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THEORY OF ADJOINT RECIPROCITY FOR ELECTROACOUSTIC TRANDUCERS

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A. D. Yaghjian

ABSTRACT

Analytical techniques for the measurement of the external characteristics of electroacoustic transducers have been developed by D. M. Kerns using a plane-wave scattering-matrix (PWSM) formulation. Foldy and Primakoff, in their classic papers on linear electroacoustic transducers, utilize a spatial impedance-matrix (SIM) formulation. Both formulations involve a continuous, linear "matrix" transformation in which reciprocity is defined as a relationship between elements of the matrix.

The first portion of the present report demonstrates that a transducer satisfying the "SIM relations" also satisfies the "PWSM equations" (but that the converse theorem does not hold), and that the alternate expressions of reciprocity are equivalent for transducers that obey both formulations.

The second portion of the report examines the equations which characterize the internal behavior of a large class of electroacoustic transducers. A linear operator approach is employed to develop a generalized reciprocity lemma which is used to establish adjoint reciprocity relations between the fields of a given transducer and its adjoint transducers. The linear operator approach facilitates the identification of self-adjoint (reciprocal or antireciprocal) transducers, and the adjoint reciprocity relations have utility in the extrapolation techniques of the PWSM formulation. An adjoint "reciprocity theorem" and "principle of reciprocity" are derived from the generalized reciprocity relations. Finally it is shown that the total power inputs for the adjoint transducers belong to the same "value class" as the original transducer.

Key words: Adjoint operators; electroacoustic transducers; reciprocity; scattering matrices.

INTRODUCTION

Analytical techniques developed by Kerns[1,2,3] for the measurement of microwave antennas have been "translated" recently by that author into corresponding techniques for the measurement of electroacoustic transducers in fluids. The basic theory is formulated with a scattering-matrix description and emphasizes the use of plane-wave spectra for the representation of acoustic fields. For a summary and for details of the theory and its variety of applications, reference should be made to Unpublished Report.[4]

Although the electromagnetic (em) theory is quite comprehensive, the acoustics applications are greatly strengthened by generalized or adjoint reciprocity relations, the derivation of which forms the principal subject of the present report. In addition, the relationship between the classical spatial impedance-matrix (SIM) description of electroacoustic transducers used by Foldy and Primakoff,[5,6] and the plane-wave scattering-matrix (PWSM) description introduced by Kerns, including the expression of generalized reciprocity in each scheme, is investigated.

Kerns begins the PWSM description by expanding the external acoustic field in a double Fourier transform representing a continuous spectrum of plane-wave modes. He then relates the amplitudes of the outgoing acoustic waves

$b(\bar{K})$ and the emergent em mode b_o to the amplitudes of the incoming acoustic waves $a(\bar{K})$ and the incident em mode a_o by the scattering equations,

$$b_o = S_{oo} a_o + \int_{\bar{L}} S_{o1}(\bar{L}) a(\bar{L})\, d\bar{L} \tag{1a}$$

$$b(\bar{K}) = S_{1o}(\bar{K}) a_o + \int_{\bar{L}} S_{11}(\bar{K},\bar{L}) a(\bar{L})\, d\bar{L}, \tag{1b}$$

in which the integrations extend over the infinite $\ell_x \ell_y$ plane. The vectors $\bar{K} = k_x \hat{i} + k_y \hat{j}$ and $\bar{L} = \ell_x \hat{i} + \ell_y \hat{j}$ may be interpreted as the "transverse" parts of the plane-wave propagation vectors $\bar{k}$ and $\bar{\ell}$. Equations 1 resolve the combined scattering matrix into four submatrices S_{o1}, S_{1o}, S_{11}, S_{oo} which represent the receiving, radiating, acoustic scattering, and em reflection properties respectively for the electroacoustic transducer.

In analogy to the 2-port cases discussed by McMillan,[7] Kerns[4] defines reciprocity, antireciprocity, and nonreciprocity for electroacoustic transducers in terms of whether the following relationships hold with the plus sign, the minus sign, or not at all:

$$\eta_o S_{o1}(\bar{K}) = \pm\, \eta(K) S_{1o}(-\bar{K}) \tag{2a}$$

$$\eta(K) S_{11}(\bar{K},\bar{L}) = \eta(L) S_{11}(-\bar{L},-\bar{K}). \tag{2b}$$

The quantities η_o and η are admittances which depend upon conventions of sign and normalization.

The classic papers of Foldy and Primakoff[5,6] use a spatial impedance-matrix formulation to characterize the external operation of linear electroacoustic transducers. The SIM relations[8] may be written,

$$V_o = Z_b I_o + \int_{A_o} h'(\bar{r}_o) u_n(\bar{r}_o)\, da_o \tag{3a}$$

$$p(\bar{r}) = h(\bar{r}) I_o + \int_{A_o} Z_o(\bar{r},\bar{r}_o) u_n(\bar{r}_o)\, da_o, \tag{3b}$$

where the four spatial impedances Z_b, h', h, and Z_o characterize a transducer by relating the "voltage"

$$V_o = a_o + b_o \tag{4a}$$

and excess pressure p on the surface A_o of the transducer (except possibly on the feed area S_o) to the "current"

$$I_o = (a_o - b_o)\eta_o \tag{4b}$$

and inward (to transducer) normal velocity u_n on that surface. Although three of the four spatial impedances depend upon the position vectors $(\bar{r},\bar{r}_o)$ to the surface A_o and all four depend upon the harmonic time frequency, they are assumed independent of the medium or sources surrounding the transducer and, as linearity implies, independent of V_o, I_o, p and u_n.

A transducer satisfying the SIM relations 3 is defined as reciprocal by Foldy and Primakoff if the following relations among the spatial impedances are satisfied:

$$h'(\bar{r}) = \pm\, h(\bar{r}) \tag{5a}$$

$$Z_o(\bar{r},\bar{r}_o) = Z_o(\bar{r}_o,\bar{r}). \tag{5b}$$

Otherwise the transducer is nonreciprocal. They do not use the term "antireciprocity" but distinguish between the plus and minus sign in Eq. 5a by considering reciprocity through electric-type coupling and magnetic-type coupling.

The first portion of the present paper demonstrates that the SIM relations 3 imply the PWSM Eqs. 1 (but that the converse theorem does not hold), and proves that the alternate expressions of reciprocity, Eqs. 2 and 5, are equivalent for transducers which obey both the SIM and the PWSM formulation.

The second portion of the paper examines the equations which characterize the internal properties of a large class of electroacoustic transducers. A linear operator approach is employed to develop a generalized electroacoustic reciprocity lemma which relates the field properties of a given transducer to those of its mathematical adjoint. The linear operator approach also facilitates the identification of self-adjoint transducers, transducers which satisfy the usual reciprocity relations 2 or 5.

The reciprocity lemma is used to establish adjoint reciprocity relations, which have utility in the "extrapolation techniques" of the PWSM formulation.[4] (Extrapolation techniques were introduced for antennas by Wacker[9] and Bowman. The method is applied and briefly described in a paper by Newell and Kerns.[10]) The "electroacoustic reciprocity theorem" and the "principle of reciprocity" for scatterers are extended to mutually adjoint transducers through use of the adjoint reciprocity relations.

Finally, power relations and the associated "value classes" are investigated for electroacoustic transducers and their adjoints. In particular, it is shown that the power input to the adjoint transducers belong to the same value class, i.e. have the same "definiteness" as the power input to the original transducer.

I. THE TRANSDUCER SYSTEM

The electroacoustic transducer system under consideration is pictured in Fig. 1. The transducer is bounded by the closed, finite surface A_o. The em source or detector, which is bounded by the closed, finite surface A_p, feeds the transducer through the area S_o common to both A_o and A_p. The transducers, which may contain static bias fields and their sources, are termed "passive" if they cannot radiate

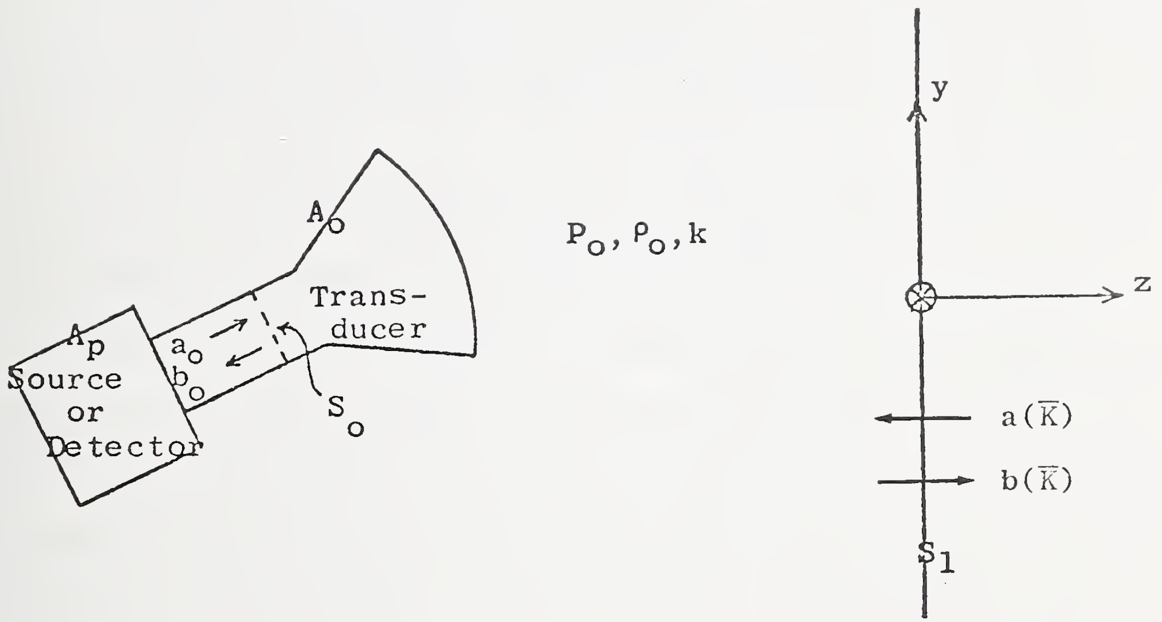

Fig. 1. Schematic of the transducer system. $A=A_o+A_p$ denotes the external surface of the electroacoustic transducer-plus-source or detector.

more power than they absorb. The present analysis, however, is not limited to passive transducers, but may include transducers which are active because they contain "gainy" material, or radiating sources, or both. (A transducer will be called source-free if its internal equations of motion are homogeneous, i.e., if they are satisfied by all fields equal to zero.)

The transducer-plus-source (or detector) is immersed in a homogeneous, isotropic, stationary fluid which extends to infinity and supports the time harmonic ($e^{-i\omega t}$) acoustic pressure-velocity field of small (first order differential) amplitude. The static pressure, static density, and propagation constant (the latter of which may be a complex function of real frequency ω to account for viscous and "expansive friction" losses) are constants in space and time denoted by P_o, ρ_o and k respectively. The fluid between the infinite xy plane S_1 (z=0) and the external surface A of the transducer-plus-source contains no acoustic sources,[11] although arbitrary sources (at frequency ω) may exist to the right of S_1.

The normal velocity on the surface A_p of the source-detector, including the feed area S_o, is assumed negligible. Electromagnetic shielding necessary for deriving the adjoint reciprocity relations is discussed in Section III.C.

The area S_o may designate the perpendicular cross section of an open or closed em waveguide (uniform and isotropic) in which a single em mode is propagating with incident amplitude a_o and emergent amplitude b_o, or simply the cross section of two wire leads when the frequency ω is low enough to allow analysis by quasi-static voltages and currents. The cross-sectional area S_o for a mode on an open waveguide (such as a Lecher line or microstrip) must extend sufficiently far from the guide to insure that the fields in the mode outside S_o contribute negligibly to the normalization integral (Eq. 11 below). The surfaces A_p, S_o, and A_o represent imaginary boundaries in that they are chosen to facilitate the theoretical analysis and need not coincide with the physical boundaries of the system. Of course, the surface $A = A_o + A_p$ must be chosen within (or on the interior boundary of) the ambient fluid.

The continuity and momentum equations, which govern the harmonic acoustic fields in the source-free fluid between the surface A and the infinite plane S_1, may be expressed as

$$k^2 p = -i\omega\rho_o \nabla\cdot\bar{u} \quad (6a)$$

$$\nabla p = i\omega\rho_o \bar{u}, \quad (6b)$$

where the "small," complex amplitudes, p and $\bar{u}$, denote the position-dependent excess pressure and fluid velocity. The effect of gravity on harmonic variation in mass density is neglected.

The excess pressure and normal velocity on the surface S_1 may be expanded in a double Fourier integral (transform) over plane-wave "voltage" and "current" spectral densities:[4]

$$p(\bar{R}) = \frac{1}{2\pi} \int_{\bar{K}} V(\bar{K}) e^{i\bar{K}\cdot\bar{R}} \, d\bar{K} \tag{7a}$$

$$u_z(\bar{R}) = -\frac{1}{2\pi} \int_{\bar{K}} I(\bar{K}) e^{i\bar{K}\cdot\bar{R}} \, d\bar{K}. \tag{7b}$$

Conversely, the plane-wave voltage and current densities,

$$V(\bar{K}) = a(\bar{K}) + b(\bar{K}) \tag{8a}$$

$$I(\bar{K}) = [a(\bar{K}) - b(\bar{K})]\eta(K), \tag{8b}$$

may be expanded in a double Fourier integral (inverse transform) over the excess pressure and normal velocity:

$$V(\bar{K}) = \frac{1}{2\pi} \int_{S_1} p(\bar{R}) e^{-i\bar{K}\cdot\bar{R}} \, d\bar{R} \tag{9a}$$

$$I(\bar{K}) = -\frac{1}{2\pi} \int_{S_1} u_z(\bar{R}) e^{-i\bar{K}\cdot\bar{R}} \, d\bar{R}. \tag{9b}$$

The vector $\bar{R}$ is confined to the xy plane S_1, and the z-component of the plane-wave admittance $\eta(K)$ equals $\gamma/\omega\rho_o$, where γ is defined as $(k^2 - K^2)^{\frac{1}{2}}$ with the sign of the radical chosen to keep the real and imaginary parts of γ positive.

The transverse em fields $(\bar{E}_t, \bar{H}_t)$ on the cross section S_o of a waveguide may be written as the sum of the transverse "electric" $\bar{e}_o$ and "magnetic" $\bar{h}_o$ components of the mode traveling to the right (incident) and left (emergent):

$$\bar{E}_t = V_o \bar{e}_o, \quad \bar{H}_t = I_o \bar{h}_o, \tag{10}$$

where V_o and I_o are defined by Eqs. 4.

(Note that here the terms "transverse electric and magnetic" are not used to denote TE and TM modes.)

If the dimensions of $\bar{e}_o$ and $\bar{h}_o$ are chosen as $(\text{meter})^{-1}$ and $(\bar{e}_o, \eta_c \bar{h}_o)$ are consistent with Maxwell's equations, V_o behaves dimensionally as volts, I_o as amperes, and η_o as an admittance (ohm^{-1}) normally chosen as a positive real number. Moreover, normalization may be expressed as a nondimensional number equal to unity, i.e.

$$\int_{S_o} (\bar{e}_o \times \bar{h}_o) \cdot \hat{n} \, da = 1, \tag{11}$$

where $\hat{n}$ is the inward normal to the transducer. (The rationalized MKS system of units is used throughout.) If the area S_o simply cuts two wire leads at quasi-static frequencies, V_o and I_o refer to conventional circuit voltages and currents which do not necessarily serve as genuine modal coefficients. In that case, Eqs. 4 become a definition of a_o and b_o.

