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NBSIR 73-243

On the Compression of a Cylinder in Contact with a Plane Surface

B. Nelson Norden

Institute for Basic Standards
National Bureau of Standards
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Final Report

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NOMENCLATURE

P_o	maximum pressure at center of contact zone
a	major axis of ellipse of contact
b	minor axis of ellipse of contact (also half-width of contact in the cylinder-plane case)
R_1, R_1'	principal radii of curvature of body 1
R_2, R_2'	principal radii of curvature of body 2
Z_1	distance from a point on body 1 to the undeformed condition point of body 2
Z_2	same terms as above for body 2 to body 1
W_1	deformation of a point on body 1
W_2	deformation of a point on body 2
δ	total deformation of bodies 1 and 2
ω	angle between planes $x_1 z$ and $x_2 z$
λ_i	parameter equal to $\frac{1 - \nu i^2}{\pi E_i}$ where νi is Poisson's ratio for body i and Ei is Young's modulus for body i
Φ	potential function at any point on the surface
e	eccentricity of the ellipse of contact $e^2 = 1 - \frac{b^2}{a^2}$
P	total load applied to produce deformation
K	complete elliptic integral of the first kind
E	complete elliptic integral of the second kind
σ	stress at some point on surface of body
h	square of the complementary modulus or $(1 - e^2)$
V	potential function used by Lundberg
M	mutual approach of remote points in two plates with cylinder between the two plates

ABSTRACT

The measurement of a diameter of a cylinder has widespread application in the metrology field and industrial sector. Since the cylinder is usually placed between two flat parallel anvils, one needs to be able to apply corrections, to account for the finite measuring force used, for the most accurate determination of a diameter of the cylinder.

An extensive literature search was conducted to assemble the equations which have been developed for deformation of a cylinder to plane contact case. There are a number of formulae depending upon the assumptions made in the development. It was immediately evident that this subject has been unexplored in depth by the metrology community, and thus no coherent treatise for practical usage has been developed.

This report is an attempt to analyze the majority of these equations and to compare their results within the force range normally encountered in the metrology field. Graphs have been developed to facilitate easy computation of the maximum compressive stress encountered in the steel cylinder-steel plane contact case and the actual deformation involved.

Since the ultimate usefulness of any formula depends upon experimental verification, we have compiled results of pertinent experiments and various empirical formulae. A complete bibliography has been included for the cylinder-plane contact case for the interested reader.

INTRODUCTION

The problem of contact between elastic bodies (male, female or neuter gender) has long been of considerable interest. Assume that two elastic solids are brought into contact at a point O as in Figure 1. If collinear forces are applied so as to press the two solids together, deformation occurs, and we expect a small contact area to replace the point of the unloaded state. If we determine the size and shape of this contact area and the distribution of normal pressure, then the interval stresses and deformation can be calculated.

The mathematical theory for the general three-dimensional contact problem was first developed by Hertz in 1881. The assumptions made are:

1) the contacting surfaces are perfectly smooth so that the actual shape can be described by a second degree equation of the form

$$z = Dx^2 + Ey^2 + Fxy.$$

2) The elastic limits of the materials are not exceeded during contact. If this occurs, then permanent deformation to the materials occurs.

3) The two bodies under examination must be isotopic.

4) Only forces which act normal to the contacting surfaces are considered. This means that there is assumed to be no frictional forces at work within the contact area.

5) The other assumption is that the contacting surfaces must be small in comparison to the entire surfaces.

Based on the above assumptions and by applying potential theory, Hertz showed that:

1) the contact area is bounded by an ellipse whose semiaxes can be calculated from the geometric parameters of the contacting bodies.

2) The normal pressure distribution over this area is:

$$p_o \left[1 - (x/a)^2 - (y/b)^2\right]^{1/2}$$

where p_o = maximum pressure at center

a = major axis of ellipse of contact

b = minor axis of ellipse.

The above assumptions are valid in the field of dimensional metrology because the materials (usually possessing finely lapped surfaces), and the measuring forces normally used are sufficient for the Hertzian equations to be accurate. In the case of surfaces that are not finely lapped, the actual deformation may differ by more than 20% from those calculated from equations.

Since the subject of deformation has such widespread impact on the field of precision metrology, we have decided to publish separate reports - (1) dealing with line contact and particularly the contact of a cylinder to a plane, and; (2) which treats the general subject of contacting bodies and derives formulae for all other major cases which should be encountered in the metrology laboratory.

An exhaustive literature search was conducted to determine equations currently in use for deformation of a cylinder to flat surface. The ultimate usefulness of deformation formulae depends on their experimental verification and, while there is an enormous amount of information available for large forces, it was found that the data is scarce for forces in the range used in measurement science. One reason for this scarcity is the degree of geometric perfection required in the test apparatus and the difficulty of measuring the small deformations reliably.

Depending upon the assumptions made, there are a number of formulae in use. Various equations will be analyzed along with the assumptions inherent in their derivations. There are basically three approaches to the problem for the deformation of a cylinder with diameter D in contact with a plane over a length L and under the action of force P:

1) the approach where a solution is generated from the general three-dimensional case of curved bodies by assigning the plane surface a radius of curvature. This is the same as replacing the plane surface with a cylindrical surface with a very large radius of curvature. The area of contact is then an elongated ellipse.
2) The approach where the contact area between a cylinder and plane is assumed to be a finite rectangle of width 2b and length L where $L \gg b$.
3) The determination of compression formula by empirical means.

GENERAL DESCRIPTION OF CONTACT PROBLEM

When two homogeneous, elastic bodies are pressed together, a certain amount of deformation will occur in each body, bounded by a curve called the curve of compression. The theory was first developed by H. Hertz [1].

Figure 1 shows two general bodies in the unstressed and undeformed state with a point of contact at O. The two surfaces have a common tangent at point O. The principal radii of curvature of the surface at the point of contact is R_1 for body 1, and R_2 for body 2. R_1' and R_2' represent the other radii of curvature of each body. The radii of curvature are measured in two planes at right angles to one another. The principal radii of curvature may be positive if the center of curvature lies within the body, and negative if the center of curvature lies outside the body. Also planes $x_1 z$ and $x_2 z$ should be chosen such that

$$\left(\frac{1}{R_1} + \frac{1}{R_2}\right) > \left(\frac{1}{R_1{}'} + \frac{1}{R_2{}'}\right)$$

The angle ω is the angle between the normal sections of the two bodies which contain the principal radii of curvature R_1 and R_2.

Figure 2 shows a cross-section of the two surfaces near the point of contact O. We must limit our analysis to the case where the dimensions of the compressed area after the bodies have been pressed together are small in comparison with the radii of curvature of bodies 1 and 2. We also assume that the surface of each body near the point of contact can be approximated by a second degree equation of the form

$$Z = Dx^2 + Ey^2 + 2\ Fxy$$

where D, E and F are arbitrary constants.

If the two bodies are pressed together by applied normal forces (Figure 3), then a deformation occurs near the original point of contact along the Z-axis. Here again, we consider only forces acting parallel to the z-axis where the distance from the z-axis is small.

The displacements at a point are w_1 and w_2 where w_1 is the deformation of point P_1 of body 1 and w_2 is the deformation of point P_2 for body 2, plane C is the original plane of tangency; z_1 is the distance from P_1 to the undeformed state, and z_2 is the distance from P_2 to the undeformed state. For points inside the contact area, we have

$$(z_1 + w_1) + (z_2 + w_2) = \delta \qquad (1)$$

where δ is the total deformation which we are so diligently seeking.

The equation for surface 1 may be written as:

$$z_1 = D_1 x^2 + E_1 y^2 + 2F_1 xy$$

and for surface 2,

$$z_2 = D_2 x^2 + E_2 y^2 + 2F_2 xy.$$

Since the sum of z_1 and z_2 enter into the equation we obtain

$$z_1 + z_2 = (D_1 + D_2)x^2 + (E_1 + E_2)y^2 + 2(F_1 + F_2)xy. \qquad (2)$$

Now Hertz showed that the axis can be transformed so that $F_1 = -F_2$, and hence, the xy term vanishes. To simplify the above equation further we replace the constants $(D_1 + D_2)$ with A and $(E_1 + E_2)$ with B thus giving,

$$z_1 + z_2 = Ax^2 + By^2$$

From equation (1) we obtain:

$$Ax^2 + By^2 + w_1 + w_2 = \delta \qquad (3)$$

The constants A and B are expressible in terms of combinations of the principal curvatures of the surfaces and the angle between the planes of curvature. These combinations have been derived by Hertz and are as follows:

$$A + B = \frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_1'} + \frac{1}{R_2} + \frac{1}{R_2'}\right) \tag{4}$$

$$B - A = \frac{1}{2}\left[\left(\frac{1}{R_1} - \frac{1}{R_1'}\right)^2 + \left(\frac{1}{R_2} - \frac{1}{R_2'}\right)^2 + 2\left(\frac{1}{R_1} - \frac{1}{R_1'}\right)\left(\frac{1}{R_2} - \frac{1}{R_2'}\right)\cos 2\omega\right]^{1/2} \tag{5}$$

Since the points within the compressed area are in contact after the compression we have:

$$w_1 + w_2 = \delta - Ax^2 - By^2$$

and since δ is the value of w_1 and w_2 at the origin (Figure 3, $x = y = 0$), we must evaluate w_1 and w_2.

The pressure P between the bodies is the resultant of a distributed pressure (P' per unit of area), over the compressed area. From Prescott [2] the values of the deformations w_1 and w_2 under the action of normal forces are:

$$w_1 = \lambda_1 \Phi (x, y) \tag{6}$$

and,

$$w_2 = \lambda_2 \Phi (x, y) \tag{7}$$

where $$\lambda_i = \left(\frac{1 - \nu i^2}{\pi E i}\right)$$

ν_i = Poisson's ratio for the ith body

E_i = Modulus of elasticity for the ith body,

and $\Phi(x, y) = \iint_A \frac{P'}{r}\, dx'dy'$ which represents the potential at a point on the surface. Here r is the distance from some point (x, y) to another point (x', y') and P' is the surface density.

By substitution in Equation 3 we obtain,

$$(\lambda_1 + \lambda_2) \iint_A \frac{P'}{r}\, dx'\, dy' = \delta - Ax^2 - By^2 \qquad (8)$$

where the subscripts 1 and 2 represent the elastic constants for bodies 1 and 2.

One important fact should be observed from Equations 6 and 7 and this is:

$$\frac{w_1}{w_2} = \frac{\lambda_1}{\lambda_2} = \frac{\left(\frac{1 - \nu_1^2}{\pi E_1}\right)}{\left(\frac{1 - \nu_2^2}{\pi E_2}\right)} \qquad (9)$$

which means if the two bodies are made of the same material $w_1 = w_2$.

The integral equation 8 allows one to compute the contact area, the pressure distribution, and the deformation of the bodies.

The problem is now to find a distribution of pressures to satisfy equation 8. Since the formula for Φ_o is a potential function due to matter distributed over the compressed area with surface density P' we see the analogy between this problem and potential theory. Hertz saw the analogy since the integral on the left side of equation 8 is of a type commonly found in potential theory, where such integrals give the potential of a distribution of charge and the potential at a point in the interior of a uniformly charged ellipsoid is a quadratic function of the coordinates.

If an ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ has a uniform charge density ρ with mass $\pi\rho abc$, then the potential within the ellipse is given by Kellog [3] as

$$\Phi(x, y, z) =$$

$$\pi\rho abc \int_0^\infty \left(1 - \frac{x^2}{a^2 + \Psi} - \frac{y^2}{b^2 + \Psi} - \frac{z^2}{c^2 + \Psi}\right) \frac{d\Psi}{((a^2 + \Psi)(b^2 + \Psi)(c^2 + \Psi))^{1/2}} \tag{10}$$

If we consider the case where the ellipsoid is very much flattened ($c \to 0$) then we have

$$\Phi(x, y) = \pi\rho abc \int_0^\infty \left(1 - \frac{x^2}{a^2 + \Psi} - \frac{y^2}{b^2 + \Psi}\right) \frac{d\Psi}{((a^2 + \Psi)(b^2 + \Psi)(\Psi))^{1/2}} \tag{11}$$

The potential due a mass density is

$$\tau(x', y') = \frac{3p}{2\pi ab} \sqrt{1 - \frac{x'^2}{a^2} + \frac{y'^2}{b^2}} \tag{12}$$

distributed over the ellipse $\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1$ in the plane $Z = 0$, where the total load P is given by,

$$P = 4/3 \, \pi\rho c \, ab. \tag{13}$$

By substituting into Equation 11 we obtain,

$$\Phi(x, y) = \frac{3P}{4} \int_0^\infty \left(1 - \frac{x^2}{a^2 + \Psi} - \frac{y^2}{b^2 + \Psi}\right) \frac{d\Psi}{((a^2 + \Psi)(b^2 + \Psi)(\Psi))^{1/2}} \tag{14}$$

From Equations 8, 11 and 14 we obtain,

$$(\lambda_1 + \lambda_2) \iint_A \frac{P'}{r} \, dx' \, dy' =$$

$$(\lambda_1 + \lambda_2) \frac{3}{4} P \int_0^\infty \left(1 - \frac{x^2}{a^2 + \Psi} - \frac{y^2}{b^2 + \Psi}\right) \frac{d\Psi}{((a^2 + \Psi)(b^2 + \Psi)(\Psi))^{1/2}} \tag{15}$$

Thus we have,

$$\iint_A \frac{P'}{r} \, dx' \, dy' = \frac{3}{4} P \int_0^\infty \left(1 - \frac{x^2}{a^2 + \Psi} - \frac{y^2}{b^2 + \Psi}\right) \frac{d\Psi}{((a^2 + \Psi)(b^2 + \Psi)(\Psi))^{1/2}} \tag{16}$$

We now substitute into Equation 8 to obtain,

$$(\lambda_1 + \lambda_2)\frac{3}{4}P\int_0^\infty\left(1 - \frac{x^2}{a^2+\Psi} - \frac{y^2}{b^2+\Psi}\right)\frac{d\Psi}{((a^2+\Psi)(b^2+\Psi)(\Psi))^{1/2}} =$$

$$\delta - Ax^2 - By^2 \tag{17}$$

Since the coefficients of 1, x^2, and y^2 must be equal in Equation 17, we have,

$$\delta = \frac{3}{4}P(\lambda_1 + \lambda_2)\int_0^\infty \frac{d\Psi}{((a^2+\Psi)(b^2+\Psi)(\Psi))^{1/2}} \tag{18}$$

$$A = \frac{3}{4}P(\lambda_1 + \lambda_2)\int_0^\infty \frac{d\Psi}{(a^2+\Psi)^{3/2}((b^2+\Psi)(\Psi))^{1/2}} \tag{19}$$

$$B = \frac{3}{4}P(\lambda_1 + \lambda_2)\int_0^\infty \frac{d\Psi}{(b^2+\Psi)^{3/2}((a^2+\Psi)(\Psi))^{1/2}} \tag{20}$$

Equations 19 and 20 determine a and b (major and minor axis of the ellipse of contact) and equation 18 determines the total deformation δ, (or the normal approach) when a and b are known.

Since the integrals in Equations 18, 19 and 20 are somewhat cumbersome, they may be expressed in terms of complete elliptic integrals where tables are readily available. Since the eccentricity (e) of any ellipse may be expressed in terms of the major and minor axis as,

$$e^2 = 1 - \frac{b^2}{a^2} \quad \text{or} \quad e = \left(1 - \frac{b^2}{a^2}\right)^{1/2}$$

we may express Equations 18, 19 and 20 in terms of the eccentricity of the contact ellipse.

