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A Method of Calibrating Two-Dimensional Reference Plates

Charles P. Reeve

Institute for Basic Standards
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A Method of Calibrating Two-dimensional Reference Plates

by

Charles P. Reeve

1. Introduction

The Dimensional Technology Section recently began investigating the problem of precisely measuring two-dimensional reference plates. The study was motivated by two factors:

(1) The awareness of a growing need for this type of calibration service.

(2) The acquisition of a sophisticated three coordinate measuring machine.

After the hardware was secured the immediate goal of the investigation became the development of a measurement algorithm. In particular, an algorithm was needed which would estimate both the mean and standard deviation of the relative (x,y) coordinates of the points on the reference plate. Additionally, it was desired to place as few restrictions as possible on the placement of the plate on the measuring machine and to include some machine parameters in the mathematical model if possible. A search of the literature revealed one model which incorporated redundant measurements and polar coordinates. An examination of the structure of this model was helpful in developing a model in rectangular coordinates.

The mathematical model which was developed is nonlinear and has a high level of redundancy. It lays no claim to being the ultimate in sophistication for this complex measurement problem, but is felt to be adequate. This report is concerned with the attainment of this immediate goal. The material is divided into two main parts, the physical model and the mathematical model. The physical model is not meant to be strongly emphasized here and therefore is described briefly. The main thrust is the presentation in detail of the mathematical model and its data reduction algorithm.

The ultimate goal of the investigation is to develop a measurement process which is under statistical control. On the way to attaining this goal, the measurement process must be applied enough times to see if unmodeled systematic errors are significantly large and can be identified and accounted for. Possible sources of such errors are machine scale inaccuries, temperature effects, and bending effects, to name a few. Hopefully, future reports will evolve as progress is made.

2. The Physical Model

2.1. Reference Plate

2.1.1. Properties

A "two-dimensional reference plate" is a rigid and flat surface on which a number of points, all in the same plane, are defined. The plate is made so that the surface containing the points is parallel to the surface upon which the plate is resting. The current facility is capable of measuring two types of plates, grid plates and ball plates. The points on a grid plate are defined by the intersection of the centers of straight lines which are engraved on the surface. The points on a ball plate are defined to be at the center of each ball.

2.1.2. Coordinate System

The plate coordinate system is coplanar with the points and is designated by (x,y) where the coordinates of the i^{th} point are (x_i,y_i). The axes of the system could reasonably be defined in a number of ways. The system which was adopted defines the coordinate axes in terms of the two points p and q which are arbitrarily chosen unless otherwise specified by the user. The x axis is defined to pass through points p and q while the y axis is defined to pass through p and be perpendicular to the x axis. Thus,

$$x_p = y_p = y_q = 0.$$

The value of x_q is defined to be positive and the positive direction of the y axis is defined such that it forms a left-handed coordinate system which is compatible with the present machine coordinate system (see fig 1).

2.2. Measuring Machine

2.2.1. Capacity

The present facility has a measuring range of 48 inches on the X axis and 24 inches on the Y axis. The vertical clearance is 12 inches. Any plate which has a horizontal dimension greater than 32 inches will not clear the measuring machine in the Y direction and thus cannot be normally rotated to the different positions required during measurement.

2.2.2. Coordinate System

The measuring machine coordinate system lies in the horizontal plane and is denoted by (X,Y). As shown in fig 1, the X axis is defined to be