II. RELATIONSHIP BETWEEN THE TWO FORMULATIONS

This section shows that the scattering-matrix equations 1 can be derived from the SIM relations 3 (but not conversely), and that the two expressions of reciprocity, Eqs. 2 and 5, are equivalent in that each expression implies the other for transducers that obey both descriptions. These results are established without reference to the internal properties or behavior of the electroacoustic transducer.

A. Derivation of the Scattering-Matrix Equations from the Spatial Impedance-Matrix Relations

Application of Green's theorem to the excess pressure (henceforth referred to as just pressure) in the fluid region between the external surface A of the transducer-plus-source or detector and the hemisphere closed on the right by the infinite plane S_1, produces

$$p(\bar{r}) = \int_A p(\bar{r}_o) \frac{\partial G(\bar{r},\bar{r}_o)}{\partial n_o} da_o - i\omega\rho_o \int_{S_1} u_z(\bar{R}_o) G(\bar{r},\bar{R}_o) d\bar{R}_o. \tag{12}$$

The pressure and Green's function in Eq. 12 satisfy the scalar wave equations,

$$\nabla^2 p + k^2 p = 0 \tag{13a}$$

$$\nabla_o^2 G(\bar{r},\bar{r}_o) + k^2 G(\bar{r},\bar{r}_o) = \delta(\bar{r}-\bar{r}_o), \tag{13b}$$

where $\delta(\bar{r}-\bar{r}_o)$ denotes the three-dimensional delta function. The boundary conditions for the Green's function have been chosen as

$$\frac{\partial G(\bar{r},\bar{r}_o)}{\partial n_o} = 0 \quad \bar{r}_o \text{ on } S_1 \text{ and } A_p - S_o \tag{14a}$$

$$G(\bar{r},\bar{r}_o) = 0 \quad \bar{r}_o \text{ on } A_o - S_o . \tag{14b}$$

Also, by means of Green's theorem, the Green's function with the prescribed boundary conditions can be shown to possess the symmetry property,

$$G(\bar{r},\bar{r}_o) = G(\bar{r}_o,\bar{r}). \tag{14c}$$

The exponential decay in lossy fluids or the radiation condition in lossless fluids insure the vanishing of the integrals over the hemisphere at infinite radius. Equation 6b has been used to relate the normal derivative of pressure on A and S_1 to the normal velocity there, i.e.

$$\hat{n}\cdot\nabla p = \frac{\partial p}{\partial n} = i\omega\rho_o u_n \quad (\hat{n} = \hat{e}_z \text{ on } S_1) \tag{15}$$

where the unit normals are directed away from the region between A and S_1. (The existence of unique mathematical solutions for p, $\bar{u}$, and G, here and in the ensuing expressions, has been assumed implicitly.)

Substitution of pressure from Eq. 3b and normal velocity from Eq. 7b into their respective integrals in Eq. 12 yields

$$p(\bar{r}) = I_o \int_A h(\bar{r}_o) \frac{\partial G(\bar{r},\bar{r}_o)}{\partial n_o} da_o + \frac{i\omega\rho_o}{2\pi} \int_{S_1} \int_{\bar{K}} I(\bar{K}) G(\bar{r},\bar{R}_o) e^{i\bar{K}\cdot\bar{R}_o} d\bar{R}\, d\bar{R}_o$$

$$+ \int_A \int_A Z_o(\bar{r}_o,\bar{r}_1) \frac{\partial G(\bar{r},\bar{r}_o)}{\partial n_o} u_n(\bar{r}_1)\, da_o\, da_1 \quad (\bar{r} \text{ on } \bar{A}). \tag{16}$$

After differentiation with respect to the normal direction on A and utilization of Eq. 15, Eq. 16 recasts into an integral equation determining u_n on the surface A in terms of the "currents" I_o and $I(\bar{K})$,

$$u_n(\bar{r}) = \int_A D(\bar{r},\bar{r}_o)\, u_n(\bar{r}_o)\, da_o + g(\bar{r}) \quad (\bar{r} \text{ on } A). \tag{17}$$

The kernel is given by

$$D(\bar{r},\bar{r}_o) = \frac{1}{i\omega\rho_o} \int_A Z_o(\bar{r}_1,\bar{r}_o) \frac{\partial^2 G(\bar{r},\bar{r}_1)}{\partial n \partial n_1} da_1, \tag{18a}$$

and the source function by

$$g(\bar{r}) = \frac{I_o}{i\omega\rho_o} \int_A h(\bar{r}_o) \frac{\partial^2 G(\bar{r},\bar{r}_o)}{\partial n \partial n_o} da_o$$

$$+ \frac{1}{2\pi} \int_{S_1} \int_{\bar{K}} I(\bar{K}) \frac{\partial G(\bar{r},\bar{R}_o)}{\partial n} e^{i\bar{K}\cdot\bar{R}_o} d\bar{K}\, d\bar{R}_o. \tag{18b}$$

Because u_n vanishes when I_o and $I(\bar{K})$ vanish, nontrivial homogeneous solutions to this two-dimensional Fredholm integral equation 17 of the second kind do not exist. Thus, a unique solution to Eq. 17 for the normal velocity u_n may be expressed formally be means of a resolvent or reciprocal kernel $\mathcal{D}$ operating with the source function g,[12]

$$u_n(\bar{r}) = g(\bar{r}) + \int_A \mathcal{D}(\bar{r},\bar{r}_o)\, g(\bar{r}_o)\, da_o \quad (\bar{r} \text{ on } A). \tag{19}$$

(The resolvent kernel $\mathcal{D}$ is expanded in Eq. 28 as a Liouville-Neumann series.)

The normal velocity in Eq. 19 may be rewritten as a linear function of I_o and $I(\bar{K})$. With the aid of Eq. 18b and the definitions,

$$F_o(\bar{r}) = \frac{1}{i\omega\rho_o} \int_A h(\bar{r}_o) \frac{\partial^2 G(\bar{r},\bar{r}_o)}{\partial n \partial n_o} da_o + \frac{1}{i\omega\rho_o} \int_A \int_A h(\bar{r}_1) \frac{\partial^2 G(\bar{r}_o,\bar{r}_1)}{\partial n_o \partial n_1} \mathcal{D}(\bar{r},\bar{r}_o)\, da_o\, da_1 \tag{20a}$$

$$F(\bar{K},\bar{r}) = \frac{1}{2\pi} \int_{S_1} \frac{\partial G(\bar{r},\bar{R}_o)}{\partial n} e^{i\bar{K}\cdot\bar{R}_o} d\bar{R}_o + \frac{1}{2\pi} \int_{S_1} \int_A \frac{\partial G(\bar{r}_o,\bar{R}_1)}{\partial n_o} \mathcal{D}(\bar{r},\bar{r}_o) e^{i\bar{K}\cdot\bar{R}_1} da_o\, d\bar{R}_1, \tag{20b}$$

Eq. 19 becomes

$$u_n(\bar{r}) = I_o F_o(\bar{r}) + \int_{\bar{K}} I(\bar{K}) F(\bar{K},\bar{r})\, d\bar{K} \quad (\bar{r} \text{ on } A). \tag{21}$$

The normal velocity in Eq. 16 can be replaced by the right side of Eq. 21 to produce

$$p(\bar{r}) = I_o A_o(\bar{r}) + \int_{\bar{K}} I(\bar{K}) A(\bar{K},\bar{r})\, d\bar{K} \quad (\bar{r} \text{ on } A). \tag{22}$$

where $A_o(\bar{r})$ and $A(\bar{K},\bar{r})$ are related to $F_o(\bar{r})$ and $F(\bar{K},\bar{r})$ by

$$A_o(\bar{r}) = \int_A h(\bar{r}_o) \frac{\partial G(\bar{r},\bar{r}_o)}{\partial n_o}\, da_o + \int_A \int_A Z_o(\bar{r}_o,\bar{r}_1) \frac{\partial G(\bar{r},\bar{r}_o)}{\partial n_o} F_o(\bar{r}_1)\, da_o\, da_1 \tag{23a}$$

$$A(\bar{K},\bar{r}) = \frac{i\omega\rho_o}{2\pi} \int_{S_1} G(\bar{r},\bar{R}_o) e^{i\bar{K}\cdot\bar{R}_o}\, d\bar{R}_o + \int_A \int_A Z_o(\bar{r}_o,\bar{r}_1) \frac{\partial G(\bar{r},\bar{r}_o)}{\partial n_o} F(\bar{K},\bar{r}_1)\, da_o\, da_1. \tag{23b}$$

One spectral[13] impedance-matrix equation results from taking the inverse Fourier transform of Eq. 22 on the plane S_1:

$$V(\bar{K}) = \frac{1}{2\pi} \int_{S_1} p(\bar{R}) e^{-i\bar{K}\cdot\bar{R}}\, d\bar{R} = \frac{I_o}{2\pi} \int_{S_1} A_o(\bar{R}) e^{-i\bar{K}\cdot\bar{R}}\, d\bar{R} + \frac{1}{2\pi} \int_{S_1} \int_{\bar{L}} I(\bar{L}) A(\bar{L},\bar{R}) e^{-i\bar{K}\cdot\bar{R}}\, d\bar{L}\, d\bar{R}, \tag{24a}$$

or

$$V(\bar{K}) = Z_{1o}(\bar{K}) I_o + \int_{\bar{L}} Z_{11}(\bar{K},\bar{L}) I(\bar{L})\, d\bar{L}, \tag{24b}$$

with the impedances Z_{1o} and Z_{11} given by

$$Z_{1o} = \frac{1}{2\pi} \int_{S_1} A_o(\bar{R}) e^{-i\bar{K}\cdot\bar{R}}\, d\bar{R}, \quad Z_{11}(\bar{K},\bar{L}) = \frac{1}{2\pi} \int_{S_1} A(\bar{L},\bar{R}) e^{-i\bar{K}\cdot\bar{R}}\, d\bar{R}. \tag{25a,b}$$

Similarly, the second spectral impedance-matrix equation is determined by substituting the normal velocity of Eq. 21 into the SIM relation 3a, which becomes

$$V_o = Z_{oo} I_o + \int_{\bar{L}} Z_{o1}(\bar{L}) I(\bar{L}) \, d\bar{L}, \qquad (26)$$

when the impedances Z_{oo} and Z_{o1} are defined as

$$Z_{oo} = Z_b + \int_A h'(\bar{r}) F_o(\bar{r}) \, da \qquad (27a)$$

$$Z_{o1}(\bar{K}) = \int_A h'(\bar{r}) F(\bar{K},\bar{r}) \, da. \qquad (27b)$$

The spectral matrix-impedance representation (Eqs. 24b and 26) combine with the definitions of the generalized voltages and currents, Eqs. 4 and 8, to form the scattering-matrix equations 1. (The derivation of S_{oo}, S_{o1}, S_{1o}, and S_{11} from Z_{oo}, Z_{o1}, Z_{1o} and Z_{11} is a straightforward, reversible, but lengthy procedure which parallels the analagous derivation for 2-ports.)

In brief, every electroacoustic transducer which satisfies the SIM relations can be described by a spectral impedance or scattering matrix. Moreover, the elements of these matrices are determinable as explicit functions of the four spatial impedances (Z_b, h', h, Z_o) that characterize the transducer.

The following investigation shows by means of a simple class of counter-examples, that the converse theorem does not hold.

Consider an electroacoustic transducer with I_o held zero but subjected to an incident acoustic field from without. Suppose also that the transducer behaves as an acoustically rigid ($u_n = 0$) scatterer when I_o is zero, and that V_o is linearly related to the pressure on A. The SIM relations 3 do not describe such a transducer because p and V_o do not vanish even though u_n and I_o do.

The same rigid transducer with zero I_o and V_o linearly related to pressure on A is described, however, by a scattering matrix. Integral equation 12 can be solved for pressure on A as a function of the normal velocity u_z, and the result substituted back into the first integral in Eq. 12. If the normal velocity is then expressed as the integral in Eq. 7b over $I(\bar{K})$, and the inverse Fourier transform in the S_1 plane is taken of the resulting equation, the impedance-matrix equations corresponding to Eqs. 24b and 26 with zero I_o appear. The impedance-matrix equations in turn convert into the scattering-matrix equations 1 with $a_o=b_o$.

Rigid scatterers may not represent the only examples of linear transducers which can not be described by the SIM relations, but they suffice to demonstrate that the PWSM description applies to a larger class of transducers than do the SIM relations.

B. Equivalence of the Alternate Forms of Reciprocity

Consider the separate definitions of reciprocity expressed by Eqs. 2 and 5. Although it was just shown that the PWSM equations 1 describe a larger class of transducers than the SIM relations 3, the alternate expressions for reciprocity prove equivalent for transducers which obey both descriptions. Equivalence is established by deriving the PWSM reciprocity relations 2 from the SIM reciprocity relations 5, and vice versa.

The spectral impedance matrices Z_{o1}, Z_{1o}, and Z_{11} can be expressed through Eqs. 27b, 25, 23, and 20 as functions of the spatial impedances h, h', Z_o, and the Green's function, provided the resolvent kernel $\mathcal{D}(\bar{r}, \bar{r}_o)$ in Eq. 19 is determined as a function of the same spatial impedances.

Successive substitution with the source function in Eq. 17 determines the solution for the resolvent kernel as the Liouville-Neumann series,[12]

$$\mathcal{D}(\bar{r},\bar{r}_o) = D(\bar{r},\bar{r}_o) + \int_A D(\bar{r},\bar{r}_1)D(\bar{r}_1,\bar{r}_o)\, da_1 + \int_A \int_A D(\bar{r},\bar{r}_2)D(\bar{r}_2,\bar{r}_1)D(\bar{r}_1,\bar{r}_o)\, da_1\, da_2 + \ldots . \quad (28)$$

In spite of its strong requirements for convergence, the Liouville-Neumann series is used in Eq. 28 instead of the Fredholm series, which converges except at the eigenvalues, because identical results are obtained using either series, but the mathematical details are shortened considerably.