From Equation 19 we obtain,

$$A = \frac{3}{4} P (\lambda_1 + \lambda_2) \int_0^\infty \frac{d\Psi}{(a^2 + \Psi)^{3/2} (a^2 - a^2 e^2 + \Psi)^{1/2} \Psi^{1/2}}$$

By multiplying the numerator and denominator by $(\frac{1}{a^2})^{5/2}$ and making the change of variable $a^2 \xi = \Psi$ we obtain,

$$A = \frac{3}{4} P (\lambda_1 + \lambda_2) \int_0^\infty \frac{d\xi}{a^3 (1 + \xi)^{3/2} (1 - e^2 + \xi)^{1/2} \xi^{1/2}}$$

or

$$Aa^3 = \frac{3}{4} P (\lambda_1 + \lambda_2) \int_0^\infty \frac{d\xi}{(1 + \xi)^{3/2} (1 - e^2 + \xi)^{1/2} \xi^{1/2}} \qquad (21)$$

From the same analysis we obtain for Equations 18 and 20,

$$Ba^3 = \frac{3}{4} P (\lambda_1 + \lambda_2) \int_0^\infty \frac{d\xi}{(1 - e^2 + \xi)^{3/2} [\xi (1 + \xi)]^{1/2}} \qquad (22)$$

and

$$\delta a = \frac{3}{4} P (\lambda_1 + \lambda_2) \int_0^\infty \frac{d\xi}{[\xi (1 + \xi)(1 - e^2 + \xi)]^{1/2}} \qquad (23)$$

By making the substitution $\xi = \cot^2 \theta$ [3], and $d\xi = -2 \cot \theta \csc^2 \theta \, d\theta$ where $\theta: \frac{\pi}{2}$ to 0 and $\xi: 0$ to ∞ we obtain,

$$Aa^3 = \frac{3}{4} P (\lambda_1 + \lambda_2) \int_{\pi/2}^0 \frac{-2 \cot \theta \csc^2 \theta \, d\theta}{(1 + \cot^2 \theta)^{3/2} (\cot^2 \theta (1 - e^2 + \cot^2 \theta)^{1/2})^{1/2}}$$

and since $(1 + \cot^2 \theta) = \csc^2 \theta$

$$= \frac{3}{4} P (\lambda_1 + \lambda_2) \int_{\pi/2}^0 \frac{-2 \csc^2 \theta}{(\csc^2 \theta)^{3/2} (\csc^2 \theta - e^2)^{1/2}} \, d\theta$$

$$= \frac{3}{4} P (\lambda_1 + \lambda_2) \int_{\pi/2}^{0} \frac{-2}{(\csc\theta)(\csc^2\theta - e^2)^{1/2}} d\theta$$

$$= \frac{3}{4} P (\lambda_1 + \lambda_2) \int_{\pi/2}^{0} \frac{-2}{(\csc\theta)\left[\csc^2\theta(1 - \frac{e^2}{\csc^2\theta})\right]^{1/2}} d\theta$$

$$= \frac{3}{4} P (\lambda_1 + \lambda_2) \int_{\pi/2}^{0} \frac{-2}{\csc^2\theta(1 - \frac{e^2}{\csc^2\theta})^{1/2}} d\theta$$

and since $\sin\theta = \frac{1}{\csc\theta}$,

$$Aa^3 = \frac{3}{4} P (\lambda_1 + \lambda_2) \int_{\pi/2}^{0} \frac{-2 \sin^2\theta}{(1 - e^2 \sin^2\theta)^{1/2}} d\theta \tag{24}$$

By the same analysis we find,

$$Ba^3 = \frac{3}{4} P (\lambda_1 + \lambda_2) \int_{\pi/2}^{0} \frac{-2 \sin^2\theta}{(1 - e^2 \sin^2\theta)^{3/2}} d\theta \tag{25}$$

and,

$$\delta a = \frac{3}{4} P (\lambda_1 + \lambda_2) \int_{\pi/2}^{0} \frac{2}{(1 - e^2 \sin^2\theta)^{1/2}} d\theta \tag{26}$$

By rearranging the above equations we obtain,

$$Aa^3 = \frac{3}{2} P (\lambda_1 + \lambda_2) \int_{0}^{\pi/2} \frac{\sin^2\theta}{(1 - e^2 \sin^2\theta)^{1/2}} d\theta \tag{27}$$

$$Ba^3 = \frac{3}{2} P (\lambda_1 + \lambda_2) \int_{0}^{\pi/2} \frac{\sin^2\theta}{(1 - e^2 \sin^2\theta)^{3/2}} d\theta \tag{28}$$

$$\delta a = \frac{3}{2} P (\lambda_1 + \lambda_2) \int_{0}^{\pi/2} \frac{1}{(1 - e^2 \sin^2\theta)^{1/2}} d\theta \tag{29}$$

Now the Legendre forms of elliptic integrals of the first and second kinds are from Boas [4],

$$F(e,\ \phi) = \int_0^{\phi} \frac{d\phi}{(1 - e^2 \sin^2 \phi)^{1/2}} \qquad \left\{ \begin{array}{l} 0 \leq e \leq 1 \\ \\ e = \sin\theta \\ 0 \leq \theta \leq \pi/2 \end{array} \right.$$

$$E\ (e,\ \phi) = \int_0^{\phi} (1 - e^2 \sin^2 \phi)^{1/2}\ d\phi$$

where e is the modulus and ϕ the amplitude of the elliptic integral. $e' = (1 - e^2)^{1/2}$ and is called the complementary modulus.

The complete elliptic integrals of the first and second kinds are the values of F and E for $\phi = \pi/2$ so that,

$$K = K(e) = F(e,\ \pi/2) = \int_0^{\pi/2} \frac{d\theta}{(1 - e^2 \sin^2 \theta)}$$

$$E = E(e) = E(e,\ \pi/2) = \int_0^{\pi/2} (1 - e^2 \sin^2 \theta)^{1/2}\ d\theta$$

There are numerous ways to evaluate the above integrals. Hastings [5], has polynomial approximations accurate to 2 parts in 10^8 which are of the form,

$$K(e) = [a_o + a_1 m_1 + \ldots + a_4 m_1^4] +$$

$$[b_o + b_1 m_1 + \ldots + b_4 m_1^4]\ \ln(1/m_1)$$

$a_o = 1.38629\ 436112$	$b_o = .5$
$a_1 = .09666\ 344259$	$b_1 = .12498\ 593597$
$a_2 = .03590\ 092383$	$b_2 = .06880\ 248576$
$a_3 = .03742\ 563713$	$b_3 = .03328\ 355346$
$a_4 = .01451\ 196212$	$b_4 = .00441\ 787012$

where $e = \sin^2 \alpha$ and $m_1 = \cos^2 \alpha$

and the approximation for the elliptic integral of the second kind is given by,

$$E(e) = [1 + a_1 m_1 + \ldots a_4 m_1^4] + [b_1 m_1 + \ldots b_4 m_1^4] \ln(\frac{1}{m_1})$$

a_1	= .44325 141463	b_1	= .24998 368310
a_2	= .06260 601220	b_2	= .09200 180037
a_3	= .04757 383546	b_3	= .04069 697526
a_4	= .01736 506451	b_4	= .00526 449639

Now since the complete elliptic integral of the first kind is,

$$K = \int_o^{\pi/2} (1 - e^2 \sin^2 \theta)^{-1/2} \, d\theta$$

we can obtain,

$$\frac{dK}{de} = e \int_o^{\pi/2} \frac{\sin^2 \theta}{(1 - e^2 \sin^2 \theta)^{3/2}} \, d\theta \tag{30}$$

and in a similar manner for the elliptic integral of the second kind we obtain,

$$\frac{dE}{de} = -e \int_o^{\pi/2} \frac{\sin^2 \theta}{(1 - e^2 \sin^2 \theta)^{1/2}} \, d\theta \tag{31}$$

and we obtain from Equations 27, 28 and 29,

$$Aa^3 = \frac{3}{2} P (\lambda_1 + \lambda_2) (\frac{dE}{de}) (-\frac{1}{e}) \tag{32}$$

$$Ba^3 = \frac{3}{2} P (\lambda_1 + \lambda_2) (\frac{dK}{de}) (\frac{1}{e}) \tag{33}$$

$$\delta a = \frac{3}{2} P (\lambda_1 + \lambda_2) (K) \tag{34}$$

We may rewrite Equation 32 as,

$$Aa^3 = \frac{3}{2} P (\lambda_1 + \lambda_2) \int_0^{\pi/2} \frac{1}{e^2} \left(\frac{1}{(1 - e^2 \sin^2\theta)^{1/2}} - \right.$$

$$\left. \frac{1}{(1 - e^2 \sin^2\theta)^{1/2}} + \frac{e^2 \sin^2\theta}{(1 - e^2 \sin^2\theta)^{1/2}} \right) d\theta$$

$$= \frac{3}{2} P (\lambda_1 + \lambda_2) (\frac{1}{e^2}) \int_0^{\pi/2} \left[\frac{-(1 - e^2 \sin^2\theta)}{(1 - e^2 \sin^2\theta)^{1/2}} + \frac{1}{(1 - e^2 \sin^2\theta)^{1/2}} \right] d\theta$$

$$= \frac{3}{2} P (\lambda_1 + \lambda_2) (\frac{1}{e^2}) \int_0^{\pi/2} \left[\frac{1}{(1 - e^2 \sin^2\theta)^{1/2}} - (1 - e^2 \sin^2\theta)^{1/2} \right] d\theta$$

$$Aa^3 = \frac{3}{2} P (\lambda_1 + \lambda_2) (\frac{1}{e^2}) [K - E] \qquad (35)$$

Equating equations 32 and 35 we obtain,

$$\frac{dE}{de} = \frac{1}{e} (E - K) \qquad (36)$$

From equation 30

$$\frac{dK}{de} = \int_0^{\pi/2} \frac{e \sin^2\theta}{(1 - e^2 \sin^2\theta)^{3/2}} d\theta$$

$$= \int_0^{\pi/2} \frac{(1 - e^2 \sin^2\theta)^{1/2} (e^2 \sin^2\theta)}{e(1 - e^2 \sin^2\theta)^{1/2} (1 - e^2 \sin^2\theta)^{3/2}} d\theta$$

$$= \int_0^{\pi/2} \frac{(1 - e^2 \sin^2\theta)^{1/2} [1 - (1 - e^2 \sin^2\theta)]}{e(1 - e^2 \sin^2\theta)^{1/2} (1 - e^2 \sin^2\theta)^{3/2}} d\theta$$

$$= \int_0^{\pi/2} \left[\frac{(1 - e^2 \sin^2\theta)^{1/2}}{e(1 - e^2 \sin^2\theta)^{1/2} (1 - e^2 \sin^2\theta)^{3/2}} - \frac{(1 - e^2 \sin^2\theta)^{1/2} (1 - e^2 \sin^2\theta)}{e(1 - e^2 \sin^2\theta)^{1/2} (1 - e^2 \sin^2)^{3/2}} \right] d\theta$$

$$= \int_0^{\pi/2} \left[- \frac{1}{e(1 - e^2 \sin^2 \theta)^{1/2}} + \frac{1}{e(1 - e^2 \sin^2 \theta)^{3/2}} \right] d\theta$$

$$\frac{dK}{de} = -\frac{1}{e} K + \frac{1}{e} \int_0^{\pi/2} \frac{d\theta}{(1 - e^2 \sin^2 \theta)^{3/2}}$$

The integral above may be reduced to the form

$$\frac{1}{e} \int_0^{\pi/2} \frac{d\theta}{(1 - e^2 \sin^2 \theta)^{3/2}} = \frac{E}{e(1 - e^2)}$$

thus,

$$\frac{dK}{de} = - \frac{1}{e} K + \frac{E}{e(1 - e^2)} = \frac{1}{e(1 - e^2)} \left[E - (1 - e^2) K \right] \tag{37}$$

By dividing Equation 32 by 33 we obtain,

$$\frac{A}{B} = \frac{\frac{-dE}{de}}{\frac{dK}{de}} \tag{38}$$

and substituting the values in Equations 36 and 37 into Equation 38, we obtain,

$$\frac{A}{B} = \frac{-(1 - e^2)(E - K)}{E - (1 - e^2)K} \tag{39}$$

We now see that for any value of the eccentricity of the contact ellipse we can obtain values for $\frac{A}{B}$, K, and $\frac{1}{e} \frac{dE}{de}$ which will allow us to compute the normal approach δ (deformation).

SPECIAL CASE OF LINE CONTACT

In the metrology field one fundamental measurement occurs frequently, i.e., the measurement of the diameter of a cylinder. Figure 4a shows a cross-section of the typical measurement between plane parallel anvils of a measuring machine. Since it is infinitely difficult to measure the object with zero force applied, Figure 4b shows the exaggerated resultant shape of the anvils and cylinder after a measuring force is applied. Since the customer usually desires to know the "unsquashed" diameter, certain corrections must be applied to account for the measurement process.

To solve this problem we shall make use of the expressions already developed to solve for the "pressure distribution" and size of the area of contact by allowing one axis of the ellipse of contact to become infinite. To determine the deformation, the contact area will be taken as being a finite rectangle with one side very much greater than the other.

The derivation will be for the case of a pair of cylinders with their axes parallel and is based on the works of Thomas and Hoersch [7], and Love [8]. The solution for a cylinder to plane contact can easily be obtained by allowing the radius of one of the cylinders to become infinite.

Line contact occurs when two cylinders rest on each other with their axes parallel (figure 5a), and when a cylinder rests on a plane. As the two cylinders are pressed together along their axes, the resulting pressure area is a narrow "rectangle" of width 2b and length L (assuming no taper in the cylinders). In other words, the area of contact is an elongated ellipse with the major axis of the ellipse equal to L and the eccentricity approaching unity.

The distribution of compressive stress along the width 2b of the surface of contact is represented by a semi-ellipse (figure 5c). The stress distribution over the ellipse of contact for the three-dimensional case will be remembered from Equation 12 as,

$$\sigma(x', y') = \frac{3P}{2\pi ab}\sqrt{1 - \frac{x'^2}{a^2} - \frac{y'^2}{b^2}}$$

where $\sigma(x', y')$ represents the stress acting at any point (x', y').

Now the integrated pressure over the surface of a finite rectangle across the minor axis of the ellipse in the plane $x = 0$ (figure 5c), is

$$\overline{P} = \int_{-b}^{+b} \sigma(0, y')\, dy'$$

$$= \frac{3P}{2\pi ab}\int_{-b}^{+b}\left(1 - \frac{y'^2}{b^2}\right)^{1/2} dy'$$

$$\overline{P} = \frac{3P}{2\pi ab}\left(\frac{\pi b}{2}\right) = \frac{3P}{4a} \tag{40}$$

Now as $a \to \infty$, let $P \to \infty$ in a manner so that P/a remains equal to a finite constant. Then the value of $\overline{P}$ is the force per unit length of the contact area. For $a = \infty$, the compressive stress at any point y is given by,

$$\sigma(y) = \frac{3P}{2\pi ab}\left(1 - \frac{y^2}{b^2}\right)^{1/2} \tag{41}$$

and by substitution of Equation 40 into Equation 41 we obtain, since $\overline{P} = P/L$,

$$\sigma(y) = \frac{2P}{\pi Lb}\left(1 - \frac{y^2}{b^2}\right)^{1/2} \tag{42}$$

We can now see that the maximum value for the stress within the area of contact will be at the center where $y = 0$ and is,

$$\sigma_{max} = \frac{2P}{\pi Lb} \tag{43}$$

The maximum compressive stress is important in any contact problem because the surface area of contact is small and very high stresses may easily be obtained with relatively light loads. If σ_{max} exceeds the microplastic yield point of the material, then permanent deformation will occur. A relationship for hardened steel relating yield stress to surface finish is shown in Figure 6.

