{n} \bar{u}_{ij} .$$

Solving this for the sum on the left-hand side, substituting the result into (4.1.6), and solving for u_i, yields our final result:

$$(4.1.7) \qquad u_i = (\sum_{\substack{j=1 \\ j\neq i}}^{n} u_{ij})/(n-2) - (\sum_{k=1}^{n} \sum_{j=k+1}^{n} \bar{u}_{kj})/(n-1)(n-2) .$$

4.2 Rank-Optimizable Objectives

The objective function to be maximized is

$$F = \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} v_{ij} x_i x_j .$$

It was seen before, that if the objective function has coefficients of the form

$$v_{ij} = v_i v_j ,$$

where v_i, v_j are nonnegative site-associated potentials (4.1.2), then ranking by potentials yields an optimal solution for each prescribed number of installations. We say therefore that such an objective function is

(4.2.1) rank-optimizable .

In order to describe other rank-optimizable functions, let ϕ be an arbitrary function of two variables. With respect to some given sequence of potentials v_i we then form the link benefits $v_{ij} = \phi(v_i, v_j)$.
Under which circumstances is the corresponding objective function rank-optimizable?

(4.2.2) Theorem: Given potentials $v_i > 0$ for each site, a function ϕ of two variables defines an objective function ,

$$v_{ij} = \phi(v_i, v_j) \quad \text{for } i \neq j ,$$

which is rank-optimizable (4.2.1) by the potentials v_i , if ϕ is monotone increasing in each variable in the positive quadrant:

$$\left.\begin{array}{l} \dfrac{\partial\phi(z_1, z_2)}{\partial z_1} \geq 0 \\[2ex] \dfrac{\partial\phi(z_1, z_2)}{\partial z_2} \geq 0 \end{array}\right. \qquad \text{for} \qquad z_1, z_2 \geq 0 .$$

Proof: Suppose the sites are numbered by decreasing rank,

$$v_1 \geq v_2 \cdots \geq v_n .$$

Then deploying EMFs at sites 1,...,k realizes the value

$$F_o = F(\{1,\ldots,k\}) = \sum_{i=1}^{k} \sum_{\substack{j=1 \\ j\neq 1}}^{k} \phi(v_i, v_j)$$

of the objective function. Let then

$$m_1 < m_2 < \ldots < m_k$$

be the site indices of any other deployment of the prescribed number k of facilities. It will realize the value

$$F = F(\{m_1, \ldots, m_k\}) = \sum_{i=1}^{k} \sum_{j=1}^{k} \phi(v_{m_i}, v_{m_k}) .$$

Clearly,

$$m_i \geq i , \; m_j \geq j ,$$

whence

$$v_{m_i} \leq v_i , \; v_{m_j} \leq v_j .$$

Thus monotonicity gives

$$\phi(v_{m_i}, v_{m_j}) \leq \phi(v_i, v_{m_j}) \leq \phi(v_i, v_j) ,$$

and therefore

$$F \leq F_o ,$$

which was to be shown.

For symmetric functions ϕ , C.R. Johnson (Postdoctoral Research Fellow, NBS Applied Mathematics Division) has derived a sufficient and necessary condition for generating a rank-optimizable objective function.

(4.2.3) <u>Theorem:</u> $v_{ij} = \phi(v_i, v_j)$ <u>is rank optimizable</u> (4.2.1) <u>in</u> v_i <u>if and only if</u>

(4.2.4) $\phi(y,z) \geq \max\{\phi(x,y), \phi(x,z)\}$ <u>whenever</u> $x \leq y \leq z$.

Proof: (4.2.4) is necessary for any function Φ. Take $n=3$, $k=2$, $v_1=z$, $v_2=y$, $v_3=x$. If rank-optimizability is to hold, the solution set must be $\{y,z\}$ rather than $\{x,y\}$ or $\{x,z\}$ so that (4.2.4) must apply.

To prove sufficiency, assume again that the sites are numbered by decreasing rank, so that we have to show that the sites $\{1,2,\ldots,k\}$ are optimal. Consider any other set

$$M = \{m_1, m_2, \ldots, m_k\} \neq \{1,2,3,\ldots,k\}$$

with

$$m_1 < m_2 < \ldots < m_k$$

so that

$$v_{m_1} \geq v_{m_2} \geq \ldots \geq v_{m_k} .$$

There exists some index $h \leq k$ which is not in M. Let then M' be the set which arises from M by removing m_k and adjoining h instead. Using the notation $F(M)$ and $F(M')$ for the total benefits accruing from deployments M and M', respectively, we have (using the symmetry of ϕ):

$$F(M) = \sum_{i=1}^{k} \sum_{\substack{j=1 \\ j\neq i}}^{k} \phi(v_{m_i}, v_{m_j}) = \sum_{i=1}^{k-1} \sum_{\substack{j=1 \\ j\neq i}}^{k-1} \phi(v_{m_i}, v_{m_j}) + 2 \sum_{i=1}^{k-1} \phi(v_{m_i}, v_{m_k})$$

$$F(M') = \sum_{i=1}^{k-1} \sum_{\substack{j=1 \\ j\neq i}}^{k-1} \phi(v_{m_i}, v_{m_j}) + 2 \sum_{i=1}^{k-1} \phi(v_{m_i}, v_h)$$

and consequently

$$(4.2.5) \qquad F(M') - F(M) = 2 \sum_{i=1}^{k-1} (\phi(v_{m_i}, v_h) - \phi(v_{m_i}, v_{m_k})) .$$

Now distinguish two cases: (I) $m_i < h$; (II) $h < m_i$.