Insertion of the original kernel D of Eq. 18a into the expression 28, Eq. 28 into Eqs. 20, Eqs. 20 into Eqs. 23, and Eqs. 23 into the impedances 25 and 27b, results, after some rearrangement, in the final expansions for Z_{o1}, Z_{1o}, and Z_{11}:

$$Z_{o1}(\bar{K}) = \frac{1}{2\pi} \int_{S_1} \int_{A} h'(\bar{r}_o) \frac{\partial G(\bar{r}_o,\bar{R})}{\partial n_o} e^{i\bar{K}\cdot\bar{R}} \, da_o \, d\bar{R}$$

$$+ \frac{1}{2\pi i\omega\rho_o} \int_{S_1} \int_{A} \int_{A} \int_{A} h'(\bar{r}_o) \frac{\partial G(\bar{r}_1,\bar{R})}{\partial n_1} Z_o(\bar{r}_2,\bar{r}_1) \frac{\partial^2 G(\bar{r}_o,\bar{r}_2)}{\partial n_o \partial n_2}$$

$$e^{i\bar{K}\cdot\bar{R}} \, da_o \, da_1 \, da_2 \, d\bar{R} + \ldots. \tag{29a}$$

$$Z_{1o}(\bar{K}) = \frac{1}{2\pi} \int_{S_1} \int_{A} h(\bar{r}_o) \frac{\partial G(\bar{R},\bar{r}_o)}{\partial n_o} e^{-i\bar{K}\cdot\bar{R}} \, da_o \, d\bar{R}$$

$$+ \frac{1}{2\pi i\omega\rho_o} \int_{S_1} \int_{A} \int_{A} \int_{A} h(\bar{r}_o) \frac{\partial G(\bar{R},\bar{r}_1)}{\partial n_1} Z_o(\bar{r}_1,\bar{r}_2) \frac{\partial^2 G(\bar{r}_2,\bar{r}_o)}{\partial n_2 \partial n_o}$$

$$e^{-i\bar{K}\cdot\bar{R}} \, da_o \, da_1 \, da_2 \, d\bar{R} + \ldots \tag{29b}$$

$$Z_{11}(\overline{K},\overline{L}) = \frac{i\omega\rho_o}{(2\pi)^2} \int_{S_1} \int_{S_1} e^{-i\overline{K}\cdot\overline{R}} G(\overline{R},\overline{R}_o) e^{i\overline{L}\cdot\overline{R}_o} \, d\overline{R}_o \, d\overline{R}$$

$$+ \frac{1}{(2\pi)^2} \int_{S_1} \int_{S_1} \int_A \int_A e^{-i\overline{K}\cdot\overline{R}} Z_o(\overline{r}_1,\overline{r}_2) \frac{\partial G(\overline{R},\overline{r}_1)}{\partial n_1} \frac{\partial G(\overline{r}_2,\overline{R}_o)}{\partial n_2}$$

$$e^{i\overline{L}\cdot\overline{R}_o} \, da_1 \, da_2 \, d\overline{R}_o \, d\overline{R} + \ldots . \quad (29c)$$

Only the first two terms in the expansions 29 are carried explicitly, but the essential "symmetry" property required to prove Eqs. 30 below is found in every term.

The symmetrical properties of the Green's function $G(\overline{r},\overline{r}_o)$ and the impedance $Z_o(\overline{r},\overline{r}_o)$, as expressed by Eqs. 14c and 5b respectively, imply from Eq. 29c that

$$Z_{11}(\overline{K},\overline{L}) = Z_{11}(-\overline{L},-\overline{K}). \quad (30a)$$

Similarly, Eqs. 14c and 5 imply from Eqs. 29a and 29b that

$$Z_{o1}(\overline{K}) = \pm Z_{1o}(-\overline{K}). \quad (30b)$$

Again, by a straightforward, reversible, but lengthy exercise which will not be included here, the impedance reciprocity relations 30 combine with the definitions 4 and 8 and the scattering-matrix equations to form the PWSM reciprocity relations 2. The ± signs in Eqs. 2a and 30b are coupled with the ones in Eq. 5a.

The completion of the proof of equivalence consists in deriving the SIM reciprocity relations 5 from the PWSM relations 2.

If the Eqs. 7 for pressure and normal velocity, the spectral impedance Eq. 24b, and the SIM Eq. 3b are inserted into the acoustical reciprocity theorem,[14]

$$\int_A (p^1 u_n^2 - p^2 u_n^1)\, da + \int_{S_1} (p^1 u_z^2 - p^2 u_z^1)\, d\bar{R} = 0, \qquad (31)$$

as applied to any two acoustic fields supported by the fluid between A and S_1, the following expression emerges:

$$\int_A \int_A u_n^1(\bar{r}_o)[Z_o(\bar{r},\bar{r}_o) - Z_o(\bar{r}_o,\bar{r})]u_n^2(\bar{r})\, da\, da_o + \int_A h(\bar{r})[I_o^1 u_n^2(\bar{r}) - I_o^2 u_n^1(\bar{r})]\, da$$

$$= \int_{\bar{L}} \int_{\bar{K}} I^1(\bar{L})[Z_{11}(\bar{K},\bar{L}) - Z_{11}(-\bar{L},-\bar{K})]I^2(-\bar{K})\, d\bar{K}\, d\bar{L} + \int_{\bar{K}} Z_{1o}(-\bar{K})[I_o^1 I^2(\bar{K}) - I_o^2 I^1(\bar{K})]\, d\bar{K}. \qquad (32)$$

Again, the exponential decay in lossy fluids or the radiation condition in lossless fluids insure the vanishing of the integral in Eq. 31 over the hemisphere at infinite radius.

The scattering reciprocity relations 2 imply the spectral impedance reciprocity relations 30, which in turn imply the SIM reciprocity relation 5b,

$$Z_o(\bar{r},\bar{r}_o) = Z_o(\bar{r}_o,\bar{r}),$$

by letting

$$I_o^1 = I_o^2 = 0$$

$$u_n^1(\bar{r}) = \delta(\bar{r}-\bar{r}_1)$$

$$u_n^2(\bar{r}) = \delta(\bar{r}-\bar{r}_2)^{15}$$

in Eq. 32.

The derivation of Eq. 5a from Eqs. 2 requires more effort. Since the first integral on both sides of Eq. 32 is zero and $Z_{1o}(-\overline{K}) = \pm Z_{o1}(\overline{K})$, Eq. 32 may be written as

$$\int_A h(\overline{r})u_n^2(\overline{r})\,da \mp \int_{\overline{K}} Z_{o1}(\overline{K})I^2(\overline{K})\,d\overline{K} = 0, \tag{33}$$

if I_o^2 is chosen zero. The additional choice of $I^2(\overline{K}) = \delta(\overline{K}-\overline{K}_o)$ and use of Eqs. 21 and 27b transforms Eq. 33 into the two-dimensional homogeneous Fredholm integral equation of the first kind,

$$\int_A [h(\overline{r}) \mp h'(\overline{r})]F(\overline{K},\overline{r})\,da = 0. \tag{34}$$

Under the assumption that the kernel $F(\overline{K},\overline{r})$ does not have zero as an eigenvalue,[16] the one continuous solution to Eq. 34 is

$$h(\overline{r}) = \pm h'(\overline{r}), \tag{35}$$

which completes the exercise to demonstrate the equivalence of the PWSM and SIM reciprocity relations.

Although the reciprocal symmetry of the spectral impedance and scattering matrices has been derived from Eqs. 29 and the symmetry of the spatial impedances, an alternate approach can be taken from the acoustical reciprocity theorem 32. Such an approach, however, encounters mathematical difficulties which discourage its use.

III. ADJOINT RECIPROCITY

For reciprocal transducers, the transmitting and receiving characteristics determine each other, the speaker and microphone response functions become proportional,[5] and scattering in one set of directions relates directly to scattering in the opposite set of directions. In general, the analysis and measurement needed to characterize an electroacoustic transducer are reduced if the transducer is known to obey the reciprocity relations. Moreover, some techniques are applicable to reciprocal transducers only. Kerns has shown, for example, that the transmitting and receiving characteristics of two unknown but identical transducers obeying reciprocity and possessing symmetry about an axial plane can be determined by a single transverse scan of one transducer by the other.[4]

In addition to its direct importance in simplifying transducer analysis and measurement, reciprocity is used in the plane-wave scattering-matrix theory to express mathematically the receiving matrix S_{01} as a function of equivalent sources within the transducer.[17] Such expressions have become especially important because of their use in the theoretical foundation of extrapolation techniques which predict gain on an arbitrary axis from measurements at reduced distances along that axis.[9,10]

For nonreciprocal transducers, the expressions for S_{o1} in terms of equivalent sources no longer apply unless there exists an operator or transducer (the adjoint transducer) with transmitting characteristics S^a_{1o} related "reciprocally" to the receiving characteristics S_{o1} of the original transducer. The remainder of the present report investigates the internal operation of a general class of linear electroacoustic transducers to prove the mathematical existence of the adjoint transducer and to determine its precise relationship to the original transducer.

Specifically, the equations describing the material properties and behavior of the transducer are written as a linear operator matrix. The operator is "transposed" to form the adjoint operator, which is compared and combined with the original operator to derive a generalized electroacoustic reciprocity lemma. The reciprocity lemma leads to the generalized reciprocal relations between the characteristic "matrices" of the mutually adjoint transducers. A similar analysis for antennas has been performed by Kerns.[18]

A. The Operator Matrix Description

Primakoff and Foldy,[6] in their original work on electroacoustic reciprocity, examined the equations which model the internal operation of a large class of transducers. They found that if certain "symmetry conditions" were satisfied by its material parameters and static fields, the transducers obeyed the SIM reciprocity relations. The present derivation begins in a similar way by writing the linear, harmonic ($e^{-i\omega t}$) equations that govern the "physics" of the material within the transducer A_o:

$$\nabla \cdot \bar{\bar{T}} + i\omega\rho_m^o \, \bar{u} + \bar{f} = 0 \tag{36}$$

$$\nabla \times \bar{E} - i\omega\bar{B} = 0 \tag{37a}$$

$$\nabla \times \bar{H} + i\omega\bar{D} - \bar{J} = 0. \tag{37b}$$

The momentum equation 36 expresses Newton's second law for stationary ($\bar{u}_o = 0$) media, and Eqs. 37 express Maxwell's equations in differential form for stationary media in which regions of electric and magnetic polarization may exist.

The static quantities, which will be distinguished by the superscript or subscript "o," are real functions of the space coordinates. All other quantities, except the real frequency ω, represent complex, space-dependent, harmonic amplitudes, which superimpose upon the static variables as products with $e^{-i\omega t}$. For example, if the total stress dyadic at a point in space and time is denoted by $\bar{\bar{T}}'(\bar{r},t)$ then $\bar{\bar{T}}'$ can be divided into a real static part $\bar{\bar{T}}_o(\bar{r})$ and a complex harmonic part $\bar{\bar{T}}(\bar{r})e^{-i\omega t}$, i.e.

$$\bar{\bar{T}}'(\bar{r},t) = \bar{\bar{T}}_o(\bar{r}) + \bar{\bar{T}}(\bar{r})e^{-i\omega t}.$$

Of course, it is the real part of $\bar{\bar{T}}'$ that corresponds to a measurable quantity. Only the complex harmonic amplitude $\bar{\bar{T}}(\bar{r})$ shows in Eq. 36.

Similarly the total mass density $\rho_m'(\bar{r},t)$ may be divided as

$$\rho_m'(\bar{r},t) = \rho_m^o(\bar{r}) + \rho_m(\bar{r})e^{-i\omega t},$$

but only the real static part occurs in Eq. 36. In fact, the mass continuity equation is not included in addition to the momentum equation 36 because it merely introduces the extra variable $\rho_m(\bar{r})$.

The other variables in Eq. 36 are $\bar{u}$, the harmonic velocity of the material, and $\bar{f}$, the total harmonic volume force. The usual harmonic em field vectors are denoted by $\bar{E}$, $\bar{H}$, $\bar{B}$, $\bar{D}$, and $\bar{J}$. All variables refer to "macroscopic" quantities which are evaluated, at least in concept, by averaging "microscopic" variables over "small" but finite volumes and time intervals.

The number of unknowns in Eqs. 36 and 37 reduces to the number of equations if the following linear constitutive relations characterize the transducer material:

$$\bar{D} = \bar{\bar{\varepsilon}}\cdot\bar{E} + \bar{\bar{\tau}}\cdot\bar{H} + \bar{\bar{\bar{\alpha}}}:\bar{\bar{S}} + \bar{P}_s \tag{38}$$

$$\bar{B} = \bar{\bar{\nu}}\cdot\bar{E} + \bar{\bar{\mu}}\cdot\bar{H} + \bar{\bar{\bar{\beta}}}:\bar{\bar{S}} + \mu_o\bar{M}_s \tag{39}$$

$$\bar{J} = \bar{\bar{\sigma}}\cdot(\bar{E} + \bar{u} \times \bar{B}_o) + \rho_e^o\,\bar{u} + \bar{J}_s \tag{40}$$

$$\bar{\bar{T}} = \bar{\bar{\bar{\bar{c}}}}:\bar{\bar{S}} - \bar{\bar{\bar{\alpha}}}'\cdot\bar{E} - \bar{\bar{\bar{\beta}}}'\cdot\bar{H} \tag{41}$$

$$\bar{f} = \rho_e^o\,\bar{E} + \bar{J} \times \bar{B}_o + \bar{f}_s, \tag{42}$$

where the strain dyadic $\overline{\overline{s}}$ is related to the velocity by

$$\overline{\overline{s}} = -\frac{1}{i\omega}\left\{\nabla\overline{u} - \frac{1}{2}(\nabla\times\overline{u}) \times \overline{\overline{I}}\right\}. \tag{43}$$

($\overline{\overline{I}}$ is the identity dyadic)

The various "dot products" involved in the vector and dyadic transformations of Eqs. 38-43 can be understood in terms of Cartesian tensor notation, in which $\overline{\overline{\varepsilon}}\cdot\overline{E}$, $\overline{\overline{\overline{\alpha}}}'\cdot\overline{E}$, $\overline{\overline{\overline{\alpha}}}:\overline{\overline{s}}$ and $\overline{\overline{\overline{\overline{c}}}}:\overline{\overline{s}}$ become $\varepsilon_{ij}E_j$, $\alpha'_{ijk}E_k$, $\alpha_{ijk}s_{jk}$, and $c_{ijk\ell}s_{k\ell}$ respectively (summation over repeated dummy indices is implied). The "n-adic" notation[19] is used wherever convenient because it preserves the familiar vector-dyadic notation for the electro-acoustic fields.[6,20]

The dyadics $\overline{\overline{\sigma}}$, $\overline{\overline{\varepsilon}}$, $\overline{\overline{\mu}}$, and $(\overline{\overline{\tau}},\overline{\overline{\nu}})$ stand for the conductivity, permittivity, permeability, and magneto-electric properties[21] of the transducer material.

The direct piezoelectric and piezomagnetic stress tri-adics,[22] $\overline{\overline{\overline{\alpha}}}$ and $\overline{\overline{\overline{\beta}}}$, may be chosen symmetric:

$$\alpha_{ijk} = \alpha_{ikj} \tag{44a}$$

$$\beta_{ijk} = \beta_{ikj}, \tag{44b}$$

since the strain dyadic is symmetric,

$$s_{jk} = \frac{-1}{2i\omega}\left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\right) = s_{kj}. \tag{45}$$

Similarly, because the stress is a symmetric dyadic ($T_{ij} = T_{ji}$) (provided there are no distributions of body torque, stress couples, or angular momentum density),[23] the converse piezoelectric and piezomagnetic stress triadics,[22] $\overline{\overline{\overline{\alpha}}}'$ and $\overline{\overline{\overline{\beta}}}'$, must possess the symmetry,

$$\alpha'_{ijk} = \alpha'_{jik} \tag{46a}$$

$$\beta'_{ijk} = \beta'_{jik}. \tag{46b}$$

The symmetry of the stress and strain dyadics also requires the Hooke's tetradic $\overline{\overline{\overline{\overline{c}}}}$ to obey the symmetry

$$c_{ijk\ell} = c_{jik\ell}, \tag{47a}$$

and allows the defined symmetry

$$c_{ijk\ell} = c_{ij\ell k}. \tag{47b}$$

The static charge distribution ρ_e^o, the static magnetic induction $\overline{B}_o$, all other static quantities, and material parameters which may depend upon frequency ω, remain independent of the values of the harmonic variables.