MEASUREMENT OF i^{th} POINT IN j^{th} POSITION

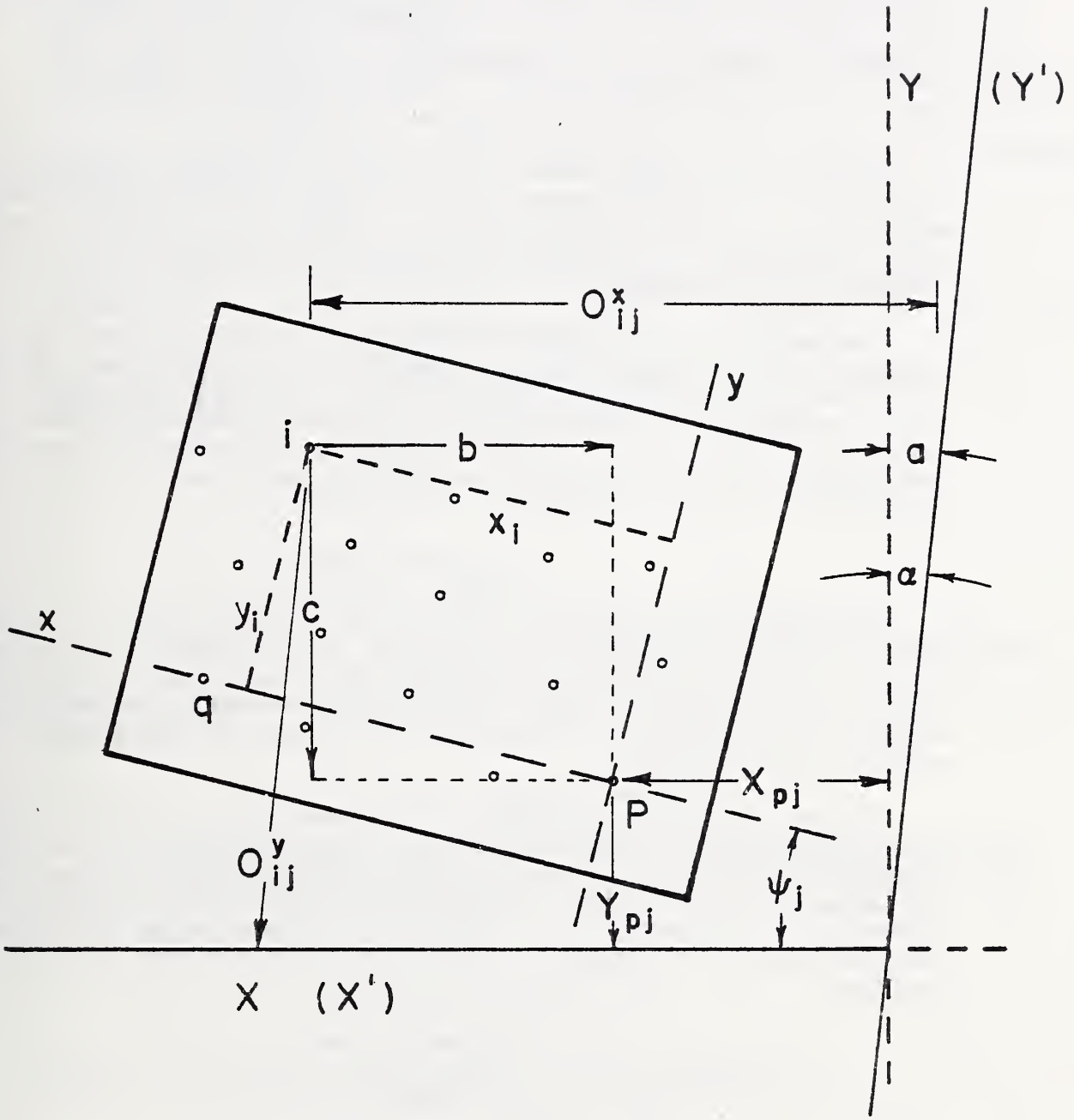

FIGURE 1

colinear with the X' machine axis of measurement and the Y axis is defined to be perpendicular to the X axis. The Y' machine axis of measurement is out-of-parallel with the Y axis by a small angle α. The Y and Y' axes intersect the colinear X and X' axes at the same point. The coordinates of the i^{th} point in the j^{th} plate position are given by (X_{ij},Y_{ij}).

2.2.3. Location of Points

The points on a grid plate are located by centering each intersection of two lines under the crosshairs of a microscope either manually or automatically. The precision with which the points can be located is dependent on the quality of the lines and the precision of the instrument which locates the intersections.

The points on a ball plate are located by centering each ball on the vertical axis of a spindle while the spindle is rotating around the equator of the ball and giving a readout of the horizontal deflection. The ball is considered to be centered when the deflection is minimized. The precision with which the ball can be centered is dependent on the roundness of the ball, the sensitivity of the indicator, and the skill of the operator.

2.3. Setup Procedures

The measurement algorithm calls for the reference plate to be measured in m different positions where $m \geq 3$. The degree of translation of the plate between positions is basically irrelevant, but the angles of rotation are important. Each rotation should be made approximately 360/m degrees (2π/m radians) clockwise so that the plate will be measured in m equally spaced positions.

Before making any measurements, the procedure for rotating the plate should be thoroughly understood so there is no danger of it being unexpectedly obstructed by any part of the measuring machine. In cases where the geometry is such that the plate cannot be measured in equally spaced intervals, it is permissible to choose an alternate method of spacing. Note that setting the angles of rotation by eyesight gives sufficient accuracy.

3. Mathematical Model

3.1. Parameters

3.1.1. Description

The mathematical model is an abstraction of the known parameters of the physical model. Parameters whose existence or form is unknown are initially assumed to have a negligible effect and are put in the category of "random measurement error".

The angle α which is associated with the out-of-perpendicularity of the measurement axes has already been described. The angles $\psi_1, \ldots, \psi_m$ represent the angles of rotation of the plate in its m positions. The angles are measured clockwise from the $\psi = 0$ position which occurs when the x axis of the plate coordinate system is aligned with the X axis of the machine coordinate system. Also associated with the m plate positions are the 2m parameters $X_{p1}, \ldots, X_{pm}$ and $Y_{p1}, \ldots, Y_{pm}$ which taken pairwise are the coordinates of point p with respect to the machine coordinate system.

Since the values x_p, y_p, and y_q were defined to be zero, they are not considered to be parameters of the system. The remaining 2n-3 coordinates of the n points relative to the (x,y) coordinate system are the n-1 values $x_1, \ldots, x_n$ (x_p excluded) and the n-2 values $y_1, \ldots, y_n$ (y_p, y_q excluded). To simplify later expressions, let the entire set of 2n+3m-2 parameters be designated by the vector β where

$$\beta = (\alpha\ X_{p1}\ Y_{p1}\ \psi_1 \cdots X_{pm}\ Y_{pm}\ \psi_m\ x_1\ y_1 \cdots x_n\ y_n)'$$

$$= (\beta_1\ \beta_2 \cdots \beta_{2n+3m-2})' .$$

A complete set of measurements gives the 2mn observations $\left\{0^X_{ij}\right\}$ and $\left\{0^Y_{ij}\right\}$ where i = 1,n and j = i,m. The system then has 2mn-2n-3m+2 degrees of freedom.

3.1.2. Initial Estimates

The nonlinear mathematical model is solved by an iterative process which requires initial estimates for all parameters. Estimates for the values X_{pj} and Y_{pj} are given by

$$X^{(0)}_{pj} = 0^X_{pj} \text{ and } Y^{(0)}_{pj} = 0^Y_{pj}$$

where j = 1,m. The corresponding ψ_j can be estimated by

$$\psi^{(0)}_j = \mathrm{Tan}^{-1}\left(\frac{0^Y_{qj} - 0^Y_{pj}}{0^X_{qj} - 0^X_{pj}}\right) + k_j\ \pi$$

where the value of k_j is dependent on the signs of the numerator and denominator as follows:

num	den	k_j
+	+	0
+	-	1
-	-	1
-	+	2 .