In case (I), $m_i < h < m_k$ as $m_k > k \geq h$, whence

$$x = v_{m_k} \leq y = v_h \leq z = v_{m_i} ,$$

and (4.2.4) gives in view of symmetry

$$\phi(v_{m_i}, v_h) = \phi(z,y) = \phi(y,z) \geq \phi(z,x) = \phi(v_{m_i}, v_{m_k}) .$$

In case (II), $h < m_i \leq m_k$, whence

$$x = v_{m_k} \leq y = v_{m_i} \leq z = v_h ,$$

and (4.2.4) gives

$$\phi(v_{m_i}, v_h) = \phi(y,z) \geq \phi(x,y) = \phi(v_{m_i}, v_{m_k}) .$$

Thus $F(M') - F(M) \geq 0$, which was to be shown.

Condition (4.2.4) is weaker than monotonicity(4.2.2). This is shown by the following example (W.A. Horn [11]) of a symmetric function ϕ which satisfies (4.2.4), but is not monotone (4.2.2).

$$\phi = \phi_1 + \phi_2 , \text{ where } \phi_1(x,y) = \begin{cases} x+y & \text{if } |x-y| \leq 1 \\ 2\min(x,y)+1 & \text{else} \end{cases} \text{ and } \phi_2(x,y) = e^{-x-y-1} .$$

Theorem (4.2.2) yields that, for instance, the benefit pattern

$$(4.2.6) \qquad v_{ij} = b + av_i + cv_j + v_i v_j ,\ i \neq j ,$$
$$a, b, c \geq 0 ,$$

is rank-optimizable. This more flexible expression should permit a closer approximation to a given set of v_{ij}'s than do the simple products $v_i v_j$ mentioned in Section 4.1.

Another rank-optimizable objective function is generated by

(4.2.7) $$\phi(x,y) = \frac{xy}{a-x}, \qquad 0 \leq x < a .$$

This function relates to the case discussed in Section 2.1, where the potentials v_i are the outgoing, non-local mail volumes at location i,

$$v_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} v_{ij} ,$$

the link volumes v_{ij} are proportional to the non-local mail volumes at the destinations j (2.1.1)

$$v_{ij} = \rho_i v_j ,$$

and volume is to be maximized. Clearly

$$v_i = \sum_{j \neq i} v_{ij} = \rho_i \sum_{j \neq i} v_j = \rho_i (V - v_i) ,$$

and therefore $\rho_i = v_i/(V-v_i)$, where V is total outgoing non-local mail volume. Thus there is indeed a functional relationship between link benefits and the potentials v_i of the form (4.2.7), which then establishes rank optimizability (4.2.1).

5. RECOMMENDATIONS

Four different approaches have been evaluated in Parts 2-4:

(1) Use sequential methods such as the maximum increment method to arrive at almost optimal networks for given numbers of installations.

(2) Use integer programming methods for determining "true" optimal networks for given numbers of installations.

(3) Use the Rhys approach employing network optimization to cost out installation expenses versus link benefits, thereby arriving at a suggested number of installations in optimal position.

(4) Approximate the link benefits by expressions in some associated site "potentials" by which to rank the sites.

Approach (2) does not appear suitable: the available mixed-integer programming packages, when applied to the problem at hand, will most probably be too expensive to run. Good algorithms specifically tailored for the problem are not yet known. The necessity of finding a "true" optimum, in view of the approximations necessarily involved in the problem formulation, seems hardly urgent enough to warrant high computer or research expenses.

The usefulness of approach (4) depends critically on the data. The calculation of the best approximation may not be cheap, and there seems to be no inherent reason why its results should be better than the ones derived by the sequential methods of (1). A low level experimental and developmental effort may nevertheless be rewarding.

Automation of the maximum increment method is definitely recommended. This method will be able to rely mainly on ODIS data and provide an efficient and reliable configuration tool for EMS-studies. Its capabilities include optimization of total transmissible volume as well as total service improvement.

A strong recommendation is to implement the Rhys approach. The advantage of this approach is that it permits the inclusion, interpretation, and evaluation of marketing information for a straight cost-benefit analysis. The approach addresses the problem of how many installations can be justified, not just how to deploy a prescribed number of them. Even in the absence of cost and marketing information, the Rhys approach can be definitely useful for parametric analyses over a wide range of critical variables.

The power of the Rhys approach is not restricted to the problem at hand, but can be used in other system planning problems. The implementation of this approach would provide a quite general planning tool of considerable potential.

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The concept of electronic message (mail) transmission has been the subject of several feasibility studies during the past decade. It requires the installation of electronic message handling facilities at selected locations. If transmission is to be via communications satellite, then any such facility can transmit to and receive from any other one. In this report, the mathematical aspects of choosing the number and locations of these facilities are examined. An inventory of solution methods is presented, along with recommendations as to which among them should be employed or developed further.

17. KEY WORDS *(six to twelve entries; alphabetical order; capitalize only the first letter of the first key word unless a proper name; separated by semicolons)* Communication; cost-benefit; deployment; electronic transmission; facility location; mail; mathematical programming; message network synthesis; network optimization; satellite; service improvement.

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