Although only transducers which obey the homogeneous equations are considered finally, the harmonic sources of polarization $\overline{P}_s$, magnetization $\overline{M}_s$ (μ_o is the permeability of free-space), current $\overline{J}_s$, and volume force $\overline{f}_s$ have been included in Eqs. 38, 39, 40, and 42 to allow insight into applications involving these sources.

A number of other assumptions underlie the development of Eqs. 38-43. The harmonic displacements (rotation as well as strain) about the static positions must be small - actually, differentials of first order - to insure the validity of the strain-velocity relationship 43. Consequently, all the harmonic field variables are assumed to be first order differentials.

In the current equation 40 the "Hall effect" of the magnetic induction on "drift" velocity is ignored, as well as harmonic variations in conductivity which might be found, for example, in a biased carbon microphone. A carbon microphone would not be described by the constitutive equation 40.

The volume forces on electric and magnetic dipoles, static currents, and harmonic variations in mass density (acted upon by gravity) have been omitted in the force Eq. 42. The effects of gravity are usually minute, and if the material that contains the dipoles or static currents is rigid or has insignificant effect on the essential behavior of the transducer, the volume forces on the dipoles and static currents are justifiably neglected also. The magnet in a dynamic speaker may experience appreciable volume forces, but because the magnet is relatively rigid, the Eqs. 36-43 describe such speakers. However, the operation of a hypothetical transducer containing dipole or static current elements which move through strongly inhomogeneous em fields would not, in general, be described by Eq. 42.

Application of Maxwell's stress tensor reveals that forces on surface charges and currents can be evaluated by substituting the appropriate surface delta functions into

Eq. 42 if nonconductors contain no static surface charge, and if the electric field just outside the surface of conductors, on which static surface charge exists, is used in the evaluation of $\rho_e^o \overline{E}$. The force term $\rho_e \overline{E}_o$ is not needed in Eq. 42 because it vanishes in nonconductors where the harmonic charge distribution ρ_e must be zero, and in conductors where the static electric field $\overline{E}_o$ must be zero. The standard condenser microphone is well described by the Eqs. 36-43.

The constitutive Eqs. 38-43 may be used to reduce the three physical "laws" 36, 37a, and 37b to functions of three vectors, the electric field $\overline{E}$, the magnetic field $\overline{H}$, and the velocity field $\overline{u}$. The final three equations are conveniently displayed in operator matrix form as

$$\begin{bmatrix} (i\omega\overline{\overline{\varepsilon}}-\overline{\overline{\sigma}}\cdot) & (\nabla\times+i\omega\overline{\overline{\tau}}\cdot) & (-\overline{\overline{\alpha}}:\nabla+(\overline{\overline{\sigma}}\times\overline{B}_o)\cdot-\rho_e^o) \\ (\nabla\times-i\omega\overline{\overline{\nu}}\cdot) & (-i\omega\overline{\overline{\mu}}\cdot) & (\overline{\overline{\overline{\beta}}}:\nabla) \\ (\nabla\cdot\overline{\overline{\overline{\alpha}}}'\cdot+\overline{B}_o\times\overline{\overline{\sigma}}\cdot-\rho_e^o) & (\nabla\cdot\overline{\overline{\overline{\beta}}}'\cdot) & \dfrac{\nabla\cdot\overline{\overline{\overline{\overline{c}}}}:\nabla}{i\omega}-(\overline{B}_o\times\overline{\overline{\sigma}}\times\overline{B}_o)\cdot+\rho_e^o\overline{B}_o\times-i\omega\rho_m^o \end{bmatrix} \begin{bmatrix} \overline{E} \\ \overline{H} \\ \overline{u} \end{bmatrix} = \begin{bmatrix} \overline{J}_s-i\omega\overline{P}_s \\ i\omega\mu_o\overline{M}_s \\ \overline{f}_s-\overline{B}_o\times\overline{J}_s \end{bmatrix}. \quad (48)$$

Since $(\nabla\times\overline{u})\times\overline{\overline{I}}$ is antisymmetric, its dot products with the symmetric $\overline{\overline{\overline{\alpha}}}$, $\overline{\overline{\overline{\beta}}}$, and $\overline{\overline{\overline{\overline{c}}}}$ vanish. All operator symbols of the 3x3 matrix containing three vectors are evaluated in succession from right to left. In full scalar matrix notation, the left side of Eq. 48 becomes a 9x9 matrix operating on a 9-element column matrix, and the right side another 9-element column matrix.

Actually, the three vector equations in Eq. 48 can be reduced to two by eliminating either $\overline{E}$ or $\overline{H}$. For example, the first equation can be solved for $\overline{E}$ and the result substituted into the second and third equations. Or the second equation can be solved for $\overline{H}$ and the result substituted into the first and third equations. However, such an approach produces little or no advantage for the purposes of the present paper.

The particular arrangement of elements in the matrix of Eq. 48 was chosen because it exhibits Maxwell's equations as a 2x2 submatrix (the upper left corner), and because the associated adjoint operator matrix also represents the equations of an electroacoustic transducer (the adjoint transducer).

B. The Adjoint Operator Matrix and an Electroacoustic Reciprocity Lemma

The development of the desired generalized reciprocity relations requires the intermediate derivation of a bilinear divergence expression which combines, essentially, the Lorentz lemma[24] of electromagnetic theory with the acoustical reciprocity theorem (Eq. 31). The theory of adjoint operators assures the existence of such a bilinear expression (called the "bilinear identity"), and provides a straightforward method for obtaining the identity from the operator matrix of Eq. 48.[25] Specifically, if L denotes the linear operator matrix of Eq. 48, the bilinear identity may be written in differential form as

$$\tilde{\phi}_1 L \phi_2 - \tilde{\phi}_2 L^a \phi_1 = \nabla \cdot \overline{F}, \qquad (49)$$

where L^a denotes the "transposed" or adjoint operator associated with L; ϕ_1 and ϕ_2 are any two continuously differentiable column matrices containing three vectors (the tilde transposes the column matrix to the corresponding row matrix);

and $\overline{F}$ is a bilinear function of ϕ_1, ϕ_2 and their derivatives. The explicit evaluation of $\nabla \cdot \overline{F}$ requires the determination of the adjoint operator L^a, which is actually defined by Eq. 49.[25] (In Appendix II, Eq. 49 is used as a starting point to write the electroacoustic fields inside the transducer in terms of the fields on the surface of the transducer.)

For a given linear operator L, the adjoint operator L^a exists uniquely (a proof of the uniqueness of L^a is given in Appendix I) and may be found by transposing the matrix elements of L, and replacing the differential operators by their adjoints.[26] Fortunately, the adjoints of the four different linear differential operators of L,

$$\nabla\times, \quad \overline{\overline{\overline{\alpha}}}:\nabla, \quad \nabla\cdot\overline{\overline{\overline{\alpha}}}'\cdot, \quad \text{and} \quad \nabla\cdot\overline{\overline{\overline{\overline{c}}}}:\nabla, \qquad (50a,b,c,d)$$

are readily determined as

$$\nabla\times, \quad -\nabla\cdot\overline{\overline{\overline{\alpha}}}_t\cdot, \quad -\overline{\overline{\overline{\alpha}}}'_t:\nabla, \quad \text{and} \quad \nabla\cdot\overline{\overline{\overline{\overline{c}}}}_t:\nabla, \qquad (51a,b,c,d)$$

respectively, with the help of the associated identities,

$$\overline{A}_1\cdot(\nabla\times\overline{A}_2)-\overline{A}_2\cdot(\nabla\times\overline{A}_1) = \nabla\cdot(\overline{A}_2\times\overline{A}_1) \qquad (52a)$$

$$\overline{A}_1\cdot\left\{\nabla\cdot(\overline{\overline{\overline{\alpha}}}\cdot\overline{A}_2)\right\} + \overline{A}_2\cdot\left\{\overline{\overline{\overline{\alpha}}}_t:\nabla\overline{A}_1\right\} = \nabla\cdot(\overline{A}_1\cdot\overline{\overline{\overline{\alpha}}}\cdot\overline{A}_2) \qquad (52b)$$

$$\overline{A}_1\cdot\left\{\nabla\cdot(\overline{\overline{\overline{\overline{c}}}}:\nabla\overline{A}_2)\right\}-\overline{A}_2\cdot\left\{\nabla\cdot(\overline{\overline{\overline{\overline{c}}}}_t:\nabla\overline{A}_1)\right\} = \nabla\cdot(\overline{A}_1\cdot\overline{\overline{\overline{\overline{c}}}}:\nabla\overline{A}_2-\overline{A}_2\cdot\overline{\overline{\overline{\overline{c}}}}_t:\nabla\overline{A}_1). \qquad (52c)$$

The subscript t denotes the transposed matrix, and $\overline{A}_1$, $\overline{A}_2$ are arbitrary, continuously differentiable vectors. In tensor notation the transpose of α_{ijk} and $c_{ijk\ell}$, for example, used in Eqs. 51, 52 and below is α_{jki} and $c_{k\ell ij}$. Note that the curl operator is self-adjoint, that the direct and converse piezoelectric and piezomagnetic operators are adjoints of each other if $\overline{\overline{\alpha}}_t = -\overline{\overline{\alpha}}'$, $\overline{\overline{\beta}}_t = -\overline{\overline{\beta}}_t'$, and that the purely acoustic operator is self-adjoint if $\overline{\overline{\overline{c}}}_t = \overline{\overline{\overline{c}}}$. These self-adjoint properties are understandable from Lanczos's theory of adjoint operators where he shows that the curl operator is self-adjoint and that the gradient and divergence operators are negative adjoints of each other.[25]

The straightforward transposition of the nondifferential elements of L completes the construction of the adjoint operator L^a:

$$L^a = \begin{bmatrix} (i\omega\overline{\overline{\varepsilon}}_t\cdot - \overline{\overline{\sigma}}_t\cdot) & (\nabla\times - i\omega\overline{\overline{\nu}}_t\cdot) & \left[-\overline{\overline{\alpha}}_t':\nabla - (\overline{\overline{\sigma}}_t\times\overline{B}_o)\cdot - \rho_e^o\right] \\ (\nabla\times + i\omega\overline{\overline{\tau}}_t\cdot) & (-i\omega\overline{\overline{\mu}}_t\cdot) & (-\overline{\overline{\beta}}_t':\nabla) \\ (\nabla\cdot\overline{\overline{\alpha}}_t\cdot - \overline{B}_o\times\overline{\overline{\sigma}}_t\cdot - \rho_e^o) & (-\nabla\cdot\overline{\overline{\beta}}_t\cdot) & \left[\frac{\nabla\cdot\overline{\overline{\overline{c}}}_t:\nabla}{i\omega} - (\overline{B}_o\times\overline{\overline{\sigma}}_t\times\overline{B}_o)\cdot - \rho_e^o\overline{B}_o\times - i\omega\rho_m^o\right] \end{bmatrix}. \tag{53}$$

Comparison of the adjoint operator L^a with L reveals that, at a given frequency ω, the adjoint operator describes the equations of a second transducer (the adjoint transducer)[27] in which the transposed parameters

$$\overline{\overline{\sigma}}_t,\ \overline{\overline{\varepsilon}}_t,\ \overline{\overline{\mu}}_t,\ -\overline{\overline{\tau}}_t,\ -\overline{\overline{\nu}}_t,\ \overline{\overline{\alpha}}_t,\ -\overline{\overline{\beta}}_t,\ \overline{\overline{\alpha}}_t',\ -\overline{\overline{\beta}}_t',\ \overline{\overline{\overline{c}}}_t,\ -\overline{B}_o,\ \rho_e^o,\ \text{and}\ \rho_m^o \tag{54a}$$

replace the original parameters

$$\overline{\overline{\sigma}},\ \overline{\overline{\varepsilon}},\ \overline{\overline{\mu}},\ \overline{\overline{\nu}},\ \overline{\overline{\tau}},\ \overline{\overline{\alpha}}',\ \overline{\overline{\beta}}',\ \overline{\overline{\alpha}},\ \overline{\overline{\beta}},\ \overline{\overline{\overline{c}}},\ \overline{B}_o,\ \rho_e^o\ \text{and}\ \rho_m^o, \tag{54b}$$

respectively. The adjoint transducer refers to a mathematically conceived transducer with hypothetical material parameters which may or may not be physically realizable. Taking the adjoint of the adjoint operator returns the adjoint operator and transducer to the original operator and transducer. When the transposed parameters 54a equal the original parameters 54b, the operators L and L^a become identical to form a "self-adjoint" operator for which the adjoint transducer and original transducer are one and the same. In that case $\overline{B}_o$ is zero ($\overline{B}_o = -\overline{B}_o$), which corresponds to the electric-type coupling of Primakoff and Foldy.[6]

The expression 53 for the adjoint operator allows the explicit evaluation of $\nabla \cdot \overline{F}$. If ϕ_2 is restricted to solutions of Eq. 48, and ϕ_1 to solutions of the adjoint equations corresponding to Eq. 48 with L^a replacing L, i.e.

$$L\phi_2 = Q, \quad L^a\phi_1 = Q^a \tag{55a,b}$$

where

$$\phi_2 = \begin{bmatrix} \overline{E} \\ \overline{H} \\ \overline{u} \end{bmatrix}, \quad \phi_1 = \begin{bmatrix} \overline{E}^a \\ \overline{H}^a \\ \overline{u}^a \end{bmatrix} \tag{56a,b}$$

and

$$Q = \begin{bmatrix} \overline{J}_s - i\omega\overline{P}_s \\ i\omega\mu_o\overline{M}_s \\ \overline{f}_s - \overline{B}_o \times \overline{J}_s \end{bmatrix}, \quad Q^a = \begin{bmatrix} \overline{J}^a_s - i\omega\overline{P}^a_s \\ i\omega\mu_o\overline{M}^a_s \\ \overline{f}^a_s + \overline{B}_o \times \overline{J}^a_s \end{bmatrix}, \tag{57a,b}$$

then $\nabla\cdot\overline{F}$ becomes, after rearrangement and use of Eqs. 41 and 43,

$$\begin{aligned}\nabla\cdot\overline{F} &= \nabla\cdot(\overline{E}\times\overline{H}^a-\overline{E}^a\times\overline{H}) - \nabla\cdot(\overline{u}^a\cdot\overline{\overline{T}}-\overline{u}\cdot\overline{\overline{T}}^a)\\ &= \overline{E}^a\cdot(\overline{J}_s-i\omega\overline{P}_s) + i\omega\mu_o\overline{H}^a\cdot\overline{M}_s+\overline{u}^a\cdot(\overline{f}_s-\overline{B}_o\times\overline{J}_s)\\ &\quad - \overline{E}\cdot(\overline{J}^a_s-i\omega\overline{P}^a_s) - i\omega\mu_o\overline{H}\cdot\overline{M}^a_s - \overline{u}\cdot(\overline{f}^a_s+\overline{B}_o\times J^a_s).\end{aligned} \tag{58}$$

When the harmonic sources are set equal to zero, application of the divergence theorem to Eq. 58 in the volume bounded by the surface A_o of the transducer yields a generalized reciprocity "lemma" between the independent electroacoustic fields of a given transducer and those of its mathematical adjoint:

$$\int_{A_o} (\overline{E}\times\overline{H}^a-\overline{E}^a\times\overline{H})\cdot\hat{n}da + \int_{A_o} (\overline{\overline{T}}^a\cdot\overline{u}-\overline{\overline{T}}\cdot\overline{u}^a)\cdot\hat{n}da = 0. \tag{59}$$

The name "generalized electroacoustic reciprocity lemma" seems appropriate for Eq. 59 since it represents a merger of the Lorentz lemma[24] of em theory and the acoustical reciprocity theorem (Eq. 31) as applied to adjoint systems. For self-adjoint transducers, i.e. transducers for which the transposed material parameters of Eq. 54a equal those of Eq. 54b, Eq. 59 reduces to the corresponding Primakoff-Foldy expression[6] in which a second set of fields on the same transducer replace the adjoint fields. Auld[20] uses this latter expression with the complex conjugate fields to derive the orthogonality relation[28] between different modes of piezoelectric waveguides and resonators. Equation 59 demonstrates that the waveguides and resonators need not be restricted to piezo-electric material in order for the modes to satisfy orthogonality.