We now need to develop an expression for the width of contact b, and by knowing b, the stress at any point can be computed. The surface of each cylinder may be represented by the equations,

$$Z_1 = B_1 y^2$$

$$Z_2 = B_2 y^2$$

From Equation 20 and using the expression developed in Equation 40 we obtain,

$$B = (\lambda_1 + \lambda_2) \frac{P}{L} \int_0^{\infty} \frac{d\Psi}{(b^2 + \Psi)^{3/2}((a^2 + \Psi)\Psi)^{1/2}} \tag{44}$$

and since the one axis of the ellipse, $a \to \infty$, the expression becomes,

$$B = (\lambda_1 + \lambda_2) \frac{P}{L} \int_0^{\infty} \frac{d\Psi}{(b^2 + \Psi)^{3/2} \Psi^{1/2}} \tag{45}$$

By using the expression in Equation 42, Thomas and Hoersch evaluated the above equation to give,

$$B = 2(\lambda_1 + \lambda_2) P/Lb^2 \text{ or}$$

$$b^2 = \frac{2(\lambda_1 + \lambda_2)P}{LB} \tag{46}$$

where b is the half-width of contact,

P is the total force,

L is the contact length.

For a pair of cylinders with their axes parallel and radii r_1 and r_2, respectively, we have

$$Z_1 = B_1 y^2 = \frac{1}{2r_1} y^2$$

$$Z_2 = B_2 y^2 = \frac{1}{2r_2} y^2$$

Thus,
$$B = B_1 + B_2 = \frac{r_1 + r_2}{2r_1 r_2}$$

So
$$b^2 = \frac{4(\lambda_1 + \lambda_2)P(r_1 r_2)}{L(r_1 + r_2)} \tag{47}$$

For a cylinder on a plane surface,

$$Z_1 = \frac{1}{2r} y^2$$

$$Z_2 = \frac{1}{\infty} = 0$$

thus,
$$B = \frac{1}{2r}$$

$$b^2 = \frac{4r(\lambda_1 + \lambda_2)P}{L} \tag{48}$$

Now since our cylindrical surfaces are described by By^2 forms, we obtain from Equation 8,

$$(\lambda_1 + \lambda_2)\iint_A \frac{P(y')}{r}\, dx'dy' = \delta - By^2 \tag{49}$$

where $r = ((y - y')^2 + x'^2)^{1/2}$ and the integral applies to the finite rectangle of contact with one side (L) very much longer than the other (2b).

We now have,

$$\Phi(0, y) = \int_{-b}^{+b}\int_{-L/2}^{+L/2} \frac{P(y')}{((y - y')^2 + x'^2)^{1/2}}\, dx'dy'$$

$$= \int_{-b}^{+b} 2p(y') \int_{0}^{L/2} \frac{1}{((y - y')^2 + x'^2)^{1/2}} \, dx'dy'$$

$$= \int_{-b}^{+b} 2p(y') \ln \left[\frac{\frac{L}{2} + ((y - y')^2 + (\frac{L}{2})^2)^{1/2}}{|y - y'|} \right] dy'$$

If we assume $(\frac{L}{2})$ is very large in relation to $(y - y')$ then,

$$\Phi(0, y) = \int_{-b}^{+b} 2p(y') \ln \left(\frac{L}{y - y'} \right) dy' \tag{50}$$

and

$$\Phi(0, 0) = \int_{-b}^{+b} 2p(y') \ln \left(\frac{L}{-y'} \right) dy' \tag{51}$$

Let us pause a moment to recap what we are doing. Equation 49 gives us the relation between the total deformation δ to the potential for points within the contact area. Since we are interested in the maximum deformation which occurs, we need to evaluate the integral in Equation 49 at the center line or where $y = 0$. Thus, Equation 51 represents that maximum potential.

Continuing, we may rewrite Equation 51 as,

$$\Phi(0, 0) = 2 \ln L \int_{-b}^{+b} p(y')dy' - \int_{-b}^{+b} p(y') \ln(y'^2) \, dy' \tag{52}$$

Since the force per unit length $\frac{P}{L} = \int_{-b}^{+b} p(y')dy'$ and from Equation 41 we may substitute into Equation 52 to obtain,

$$\Phi(0, 0) = 2 \frac{P}{L} \ln L - \frac{2P}{\pi b L} \int_{-b}^{+b} \left(1 - \frac{y'^2}{b^2}\right)^{1/2} \ln(y'^2)dy' \tag{53}$$

To evaluate the integral in the above relationship, we make the substitution $\frac{y}{b} = \sin \theta$ and then $dy = b \cos \theta \, d\theta$ to obtain,

$$= \int_{-\pi/2}^{+\pi/2} (1 - \sin^2\theta)^{1/2} \left[\ln(b^2\sin^2\theta)\right](b\cos\theta)\, d\theta$$

$$= b \int_{-\pi/2}^{+\pi/2} \cos^2\theta \left[\ln(b^2\sin^2\theta)\right] d\theta$$

$$= b(2\ln b) \int_{-\pi/2}^{+\pi/2} \cos^2\theta\, d\theta + 2b \int_{-\pi/2}^{+\pi/2} \cos^2\theta \ln|\sin\theta|\, d\theta$$

we know that,

$$\int_{-\pi/2}^{+\pi/2} \cos^2\theta\, d\theta = \frac{+\pi}{2}$$

and the integral

$$I = \int_{-\pi/2}^{+\pi/2} \cos^2\theta \ln|\sin\theta|\, d\theta$$

has been evaluated by Birens de Haan [9], to give

$$I = \frac{-\pi}{4} (1 + \ln 4)$$

So we obtain the value for the total integral as,

$$= b(2\ln b)\frac{\pi}{2} + 2b\left(-\frac{\pi}{4} - \frac{\pi}{4}\ln 4\right),$$

$$= \pi b\left(\ln b - \frac{1}{2} + \frac{\ln 4}{2}\right)$$

Thus by substituting into Equation 53 we obtain

$$\Phi(0, 0) = 2\frac{P}{L}\ln L - \frac{2P}{\pi b L}\left[\pi b\left(\ln b - \frac{(1 + \ln 4)}{2}\right)\right]$$

$$= 2\frac{P}{L}\left[\ln L - \ln b + \frac{1 + \ln 4}{2}\right]$$

$$\Phi(o, o) = 2\frac{P}{L}\left[\ln L - \ln b + 1.193145\right]$$

Since

$$\delta = (\lambda_1 + \lambda_2)\iint_A \frac{P(y')}{r}\, dx'dy' + By^2$$

and $\Phi(o, o)$ gives us the potential at the center of pressure zone we have,

$$\delta = (\lambda_1 + \lambda_2)\, 2\,\frac{P}{L}\left[\ln L - \ln b + 1.193145\right]$$

as the total deformation of a pair of cylinders with their axes parallel or a cylinder on a flat surface. The value of b (half-width of contact) is substituted into equation 54 to obtain the appropriate answer for any particular case.

Since equation 48 gives us the value of b as,

$$b = \left[\frac{4r(\lambda_1 + \lambda_2)P}{L}\right]^{1/2}$$

we have

$$\delta = (\lambda_1 + \lambda_2)\, 2\,\frac{P}{L}\left[\ln L - \frac{1}{2}\ln\left(\frac{4r(\lambda_1 + \lambda_2)P}{L}\right) + 1.193145\right]$$

$$= \frac{P}{L}(\lambda_1 + \lambda_2)\left[2\ln L - \ln\left(\frac{4r(\lambda_1 + \lambda_2)P}{L}\right) + 2.38629\right]$$

$$= \frac{P}{L}(\lambda_1 + \lambda_2)\left[2\ln L + \ln\left(\frac{L}{4r(\lambda_1 + \lambda_2)P}\right) + 2.38629\right]$$

$$\delta = \frac{P}{L}(\lambda_1 + \lambda_2)\left[\ln\left(\frac{L^3}{4(\lambda_1 + \lambda_2)Pr}\right) + 2.38629\right] \qquad (54)$$

where P = total measuring force

L = length of contact between plane and cylid

$\lambda_i = 1 - \nu_1^{\,2}/\pi E_1$

r = radius of cylinder

δ = total deformation of cylinder and plane.

Another form of Equation 54 can be obtained by not evaluating the term $\left(\frac{1 + \ln 4}{2}\right)$ which gives,

$$\delta = \frac{P}{L}(\lambda_1 + \lambda_2) \left[1.00 + \ln \frac{L^3}{(\lambda_1 + \lambda_2)Pr}\right] \tag{55}$$

Remember this formula is for the case of a cylinder to plane on only one side. If both sides are desired, such as the case in Figure 4, then

$$\delta_T = 2\delta$$

If the lengths of the lines of contact are not equal or if the material of either cylinder or plane are different, then the total deformation will be,

$$\delta_T = \delta_1 + \delta_2$$

A similar approach for the computation of the deformation of a cylinder to plane has been obtained from correspondence with Bob Fergusson [10]. The basis of the work is a paper by Airey [11] in which formulae are given for the solutions of elliptic integrals when the eccentricity (e) approaches unity.

From Equations 4 and 5 we have,

$$A + B = \frac{1}{D_1} + \frac{1}{D_1'} + \frac{1}{D_2} + \frac{1}{D_2'}$$

$$(B - A)^2 = \left(\frac{1}{D_1} - \frac{1}{D_1'}\right)^2 + \left(\frac{1}{D_2} - \frac{1}{D_2'}\right)^2 + \left(\frac{1}{D_1} - \frac{1}{D_1'}\right)\left(\frac{1}{D_2} - \frac{1}{D_2'}\right) \cos 2\omega$$

where D_1 and D_1' are the two principal diameters of body 1,

D_2 and D_2' are the two principal diameters of body 2.

In the case of a cylinder to plane, if body 1 is the cylinder and body 2 is the plane, then

D_1 = Diameter of cylinder

$D_1' = \infty$

$D_2 = \infty$ if the plane is truly a flat surface

$D_2' = \infty$

Since we have shown for the case of a cylinder to plane

$$B = \frac{1}{2r_1} = \frac{1}{D_1}$$

Thus the parameter $\frac{A}{B} = \frac{A}{\frac{1}{D_1}} = AD_1$

From the previous Equation 38, we know

$$\frac{A}{B} = \frac{-\frac{dE}{de}}{\frac{dK}{de}}$$

where E is the complete elliptic integral of the second kind

K is the complete elliptic integral of the first kind

e is the eccentricity of the "ellipse" of contact which approaches unity

In Airey's work [11], expressions were given for evaluation of the elliptic integrals when the eccentricity approaches one, viz

$$K(e^2) = \ln \frac{4}{\sqrt{h}} \cdot K_1 - K_2 \tag{56}$$

where $K(e^2)$ = complete elliptic integral of first kind

e^2 = square of modulus e

h = square of the complementary modulus or $(1 - e^2)$

$$K_1 = \left\{ 1 + \frac{1^2}{2^2} \cdot h + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} \cdot h^2 + \cdot \cdot \cdot \right\}$$

$$K_2 = \left\{ \frac{1^2}{2^2} \left(\frac{2}{1} - \frac{2}{2}\right) \cdot h + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} \left(\frac{2}{1} - \frac{2}{2} + \frac{2}{3} - \frac{2}{4}\right) \cdot h^2 + \cdot \cdot \cdot \right\}$$

and $$E(e^2) = \ln \frac{4}{\sqrt{h}} \cdot E_1 + E_2 \tag{57}$$

where $E(e^2)$ = complete elliptic integral of second kind

$$E_1 = \left\{ \frac{1}{2} \cdot h + \frac{1^2 \cdot 3}{2^2 \cdot 4} \cdot h^2 + \cdot \cdot \cdot \right\}$$

$$E_2 = \left\{ 1 - \frac{1}{2}\left(\frac{1}{1} - \frac{1}{2}\right) \cdot h - \frac{1^2 \cdot 3}{2^2 \cdot 4} \cdot \left(\frac{2}{1} - \frac{2}{2} + \frac{1}{3} - \frac{1}{4}\right) \cdot h^2 - \cdot \cdot \cdot \right\}$$

Thus as $e \to 1$

$$k(e^2) = \ln \frac{4}{\sqrt{(1 - e^2)}} \tag{58}$$

From Equation 37 and using Airey's expressions when $e \to 1$

$$\frac{dK}{dE} = - \frac{1}{e}K + \frac{E}{e(1 - e^2)}$$

which reduces to

$$\frac{dK}{dE} = \frac{1}{(1 - e^2)} \tag{59}$$

and

$$\frac{1}{h} \frac{A}{B} = - \frac{1}{e} \frac{dE}{de} \tag{60}$$

Since from previous derivations, namely Eq. 32 where

$$A\ a^3 = \frac{3}{2}P\ (\lambda_1 + \lambda_2) \left(\frac{dE}{de}\right)\left(- \frac{1}{e}\right)$$

We substitute (6) into the above to obtain

$$A\ a^3 = \frac{3}{2}P\ (\lambda_1 + \lambda_2) \left(\frac{1}{h} \frac{A}{B}\right)$$

and

$$A\ a^3 = \frac{3}{2}P\ (\lambda_1 + \lambda_2) \frac{1}{h} (AD_1)$$

$$a^3 = \frac{3}{2} (\lambda_1 + \lambda_2) \frac{PD_1}{h}$$

or

$$h = \frac{3}{2} (\lambda_1 + \lambda_2) \frac{PD_1}{a^3}$$

Now since $a = \frac{L}{2}$ (the ellipse of contact approach where the major axis of the ellipse is a) we have

$$h = 12\ (\lambda_1 + \lambda_2) \frac{PD_1}{L^3}$$

From Equation 34

$$\delta a = \frac{3}{2}P\ (\lambda_1 + \lambda_2)\ (K)$$

$$= \frac{3}{2}\,\frac{P}{a}\ (\lambda_1 + \lambda_2)\ \ln \frac{4}{\sqrt{h}}$$

Since $\ln \frac{4}{\sqrt{h}} = \frac{1}{2}\ \ln 16 - \frac{1}{2}\ \ln h = \frac{1}{2}\ \ln \left(\frac{16}{h}\right)$

$$\delta = \frac{3}{2}\,\frac{P}{a}\ (\lambda_1 + \lambda_2)\ \left(\frac{1}{2}\ \ln \left(\frac{16\ L^3}{12(\lambda_1 + \lambda_2)PD}\right)\right)$$

and since $a = L/2$

$$\delta = \frac{3}{2}\ (\lambda_1 + \lambda_2)\ \frac{P}{L}\ \ln \left(\frac{4\ L^3}{3(\lambda_1 + \lambda_2)PD}\right) \tag{61}$$

$$\delta = \frac{3}{2}\ (\lambda_1 + \lambda_2)\ \frac{P}{L}\ \ln \frac{4\ L^3}{3(\lambda_1 + \lambda_2)PD}$$

since $D = 2r$

$$\delta = \frac{3}{2}\ (\lambda_1 + \lambda_2)\ \frac{P}{L}\ \ln \frac{4\ L^3}{3(2)(\lambda_1 + \lambda_2)Pr}$$

$$\ln \frac{4\ L^3}{6(\lambda_1 + \lambda_2)Pr} = \ln \frac{4}{6} + \ln \frac{L^3}{(\lambda_1 + \lambda_2)Pr}$$

$$\ln \frac{4}{6} = \ln 4 - \ln 6 = 1.38629 - 1.79176 = -.40547$$

$$\delta = \frac{3}{2}\ (\lambda_1 + \lambda_2)\ \frac{P}{L}\ \left[-\ .40547 + \ln \frac{L^3}{(\lambda_1 + \lambda_2)Pr}\right]$$

$$\delta = (\lambda_1 + \lambda_2)\ \frac{P}{L}\ \left[-.608205 + 1.5\ \ln \frac{L^3}{(\lambda_1 + \lambda_2)Pr}\right] \tag{62}$$

This expression is for the contact case of one side of the cylinder pressed against a plane. If the normal approach (deformation) for the cylinder is desired then

$$\delta_{Total} = \delta_1 + \delta_2$$

If the materials of body 1 (cylinder) are the same as body 2 (plane) then $\lambda_1 = \lambda_2$ and Eq. 61 becomes

$$\delta = 3\,(\lambda_o)\,\frac{P}{L}\,\ln\,\frac{2\,L^3}{3\,\lambda_o\,PD}$$

where $\lambda_o = (\lambda_1 + \lambda_2) = 2\lambda,$

P = Total load applied

L = Length of contract

D = Diameter of cylinder

Another approach for the cylinder-plane case was undertaken by Lundberg [12] in 1939 in which the contacting bodies were treated as elastic half-spaces. Lundberg obtained approximations for the deformation by assuming an elliptical pressure distribution in the narrow dimension (2b in Figure 5b) and constant in the length of contact dimension (L in Figure 5b).