Since α is normally very small, it can be estimated zero. The remaining 2n-3 plate coordinates can be estimated from the measurements made with the plate in the first position by

$$x_i^{(0)} = (0_{i1}^X - X_{p1}^{(0)})\cos \psi_1^{(0)} + (0_{i1}^Y - Y_{p1}^{(0)})\sin \psi_1^{(0)}$$

$$y_i^{(0)} = -(0_{i1}^X - X_{p1}^{(0)})\sin \psi_1^{(0)} + (0_{i1}^Y - Y_{p1}^{(0)})\cos \psi_1^{(0)}.$$

3.2. Measurement Equations

Let 0_{ij}^X and 0_{ij}^Y denote the observed and, if necessary, corrected coordinates of the i^{th} point in the j^{th} position relative to the (X,Y) coordinate system. Then, as shown in fig 1,

$$0_{ij}^X = b + X_{pj} + a + \varepsilon_{ij}^X \quad \text{and}$$

$$0_{ij}^Y = (c + Y_{pj}) \sec \alpha + \varepsilon_{ij}^Y$$

where $b = x_i \cos \psi_j - y_i \sin \psi_j$,

$c = x_i \sin \psi_j + y_i \cos \psi_j$, and

$a = (c + Y_{pj}) \tan \alpha$.

The values ε_{ij}^X and ε_{ij}^Y are independent error values from a distribution with mean zero and variance σ^2. The 2mn observational equations are given by

$$O^X_{ij} = f^X_{ij} + \varepsilon^X_{ij}$$

$$O^Y_{ij} = f^Y_{ij} + \varepsilon^Y_{ij}$$

where i = 1, n and j = 1, m. The regression function f is defined according to the value of i as follows:

(1) if $i \neq p$ and $i \neq q$,

$$f^X_{ij} = x_i \cos \psi_j - y_i \sin \psi_j + X_{pj}$$

$$+(x_i \sin \psi_j + y_i \cos \psi_j + Y_{pj}) \tan \alpha$$

$$f^Y_{ij} = (x_i \sin \psi_j + y_i \cos \psi_j + Y_{pj}) \sec \alpha \ .$$

(2) if i = p,

$$f^X_{ij} = X_{pj} + Y_{pj} \tan \alpha$$

$$f^Y_{ij} = Y_{pj} \sec \alpha \ .$$

(3) if i = q,

$$f^X_{ij} = x_i \cos \psi_j + X_{pj} + (x_i \sin \psi_j + Y_{pj}) \tan \alpha$$

$$f^Y_{ij} = (x_i \sin \psi_j + Y_{pj}) \sec \alpha \ .$$

In vector notation

$$\vec{O} = \vec{f}\ (\alpha, X_{p1}, Y_{p1}, \psi_1, \ldots, x_n, y_n) + \vec{\varepsilon}$$

$$= \vec{f}\ (\beta) + \vec{\varepsilon} \ .$$

3.3. Method of Solution

This nonlinear model can be solved by the Gauss-Seidel method as described by Himmelblau [5][1]. The model is first linearized by expanding it about initial guesses for its parameters in a truncated Taylor series. The approximations are given by

$$f_{ij}^X = (f_{ij}^X)_0 + \sum_{k=1}^{2n+3m-2} \left(\frac{\partial f_{ij}^X}{\partial \beta_k} \right)_0 (\beta_k - \beta_k^{(0)})$$

$$= (f_{ij}^X)_0 + \sum_{k=1}^{2n+3m-2} \left(\frac{\partial f_{ij}^X}{\partial \beta_k} \right)_0 \delta_k^{(0)} \quad , \text{ and}$$

$$f_{ij}^Y = (f_{ij}^Y)_0 + \sum_{k=1}^{2n+3m-2} \left(\frac{\partial f_{ij}^Y}{\partial \beta_k} \right)_0 (\beta_k - \beta_k^{(0)})$$

$$= (f_{ij}^Y)_0 + \sum_{k=1}^{2n+3m-2} \left(\frac{\partial f_{ij}^Y}{\partial \beta_k} \right)_0 \delta_k^{(0)}$$

where $\delta_k^{(0)} = \beta_k - \beta_k^{(0)}$. The $\beta_k^{(0)}$ are the initial estimates for the parameters, and the subscripts on the functions and partial derivatives indicate they are evaluated using those initial estimates. (The expressions for the partial derivatives are given in Appendix A.) Let

$$(z_{ij}^X)_0 = 0_{ij}^X - (f_{ij}^X)_0 \quad \text{and}$$

$$(z_{ij}^Y)_0 = 0_{ij}^Y - (f_{ij}^Y)_0 \ .$$

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[1] Figures in brackets designate listing in the bibliography of the paper.