A more general expression similar to Eq. 59 may be derived from the equations represented by the operator L. Each row of the operator L in Eq. 48 could be multiplied by an arbitrary constant without affecting the essential content of the equations. Specifically, if the first, second, and third row of L are multiplied by arbitrary complex constants, c_1, c_2 and c_3 respectively, Eq. 59 is replaced by

$$\int_{A_o} [c_1(\bar{E}\times\bar{H}^a)-c_2(\bar{E}^a\times\bar{H})]\cdot\hat{n}da + c_3 \int_{A_o} (\bar{\bar{T}}^a\cdot\bar{u}-\bar{\bar{T}}\cdot\bar{u}^a)\cdot\hat{n}da = 0. \tag{60}$$

Although Eq. 60 has mathematical validity and interest, the constants c_1, c_2 and c_3 must be restricted if Eq. 60 is to be used in the derivation of the generalized reciprocity relations between a given transducer and its adjoint transducer. First, because the right side of Eq. 60 is zero, the constant c_1, can be set equal to unity without loss of generality. Second, c_2 must equal $c_1 = 1$ to accomplish the derivation of the scattering-matrix reciprocity relations (Eqs. 76 below). And third, if the adjoint operator underlying Eq. 60 is required to describe the fields of a second transducer (the adjoint transducer), comparison of Eq. 53 with 48 (when the constants c_1, c_2 and c_3 are included) shows that for $c_1 = c_2 = 1$, the only allowable values of c_3 are ± 1 unless both $\overline{B}_o$ and ρ_e^o are zero. If both $\overline{B}_o$ and ρ_e^o vanish throughout the transducers, the adjoint operator represents an adjoint transducer for all values of c_3.[29]

Equation 59 already represents the case of $c_3 = +1$, where the relationship between the adjoint transducer parameters and those of the original transducer is given by Eqs. 54.

For $c_3 = -1$, the operator L in Eq. 48 changes to L^-, the superscript (-) indicating that the third row of L has been multiplied by a minus sign. The operator adjoint to L^- describes an adjoint transducer in which the transposed parameters

$$\overline{\overline{\sigma}}_t, \overline{\overline{\varepsilon}}_t, \overline{\overline{\mu}}_t, -\overline{\overline{\tau}}_t, -\overline{\overline{\nu}}_t, -\overline{\overline{\overline{\alpha}}}_t, \overline{\overline{\overline{\beta}}}_t, -\overline{\overline{\overline{\alpha}}}'_t, \overline{\overline{\overline{\beta}}}'_t, \overline{\overline{\overline{\overline{c}}}}_t, \overline{B}_o, -\rho_e^o \text{ and } \rho_m^o, \qquad (61)$$

respectively, replace the original parameters listed in Eq. 54b. The transducer of Eq. 54a ($c_3 = 1$) converts to the transducer of Eq. 61 ($c_3 = -1$), and vice versa, by reversing the signs of the off-diagonal elements in the third row and column of L^a.

As mentioned above for L, when the transposed parameters 61 assume the values of the original parameters 54b, the operator L^- and its adjoint L^{-a} become identical to form a self-adjoint operator for which the adjoint and original transducer are one and the same. In that case, ρ_e^o must be zero ($\rho_e^o = -\rho_e^o$), which corresponds to the magnetic-type coupling of Primakoff and Foldy.[6]

The equations represented by the operators L and L^- describe the same transducer. However, the equations represented by the adjoints L^a and L^{-a} refer, in general, to different adjoint transducers. (Physical sources associated with the given transducers show up as inhomogeneous terms in the operator equations. However, it should be noted that the same physical sources show up as different inhomogeneous terms in the L and L^- equations.) Section III.C. shows that a transducer self-adjoint with respect to L satisfies reciprocity, while a transducer self-adjoint with respect to L^- satisfies antireciprocity.

C. Generalized Reciprocity Relations

The derivation of the desired reciprocity relations can now be accomplished with the help of the reciprocity lemma 60. It will also be demonstrated that the SIM relations 3 and PWSM equations 1 result as a by-product of the reciprocity derivation. Such a demonstration is included simply to prove that the two matrix descriptions of a transducer, which satisfies the linear equations 36-43, need not be postulated as a definition of linearity, but follow from the linearity of equations 36-43.

The reciprocity lemma 60 for adjoint transducers may be written as

$$\int_{A_o} [(\bar{E}^a \times \bar{H}) - (\bar{E} \times \bar{H}^a)] \cdot \hat{n} da \pm \int_{A_o} (p^a u_n - p u_n^a) da = 0, \quad (62)$$

where the plus sign refers to the operator L and the minus sign to L^-. The surface A_o of the transducer has been taken in the surrounding fluid, so

$$-\hat{n}\cdot\bar{\bar{T}}\cdot\hat{n} = p$$

$$-\hat{n}\cdot\bar{\bar{T}}^a\cdot\hat{n} = p^a,$$

except on the feed area S_o where the contribution to the integral is assumed negligible.

The integral over A_o of the em fields expands to

$$V_o^a I_o - V_o I_o^a + \int_{A_o - S_o} [(\bar{E}^a \times \bar{H}) - (\bar{E} \times \bar{H}^a)]\cdot\hat{n}da \qquad (63)$$

($\hat{n}$ points into the transducer),

after substitutions of the modal fields normalized to unity by Eq. 11. The contribution from the integral over $A_o - S_o$ vanishes if either of the following conditions is satisfied:

1) The surface $A_o - S_o$ of the transducer (but not necessarily the surface $A_p - S_o$ of the power supply) is electromagnetically shielded so that $\bar{E}$, $\bar{E}^a$ or $\bar{H}$, $\bar{H}^a$ are zero on $A_o - S_o$. (Prescribed boundary conditions on the adjoint transducer are chosen to follow those of the original transducer.)

2) The surface $A_p - S_o$ of the power supply (but not necessarily the surface $A_o - S_o$ of the transducer) is electromagnetically shielded and no em sources[30] exist external to the transducer-plus-source (or detector) A. Under these conditions, the surface $A_o - S_o$ may be enlarged to a surface of infinite radius where the Sommerfeld radiation condition or the exponential decay in electromagnetically lossy media demands that the integral be zero.

If the transducer operates at quasi-static frequencies so that the em fields outside the transducer obey

$$\bar{E} = -\nabla\psi, \quad \bar{E}^a = -\nabla\psi^a \tag{64a}$$

$$\nabla\times\bar{H} = \bar{J}, \quad \nabla\times\bar{H}^a = \bar{J}^a, \tag{64b}$$

modal theory may not apply, but substitution of Eqs. 64 into the em integral of Eq. 62 shows that the contribution over A_o-S_o still vanishes. That is, no shielding is required at quasi-static frequencies of operation for the expression

$$V_o^a I_o - V_o I_o^a$$

to represent the em term of Eq. 62.

With the em integral over A_o-S_o zero, Eq. 62 becomes

$$V_o^a I_o - V_o I_o^a \pm \int_{A_o} (p^a u_n - p u_n^a)\, da = 0. \tag{65}$$

Because the fields of the transducer and its adjoint exist independently and the velocity-current variables of each transducer may assume arbitrary values, Eq. 65 transforms into the SIM relations by first choosing

$$I_{o1}^a = 1, \quad u_{n1}^a(\bar{r}) = 0, \tag{66a}$$

which gives

$$V_o = V_{o1}^a I_o \pm \int_{A_o} p_1^a(\bar{r}) u_n(\bar{r})\, da; \tag{67a}$$

then choosing

$$I_{o2}^a = 0, \quad u_{n2}^a(\bar{r}) = \delta(\bar{r}-\bar{r}'), \tag{66b}$$

which gives

$$p(\bar{r}) = \pm V^a_{o2}(\bar{r}) I_o + \int_{A_o} p^a_2(\bar{r},\bar{r}_o) u_n(\bar{r}_o)\, da. \tag{67b}$$

Comparison of Eqs. 67 with Eqs. 3 shows that the various spatial impedances equate as

$$Z_b = V^a_{o1}, \qquad h'(\bar{r}) = \pm p^a_1(\bar{r}),$$
$$h(\bar{r}) = \pm V^a_{o2}(\bar{r}), \quad Z_o(\bar{r},\bar{r}_o) = p^a_2(\bar{r},\bar{r}_o). \tag{68}$$

The PWSM equations 1 are disclosed in a similar manner[31] once the acoustical reciprocity theorem 31, Eqs. 7, and Eqs. 4 and 8 convert Eq. 65 to

$$\eta_o(b_o a^a_o - b^a_o a_o) \pm \int_{\bar{K}} \eta(K)[a(\bar{K}) b^a(-\bar{K}) - a^a(-\bar{K}) b(\bar{K})]\, d\bar{K} = 0. \tag{69}$$

Since the values of a_o, a and a^a_o, a^a may be designated arbitrarily and independently of each other, one may choose

$$a^a_{o1} = 1, \quad a^a_1(\bar{K}) = 0, \tag{70a}$$

and

$$a^a_{o2} = 0, \quad a^a_2(\bar{K}) = \delta(\bar{K} + \bar{L}), \tag{70b}$$

to extract the PWSM equations from Eq. 69:

$$b_o = b^a_{o1} a_o \mp \frac{1}{\eta_o} \int_{\bar{L}} \eta(L)\, b^a_1(-\bar{L}) a(\bar{L})\, d\bar{L} \tag{71a}$$

$$b(\bar{K}) = \mp \frac{\eta_o}{\eta(K)}\, b^a_{o2}(\bar{K}) a_o + \frac{1}{\eta(K)} \int_{\bar{L}} \eta(L) b^a_2(\bar{K},-\bar{L}) a(\bar{L})\, d\bar{L}. \tag{71b}$$

Equations 71 coincide with Eqs. 1 under the replacements,

$$S_{oo} = b_{o1}^{a}, \quad S_{o1}(\bar{L}) = \mp \eta(L) b_{1}^{a}(-\bar{L})/\eta_{o},$$
$$S_{1o}(\bar{K}) = \mp \eta_{o} b_{o2}^{a}(\bar{K})/\eta(K), \; S_{11} = \eta(L) b_{2}^{a}(\bar{K},-\bar{L})/\eta(K). \tag{72}$$

If the conditions of Eqs. 66 and 70 are applied to the given transducer rather than its adjoint transducers, Eqs. 65 and 69 produce Eqs. 67 and 71 with the given transducer and adjoint transducers interchanged. That is, the SIM relations and the PWSM equations describe the adjoint transducers as well as the original transducer.

The generalized reciprocity relations between a given transducer and its adjoint transducers may also be extracted from Eqs. 65 and 69 by eliminating the pressures and voltages of Eqs. 65 with the SIM relations, and the outgoing amplitudes of Eqs. 69 with the PWSM equations. Such a procedure yields

$$I_{o}^{a}(Z_{b}^{a}-Z_{b}) I_{o} + I_{o}\int_{A_o} [h^{a'}(\bar{r}) \mp h(\bar{r})] u_{n}^{a}(\bar{r})\, da - I_{o}^{a}\int_{A_o} [h'(\bar{r}) \mp h^{a}(\bar{r})] u_{n}(\bar{r})\, da$$
$$\pm \int_{A_o}\int_{A_o} u_{n}^{a}(\bar{r}) [Z_{o}^{a}(\bar{r}_{o},\bar{r}) - Z_{o}(\bar{r},\bar{r}_{o})] u_{n}(\bar{r}_{o})\, da_{o}\, da = 0 \tag{73}$$

for Eq. 65, and

$$\eta_o a_o^a (S_{oo}^a - S_{oo}) a_o + a_o \int_{\bar{L}} [\eta_o S_{o1}^a(\bar{L}) \mp \eta(L) S_{1o}(-\bar{L})] a^a(\bar{L})\, d\bar{L}$$

$$+ a_o^a \int_{\bar{L}} [\eta_o S_{o1}(\bar{L}) \mp \eta(L) S_{1o}^a(-\bar{L})] a(\bar{L})\, d\bar{L} \tag{74}$$

$$\pm \int_{\bar{L}} \int_{\bar{K}} a^a(\bar{L}) [\eta(K) S_{11}^a(-\bar{K},\bar{L}) - \eta(L) S_{11}(-\bar{L},\bar{K})] a(\bar{K})\, d\bar{K}\, d\bar{L} = 0$$

for Eq. 69.

Since they must hold for arbitrary values of the inputs, Eqs. 73 and 74 imply

$$Z_b^a = Z_b \tag{75a}$$

$$h^{a'}(\bar{r}) = \pm h(\bar{r}) \tag{75b}$$

$$h^a(\bar{r}) = \pm h'(\bar{r}) \tag{75c}$$

$$Z_o^a(\bar{r}_o,\bar{r}) = Z_o(\bar{r},\bar{r}_o) \tag{75d}$$

and

$$S_{oo}^a = S_{oo} \tag{76a}$$

$$\eta_o S_{o1}^a(\bar{K}) = \pm \eta(K) S_{1o}(-\bar{K}) \tag{76b}$$

$$\eta(K) S_{1o}^a(\bar{K}) = \pm \eta_o S_{o1}(-\bar{K}) \tag{76c}$$

$$\eta(K) S_{11}^a(-\bar{K},\bar{L}) = \eta(L) S_{11}(-\bar{L},\bar{K}). \tag{76d}$$

With the substitution of pressure and normal velocity from Eqs. 7 and the help of the spectral impedance equations 24b and 26, Eq. 65 shows that the spectral impedance matrices also satisfy

$$Z^{a}_{oo} = Z_{oo} \tag{77a}$$

$$Z^{a}_{o1}(\bar{K}) = \pm\, Z_{1o}(-\bar{K}) \tag{77b}$$

$$Z^{a}_{1o}(\bar{K}) = \pm\, Z_{o1}(-\bar{K}) \tag{77c}$$

$$Z^{a}_{11}(-\bar{K},\bar{L}) = Z_{11}(-\bar{L},\bar{K}). \tag{77d}$$

The procedure and conclusions of Section II.B demonstrates the equivalence of the three separate expressions of reciprocity (Eqs. 75, 76, 77). Either the "b" or "c" equation of Eqs. 75-77 may be considered redundant since the entire derivation could be repeated with the given and adjoint transducers interchanged.