Recalling Eq. 1 and Figure 3 we have

$$w_1 + w_2 + Z_1 + Z_2 = \delta$$

where δ is the approaching distance of the points at the surface of the bodies.

$$Z_1 = \frac{4(1 - \nu_1^{\,2})}{E}\,V \qquad Z_2 = \frac{4(1 - \nu_1^{\,2})}{E}\,V$$

where V is the potential function

$$V = F(x,y) = \frac{1}{4}\left(\frac{\theta_1\theta_2}{\theta_1 + \theta_2}\right)\left[\delta - Z_1 - Z_2\right]$$

where $\theta_i = \frac{E_i}{1 - \nu_i^2}$

Now $F(x,y) = \frac{1}{4\pi} \phi\ (x,y) = \frac{1}{4\pi} \int \int_A \frac{P(r')}{r}\ dx'dy'$

where $r' = \left[(x - x')^2 + (y - y')^2 \right]^{1/2}$

and if $w_1 = F_1(x,y)$

$w_2 = F_2(x,y)$

that is some function of x and y.

Then we have

$$F(x,y) = \frac{1}{4} \left(\frac{\theta_1 \theta_2}{\theta_1 + \theta_2} \right) \left[\delta - F_1(x,y) - F_2(x,y) \right]$$

Do not confuse the potential $F(x,y)$ with $F_1(x,y)$ and $F_2(x,y)$. Since the total deformation is

$$\delta = 4 \left(\frac{1}{\theta_1} + \frac{1}{\theta_2} \right) F\ (x_o, y_o) \qquad (63)$$

where (x_o, y_o) represents the point where the deformation is largest we have

$$w_1 + w_2 = F_1(x,y) + F_2(x,y) = 4\left(\frac{1}{\theta_1} + \frac{1}{\theta_2} \right) \left[F(x_o, y_o) - F(x,y) \right] \qquad (64)$$

Thus we see the relationships between the pressure distribution and the total deformation is expressed by Eqs. 63 and 64.

The interested reader is invited to follow Lundburg's analysis but for brevity the results will only be given here.

The relationship for the displacement of the surface of the plane under the action of an elliptical pressure distribution when $b \ll a$ is

$$F_1(x,o) + F_2(x,o) = \left(\frac{1}{\theta_1} + \frac{1}{\theta_2}\right)\frac{P}{L}\frac{1}{\pi}\left(1.1932 + \ln\frac{L}{2b}\right) \tag{65}$$

where $F_1(x,o) + F_2(x,o)$ = depression of the center of the contact area (y = o)

$$\theta_i = \frac{E_i}{1 - \nu_i^2}$$

P = Total Load

L = Length of contact

b = Hertzian half-width

ν_i = Poisson's ratio

E_i = Modulus of elasticity

The relationship for the total deformation δ is

$$\delta = \left(\frac{1}{\theta_1} + \frac{1}{\theta_2}\right)\frac{P}{L}\frac{2}{\pi}\left[1.8864 + \ln\frac{L}{2b}\right] \tag{66}$$

This equation is based upon an approximation as a function of $\ln\left(\frac{2b}{L}\right)$ and is accurate for all values of $\left(\frac{2b}{L}\right)$ except when $\left(\frac{2b}{L}\right) \to 1$. In this case the accuracy is approximately four percent.

Since we have shown that

$$b = \left(\frac{4r(\lambda_1 + \lambda_2)P}{L}\right)^{1/2}$$

we may substitute into Eqs. 65 and 66 to obtain

$$\delta = (\lambda_1 + \lambda_2)\frac{2D}{L}\left[1.8864 + \ln L - \ln 2b\right]$$

$$= (\lambda_1 + \lambda_2)\frac{P}{L}\left[2(1.8864) + 2\ln L - 2\ln 2b\right]$$

$$= (\lambda_1 + \lambda_2)\frac{P}{L}\left[3.7728 + 2\ln L - 2\ln 2 - 2\ln b\right]$$

$$= (\lambda_1 + \lambda_2)\frac{P}{L}\left[3.7728 - 1.3863 + 2\ln L - 2(\tfrac{1}{2})\ln\frac{4r(\lambda_1 + \lambda_2)P}{L}\right]$$

$$= (\lambda_1 + \lambda_2)\frac{P}{L}\left[2.3865 + \ln L^2 + \ln\frac{L}{4r(\lambda_1 + \lambda_2)P}\right]$$

$$= (\lambda_1 + \lambda_2) \frac{P}{L} \left[2.3865 + \ln (.25) + \ln \frac{L^3}{(\lambda_1 + \lambda_2)Pr} \right]$$

$$\delta = (\lambda_1 + \lambda_2) \frac{P}{L} \left[1.0000 + \ln \frac{L^3}{(\lambda_1 + \lambda_2)Pr} \right] \quad (67)$$

This δ gives us the normal approach for one side of a cylinder-plane contact.

and

$$F = (\lambda_1 + \lambda_2) \frac{P}{L} \left[1.1932 + \ln L - \ln 2b \right]$$

$$= (\lambda_1 + \lambda_2) \frac{P}{L} \left[1.1932 + \ln L - \ln 2 - \frac{1}{2} \ln \frac{4r(\lambda_1 + \lambda_2)P}{L} \right]$$

$$= (\lambda_1 + \lambda_2) \frac{P}{L} \left[1.11932 - .69315 + \frac{1}{2} \ln \frac{L}{4r(\lambda_1 + \lambda_2)P} \right]$$

$$F = (\lambda_1 + \lambda_2) \frac{P}{L} \left[.5 + \frac{1}{2} \ln \frac{L^3}{4r(\lambda_1 + \lambda_2)P} \right]$$

$$F = (\lambda_1 + \lambda_2) \frac{P}{L} \left[-.193145 + \frac{1}{2} \ln \frac{L^3}{(\lambda_1 + \lambda_2)Pr} \right] \quad (68)$$

This equation gives us the deformation of the plane surface.

In 1933 Weber [13] considered the deformation of a circular cylinder loaded on a side by pressure distributed elliptically in the narrow dimension and constant along the longitudinal axis of the cylinder.

The assumption made here was that the Hertzian half-width $b \ll$ radius of cylinder r. The expression for the deformation of the cylinder (one-sided) is

$$V_c = \frac{1 - \nu^2}{\pi E} \frac{P}{L} 2 \left(\ln \frac{4r}{b} - 1 \right) \quad (69)$$

where b is Hertzian half-width

This equation assumes that both contacting bodies are of the same material. If they are not we may rewrite the above relationship as

$$V_c = 2 \left(\frac{1 - \nu^2}{\pi E} \right) \frac{P}{L} \left(\ln \frac{4r}{b} - \frac{1}{2} \right)$$

$$= \left(\frac{1-\nu_1^2}{\pi E_1} + \frac{1-\nu_2^2}{\pi E_2}\right)\frac{P}{L}\left(\ln\frac{4r}{b} - \frac{1}{2}\right)$$

Now

$$\ln\frac{4r}{b} = \ln 4 + \ln r - \ln b$$

$$= \ln 4 + \ln r - \frac{1}{2}\ln\frac{4r(\lambda_1 + \lambda_2)P}{L}$$

$$= \ln 4 + \ln r + \frac{1}{2}\ln\frac{L}{4r(\lambda_1 + \lambda_2)P}$$

$$= 1.38629 + \frac{1}{2}\ln r^2 + \frac{1}{2}\ln\frac{L}{4r(\lambda_1 + \lambda_2)P}$$

$$= 1.38629 + \frac{1}{2}\ln\frac{Lr}{4(\lambda_1 + \lambda_2)P}$$

$$= 1.38629 + \frac{1}{2}\ln(.25) + \frac{1}{2}\ln\frac{Lr}{(\lambda_1 + \lambda_2)P}$$

$$\ln\frac{4r}{b} = .693145 + \frac{1}{2}\ln\frac{Lr}{(\lambda_1 + \lambda_2)P}$$

So we have

$$V_c = \left(\lambda_1 + \lambda_2\right)\frac{P}{L}\left[.193145 + \frac{1}{2}\ln\frac{Lr}{(\lambda_1 + \lambda_2)P}\right] \tag{70}$$

Weber also obtained an expressed for the deformation of the plane surface as

$$V_F = \frac{1-\nu^2}{\pi E}\left(\frac{P}{L}\right)\left(1 + 2\ln\frac{L}{\pi b}\right) \tag{71}$$

Using the same logic as before in the case of the cylinder

$$= 2\left(\frac{1-\nu^2}{\pi E}\right)\left(\frac{P}{L}\right)\left(\frac{1}{2} + \ln\frac{L}{\pi b}\right)$$

$$V_F = \left(\frac{1-\nu_1^2}{\pi E_1} + \frac{1-\nu_2^2}{\pi E_2}\right)\frac{P}{L}\left[\frac{1}{2} + \ln\frac{L}{\pi b}\right]$$

$$\ln \frac{L}{\pi b} = \ln L - \ln\pi - \ln b$$

$$= \ln L - \ln\pi - \frac{1}{2}\ln \frac{4r(\lambda_1 + \lambda_2)P}{L}$$

$$= \frac{1}{2}\ln L^2 - \ln\pi + \frac{1}{2}\ln \frac{L}{4r(\lambda_1 + \lambda_2)P}$$

$$= -\ln\pi + \frac{1}{2}\ln \frac{L^3}{4r(\lambda_1 + \lambda_2)P}$$

$$= -1.1447 + \frac{1}{2}\ln(.25) + \frac{1}{2}\ln \frac{L^3}{(\lambda_1 + \lambda_2)Pr}$$

$$= -1.1447 - .69315 + \frac{1}{2}\ln \frac{L^3}{(\lambda_1 + \lambda_2)Pr}$$

$$\ln \frac{L}{\pi b} = -1.83787 + \frac{1}{2}\ln \frac{L^2}{(\lambda_1 + \lambda_2)Pr}$$

So

$$V_F = (\lambda_1 + \lambda_2)\frac{P}{L}\left[-1.33787 + \frac{1}{2}\ln \frac{L^3}{(\lambda_1 + \lambda_2)Pr}\right] \tag{72}$$

To obtain the total deformation for the case of a cylinder between two anvils we have

$$\delta_{Total} = 2(V_c + V_F)$$

$$= 2\left[(\lambda_1 + \lambda_2)\frac{P}{L}\left[.193145 + \frac{1}{2}\ln \frac{Lr}{(\lambda_1 + \lambda_2)P} - 1.33787 + \frac{1}{2}\ln \frac{L^3}{(\lambda_1 + \lambda_2)Pr}\right]\right]$$

$$= 2\left[(\lambda_1 + \lambda_2)\frac{P}{L}\left[-1.14473 + \frac{1}{2}\ln \frac{L^4}{(\lambda_1 + \lambda_2)^2P^2}\right]\right]$$

$$\delta = 2\left[(\lambda_1 + \lambda_2)\frac{P}{L}\left(-1.14473 + \ln \frac{L^2}{(\lambda_1 + \lambda_2)P}\right)\right] \tag{73}$$

Various other relationships for deformation of circular cylinders have been derived by Dörr [14], Föppl [15], Kovalsky [16], and DinniK [17]. Only the pertinent results will be given here.

Dörr calculated the displacement of the points of initial contact by assuming a smooth circular cylinder of radius r was compressed between smooth "rigid" planes from the relationship

$$\delta_{\text{one-sided}} = \frac{2(1 - \nu_1^2)}{\pi E} \left(\frac{P}{L}\right) \left[\ln \frac{4R}{b} - \frac{1}{2}\right]$$

where $b = \left(4r\ (\lambda_1 + \lambda_2)\ \frac{P}{L}\right)^{1/2}$

or

$$\delta = \frac{2(1 - \nu_1^2)}{\pi E} \quad \frac{P}{L} \left[.193145 + \frac{1}{2} \ln \frac{LR}{(\lambda_1 + \lambda_2)P}\right] \tag{74}$$

Thus the change in diameter of the cylinder according to Dörr would be

$$\delta_{\text{Total}} = 2\delta$$

According to Roark [18] a similar expression for the total compression of of a cylinder between two rigid planes was derived by Föppl. It is

$$\delta_{\text{Total}} = 4 \left(\frac{1 - \nu^2}{\pi E}\right) \frac{P}{L} \left[\frac{1}{3} + \ln \frac{2r}{b}\right]$$

where b is Hertzian half-width

$$\delta_{\text{one-side}} = 2 \left(\frac{1 - \nu^2}{\pi E}\right) \frac{P}{L} \left[.3333 + \frac{1}{2} \ln \frac{LR}{(\lambda_1 + \lambda_2)P}\right] \tag{75}$$

Roark also gives an expression for the mutual approach of remoted points in two plates as

$$M = 4 \left(\frac{1 - \nu^2}{\pi E}\right) \frac{P}{L} \ln \frac{\pi E L^2}{P(1 - \nu^2)} \tag{76}$$

When a cylinder is compressed between two parallel planes.

Kovalsky, in 1940, derived an expression for the deformation of a circular cylinder of finite length loaded from two sides by pressure distributed elliptically across the contact width. The change in diameter parallel to the direction of the force was

$$\delta_{Tot} = \frac{4(1 - \nu^2)}{\pi E} \frac{P}{L} \left[\ln \frac{2r}{b} + 0.407 \right] \tag{77}$$

Again ν = Poisson ratio for the cylinder

E = Modulus of elasticity for cylinder

P = Total load

L = Length of contact

r = Radius of cylinder

b = Hertzian half width

Dinnik also calculated the change in diameter of a circular cylinder but by assuming the pressure distribution across the contact width was parabolic in nature. His expression was

$$\delta_{Tot} = 4 \left(\frac{1 - \nu^2}{\pi E} \right) \frac{P}{L} \left[\ln \frac{2r}{b} + \frac{1}{3} \right] \tag{78}$$

Since Lundberg calculated the displacement of the surface of an elastic half-space under the action of an elliptical pressure distribution from the relation

$$\Delta = 2 \left(\frac{1 - \nu_2^2}{\pi E_2} \right) \frac{P}{L} \left[1.1932 + \ln \frac{L}{2b} \right]$$

where Δ = depression at the center of pressure zone.