Then the linearized model is given by the mn pairs of equations,

$$\vdots$$

$$(z_{ij}^X)_0 = \sum_{k=1}^{2n+3m-2} \left(\frac{\partial f_{ij}^X}{\partial \beta_k} \right)_0 \delta_k^{(0)}$$

$$(z_{ij}^Y)_0 = \sum_{k=1}^{2n+3m-2} \left(\frac{\partial f_{ij}^Y}{\partial \beta_k} \right)_0 \delta_k^{(0)}$$

$$\vdots$$

where $\text{Var}\,[(z_{ij}^X)_0] = \text{Var}\,[(z_{ij}^Y)_0] = \sigma^2$. Let

$$z_0 = \begin{bmatrix} \vdots \\ (z_{ij}^X)_0 \\ \\ (z_{ij}^Y)_0 \\ \vdots \end{bmatrix}, \quad \delta^{(0)} = \begin{bmatrix} \delta_1^{(0)} \\ \delta_2^{(0)} \\ \vdots \\ \delta_{2n+3m-2}^{(0)} \end{bmatrix}, \text{ and}$$

$$T_0 = \begin{bmatrix} \vdots & & \vdots \\ \left(\frac{\partial f_{ij}^X}{\partial \beta_1} \right)_0 & \cdots & \left(\frac{\partial f_{ij}^X}{\partial \beta_{2n+3m-2}} \right)_0 \\ \left(\frac{\partial f_{ij}^Y}{\partial \beta_1} \right)_0 & \cdots & \left(\frac{\partial f_{ij}^Y}{\partial \beta_{2n+3m-2}} \right)_0 \\ \vdots & & \vdots \end{bmatrix}$$

The least squares estimation then takes the form

$$E\,(z_0) = T_0\, \delta^{(0)} \quad \text{where}$$

$$\text{Var}\,(z_0) = \sigma^2\, I\ .$$

The normal equations take the form

$$T_0' T_0 \delta^{(0)} = T_0' z_0 .$$

After solving for the vector of unknowns $\delta^{(0)}$, the new approximations for the parameters are given by

$$\beta^{(1)} = \beta^{(0)} + \delta^{(0)} .$$

The calculation is then repeated with the new approximations substituted into z_1 and T_1 to give the estimates $\delta^{(1)}$. The iterations are continued according to the recursion relation

$$\beta^{(k+1)} = \beta^{(k)} + \delta^{(k)}$$

until for some $k=k_0$, the vector $\delta^{(k_0)}$ is sufficiently near zero. The final estimates are then given by

$$\beta^{(k_0+1)} = \beta^{(k_0)} + \delta^{(k_0)} .$$

The number of iterations needed is, of course, dependent on the closeness of the initial approximations to their true values and on the strictness of the convergence criterion. One criterion is that for some small value ω,

$$\left| \delta_i^{(k)} \right| < \omega \qquad \text{for } i=1,2n+3m-2.$$

(The computer algorithm for storing the matrices and solving the normal equations is given in Appendix B.)

3.4. Error Analysis

The variance-covariance matrix of the estimates is given by

$$(T_{k_0}' T_{k_0})^{-1} = M_{k_0} .$$

The predicted values of the observations are given by

$$\hat{z}_{k_0} = T_{k_0} \delta^{(k_0)}$$

and the deviations by

$$d = z_{k_0} - \hat{z}_{k_0} .$$

The estimate of σ is given by

$$s = \sqrt{d'd/(2mn-2n-3m+2)}$$

and the approximate standard deviation of the estimates by

$$s_{\beta_i} = s\sqrt{(M_{k_o})_{ii}} \text{ where } i=1,2n+3m-2.$$

The estimate of the standard deviation of a single measurement, s, is likely to include effects from systematic errors such as machine scale inaccuracies, thermal expansion, and elasticity of the plate and machine. With a machine of high quality and a carefully controlled environment, these errors may be small enough to pass as random errors. In that case, the uncertainty of the calibrated values can be taken to be the three standard deviation limit for random error,

$$3s_{\beta_i}\sqrt{(M_{k_0})_{ii}} .$$

If additional systematic errors are determined to be significant, the estimated limit, E_i, to these errors can be added giving a total uncertainty of the estimates of

$$3s_{\beta_i}\sqrt{(M_{k_0})_{ii}} + E_i .$$

4. The Incomplete Mathematical Model

Plates whose dimensions exceed the measuring range of the X' and Y' axes can still be calibrated if each point on the plate can be measured in at least two positions. The complete model is modified to become an incomplete model where the measurements of off-scale points are omitted. In the matrix T_k and vector z_k the row elements corresponding to omitted measurements are set to zero, or in effect, the omitted measurements are weighted zero. The calculations then proceed as usual except that the degrees of freedom are reduced. Suppose that t is the number of times a

point is omitted from measurement, then the total number of measurements is decreased by 2t, thus the degrees of freedom are reduced to

$$2mn-2n-3m+2-2t.$$

5. Example

This measurement process has been successfully implemented in the calibration of ball plates. One of the first test calibrations involved a plate with 34 one-half inch balls mounted in an irregular pattern within a 30 x 18 inch rectangle. The plate was measured in four positions roughly 90° apart, but not every ball could be measured in each position. The incomplete model was used with the following parameters:

n = 34	(number of points)
m = 4	(number of positions)
t = 10	(total number of points omitted)
df = 174	(degrees of freedom)
p = 1	(point used for origin)
q = 11	(other point used for x axis).

The computed standard deviation of a single measurement was 18.8 microinches. The numbers in table 1 show how the estimates for six selected parameters converged during three iterations. If the initial estimates had not been so close, it may have taken an additional iteration to converge. The convergence criterion was:

$$|\delta_i| \leq .0000005 \text{ for } i = 1,2n+3m-2.$$

The uncertainty of the computed values was taken to be the three standard deviation limit for random error.

6. Conclusion

The idea of using redundant measurements to calibrate two-dimensional plates came about after reading of the experience that Brown [2] had with his two-dimensional comparator which made radial measurements. The incorporation of his ideas into a system with X and Y measurements seems to have been successful.