When a transducer is self-adjoint with respect to L or L^-, Eqs. 75-77 reduce to the original reciprocity relations of Eqs. 2, 5, and 30. Reciprocity defined by both Kerns and Foldy-Primakoff is satisfied by an electroacoustic transducer that exhibits electric-type coupling (e.g. the condenser microphone) to form a self-adjoint operator L. Antireciprocity is satisfied by a transducer that exhibits magnetic-type coupling (e.g. the dynamic speaker) to form a self-adjoint operator L^-. If the static fields as well as the piezoelectric and piezomagnetic coupling coefficients vanish, comparison of

Eqs. 54a with 61 reveals that both adjoint transducers become identical to the original transducer, and by Eqs. 75-77, the transmitting and receiving parameters h, h', S_{1o}, S_{o1}, Z_{1o}, Z_{o1} also vanish. In other words, there is no electroacoustic coupling - a result that the constitutive relations 38-42 confirm.

D. Adjoint Reciprocity Theorems

The generalized reciprocity relations 75-77 may find expression in a variety of derivative "reciprocity theorems and principles." As a consequence of Eqs. 75, 76, or 77, the "electroacoustic reciprocity theorem"[5,32] for reciprocal transducers extends to mutually adjoint transducers by the relating of the microphone response of the given transducer to the speaker response of its adjoint transducers (and vice versa). Similarly, the "principle of reciprocity"[33] that relates scattered pressure at a point B from a point source at A to scattered pressure at point A when the point source is moved to B may be extended to mutually adjoint tranducers. The following derivation of the aforementioned results are performed simply and in great generality by using the Kerns PWSM description.

The standard speaker response $S(\overline{r})$ of a given transducer is defined as the ratio of the pressure at a point $\overline{r}$ in the ambient fluid to the transducer input current I_o,[5]

$$S(\bar{r}) \equiv \frac{p(\bar{r})}{I_o}. \qquad (78)$$

Because incoming acoustic waves are assumed zero, i.e. $a(\bar{K}) = 0$, the pressure in Eq. 7a may be expressed in terms of $b(\bar{K}) = V(\bar{K})$ alone,

$$p(\bar{r}) = \frac{1}{2\pi} \int_{\bar{K}} b(\bar{K}) e^{i\gamma z} e^{i\bar{K}\cdot\bar{r}} \, d\bar{K}. \qquad (79)$$

The factor $e^{i\gamma z}$ appearing in the integral of Eq. 79 allows the z-coordinate of $\bar{r}$ to assume values other than zero. (The fluid must contain no sources or inhomogenities between z = 0 and the plane of integration.) The PWSM equations 1, which reduce to

$$b_o = S_{oo} a_o \qquad (80a)$$

$$b(\bar{K}) = S_{1o}(\bar{K}) a_o \qquad (80b)$$

when $a(\bar{K}) = 0$, combine with Eq. 4b to yield the following expression for $b(\bar{K})$ in terms of I_o:

$$b(\bar{K}) = \frac{S_{1o}(\bar{K}) \, I_o}{(1-S_{oo}) \, \eta_o}. \qquad (81)$$

After substitution of $b(\bar{K})$ from Eq. 81 into Eq. 79, the speaker response 78 takes the final form of

$$S(\bar{r}) = \frac{1}{2\pi(1-S_{oo}) \, \eta_o} \int_{\bar{K}} S_{1o}(\bar{K}) e^{i\bar{k}\cdot\bar{r}} \, d\bar{K}, \qquad (82)$$

$$\bar{k} = \bar{K} + \gamma\hat{e}_z.$$

The adjoint microphone response is defined as the ratio of the open-circuit voltage V_o for an adjoint transducer in the presence of a spherical pressure wave centered at point $\overline{r}$ in the fluid to the pressure in the spherical wave at a reference point $\overline{r}_c$,[5]

$$M^a(\overline{r}) \equiv \frac{V_o}{p_r(\overline{r}_c)}, \tag{83}$$

$$p_r(\overline{r}_c) = \frac{Ae^{ik|\overline{r}_c-\overline{r}|}}{|\overline{r}_c-\overline{r}|}. \tag{84}$$

(The amplitude of the spherical wave is denoted by A.) Since I_o is zero, a_o equals b_o, and V_o is equal to $2a_o$. The scattering-matrix equation 1a for an adjoint transducer relates a_o to $a(\overline{K})$,

$$a_o = \frac{1}{(1-S^a_{oo})} \int_{\overline{K}} S^a_{o1}(\overline{K})a(\overline{K})\, d\overline{K}. \tag{85}$$

The incoming amplitudes $a(\overline{K})$ are given by Eq. 9a as

$$a(\overline{K}) = V(\overline{K}) = \frac{1}{2\pi} \int_{S_1} p_r(\overline{R}_c)e^{-i\overline{K}\cdot\overline{R}_c}\, d\overline{R}_c. \tag{86}$$

Replacement of the pressure in Eq. 86 by its value given in Eq. 84 enables the explicit determination of $a(\overline{K})$,

$$a(\overline{K}) = \frac{Ai}{\gamma} e^{i\gamma z}e^{-i\overline{K}\cdot\overline{R}}, \tag{87}$$

which combines with Eqs. 85, 84, and 83 to produce the final form of the adjoint microphone response,

$$M^a(\overline{r}) = \frac{2id}{(1-S^a_{oo})e^{ikd}} \int_{\overline{K}} \frac{S^a_{o1}(-\overline{K})}{\gamma} e^{i\overline{K}\cdot\overline{r}} d\overline{K}, \quad d = |\overline{r}_c - \overline{r}|. \tag{88}$$

Since the adjoint reciprocity relations require that

$$S^a_{oo} = S_{oo}$$

$$\eta_o S^a_{o1}(-\overline{K}) = \frac{\pm\gamma}{\omega\rho_o} S_{1o}(\overline{K}),$$

Eqs. 82 and 88 show that

$$\left|\frac{M^a(\overline{r})}{S(\overline{r})}\right| = \frac{4\pi d}{\omega\rho_o}, \tag{89}$$

or that the magnitude of the ratio of microphone response at $\overline{r}$ of the adjoint transducers to the speaker response at $\overline{r}$ of the original transducer is a constant independent of the position $\overline{r}$ and of the particular transducer involved. The constant could also be made independent of the distance d to the reference point $\overline{r}_c$ if the microphone response were "normalized" to the amplitude A of the spherical wave. If the transducer is self-adjoint (reciprocal or antireciprocal) Eq. 89 becomes identical to the standard electroacoustic reciprocity theorem.[32]

Next, consider the derivation of the adjoint "principle of reciprocity" which applies to the scattering of a spherical wave from a given scatterer (or transducer terminated in a passive load). The scattered pressure $p_{sc}(\overline{r}_B,\overline{r}_A)$ at a point $\overline{r}_B$

in the fluid caused by a spherical wave centered at a point $\bar{r}_A$ in the fluid is given, as in Eq. 79, by

$$p_{sc}(\bar{r}_B,\bar{r}_A) = \frac{1}{2\pi}\int_{\bar{K}} b(\bar{K}) e^{i\bar{K}\cdot\bar{r}}\, d\bar{K}, \tag{90}$$

where $b(\bar{K})$ may be found in terms of $a(\bar{K})$ from the scattering equations 1:

$$b(\bar{K}) = \int_{\bar{L}} \left\{\frac{S_{1o}(\bar{K})S_{o1}(\bar{L})\Gamma_T}{(1-\Gamma_T S_{oo})} + S_{11}(\bar{K},\bar{L})\right\} a(\bar{L})\, d\bar{L}. \tag{91}$$

The reflection coefficient Γ_T is related to the terminating load $Z_T = -V_o/I_o$ by

$$\Gamma_T = \frac{Z_T\eta_o - 1}{Z_T\eta_o + 1}. \tag{92}$$

Since the source is a spherical pressure wave centered at $\bar{r}_A$, the amplitudes $a(\bar{L})$ take the same form as in Eq. 87, i.e.

$$a(\bar{L}) = \frac{A_i}{\gamma(L)} e^{i\gamma(L) z_A} e^{-i\bar{L}\cdot\bar{R}_{A_o}}. \tag{93}$$

Substitution of Eqs. 93 and 91 into 90 results in the scattered pressure

$$p_{sc}(\bar{r}_B,\bar{r}_A) = \frac{A_i}{2\pi}\int_{\bar{K}}\int_{\bar{L}} \frac{\left\{\frac{S_{1o}(\bar{K})S_{o1}(-\bar{L})\Gamma_T}{1-\Gamma_T S_{oo}} + S_{11}(\bar{K},-\bar{L})\right\}}{\gamma(L)} e^{i\bar{K}\cdot\bar{r}_B} e^{i\bar{\ell}\cdot\bar{r}_A}\, d\bar{L}\, d\bar{K}. \tag{94}$$

If the position of the source and observer is interchanged and the given transducer is replaced by an adjoint transducer, the scattered pressure at $\bar{r}_A$ is found in a similar manner to be

$$p^a_{sc}(\bar{r}_A,\bar{r}_B) = \frac{Ai}{2\pi}\int_{\bar{K}}\int_{\bar{L}} \frac{\left\{\dfrac{S^a_{1o}(\bar{K})S^a_{o1}(-\bar{L})\Gamma_T}{1-\Gamma_T S^a_{oo}} + S^a_{11}(\bar{K},-\bar{L})\right\}}{\gamma(L)} e^{i\bar{k}\cdot\bar{r}_A} e^{i\bar{\ell}\cdot\bar{r}_B}\, d\bar{L}\, d\bar{K}. \tag{95}$$

(The termination for the adjoint transducers is kept at Z_T.)

The adjoint reciprocity relations 76 demonstrate from Eqs. 94 and 95 that

$$p_{sc}(\bar{r}_B,\bar{r}_A) = p^a_{sc}(\bar{r}_A,\bar{r}_B), \tag{96}$$

which completes the proof of the adjoint principle of reciprocity. (Note that the bracketed quantity in the integrands of Eqs. 94 and 95 represents an equivalent scattering matrix with reciprocal properties identical to those of S_{11}.) That is, the scattered pressure measured at a point $\bar{r}_B$ from a point source at $\bar{r}_A$ is equal to the scattered pressure measured at the point $\bar{r}_A$ when the point source is moved to $\bar{r}_B$ and an adjoint transducer replaces the given transducer. If the transducer is self-adjoint (reciprocal or antireciprocal) Eq. 96 expresses the usual principle of reciprocity[33] for scatterers.

E. Power Relations

For harmonic fields, the time-average em and acoustic power flow per unit area is given by $\frac{1}{2}\mathrm{Re}[\bar{E}\times\bar{H}^*]$ and $-\frac{1}{2}\mathrm{Re}[\bar{\bar{T}}\cdot\bar{u}^*]$ respectively. (The asterisk denotes the complex conjugate.) The total time-average power P flowing into the electroacoustic transducer through its surface A_o is, therefore, given by

$$P = \frac{1}{2}\,\mathrm{Re}\oint_{A_o} [\bar{E}\times\bar{H}^* - \bar{\bar{T}}\cdot\bar{u}^*]\cdot\hat{n}da, \qquad (97)$$

where $\hat{n}$ denotes the unit normal into the transducer. Since the surface A_o lies in or on the boundary of the ambient fluid,

$$\hat{n}\cdot\bar{\bar{T}}\cdot\bar{u}^* = -pu_n{}^*,$$

and the power equation 97 becomes

$$P = \frac{1}{2}\,\mathrm{Re}\oint_{A_o} [(\bar{E}\times\bar{H}^*)\cdot\hat{n} + pu_n{}^*]\ da. \qquad (98)$$

If the transducer is electromagnetically shielded, the em part of the integral 98 evaluates simply as $V_oI_o{}^*$ by expanding $\bar{E}$ and $\bar{H}^*$ in terms of the waveguide mode, i.e. from Eqs. 10

$$\bar{E}_t = V_o\bar{e}_o,\quad \bar{H}^*_t = I^*_o\,\bar{h}_o,$$

and making use of the normalization equation 11. (Waveguide theory shows that the basis fields $(\bar{e}_o, \bar{h}_o)$ of a propagating mode in a uniform, isotropic guide may be chosen real.)

The SIM relations 3 convert the power expression 98 to an Hermitian form containing continuous ($u_n(\bar{r})$) and discrete (I_o) variables:

$$P = I_o(Z_b+Z_b^*) + \int_{A_o} [I_o(h'^*(\bar{r})+h(\bar{r}))u_n^*(\bar{r}) + I_o^*(h'(\bar{r})+h^*(\bar{r}))u_n(\bar{r})]\, da$$
$$+ \int_{A_o}\int_{A_o} u_n(\bar{r}_o)[Z_o(\bar{r},\bar{r}_o) + Z_o^*(\bar{r}_o,\bar{r})]u_n^*(\bar{r})\, da_o\, da. \quad (99)$$

(The taking of the real part has been accomplished by adding each term to its complex conjugate.)

A transducer may be classified as lossy, lossless, or "gainy" if, for all values of u_n and I_o (except u_n, I_o all zero), the power P is positive, zero, or negative, i.e. if the Hermitian form 99 is positive definite, zero, or negative definite.[34]

Mercer's theorem,[35] which can be generalized to provide necessary and sufficient conditions on the spacial impedances Z_b, h, h', and Z_o for the power P to belong to one of the "definite" value classes, reduces to very simple relationships among the spacial impedances when the transducer is lossless. In the lossless case, i.e. when the power expression 99 is identically zero for all values of u_n and I_o, various combinations of I_o equal to zero and $u_n(\bar{r})$ equal to zero or a sum of delta functions can be chosen to prove that the spatial impedances satisfy the following equalities:

$$Z_b + Z_b^* = 0 \quad \text{(i.e. } Z_b \text{ is imaginary)} \tag{100a}$$

$$h'(\bar{r}) + h^*(\bar{r}) = 0 \tag{100b}$$

$$Z_o(\bar{r},\bar{r}_o) + Z_o^*(\bar{r}_o,\bar{r}) = 0. \tag{100c}$$

A reciprocal or antireciprocal transducer also satisfies Eqs. 5, which combine with Eqs. 100 to show that Z_o (as well as Z_b) is imaginary, and that both h, h' are imaginary or real depending upon whether the transducer is reciprocal or antireciprocal respectively. Equations 29 may be used to demonstrate that no simple relations similar to Eqs. 100 exist among the spectral impedance or scattering matrices of lossless transducers.

Substitution of the adjoint relations 75 for the spacial impedances into Eq. 99 yields an expression (similar to Eq. 99) which can be manipulated slightly to prove that the power input to both adjoint transducers (one associated with L, the other with L^-) belong to the same value class, i.e. have the same definite or indefinite form, as the power input to the original transducer. However, the power input to the adjoint transducers does not, necessarily remain the same as the power input to the original transducer for given excitation I_o and $u_n(\bar{r})$.