We may write the normal approach of a distant point in the elastic plane to the axis of the cylinder as the sum

where

$$\delta = \frac{1}{2} \delta_{Tot} + \Delta$$

Another relationship was obtained from a Canadian report which gives the compression at the point of contact of a cylinder and a plane as

$$\delta = \frac{FQ}{a}\left[1 + \ln \frac{2a}{QFW}\right]$$

where F = Applied measuring force

a = Length of contact

w = Diameter of the cylinder

$$Q = \frac{1 + \nu_1}{\pi E_1} + \frac{1 + \nu_2}{\pi E_2}$$

if we replace L = a and 2r = w then

$$\delta = \frac{FQ}{L}\left[1 + \ln \frac{L}{QFr}\right] \tag{79}$$

Thus the total deformation for a cylinder between two planes would be

$$\delta_{Total} = \delta_1 + \delta_2$$

In a paper by Loo [19] the problem of line contact between two infinitely long isotropic, elastic circular cylinders of radius R_1 and R_2 under the action of a compressive force P (where P = load per unit longitudinal length) is discussed.

$$\delta = P\left[K_1 \ln 2R_1 + K_2 \ln 2R_2 + (K_1 + K_2) \ln \left(\frac{4B}{P}\right)^{1/2} - \frac{1}{2}(K_1 + K_2)\right] \tag{80}$$

where δ = Normal approach of the two cylinders

P = Load per unit length P_{Total}/contact length

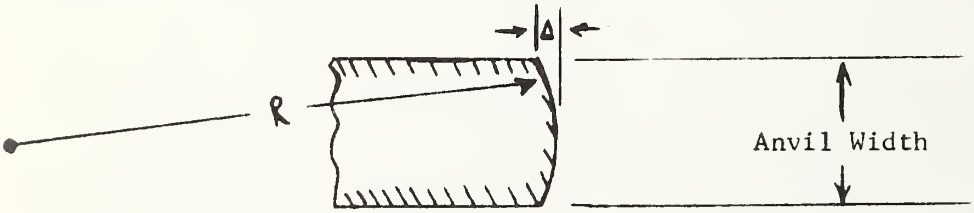
$$K_i = \frac{2(1 - \nu_i^2)}{\pi E_i}$$

ν_i = Poisson's ratio for the i^{th} body

E_i = Modulus of elasticity for the i^{th} body

$$B = \frac{1}{2(\ {}_1 + k_2)}\left[\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + \left(\frac{K_1}{R_1{}^2} + \frac{K_2}{R_2{}^2}\right)P\right] \tag{81}$$

In the case of a cylinder to a plane we may replace the plane surface with a cylindrical surface with a very large radius of curvature. In the real world of metrology this may well be the case anyway because any "flat" measuring surface is not perfectly flat. In Figure 4, if one of the contacting anvils has a very slight curvature and its surface can be approximated by the arc of a circle, then we can solve for the radius by the sagitta formula. For example, if the anvil surface is 0.5μ" higher at the center than the ends and the anvil width is 0.375 inch then



$$\text{Radius} = \frac{\left(\frac{\text{Anvil Width}}{2}\right)^2}{2 \cdot \Delta}$$

$$= \frac{\left(\frac{.375}{2}\right)^2}{2(.5)(10^{-6})}$$

Radius = 35156 inches

Thus we may solve for the deformation by plugging this value back into Equation 80. To obtain the total deformation for the case of a cylinder between two planes

$$\delta_{Total} = \delta_1 + \delta_2$$

EXPERIMENTAL VERIFICATION

Since the validity of any theoretical formulation is dependent upon experimental verification, we have performed a literature search for verification of the formula describing the deformation of a cylinder between two planes. Bochman [20] performed his dissentation on this subject along with measurement of the deformation of steel balls. He began by assuming the oblateness of a cylinder between two planes increases in proportion to the load and is influenced by the size of the measuring surfaces. An elaborate lever system was devised for amplifying the deformation and a Zeiss optimeter was used as the readout device. The contacting anvils were hardened and finished to "end-standard quality". The optimeter was calibrated with end standards but could only be estimated to 0.1μ ($4\mu''$) with a measuring accuracy of approximately 0.2μ ($8\mu''$). The cylinders used in the test were steel wires (used to measure the pitch diameter of external threads) with nominal diameter 0.18mm to 5.70mm. The applied load varied from 1 Kg to 10 Kg in some cases. Anvil pairs used were 5.92, 8.03, and 14.95mm in length.

The contact case where the wires were shorter than the measuring surfaces was also examined. The test results proved that the length of the wire exceeding the size of the measuring surface has no effect on the amount of deformation. In other words it is immaterial whether the length of the wire exceeds that of the measuring anvil or whether the measuring anvil is larger.

The values for oblateness for the 1 Kg load were plotted against the cube root of the curvature (reciprocal cylinder radius) with the result being linear. Thus he concluded the proportionality equation was

$$M = C \times 10^{-3} \frac{P}{L} \sqrt[3]{1/D} \ \text{mm} \qquad (82)$$

where M = oblateness in mm

C = Constant of proportionality

P = Applied load in Kg

L = Contact length in mm

D = Cylinder diameter in mm

The factor C was determined as 0.9228 from 35 test series. The mean error of the individual value of C was $\pm 1.99\%$ and the maximum deviations from the mean were + 3.6% and - 5.1%.

By rewriting the equation we obtain

$$2\delta = 0.2207 \frac{P}{L} D^{-1/3} \text{ microinches}$$

where 2δ = oblateness in microinches (10^{-6} inches) of a steel cylinder when compressed between two flat steel planes

P = Measuring force in pounds

L = Contact length in inches

D = Cylinder diameter in inches

In a 1921 report by the National Physical Laboratory a test was conducted on the compression of a 0.05 inch steel cylinder between a pair of flat measuring anvils 0.25 inch in diameter. The results of that experiment are shown in Figure 7 along with the results of Equations 55,62 and 82 from this paper. There is no mention of the uncertainty in the NPL experiments. We see from Figure 7 that the data from NPL agrees quite closely with Equation 62.

In a more recent study of line contact deformation by Fergusson [6] the results of a series of tests on hardened steel rollers between parallel plates was discussed. From Equation 61 the equation for two bodies of the same material is

$$\delta = 3(\lambda_o) \frac{P}{L} \ln \frac{2L^3}{3\lambda_o PD}$$

Thus the total deformation for the case of a cylinder between two planes would be

$$\delta_{Total} = 2\delta$$

From communication with Fergusson a number of tests were performed and an empirical relationship was found which agreed within 7% of those values given by the theoretical equation above for values of force from .2 lb to 16 lb, values of diameter .01 inch to 10 inch, and values of L from .01 to 1 inch. The relation is

$$\delta = 0.00534\ L^{-0.765}\ D^{-0.053}\ P^{0.94}\ (10^{-4})\ \text{inch} \qquad (83)$$

where δ = Deformation

L = Contact length in inches

D = Cylinder diameter

P = Total force in lbs

Again the total deformation would be 2δ.

The experimental results for a 1.0 inch cylinder between flat, parallel planes 0.118 inch in width were summarized by still another empirical formula.

$$\delta = 0.0091\ L^{-0.73}\ D^{-0.08}\ P^{0.6666}(10^{-4})\ \text{inch} \qquad (84)$$

Many factors have an effect upon the deformation in a cylinder to plane case such as the type of surface finish and the parallelism of the plane surfaces.

In the most recent measurements on the compression of a roller between parallel planes Thwaite [21] devised an apparatus consisting of a cylindrical shaft with an air bearing for loading coupled with capacitance probes to sense the compression. The test cylinder and flat were made of Cr - Mn hardened tool steel. The plane was flat within 0.025μm (1μ") and the cylinder was straight to 0.025μm. Both plane and cylinder possessed a surface roughness of 0.01μm (.4μ") center line average.

The entire system possessed a resolution of 0.003μm (.12μ"). Three series of measurements were made on a 6.35mm diameter cylinder in contact over a length of 9.525mm for the load/contact length range of 0.05 to 0.4 Kg/mm. The mean observed slope differed by approximately 5% from the equation based on the finite rectangle, that is Equation 55

$$2\delta = \frac{2P}{L}(\lambda_1 + \lambda_2)\left[1.0 + \ln \frac{L^3}{(\lambda_1 + \lambda_2)Pr}\right]$$

while the values predicted by the Bochman and Berndt [20] relationship differ from Thwaite's slope by 45%.

Thwaite concludes that within the load range tested the equation based on the finite rectangle gives the best approximation to compute the compression involved in the case of a cylinder between two planes. That is, the finite rectangle relation seems to be the one that would be of more practical use in the metrology field.

CONCLUSIONS

Since the contacting case of a cylinder between two planes has such widespread application in the science of metrology one needs to be able to apply elastic corrections for the most precise determination of a diameter of the cylinder. There are many prediction formulae depending upon the assumptions made.

This paper has assembled the majority of the equations in existence in the literature and a master computer program was written to evaluate their agreement over the range normally used in precision metrology. Figures 8,9 and 10 give the results of three equations 62,55 and 82 which are representative of the three basic types of cylinder-plane equations discussed on page 4.

Eight of the equations were evaluated over the range 0.5 - 2.5 lb force and cylinder diameter varying from .001 to 1.00 inch. The slopes (micro-inches/lb) of five theoretical equations are plotted in Figure 11. The slopes of three empirical equations over the same range are shown in Figure 12. To obtain the deformation for any one equation

a) Find the diameter of the cylinder,

b) Read the corresponding slope from the vertical axis, and

c) Multiply that slope by the measuring force in pounds to obtain the total deformation.

Note: Keep in mind that these slopes are for the total deformation of a cylinder between two plane surface. For example the values are twice the values given by the equations listed. Also the values for Poisson's ratio and Young's Modulus are for 52100 bearing steel in which

$$\nu = 0.295 \text{ and } E = 29.0 \times 10^6 \text{ psi.}$$

The length of contact between cylinder and plane is 0.375 inches in all cases.

One important fact to remember in all contact problems is to keep the measuring force less than that force which will cause permanent deformation to the object under test. For the case of a cylinder against a plane a nomograph has been developed for easy computation of the stress (Figure 13).

In certain cases where the compression needs to be known to better than 5% then the only practical way to account for surface finish, the actual material parameters, and the exact geometry of the cylinder and flat would be to determine the compression experimentally. For the load range normally used in the metrology field it appears that Equation 55, based on the finite-rectangle approach, gives the best approximation to the compression obtained between a cylinder and flat (Figure 14).

On a practical basis, since most line compression is small an error of 10 - 30% is not enough to give one insomnia because the measurement uncertainty (standard deviation) will likely be the same order of magnitude if not larger than the actual deformation of the system.

SUMMARY OF EQUATIONS

b = Hertzian half-width of contact

$$b = \sqrt{\frac{4R(\lambda_1 + \lambda_2)P}{L}}$$

• Equations for Compression of Cylinder and Plane in Contact

$$(55) \quad \delta = \frac{P}{L}(\lambda_1 + \lambda_2)\left[1.00 + \ln \frac{L^3}{(\lambda_1 + \lambda_2)PR}\right]$$

$$(62) \quad \delta = \frac{P}{L}(\lambda_1 + \lambda_2)\left[-.608205 + 1.5 \ln \frac{L^3}{(\lambda_1 + \lambda_2)PR}\right]$$

$$(79) \quad \delta = \frac{P}{L}\left(\frac{1 + \nu_1}{\pi E_1} + \frac{1 + \nu_2}{\pi E_2}\right)\left[1.00 + \ln \frac{L}{PR\left(\frac{1 + \nu_1}{\pi E_1} + \frac{1 + \nu_2}{\pi E_2}\right)}\right]$$

Empirical (Steel Cylinder--Steel Flat)

$$(82)^2 \quad \delta = 0.2207 \frac{P}{L} D^{-1/3} (10^{-6}) \text{ inches}$$

$$(83) \quad \delta = 0.00534\, L^{-.765} D^{-.053} P^{.940} (10^{-4}) \text{ inches}$$

$$(84) \quad \delta = 0.0091\, L^{-.73} D^{-.08} P^{.6666} (10^{-4}) \text{ inches}$$

- Equations for Deflection of Elastic Half-Space

(68) $$F = \frac{P}{L}(\lambda_1 + \lambda_2)\left[-.193145 + \frac{1}{2}\ln\frac{L^3}{(\lambda_1 + \lambda_2)PR}\right]$$

(72) $$\delta_F = \frac{P}{L}(\lambda_1 + \lambda_2)\left[-1.33787 + \frac{1}{2}\ln\frac{L^3}{(\lambda_1 + \lambda_2)PR}\right]$$

- Equations for Compression of A Cylinder Between "Rigid" Planes

(70) $$\delta_c = 2\frac{P}{L}\left(\frac{1-\nu^2}{\pi E}\right)\left[.193145 + \frac{1}{2}\ln\frac{LR}{(\lambda_1 + \lambda_2)P}\right]$$

(75) $$\delta_c = 2\frac{P}{L}\left(\frac{1-\nu^2}{\pi E}\right)\left[.3333 + \frac{1}{2}\ln\frac{LR}{(\lambda_1 + \lambda_2)P}\right]$$

(77) $$\delta_c = 2\frac{P}{L}\left(\frac{1-\nu^2}{\pi E}\right)\left[.407 + \ln\frac{2R}{b}\right]$$

(78) $$\delta_c = 2\frac{P}{L}\left(\frac{1-\nu^2}{\pi E}\right)\left[.333 + \ln\frac{2R}{b}\right]$$

- Equations for Normal Approach of Two Planes with Cylinder Between the Planes

(73) $$\delta = \frac{P}{L}(\lambda_1 + \lambda_2)\left[-1.14473 + \ln\frac{L^2}{(\lambda_1 + \lambda_2)P}\right]$$

Equation 73 is the sum of Eqs. 70 & 72

(76) $$M = 4\frac{P}{L}\left(\frac{1-\nu^2}{\pi E}\right)\ln\frac{\pi E L^2}{P(1-\nu^2)}$$