As of now this method has been applied to three different ball plates. The standard deviation of a single measurement has ranged between 15 and 18 microinches with the standard deviation of the ball coordinates being

slightly less. If this level of precision can be maintained then this method will have significantly increased the precision with which ball plates can be calibrated. Hopefully, the same will hold true for other types of two-dimensional plates.

I wish to acknowledge three people who were of great assistance to me in the development of the methods described in this paper. I thank Dr. Fred Johnson of the Applied Mathematics Division, a one-time associate of Duane Brown, for being kind enough to share with me the ideas behind the sparse matrix algorithm. I also thank Dr. Chris E. Kuyatt of the Optical Physics Division for his most helpful suggestions on linearizing the model and solving by iteration. I finally thank Dr. John A. Simpson, also of the Optical Physics Division, for his initial ideas which got this project off the ground, and for his encouragement which helped me through the early stages of its development.

	x_{21} (inches)	y_{21} (inches)	X_{p2} (inches)	Y_{p2} (inches)	ψ_2	α
initial estimate	25.994200^{0}	8.510790^{0}	-22.600000^{0}	5.500000^{0}	269° 51' 47.24"	0° 0' 0.00"
first iteration	$-.000180^{6}$	$.000559^{2}$	$-.024693^{1}$	$.135834^{6}$	0° 7' 38.49"	-0° 0' 0.18"
second iteration	$.000023^{8}$	$.000011^{0}$	$.000015^{6}$	$-.000026^{3}$	-0° 0' 0.04"	0° 0' 0.00"
third iteration	$.000000^{0}$	$.000000^{2}$	$.000000^{1}$	$-.000000^{1}$	0° 0' 0.00"	0° 0' 0.00"
final estimate	25.994043^{2}	8.511360^{4}	-22.624677^{4}	5.635808^{2}	269° 59' 25.69"	-0° 0' 0.18"
standard deviation	$.000016^{3}$	$.000015^{4}$	$.000013^{7}$	$.000012^{6}$	0° 0' 0.15"	0° 0' 0.08"

Table 1.
Convergence behavior of nonlinear least squares estimates.

Appendix A

Partial Derivatives of the Regression Functions

The partial derivatives of the functions f^X_{ij} and f^Y_{ij} are given in terms of the original parameters of the problem as follows where j=1,m:

(1) i=1,n where $i \neq p$ and $i \neq q$

a. $$\frac{\partial f^X_{ij}}{\partial \alpha} = (x_i \sin \psi_j + y_i \cos \psi_j + Y_{pj}) \sec^2 \alpha$$

b. $$\frac{\partial f^X_{ij}}{\partial X_{pj}} = 1$$

c. $$\frac{\partial f^X_{ij}}{\partial Y_{pj}} = \tan \alpha$$

d. $$\frac{\partial f^X_{ij}}{\partial \psi_j} = -x_i \sin \psi_j - y_i \cos \psi_j + x_i \tan \alpha \cos \psi_j - y_i \tan \alpha \sin \psi_j$$

e. $$\frac{\partial f^X_{ij}}{\partial x_i} = \cos \psi_j + \tan \alpha \sin \psi_j$$

f. $$\frac{\partial f^X_{ij}}{\partial y_i} = -\sin \psi_j + \tan \alpha \cos \psi_j$$

g. $$\frac{\partial f^Y_{ij}}{\partial \alpha} = (x_i \sin \psi_j + y_i \cos \psi_j + Y_{pj}) \sec \alpha \tan \alpha$$

h. $$\frac{\partial f^Y_{ij}}{\partial X_{pj}} = 0$$

i. $$\frac{\partial f^Y_{ij}}{\partial Y_{pj}} = \sec \alpha$$

j. $$\frac{\partial f^Y_{ij}}{\partial \psi_j} = x_i \sec \alpha \cos \psi_j - y_i \sec \alpha \sin \psi_j$$

k. $\frac{\partial f^{Y}_{ij}}{\partial x_i} = \sec \alpha \sin \psi_j$

l. $\frac{\partial f^{Y}_{ij}}{\partial y_i} = \sec \alpha \cos \psi_j$

(2) i=p

a. $\frac{\partial f^{X}_{ij}}{\partial \alpha} = Y_{pj} \sec^2 \alpha$

b. $\frac{\partial f^{X}_{ij}}{\partial X_{pj}} = 1$

c. $\frac{\partial f^{X}_{ij}}{\partial Y_{pj}} = \tan \alpha$

d. $\frac{\partial f^{X}_{ij}}{\partial \psi_j} = 0$

e. $\frac{\partial f^{Y}_{ij}}{\partial \alpha} = Y_{pj} \sec \alpha \tan \alpha$

f. $\frac{\partial f^{Y}_{ij}}{\partial X_{pj}} = 0$

g. $\frac{\partial f^{Y}_{ij}}{\partial Y_{pj}} = \sec \alpha$

h. $\frac{\partial f^{Y}_{ij}}{\partial \psi_j} = 0$

(3) i=q

a. $\frac{\partial f^{X}_{ij}}{\partial \alpha} = (x_i \sin \psi_j + Y_{pj}) \sec^2 \alpha$

b. $\frac{\partial f_{ij}^{X}}{\partial X_{pj}} = 1$

c. $\frac{\partial f_{ij}^{X}}{\partial Y_{pj}} = \tan \alpha$

d. $\frac{\partial f_{ij}^{X}}{\partial \psi_{j}} = x_i \, (-\sin \psi_j + \tan \alpha \cos \psi_j)$

e. $\frac{\partial f_{ij}^{X}}{\partial x_{i}} = \cos \psi_j + \tan \alpha \sin \psi_j$

f. $\frac{\partial f_{ij}^{Y}}{\partial \alpha} = (x_i \sin \psi_j + Y_{pj}) \sec \alpha \tan \alpha$

g. $\frac{\partial f_{ij}^{Y}}{\partial X_{pj}} = 0$

h. $\frac{\partial f_{ij}^{Y}}{\partial Y_{pj}} = \sec \alpha$

i. $\frac{\partial f_{ij}^{Y}}{\partial \psi_{j}} = x_i \sec \alpha \cos \psi_j$

j. $\frac{\partial f_{ij}^{Y}}{\partial x_{i}} = \sec \alpha \sin \psi_j$.