Finally, consider the problem of expressing the power relation 97 in terms of the internal fields and material parameters of the electroacoustic transducer. The divergence theorem transforms the surface integral 97 into a volume integral over the volume V_o of the transducer to produce,

$$P = \int_{V_o} [\bar{H}^* \cdot (\nabla \times E) + \bar{H} \cdot (\nabla \times \bar{E}^*) - \bar{E} \cdot (\nabla \times \bar{H}^*) - \bar{E}^* \cdot (\nabla \times \bar{H}) - (\nabla \cdot \bar{\bar{T}}) \cdot \bar{u}^* - (\nabla \cdot \bar{\bar{T}}^*) \cdot \bar{u} - \bar{\bar{T}} : \nabla \bar{u}^* - \bar{\bar{T}}^* : \nabla \bar{u}] \, dV, \tag{101}$$

after the divergence is expanded, and the taking of the real part is accomplished by adding each term to its complex conjugate. The equations 36-43 allows the integrand P_V of the power integral 101 to be written as an Hermitian form,

$$P_v = [\overline{H}^*, \overline{E}^*, \overline{\overline{s}}^*, \overline{u}^*] \begin{bmatrix} i\omega(\overline{\overline{\mu}} - \overline{\overline{\mu}}_t^*) & i\omega(\overline{\overline{\nu}} - \overline{\overline{\tau}}_t^*) & i\omega(\overline{\overline{\overline{\beta}}} - \overline{\overline{\overline{\beta}}}_t'^*) & 0 \\ i\omega(\overline{\overline{\tau}} - \overline{\overline{\nu}}_t^*) & [i\omega(\overline{\overline{\varepsilon}} - \overline{\overline{\varepsilon}}_t^*) - (\overline{\overline{\sigma}} + \overline{\overline{\sigma}}_t^*)] & i\omega(\overline{\overline{\overline{\alpha}}} - \overline{\overline{\overline{\alpha}}}_t') & \overline{B}_o \times (\overline{\overline{\sigma}} + \overline{\overline{\sigma}}_t^*) \\ i\omega(\overline{\overline{\overline{\beta}}}' - \overline{\overline{\overline{\beta}}}^*) & i\omega(\overline{\overline{\overline{\alpha}}}' - \overline{\overline{\overline{\alpha}}}_t^*) & -i\omega(\overline{\overline{\overline{\overline{c}}}} - \overline{\overline{\overline{\overline{c}}}}_t^*) & 0 \\ 0 & -\overline{B}_o \times (\overline{\overline{\sigma}} + \overline{\overline{\sigma}}_t^*) & 0 & \overline{B}_o \times (\overline{\overline{\sigma}} + \overline{\overline{\sigma}}_t^*) \times \overline{B}_o \end{bmatrix} \begin{bmatrix} \overline{H} \\ \overline{E} \\ \overline{\overline{s}} \\ \overline{u} \end{bmatrix}, \tag{102}$$

or simply

$$P_v = \tilde{X}^* \mathcal{H} X. \tag{103}$$

P_v may be interpreted as power absorbed per unit volume of the transducer.

A sufficient condition for the total time-average power input to be positive definite, zero, or negative definite is for the Hermitian integrand 102 to be correspondingly positive definite, zero, or negative definite. In particular, if P_v is zero, the transducer is lossless, $\mathcal{H}$ must be zero (provided $\overline{H}$, $\overline{E}$, $\overline{\overline{s}}$ and $\overline{u}$ may be prescribed independently at a point) and therefore the material parameters satisfy the following simple relationships:

$$\overline{\overline{\sigma}}=-\overline{\overline{\sigma}}_t^*,\ \overline{\overline{\varepsilon}}=\overline{\overline{\varepsilon}}_t^*,\ \overline{\overline{\mu}}=\overline{\overline{\mu}}_t^*,\ \overline{\overline{\nu}}=\overline{\overline{\tau}}_t^*,\ \overline{\overline{\overline{\overline{c}}}}=\overline{\overline{\overline{\overline{c}}}}_t^*,\ \overline{\overline{\overline{\alpha}}}'=\overline{\overline{\overline{\alpha}}}_t^*,\ \text{and}\ \overline{\overline{\overline{\beta}}}'=\overline{\overline{\overline{\beta}}}_t^*. \qquad (104)$$

If, in addition, the transducer is reciprocal or anti-reciprocal, Eqs. 54 and 61 show that the material parameters also satisfy

$$\overline{\overline{\sigma}}=\overline{\overline{\sigma}}_t,\ \overline{\overline{\varepsilon}}=\overline{\overline{\varepsilon}}_t,\ \overline{\overline{\mu}}=\overline{\overline{\mu}}_t,\ \overline{\overline{\nu}}=\overline{\overline{\tau}}_t,\ \overline{\overline{\overline{\overline{c}}}}=\overline{\overline{\overline{\overline{c}}}}_t,\ \overline{\overline{\overline{\alpha}}}'=\pm\overline{\overline{\overline{\alpha}}}_t,\ \text{and}\ \overline{\overline{\overline{\beta}}}'=\mp\overline{\overline{\overline{\beta}}}_t^*, \qquad (105)$$

where the upper and lower sign in the last two equations of 105 refer to reciprocal and antireciprocal transducers respectively. Equations 104 and 105 together imply that a lossless (in the sense of Eq. 104) reciprocal or antireciprocal transducer possesses real $\overline{\overline{\sigma}}$, $\overline{\overline{\varepsilon}}$, $\overline{\overline{\mu}}$, $\overline{\overline{\overline{\overline{c}}}}$, and imaginary $\overline{\overline{\nu}}$ and $\overline{\overline{\tau}}$. The piezoelectric triadics $\overline{\overline{\overline{\alpha}}},\overline{\overline{\overline{\alpha}}}'$ are real for reciprocal and imaginary for antireciprocal lossless transducers. Conversely, the piezomagnetic triadics $\overline{\overline{\overline{\beta}}},\overline{\overline{\overline{\beta}}}'$ are imaginary for reciprocal and real for antireciprocal lossless transducers.

Equations 54 and 61 may also be used to demonstrate from Eqs. 102 and 101 that the total power inputs for the two transducers adjoint to the original transducer belong to the same value class as the original transducer--a result which agrees with the previous analysis on the spacial impedance power relation 99.

APPENDIX I

Uniqueness of the Adjoint Operator

The adjoint to the electroacoustic operator L was found in Section III.B. And, in fact, the existence of a linear, differential adjoint to any linear differential operator can be demonstrated in a straightforward manner (see, e.g., Section 4.17 of Lanczos.[25]

Similarly, the proof of uniqueness of the adjoint operator when the bilinear concomitant ($\overline{F}$) is zero can be found as a standard theorem in many functional analysis books.[36] However, as far as the author is aware, the extension of the uniqueness proof to include nonzero $\overline{F}$ has not been given in the literature. For the sake of completeness, and because the proof for differential operators becomes extremely simple using generalized functions (in particular, the delta function) the proof of uniqueness will be given here explicitly.

Consider the equation for defining the adjoint operator L^a to the given operator L:

$$\phi_i^a L_{ij}\phi_j - \phi_i L_{ij}^a \phi_j^a = \nabla\cdot\overline{F}(\phi,\phi^a),$$

or simply

$$\tilde{\phi}^a(L\phi) - \tilde{\phi}(L^a\phi^a) = \nabla\cdot\overline{F}(\phi,\phi^a), \qquad \text{(A1)}$$

where

1) L is a given linear differential operator.

2) ϕ, ϕ^a, $L\phi$, and $L^a\phi^a$ are n-dimensional vector functions of position.

3) Equation A1 holds for all ϕ and ϕ^a operable upon by L.

4) $L^a\phi^a$ is assumed to exist for the allowable ϕ^a.

5) the differentiable 3-dimensional vector function $\overline{F}$ is a bilinear function of the components of ϕ, ϕ^a and their derivatives.

The purpose of this appendix is to show that the operator L^a defined by Equation A1 under the given conditions is a unique, linear, differential operator.

Linearity will be proven first. A vector function ϕ_o^a can be added to ϕ^a in Equation A1 to give

$$(\tilde{\phi}^a+\tilde{\phi}_o^a)L\phi-\tilde{\phi}[L^a(\phi^a+\phi_o^a)] = \nabla\cdot\overline{F}(\phi,\phi^a+\phi_o^a) = \nabla\cdot[\overline{F}(\phi,\phi^a) + \overline{F}(\phi,\phi_o^a)]. \quad \text{(A2)}$$

The last quantity in A2 is a consequence of the bilinear nature of $\overline{F}$. From Eq. A1 it is also true that

$$\tilde{\phi}_o^a(L\overline{\phi}) - \tilde{\phi}(L^a\phi_o^a) = \nabla\cdot\overline{F}(\phi,\phi_o^a). \quad \text{(A3)}$$

Subtraction of Eqs. A1 and A3 from A2 yields

$$\tilde{\phi}[L^a(\phi^a+\phi_o^a) - (L^a\phi^a+L^a\phi_o^a)] = 0,$$

or

$$L^a(\phi^a+\phi_o^a) = L^a\phi^a + L^a\phi,$$
$$(L^a \text{ is linear}) \tag{A4}$$

since the components of ϕ can be chosen independently.

To prove uniqueness assume for the moment a second operator L_2^a and a second function $\overline{F}_2$ that satisfies equation A1. Then

$$\tilde{\phi}[(L_2^a-L^a)\phi^a] = \nabla\cdot(\overline{F}_2-\overline{F}). \tag{A5}$$

Application of the divergence theorem to Eq. A5 in the volume V_o (of the transducer in the present paper) bounded by the surface A_o leads to

$$\int_{V_o} \tilde{\phi}[L_2^a-L^a)\phi^a]\ dV = \int_{A_o} (\overline{F}_2-\overline{F})\cdot\hat{n}\ da, \tag{A6}$$

assuming, of course, that these integrals exist. The surface integral in Eq. A6 depends only upon the values of ϕ,ϕ^a and their derivatives on A_o. Thus ϕ can be replaced by $\phi+a\delta(\overline{r}-\overline{r}_o)$ without changing the right-hand side of Eq. A6, i.e.

$$\int_{V_o} [\tilde{\phi}+\tilde{a}\delta(\overline{r}-\overline{r}_o)](L_2^a-L^a)\phi^a\ dV = \int_{A_o} (\overline{F}_2-\overline{F})\cdot\hat{n}\ da, \tag{A7}$$

where a is an arbitrary, constant, n-dimensional vector. Subtraction of (A6) from (A7) yields

$$(L_2^a-L^a)\phi^a(\overline{r}_o) = 0. \tag{A8}$$

Since Eq. (A8) holds for all allowable ϕ^a, by definition

$$L_2^a = L^a.$$

Thus the adjoint operator is unique, as well as linear. The differential nature of the adjoint operator is revealed in the demonstration of existence (See Section 4.17 of Lanczos[25]).

Although it has been assumed that ϕ, ϕ^a are functions of the three position variables (x, y, z) and $\overline{F}$ is a 3-dimensional vector, the proof holds for an arbitrary number of dimensions. For an M-dimensional space the right hand side of Eq. A1 would merely be replaced by

$$\sum_{\alpha=1}^{M} \frac{\partial F_\alpha(\phi, \phi^a)}{\partial x_\alpha}.$$

APPENDIX II

Direct Integration of the Field Equations

Equation 49 of the main text can be used to derive an expression for the electroacoustic fields at an interior point of the transducer in terms of the volume sources and the fields on the surface of the transducer. The derivation utilizes the Green's function to the adjoint operator and proceeds in a fashion similar to that outlined by Morse and Feshbach,[19] Section 7.5. Stratton[37] contains an analagous derivation for purely electromagnetic fields in homogeneous, isotropic, nonconductive media. A lucid discussion of the em expressions may be found in a more recent article by Tai.[38] The purely acoustical derivation for a homogeneous, isotropic, nonviscous fluid involves simply a direct application of Green's theorem (see, e.g., Section II.A, Eq. 12).

For an electroacoustic transducer which obeys Eqs. 36 - 43, the bilinear identify 49 may be written in tensor notation as

$$\phi_i^a L_{ij} \phi_j - \phi_i L_{ij}^a \phi_j^a = \nabla \cdot \bar{F}(\phi, \phi^a), \qquad \text{(A9)}$$

where the operator L_{ij} and its adjoint L_{ij}^a are given in Eqs. 48 and 53 respectively, ϕ is defined by

$$\phi \equiv \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \phi_9 \end{Bmatrix} = \begin{Bmatrix} E_x \\ E_y \\ E_z \\ H_x \\ H_y \\ H_z \\ u_x \\ u_y \\ u_z \end{Bmatrix} = \begin{Bmatrix} \overline{E} \\ \overline{H} \\ \overline{u} \end{Bmatrix},$$

and $\overline{F}$ is given in Eq. 58.

Sources within the transducer may be expressed as in Eqs. 55-57, i.e.

$$L_{ij} \phi_j = Q_i . \tag{A10}$$

Moreover, let ϕ_j^a represent the Green's function to the adjoint operator, i.e.

$$\begin{aligned} L^a_{ij} G^a_{j\ell} &= \delta_{i\ell} \delta(\overline{r}-\overline{r}') \\ G_{j\ell}^{a} &\equiv \phi_j^a(\ell) . \end{aligned} \qquad \begin{aligned} i&=1,2\cdots 9 \\ \ell&=1,2\cdots 9 \end{aligned} \tag{A11}$$

Substitution of Eqs. A10 and A11 into Eq. A9 and integration over the volume V_o of the transducer yields the desired expression for ϕ_ℓ.

$$\phi_\ell(\overline{r}') = \int_{V_o} G^a_{i\ell} Q_i \, dV - \int_{A_o} \hat{n}\cdot\overline{F}(\phi, G^a_\ell) da \tag{A12}$$

Equation A12 determines the rectangular components of $\overline{E}$, $\overline{H}$, and $\overline{u}$ inside V_o in terms of the volume sources ($\overline{J}_s$, $\overline{P}_s$,

$\overline{M}_s$, $\overline{f}_s$), the fields ($\overline{E}$, $\overline{H}$, $\overline{u}$, $\overline{\overline{T}}$) on the surface A_o, and the appropriate Green's function. It can be written in more explicit form as

$$\begin{pmatrix} \overline{E} \\ \overline{H} \\ \overline{u} \end{pmatrix} = \int_{V_o} \begin{pmatrix} G^a_{11} & G^a_{21} \cdots G^a_{91} \\ G^a_{12} & G^a_{22} \cdots \\ \vdots & \\ G^a_{19} & G^a_{29} \cdots G^a_{99} \end{pmatrix} \begin{pmatrix} \overline{J}_s - i\omega \overline{P}_s \\ i\omega\mu_o \overline{M}_s \\ \overline{f}_s - \overline{B}_o \times \overline{J}_s \end{pmatrix} dV - \int_{A_o} \hat{n} \cdot \begin{pmatrix} \overline{F}(\phi, G^a_1) \\ \overline{F}(\phi, G^a_2) \\ \vdots \\ \overline{F}(\phi, G^a_9) \end{pmatrix} da, \tag{A13}$$

where

$$L^a_{ij} \; G^a_{j\ell} = \delta_{i\ell}\delta(\overline{r}-\overline{r}')$$

and

$$\overline{F}(\phi, G^a_\ell) = \overline{E}\times\overline{H}^a_\ell - \overline{E}^a_\ell\times\overline{H} - \overline{u}^a_\ell\cdot\overline{\overline{T}} + \overline{u}\cdot\overline{\overline{T}}^a_\ell$$

$$G^a_\ell = \begin{pmatrix} \overline{E}^a_\ell \\ \overline{H}^a_\ell \\ \overline{u}^a_\ell \end{pmatrix}, \quad \overline{\overline{T}}^a_\ell = -\left(\frac{\overline{\overline{\overline{\overline{c}}}}:\nabla u^a_\ell}{i\omega} + \overline{\overline{\overline{\alpha}}}' \cdot \overline{E}^a_\ell + \overline{\overline{\overline{\beta}}}' \cdot \overline{H}^a_\ell\right).$$

The remaining fields may be found directly from Eqs. 38-43. Expression A13 seems quite complicated because it applies to such a general linear medium. However, for most transducers all but a few of the material parameters are zero and the expression simplifies greatly.