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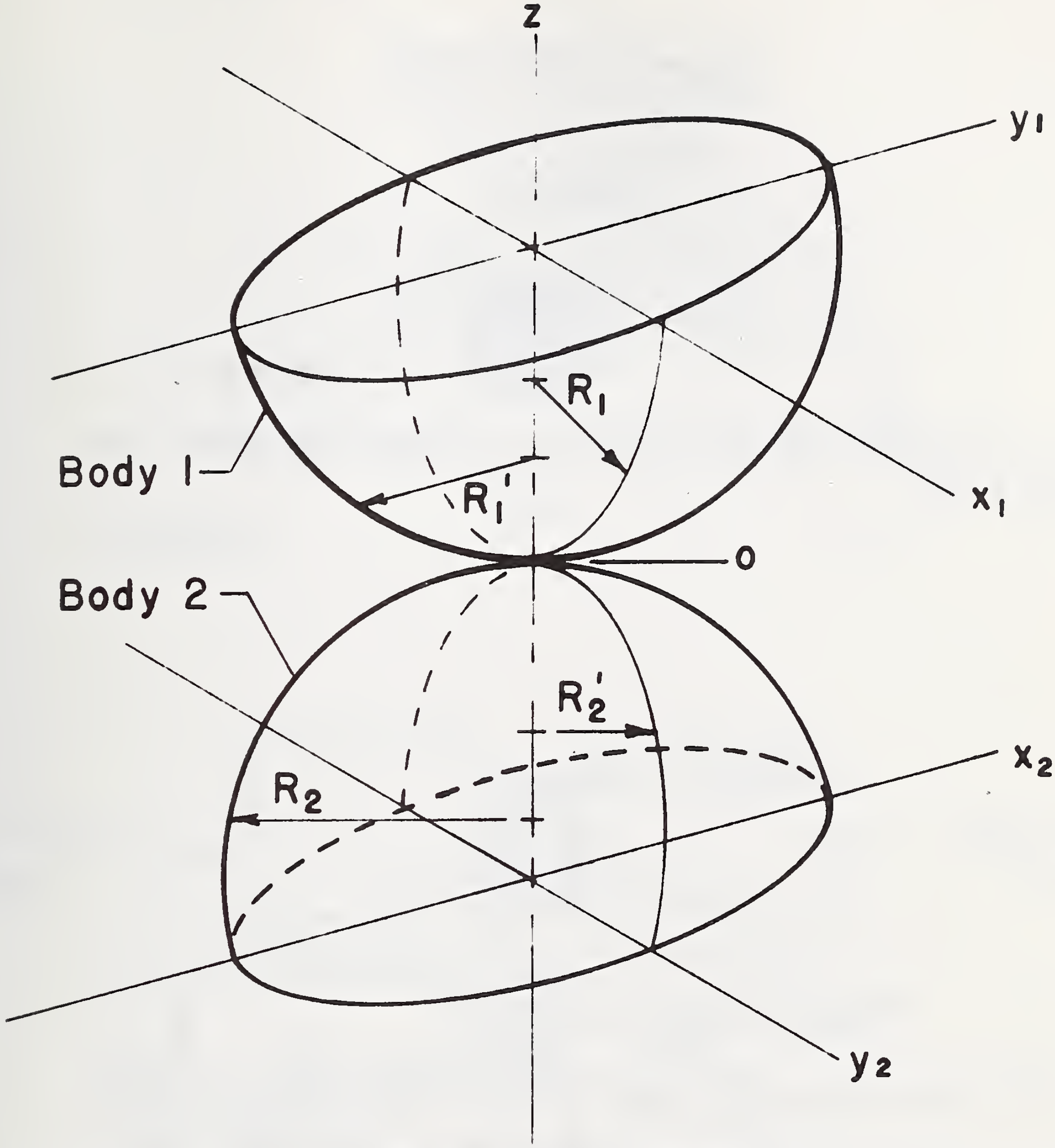
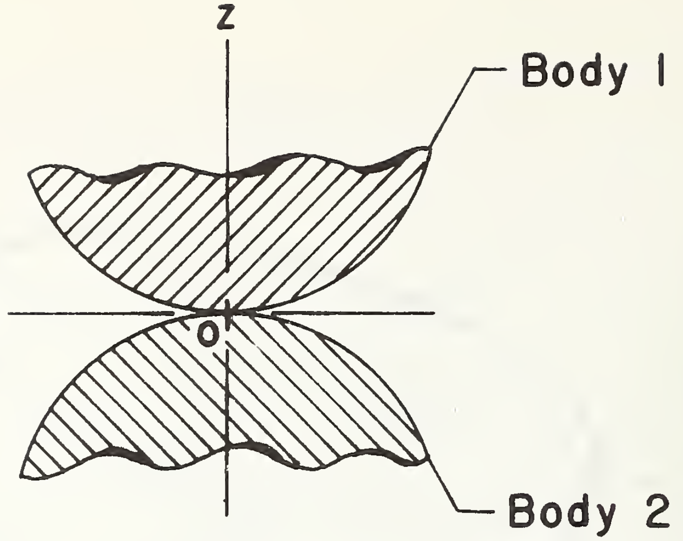


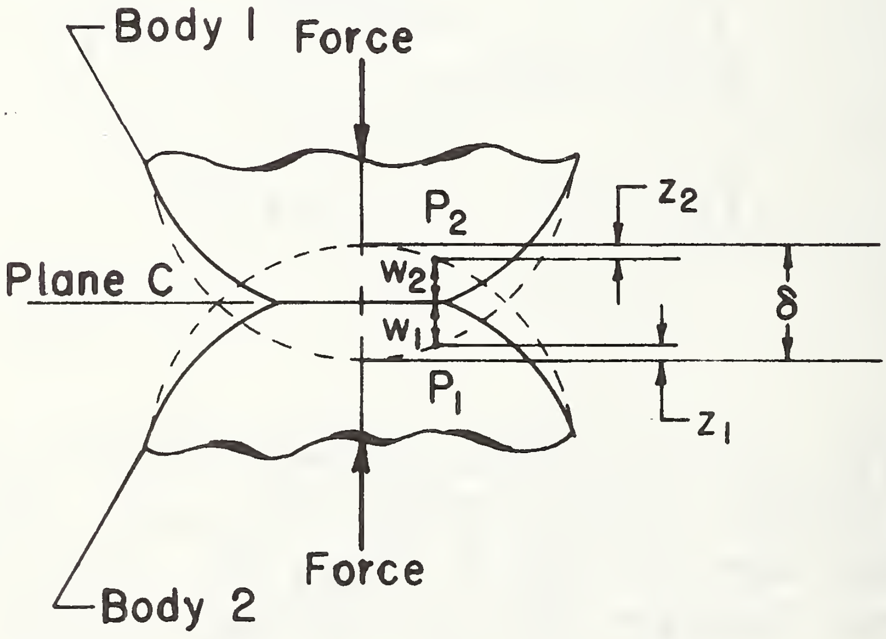
Geometry of the contact between elliptic paraboloids with principal axes of body 1 (x_1, y_1, z) and the principal axes of body 2 (x_2, y_2, z). The angle ω is the angle between the $x_1 z$ and $x_2 z$ planes, i.e. the principal radii of curvatures R_1 and R_2.

FIGURE 1



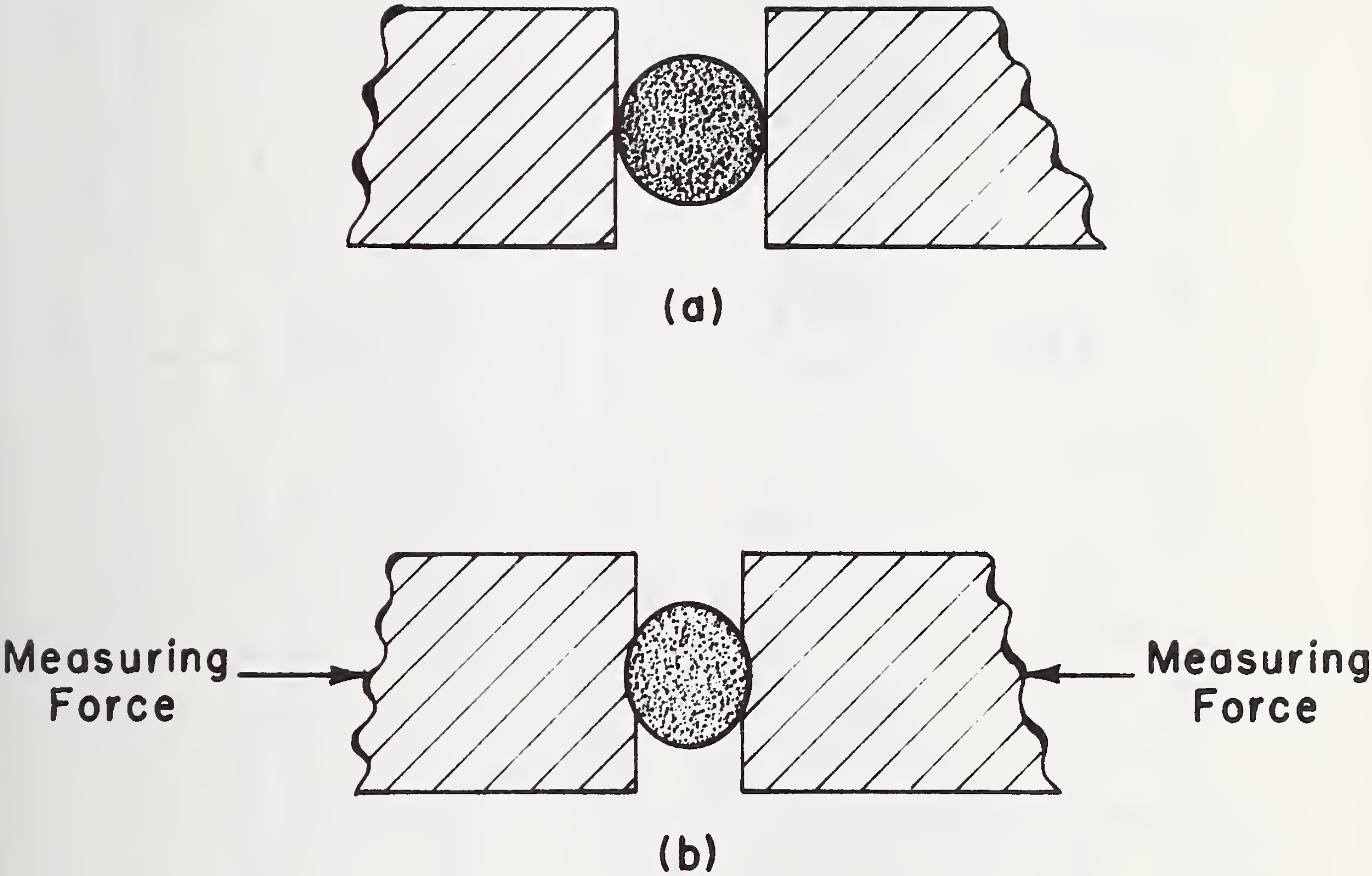
Cross-section of two surfaces near the point of contact O.

FIGURE 2



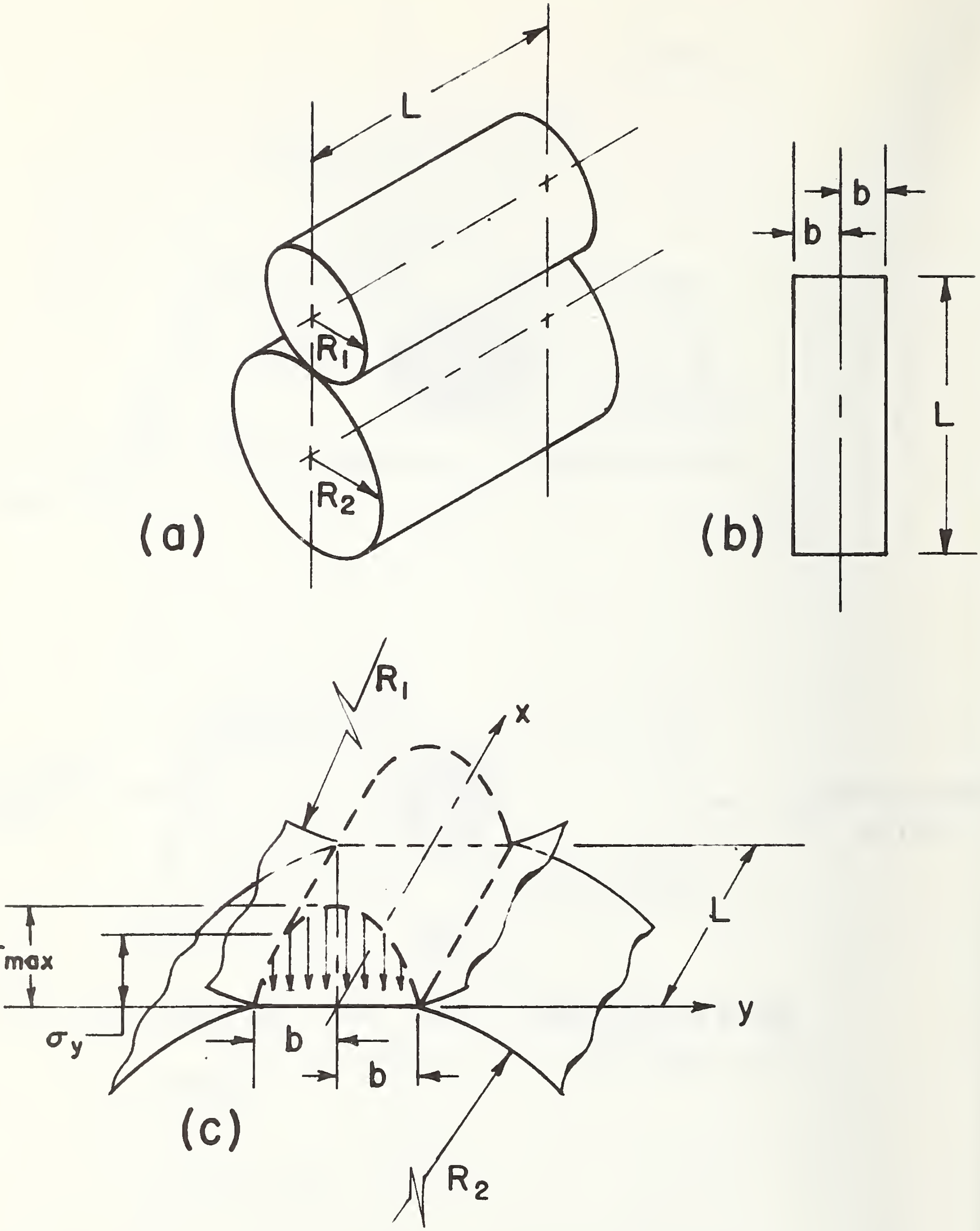
Geometry of deformed bodies. Broken lines show the surface as they would be in the absence of deformation. Continuous lines show the surfaces of the deformed bodies

FIGURE 3



MEASUREMENT OF THE DIAMETER OF A CYLINDER

FIGURE 4



Contact geometry of two parallel cylinders.