Appendix B

Computer Reduction using Sparse Matrix Algorithm

The normal equations which were given in section 3.3 can be written

$$T^t T\ \delta = T^t z$$

where t instead of ' is used to indicate transposition. Let $N = T^t T$ and $b = T^t z$. Then the normal equations are given by

$$N\ \delta = b$$

where T is 2mn x (2n+3m+1),
N is (2n+3m+1) x (2n+3m+1),
z is 2mn x 1,
b is (2n+3m+1) x 1, and
δ is (2n+3m+1) x 1.

Recall that the three parameters x_p, y_p, and y_q were excluded from the original model. For the purpose of the computer reduction, it is more convenient to reinsert these in their proper columns in the T matrix and set all elements of those columns equal to zero. When the N matrix is computed, the three rows and columns corresponding to the three parameters will contain only zeros. The three diagonal elements are then set equal to one. The three corresponding elements in the $T^t z$ vector are automatically forced to be zero. The result of this manipulation is that the estimates of x_p, y_p, and y_q are forced to be exactly zero, and they appear in their natural order in the computer printout.

The solution to the normal equations is given by

$$\delta = N^{-1} b = Mb.$$

If the normal equations are formed and solved by the usual full matrix methods on the computer, then the storage required for the six vectors and matrices given above is $24n^2 + 220n + 364$ for m = 4 (which is generally the case). The storage can be computed for several values of n as shown in the following table.

n	storage required
4	1,628
25	20,864
100	262,364
400	3,928,364

Obviously for large values of n the storage requirements get out of hand. Fortunately, the matrices T and N are sparse, that is, they have only a small percentage of nonzero elements. If they are properly partitioned then their submatrices which are of block diagonal form can be compacted so that only the blocks are carried in storage. An example is shown in figure 2 where n = 8 and m = 3. The matrix T can be written as $T = (\dot{T}\ \ddot{T})$ where $\dot{T}$ is 2mn x (3m+1) and $\ddot{T}$ is 2mn x 2n. Then

$$T^tT = \begin{pmatrix} \dot{T}^t \\ \\ \ddot{T}^t \end{pmatrix} (\dot{T}\ \ddot{T}) = \begin{pmatrix} \dot{T}^t\dot{T} & \dot{T}^t\ddot{T} \\ \\ \ddot{T}^t\dot{T} & \ddot{T}^t\ddot{T} \end{pmatrix} = \begin{pmatrix} \dot{N} & \bar{N} \\ \\ \bar{N}^t & \ddot{N} \end{pmatrix} \quad \text{and}$$

$$T^tz = \begin{pmatrix} \dot{T}^t \\ \\ \ddot{T}^t \end{pmatrix} z = \begin{pmatrix} \dot{T}^tz \\ \\ \ddot{T}^tz \end{pmatrix} = \begin{pmatrix} \dot{b} \\ \\ \ddot{b} \end{pmatrix} .$$

The normal equations can then be written in the partitioned form

$$\begin{pmatrix} \dot{N} & \bar{N} \\ \\ \bar{N}^t & \ddot{N} \end{pmatrix} \begin{pmatrix} \dot{\delta} \\ \\ \ddot{\delta} \end{pmatrix} = \begin{pmatrix} \dot{b} \\ \\ \ddot{b} \end{pmatrix}$$

where $\dot{N}$ is a (3m+1) x (3m+1) matrix with full first row column and m 3x3 blocks,