If the sources are zero and the boundary conditions on G^a_ℓ are specified as tangential $\overline{E}^a_{\ell,tan} = 0$ and $\overline{\overline{T}}^a_\ell \cdot \hat{n} = 0$ on A_o, then the electroacoustic fields are determined solely by $\overline{E}_{tan}$ and $\overline{\overline{T}}\cdot\hat{n}$ on the surface A_o. A similar statement holds for zero $(\overline{E}^a_{\ell,tan}, \hat{n}\cdot(\overline{u}^a_\ell\cdot\overline{\overline{T}})]$, $[\overline{H}^a_{\ell,tan}, \overline{\overline{T}}^a_\ell\cdot\hat{n}]$ or $[\overline{H}^a_{\ell,tan}, \hat{n}\cdot(\overline{u}^a_\ell\cdot\overline{\overline{T}})]$. Of course, it is assumed that the Green's functions exist under the specified boundary conditions.

If all electroacoustic coupling is set equal to zero, the transducer equations decouple into Maxwell's equations and the momentum equation for inhomogeneous, anisotropic material. Then Eq. A13 may also be decoupled and rearranged to give

$$\underline{\overline{E}(\overline{r}') = \int_{V_o} \overline{F}_E\cdot\overline{\overline{G}}^a_E \, dV - \frac{1}{i\omega}\oint_{A_o} (\hat{n}\times\overline{E})\cdot(\overline{\overline{\mu}}_t^{-1}\cdot\nabla\times\overline{\overline{G}}^a_E)\, da} \qquad \text{(A14)}$$

with

$$\overline{H}(\overline{r}') = \overline{\overline{\mu}}^{-1}\cdot\left[\frac{\nabla\times\overline{E}}{i\omega} - \overline{\overline{\nu}}\cdot\overline{E}\right] - \mu_o\overline{\overline{\mu}}^{-1}\cdot\overline{M}_s$$

and the dyadic Green's function $\overline{\overline{G}}^a_E$ satisfying

$$\nabla\times\left[\frac{\overline{\overline{\mu}}_t^{-1}}{i\omega}\cdot\nabla\times\overline{\overline{G}}^a_E\right] + \overline{\overline{b}}_t\cdot\nabla\times\overline{\overline{G}}^a_E + \nabla\times(\overline{\overline{d}}_t\cdot\overline{\overline{G}}^a_E) + \overline{\overline{e}}_t\cdot\overline{\overline{G}}^a_E = \delta(\overline{r}-\overline{r}')\overline{\overline{I}} \; ,$$

$$\hat{n}\times\overline{\overline{G}}^a_E = 0 \text{ on } A_o$$

where

$$\overline{\overline{b}} = -\overline{\overline{\mu}}^{-1}\cdot\overline{\overline{\nu}}, \quad \overline{\overline{d}} = \overline{\overline{\tau}}\cdot\overline{\overline{\mu}}^{-1}, \quad \overline{\overline{e}} = i\omega\overline{\overline{\varepsilon}} - \overline{\overline{\sigma}}_e - i\omega\overline{\overline{\tau}}\cdot\overline{\overline{\mu}}^{-1}\cdot\overline{\overline{\nu}}$$

$$\overline{F}_E = \overline{J}_s - i\omega\overline{P}_s + i\omega\mu_o\overline{\overline{\tau}}\cdot\overline{\overline{\mu}}^{-1}\cdot\overline{M}_s + \mu_o\,\nabla\times(\overline{\overline{\mu}}^{-1}\cdot\overline{M}_s).$$

$$\overline{H}(\overline{r}') = \int_{V_o} \overline{F}_H \cdot \overline{\overline{G}}{}^a_H \, dV - \oint_{A_o} (\hat{n} \times \overline{H}) \cdot (\overline{\overline{a}}{}_t^{-1} \cdot \nabla \times \overline{\overline{G}}{}^a_H) \, da \tag{A15}$$

with

$$\overline{E}(\overline{r}') = \overline{\overline{a}}{}^{-1} \cdot (\nabla \times \overline{H} + i\omega \overline{\overline{\tau}}) + \overline{\overline{a}}{}^{-1} \cdot (\overline{J}_s - i\omega \overline{P}_s)$$

and the dyadic Green's function $\overline{\overline{G}}{}^a_H$ satisfying

$$\nabla \times (\overline{\overline{a}}{}_t^{-1} \cdot \nabla \times \overline{\overline{G}}{}^a_H) + \overline{\overline{b}}{}'_t \cdot \nabla \times \overline{\overline{G}}{}^a_H + \nabla \times (\overline{\overline{d}}{}'_t \cdot \overline{\overline{G}}{}^a_H) + \overline{\overline{e}}{}'_t \cdot \overline{\overline{G}}{}^a_H = \delta(\overline{r} - \overline{r}')\overline{\overline{I}}$$

$$\hat{n} \times \overline{\overline{G}}{}^a_H = 0 \text{ on } A_o$$

where

$$\overline{\overline{a}} = \overline{\overline{\sigma}}_e - i\omega\overline{\overline{\varepsilon}}, \quad \overline{\overline{b}}' = i\omega \overline{\overline{a}}{}^{-1} \cdot \overline{\overline{\tau}}, \quad \overline{\overline{d}}' = -i\omega \overline{\overline{\nu}} \cdot \overline{\overline{a}}{}^{-1}$$

$$\overline{\overline{e}}' = -i\omega(\overline{\overline{\mu}} + i\omega \overline{\overline{\nu}} \cdot \overline{\overline{a}}{}^{-1} \cdot \overline{\overline{\tau}})$$

$$\overline{F}_H = i\omega\mu_o \overline{M}_s + i\omega \overline{\overline{\nu}} \cdot (\overline{J}_s - i\omega \overline{P}_s) - \nabla \times (\overline{J}_s - i\omega \overline{P}_s).$$

$$\overline{u}(\overline{r}') = \int_{V_o} \overline{f}_s \cdot \overline{\overline{G}}{}^a_u \, dV + \frac{1}{i\omega} \int_{A_o} \left\{ (\overline{\overline{T}} \cdot \hat{n}) \cdot \overline{\overline{G}}{}^a_u + \hat{n} \cdot \left(\frac{\overline{u} \cdot \overline{\overline{\overline{\overline{c}}}}_t}{i\omega} : \nabla \overline{\overline{G}}{}^a_u \right) \right\} \tag{A16}$$

with

$$\overline{\overline{T}}(\overline{r}') = -\frac{1}{i\omega} \overline{\overline{\overline{\overline{c}}}} : \nabla \overline{u}$$

and the dyadic Green's function $\overline{\overline{G}}{}^a_u$ satisfying

$$\frac{1}{i\omega} \nabla \cdot \overline{\overline{\overline{\overline{c}}}}_t : \nabla \overline{\overline{G}}{}^a_u - i\omega \rho^o_m \overline{\overline{G}}{}^a_u = \delta(\overline{r} - \overline{r}')\overline{\overline{I}}.$$

Equation A14 and A15 represent generalizations of formulas (19) and (20) in Stratton[37] to inhomogeneous, anisotropic, magneto-electric, conductive material. Note that the electromagnetic fields inside a sourceless volume is still determined by the tangential electric field alone or the tangential magnetic field alone on the surface of the volume. When the volume includes all space, and all sources lie in a region of finite extent, the surface integrals in Eqs. A14-A16 vanish. (The volume integration which is left in Eq. A14 is similar to that derived recently by Kong[39] for "bianisotropic", i.e., magneto-electric, media.)

Equation A16 shows that $\bar{u}$ and $\bar{\bar{T}}$ inside a sourceless acoustic medium are determined by $\bar{\bar{T}}\cdot\hat{n}$ alone or by $\hat{n}\cdot(\bar{u}\cdot\bar{\bar{\bar{\bar{c}}}})$ alone on the surface A_o provided $\bar{\bar{T}}^a_{ui}\cdot\hat{n}$ or $\hat{n}\cdot(\bar{u}^a_{ui}\cdot\bar{\bar{\bar{\bar{c}}}})$ (i=1,2,3) are set equal to zero on the surface A_o respectively.

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[8] Foldy and Primakoff refer to Eqs. 3 as the "linearity relations." Here they are referred to as the "spacial impedance-matrix relations" to avoid confusion with the spectral scattering-matrix equations 1, which are also linear.

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[11] Between A and S_1 there can be finite regions of either acoustically rigid material or material in which volume forces are zero and stress is linearly related to strain by a symmetric Hooke's tensor. However, the reference plane S_1 must remain entirely in the homogeneous ambient fluid.

[12] R. Courant and D. Hilbert, Methods of Mathematical Physics (Interscience, New York, 1953), Vol. 1, Chap. 3, pp. 140-142.

[13] Note the difference between "spectral" and "spacial impedance matrix."

[14] The acoustical reciprocity theorem derives from the divergence theorem and Eqs. 6.

[15] Here, the delta functions are two-dimensional since they depend only upon the two parameters needed to specify the surface A.

[16] Physically, the assumption that $F(\overline{K},\overline{r})$ does not have zero as an eigenvalue is necessary to insure a unique value of V_o for given I_o and $I(\overline{K})$. Mathematically, the assumption gains support from the facts that $F(\overline{K},\overline{r})$ is nondegenerate (nonseparable) and the infinite domain of the double variable $\overline{K}$ is equal to or greater than the domain of the two parameters in $\overline{r}$ describing the surface A.

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[26] Determination of adjoint differential operators as implied by the bilinear identity 49 is not synonymous with the interchanging of rows and columns of the Cartesian matrix form of the operator. Contrast, for example, Lanczos. Ref. 25 with Auld, Ref. 20.

[27] The terms "adjoint operator" and "adjoint transducer" refer to two distinct mathematical concepts. The adjoint of a given operator can be proven to exist

uniquely, (see Appendix I), whereas the adjoint transducer finds meaning (mathematically) only because the elements of the operator L could be arranged in the special way that allowed the equations of the adjoint operator to coincide in form to those of a second (physically hypothetical) transducer.

[28] J.A. Lewis, "The Effect of Driving Electrode Shape on the Electrical Properties of Piezoelectric Crystals," Bell Syst. Tech. J., 40, No. 5, pp. 1259-1280, Sept. 1961.

[29] The restriction to adjoint operators that describe an adjoint transducer is unnecessary for the development or utility (at least in some applications) of the generalized reciprocity relations. The scattering-matrix description, which depends upon the equations of the fluid outside the transducer and upon the linearity of the transducer, remains valid for an adjoint operator regardless of whether the adjoint operator represents a transducer. The somewhat unnecessary restriction to adjoint transducers is introduced simply to allow the comparison of the given transducer to a second transducer rather than to an abstract linear system alone, and also to facilitate the discussion of self-adjoint transducers.

[30] Regions of linear em material are permitted outside A.

[31] Alternatively, the results of Section II.A could be used to derive the PWSM equations 1 directly from the SIM Eqs. 67.

[32] American Standard of Acoustical Terminology (American Standards Association, New York, 1960), S1.1-1960, Sect. 7.30, pp. 32-33.

[33] P.M. Morse and K.U. Ingard, Theoretical Acoustics (McGraw-Hill, New York, 1968), Chap. 8, p. 405. Adjoint reciprocity principles for acoustic scatterers derive entirely from a scattering matrix having the symmetry properties of S_{11} (Eq. 76d). The analagous symmetry or "reciprocity theorem" exists for the scattered plane wave amplitudes of quantum mechanics (See Ref. 19, part 2, Chap. 9, pp. 1130-1131).

[34] L. Mirsky, An Introduction to Linear Algebra (Oxford, London, 1961), Chap. 13.

[35] J. Mercer, "Functions of Positive and Negative Type, and Their Connection with the Theory of Integral Equations," Proc. Roy. Soc., Ser. A, 83, No. A559, pp. 69-70, Nov. 1909.

[36] See, e.g., M.H. Stone, Linear Transformations in Hilbert Space and their Applications to Analysis, (American Mathematical Society Colloquium Publications, New York, 1932), 15, Theorem 2.6.

[37] J.A. Stratton, Electromagnetic Theory, (McGraw-Hill, New York, 1941), Section 8.14).

[38] C.T. Tai, "Kirchoff Theory: Scalar, Vector, or Dyadic?", IEEE Trans. Antennas Propagat., 20, No. 1, pp. 114-115, Jan. 1972.

[39] J.A. Kong, "Theorems of Bianisotropic Media," Proc. IEEE, 60, No. 1, pp. 1036-1046, Sept. 1972.

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16. ABSTRACT (A 200-word or less factual summary of most significant information. If document includes a significant bibliography or literature survey, mention it here.)

Analytical techniques for the measurement of the external characteristics of electroacoustic transducers have been developed by D. M. Kerns using a plane-wave scattering -matrix (PWSM) formulation. Foldy and Primakoff, in their classic papers on linear electroacoustic transducers, utilize a spatial impedance-matrix (SIM) formulation. Both formulations involve a continuous, linear "matrix" transformation in which reciprocity is defined as a relationship between elements of the matrix.

The first portion of the present report demonstrates that a transducer satisfying the "SIM relations" also satisfies the "PWSM equations" (but that the converse theorem does not hold), and that the alternate expressions of reciprocity are equivalent for transducers that obey both formulations.

The second portion of the report examines the equations which characterize the internal behavior of a large class of electroacoustic transducers. A linear operator approach is employed to develop a generalized reciprocity lemma which is used to establish adjoint reciprocity relations between the fields of a given transducer and its adjoint transducers. The linear operator approach facilitates the identification of self-adjoint (reciprocal or antireciprocal) transducers, and the adjoint reciprocity relations have utility in the extrapolation techniques of the PWSM formulation. An adjoint "reciprocity theorem" and "principle of reciprocity" are derived from the generalized reciprocity relations. Finally it is shown that the total power inputs for the adjoint transducers belong to the same "value class" as the original transducer.

17. KEY WORDS (Alphabetical order, separated by semicolons)

Adjoint operators; electroacoustic transducers; reciprocity; scattering matrices.

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