FIGURE 5

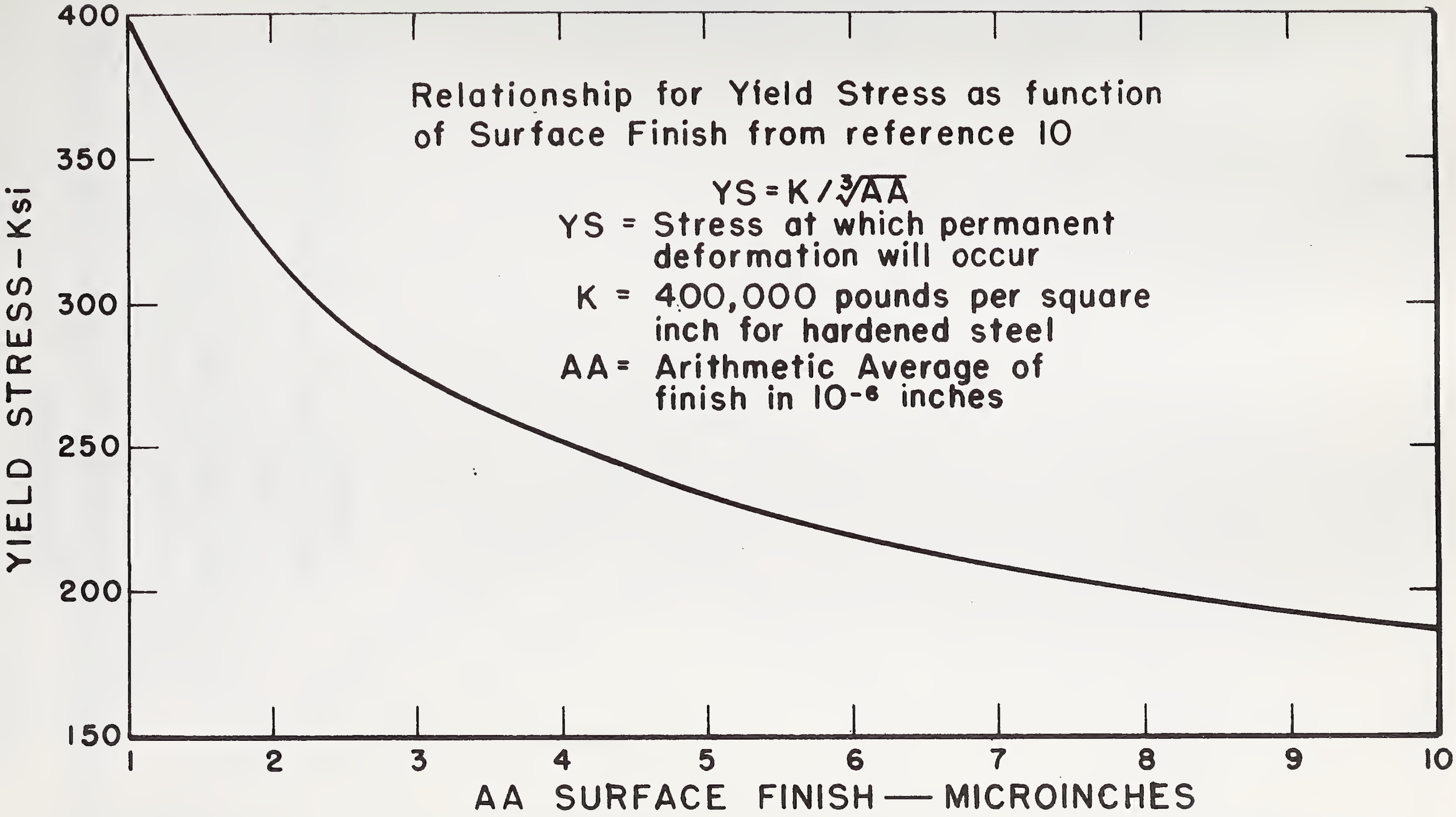
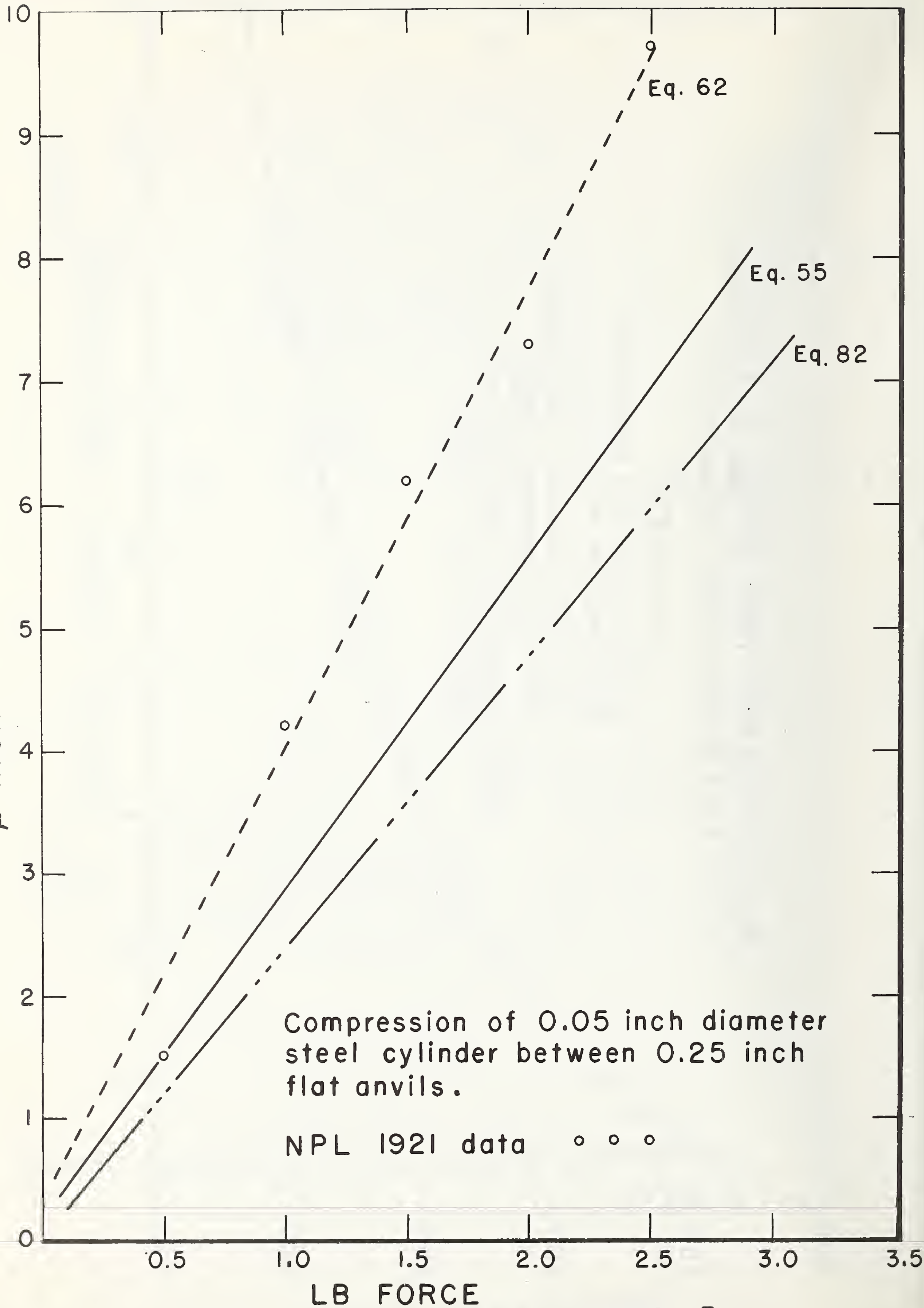

YIELD STRESS-Ksi
400
350
300
250
200
150
1
2
3
4
5
6
7
8
9
10
AA SURFACE FINISH — MICROINCHES
Relationship for Yield Stress as function of Surface Finish from reference 10
$YS = K / \sqrt[3]{AA}$
YS = Stress at which permanent deformation will occur
K = 400,000 pounds per square inch for hardened steel
AA = Arithmetic Average of finish in 10^{-6} inches

FIGURE 6



Eq. 62
Eq. 55
Eq. 82
Compression of 0.05 inch diameter steel cylinder between 0.25 inch flat anvils.
NPL 1921 data
μ INCH
LB FORCE
0
1
2
3
4
5
6
7
8
9
10
0.5
1.0
1.5
2.0
2.5
3.0
3.5

FIGURE 7

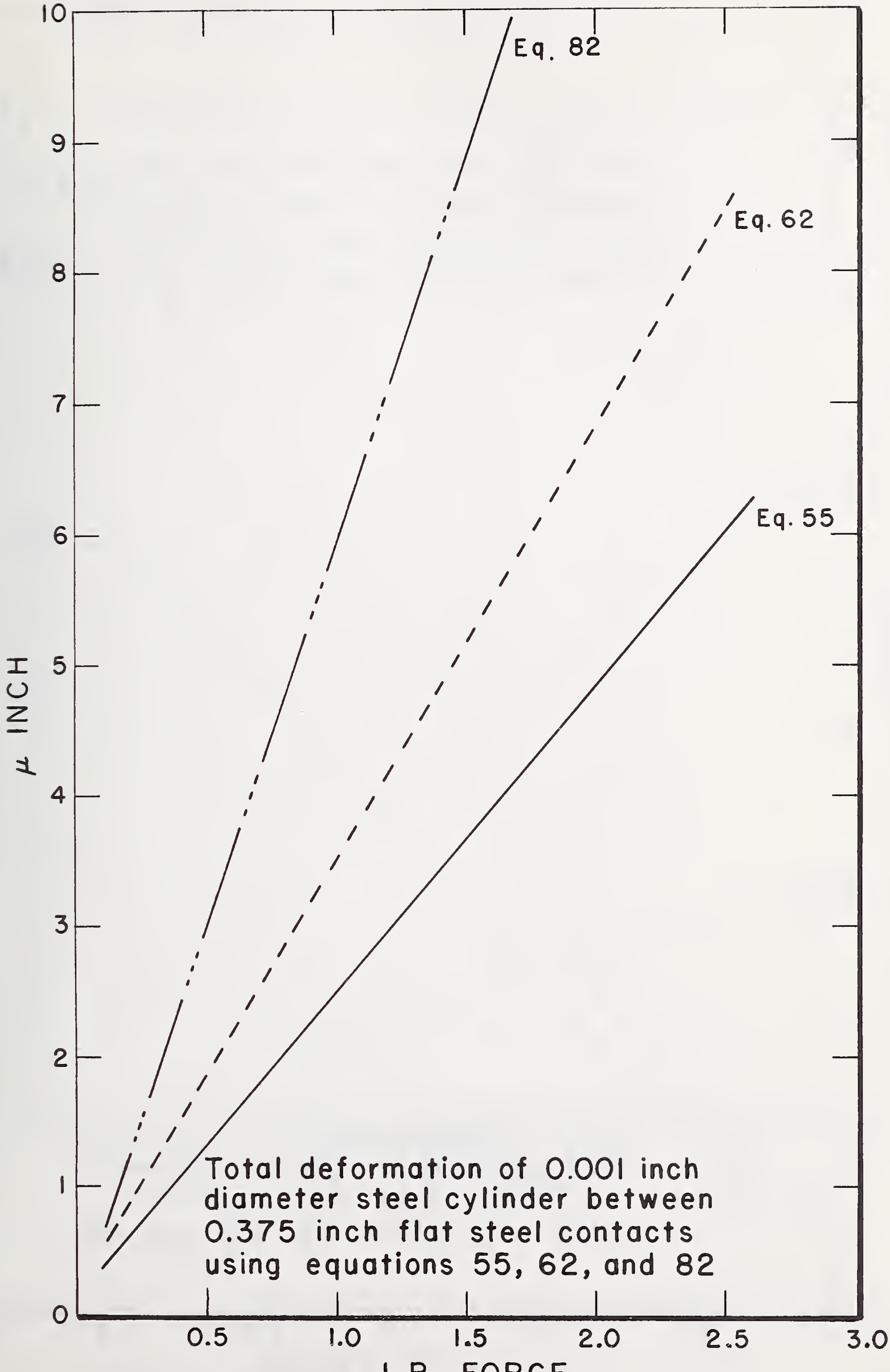
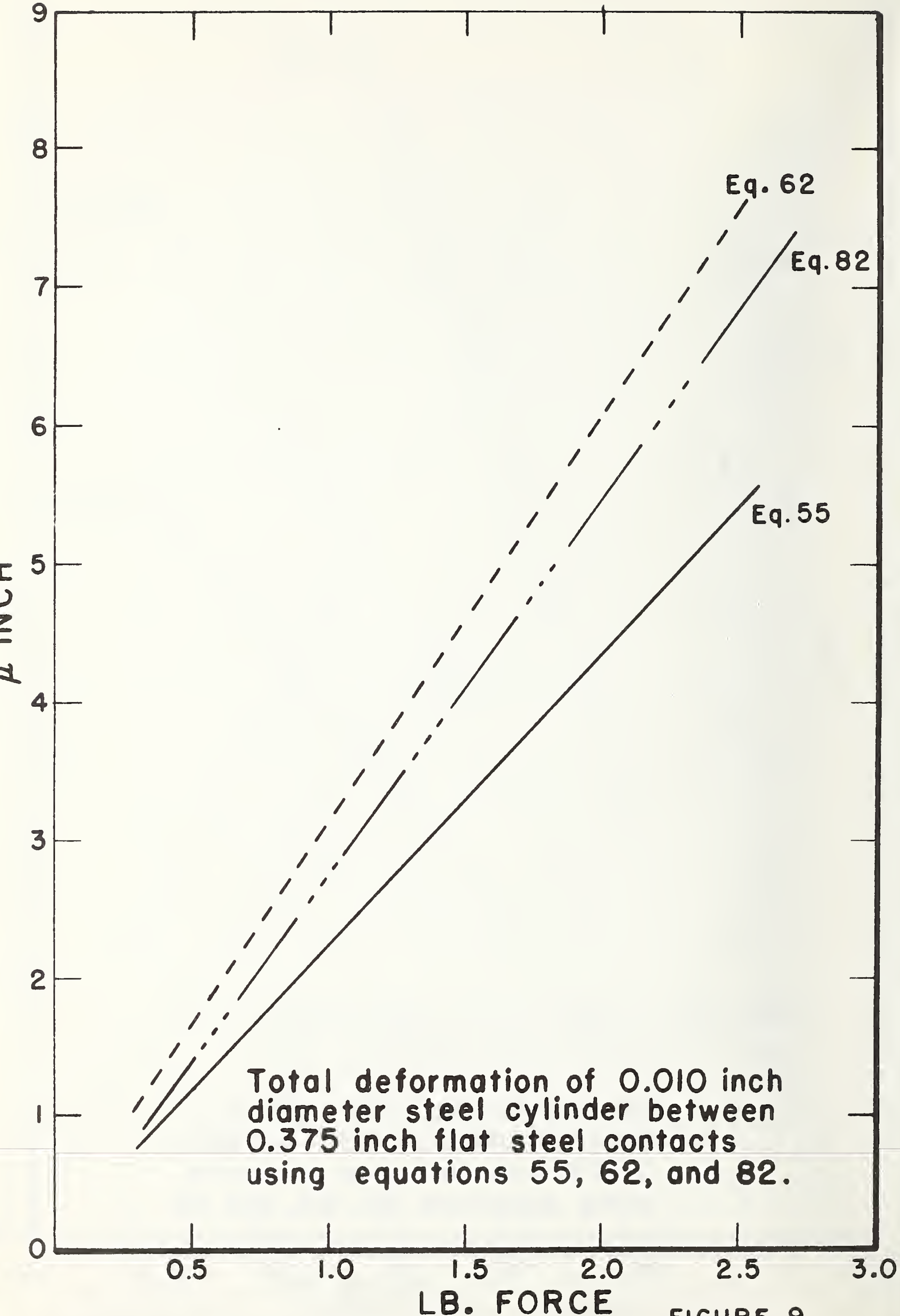
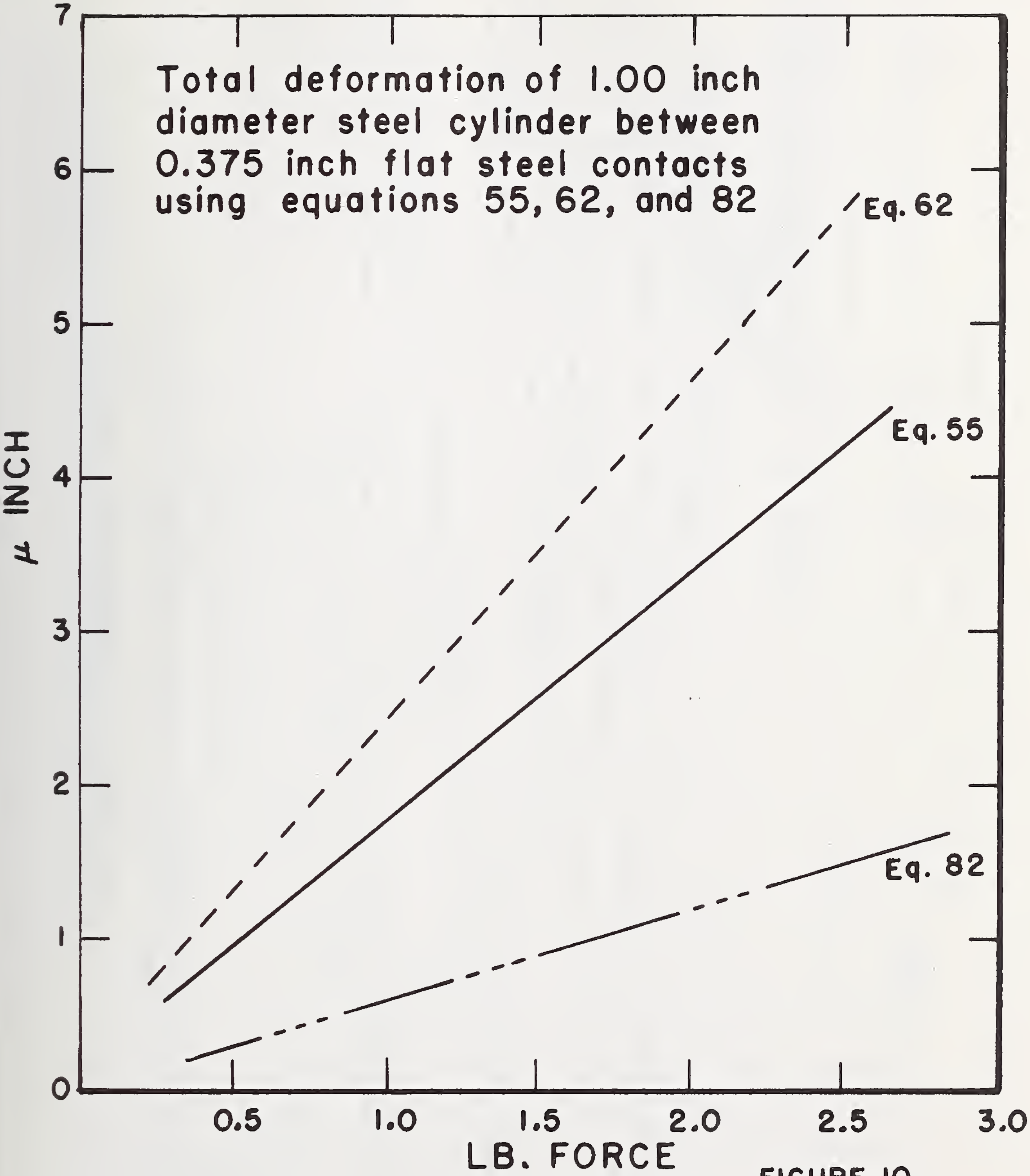