$\bar{N}$ is a (3m+1) x 2n full matrix,

$\ddot{N}$ is a 2n x 2n block diagonal matrix of n 2x2 blocks,

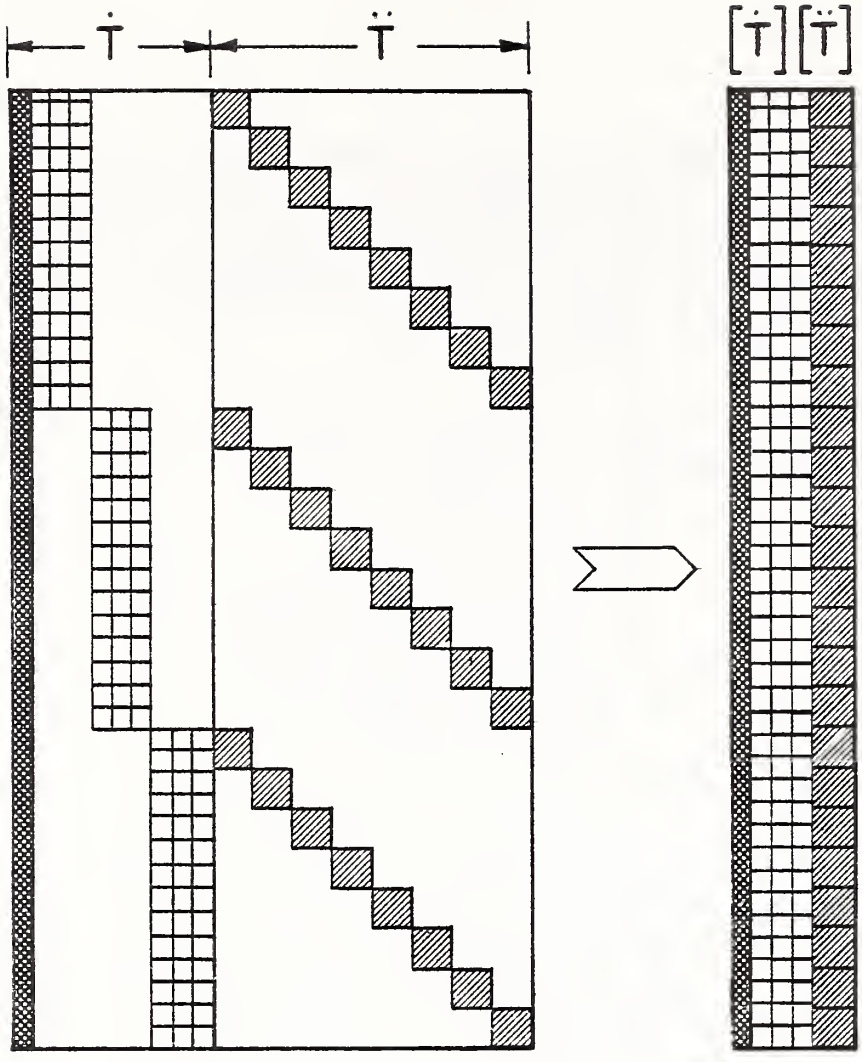
and the δ vector is partitioned to be compatible with the partitioning of N.

Let $N^{-1} = M = \begin{pmatrix} \dot{M} & \bar{M} \\ \\ \bar{M}^t & \ddot{M} \end{pmatrix}$. The equations for the partitioned inverse are

$$\dot{M} = (\dot{N} - \bar{N}\ddot{N}^{-1}\bar{N}^t)^{-1} ,$$

$$\bar{M} = -\dot{M}\bar{N}\ddot{N}^{-1}, \text{ and}$$

$$\ddot{M} = \ddot{N}^{-1} + \ddot{N}^{-1}\bar{N}^t\dot{M}\bar{N}\ddot{N}^{-1}.$$



COMPACTING OF SPARSE MATRICES FOR COMPUTER STORAGE

FULL MATRIX ⟹ COMPUTER STORAGE

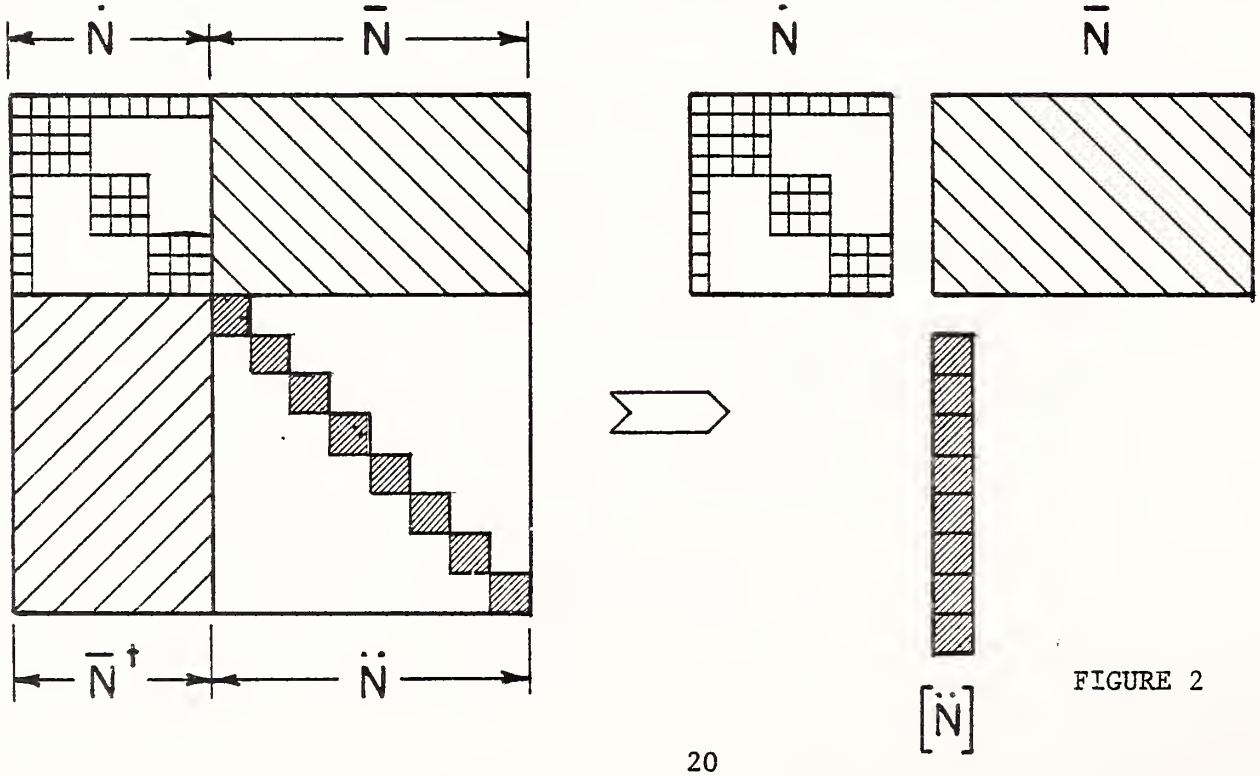


FIGURE 2

The equations for $\dot{\delta}$ and $\ddot{\delta}$ are

$$\dot{\delta} = \dot{M}\dot{b} + \bar{M}\ddot{b}$$

$$= \dot{M}(\dot{b} - \bar{N}\ddot{N}^{-1}\ddot{b}), \text{ and}$$

$$\ddot{\delta} = \bar{M}^t\dot{b} + \ddot{M}\ddot{b}$$

$$= \ddot{N}^{-1}\ddot{b} + \ddot{N}^{-1}\bar{N}^t\dot{M}(\bar{N}\ddot{N}^{-1}\ddot{b} - \dot{b}).$$

The sequence of computations and the required storage are given below. The matrices which are carried in compact form are bracketed.

1. Input		Dimension	Storage
1.1	$[\dot{T}]$	2mnx4	8mn
1.2	$[\ddot{T}]$	2mnx2	4mn
1.3	z	2mnx1	2mn
		Subtotal:	14mn

2. Compute N and b			
2.1	$\dot{N} = [\dot{T}^t][\dot{T}]$	(3m+1)x(3m+1)	$9m^2+6m+1$
2.2	$\bar{N} = [\dot{T}^t][\ddot{T}]$	(3m+1)x2n	6mn+2n
2.3	$[\ddot{N}] = [\ddot{T}^t][\ddot{T}]$	2nx2	4n
2.4	$\dot{b} = [\dot{T}^t]z$	(3m+1)x1	3m+1
2.5	$\ddot{b} = [\ddot{T}^t]z$	2nx1	2n
		Subtotal:	$9m^2+6mn+9m+8n+2$

3. Compute $\dot{M}$			
3.1	$[A]=[\ddot{N}^{-1}]$	(store in $\ddot{N}$)	-
3.2	$B = \bar{N}[A]$	(3m+1)x2n	6mn+2n
3.3	$C = B\bar{N}^t$	(3m+1)x(3m+1)	$9m^2+6m+1$
3.4	$C = \dot{N} - C$	-	-
3.5	$\dot{M} = C^{-1}$	(store in C)	-
		Subtotal:	$9m^2+6mn+6m+2n+1$

4. Compute $\dot{\delta}$

4.1 $D = [A]\ddot{b}$	$2n \times 1$		$2n$
4.2 $E = \bar{N}D$	$(3m+1) \times 1$		$3m+1$
4.3 $\dot{\delta} = \dot{M}(\dot{b} - E)$	$(3m+1) \times 1$		$3m+1$
		Subtotal:	$6mn+2n+2$

5. Compute $\ddot{\delta}$

5.1 $F = B^{t}\dot{M}$	$2n \times (3m+1)$		$6mn+2n$
5.2 $E = E - \dot{b}$	-		-
5.3 $\ddot{\delta} = D + FE$	$2n \times 1$		$2n$
		Subtotal:	$6mn+4n$

6. Compute diagonal elements of $\ddot{M}$

6.1 $G = \text{diag}(FB)$	$2n \times 1$		$2n$
6.2 $\text{diag}(\ddot{M}) =$ $\text{diag}([\ddot{N}^{-1}]) + G$	(store in G)		-
		Subtotal:	$2n$

Additional storage for inversion of the C matrix:	$(3m+1) \times (3m+1)$	$9m^2+6m+1$
Grand Total Storage:		$27m^2+32mn+27m+18n+6$

Note that the inversion of $\ddot{N}$ involves only the simple inversion of several 2x2 matrices.

The following table compares the storage required for the sparse matrix algorithm with the storage required for the full matrix algorithm for several values of n with m = 4.

	storage required		ratio
n	sparse	full	sparse/full
4	1,130	1,628	.694
25	4,196	20,864	.201
100	15,146	262,364	.058
400	58,946	3,928,364	.015

Additional storage is required by the computer algorithm for some single variables and a few other vectors, but it is only a small fraction of the above amounts. Using the sparse matrix algorithm, the present computer facilities at the National Bureau of Standards can handle values of n at least as large as 400.

This method of partitioning and solving the matrix equations is given in a form similar to the one which was worked out by Brown [1]. There can be no doubt of its value because of the tremendous savings in computer storage that it affords.

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16. ABSTRACT *(A 200-word or less factual summary of most significant information. If document includes a significant bibliography or literature survey, mention it here.)*

With the acquisition of a three coordinate measuring machine the Dimensional Technology Section began investigating the problem of precisely measuring two-dimensional reference plates. The algorithm which was developed for this measurement process is highly redundant and incorporates a nonlinear mathematical model. Both the mean and standard deviation of the relative (x,y) coordinates of the points on the reference plate are estimated.

The long term goal of the investigation is to use this algorithm as a tool in developing a measurement process which is under statistical control. On the way to that goal certain problems and pitfalls associated with two-dimensional measurements are sure to arise and will be examined. It is hoped that some light will be shed on areas such as long term stability of measuring machine performance, measuring machine characterization, temperature problems, and the stability of two-dimensional standards.

17. KEY WORDS *(six to twelve entries; alphabetical order; capitalize only the first letter of the first key word unless a proper name; separated by semicolons)* Calibration; convergence criterion; Gauss-Seidel method; incomplete model; least squares estimation; measurement algorithm; nonlinear mathematical model; sparse matrix algorithm; standard deviation; systematic errors; two-dimensional reference plates.

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