10
9
8
7
6
5
4
3
2
1
0
μ INCH
Eq. 82
Eq. 62
Eq. 55
Total deformation of 0.001 inch
diameter steel cylinder between
0.375 inch flat steel contacts
using equations 55, 62, and 82
0.5
1.0
1.5
2.0
2.5
3.0
LB. FORCE

FIGURE 8



9
8
7
6
5
4
3
2
1
0
μ INCH
Eq. 62
Eq. 82
Eq. 55
Total deformation of 0.010 inch diameter steel cylinder between 0.375 inch flat steel contacts using equations 55, 62, and 82.
0.5
1.0
1.5
2.0
2.5
3.0
LB. FORCE

FIGURE 9


Total deformation of 1.00 inch diameter steel cylinder between 0.375 inch flat steel contacts using equations 55, 62, and 82
Eq. 62
Eq. 55
Eq. 82
μ INCH
0
1
2
3
4
5
6
7
0.5
1.0
1.5
2.0
2.5
3.0
LB. FORCE
FIGURE 10

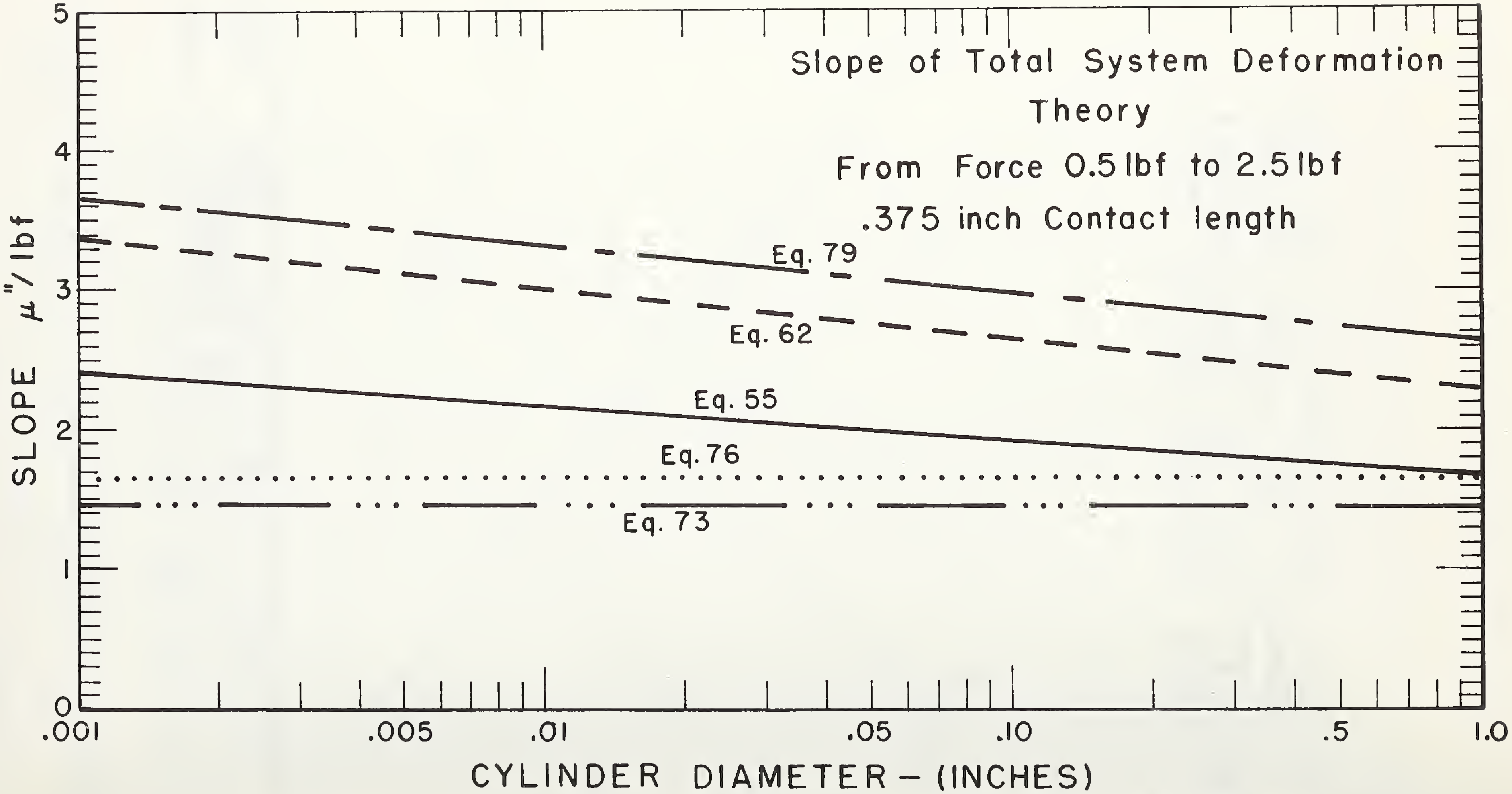
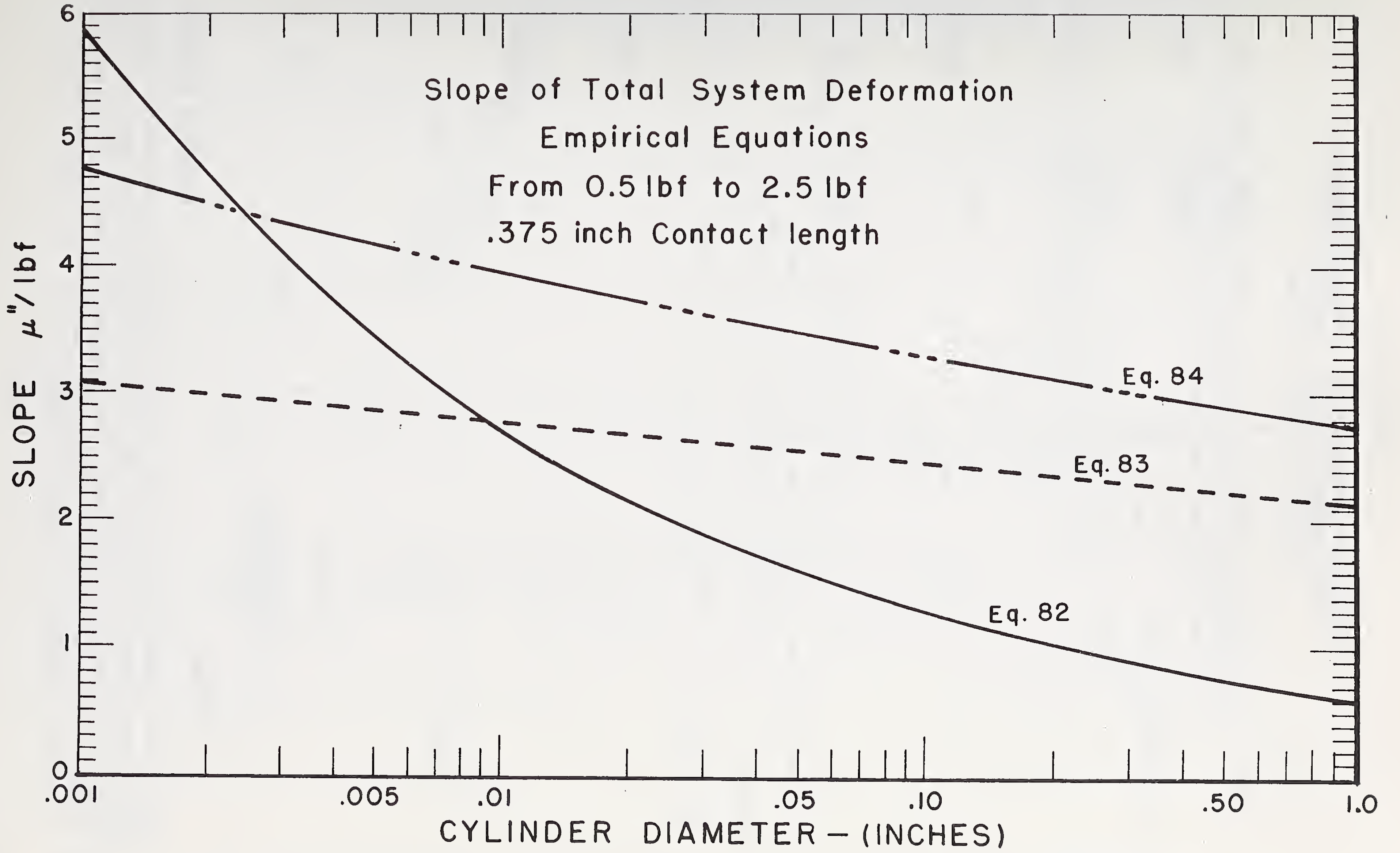

Slope of Total System Deformation
Theory
From Force 0.5 lbf to 2.5 lbf
.375 inch Contact length
Eq. 79
Eq. 62
Eq. 55
Eq. 76
Eq. 73
SLOPE μ"/lbf
5
4
3
2
1
0
.001
.005
.01
.05
.10
.5
1.0
CYLINDER DIAMETER – (INCHES)

FIGURE 11



Slope of Total System Deformation
Empirical Equations
From 0.5 lbf to 2.5 lbf
.375 inch Contact length
Eq. 84
Eq. 83
Eq. 82
SLOPE μ''/lbf
0
1
2
3
4
5
6
.001
.005
.01
.05
.10
.50
1.0
CYLINDER DIAMETER — (INCHES)

FIGURE 12

FIGURE 13 (a)

NOMOGRAM FOR COMPUTATION OF MAXIMUM STRESS ENCOUNTERED IN CONTACT PROBLEM OF A CYLINDER AGAINST A PLANE

Purpose: To determine whether the force one uses in the measurement of the diameter of a steel cylinder will exceed the yield point and cause permanent deformation to the cylinder.

Governing Equation: $$\sigma_{max} = \frac{2P}{\pi L b}$$

where P = measuring force

L = contact length between cylinder and plane

b = Hertzian half-width of contact

$$b = \left[\frac{4R(\lambda_1+\lambda_2)P}{L}\right]^{1/2}$$

R = Radius of cylinder

$$\lambda_i = \frac{1 - \nu_i^2}{\pi E_i}$$

ν_i = Poisson ratio for one material

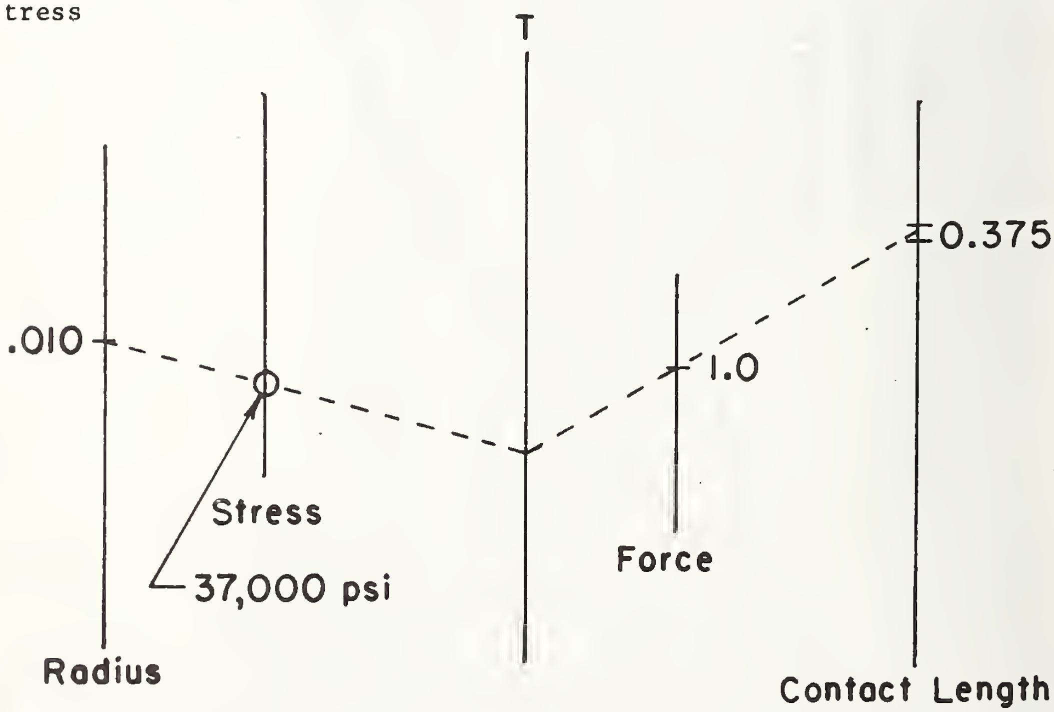
E_i = Modulus of elasticity for one material.

The nomograph was developed for 52100 steel in which

$\nu = 0.295$ and $E = 29.0 \times 10^6$ psi

Example:

If Force = 1 lb, Contact Length = 0.375 inch, Radius = 0.01 inch--Find maximum stress



(1) Locate measuring force and length of contact on the appropriate scales and connect these points until the turning axis T is intersected; (2) Locate the radius and connect with the intersected T axis; (3) Read maximum stress to be expected within the contact area; (4) Compare maximum stress read from nomograph with curve in Figure 6.

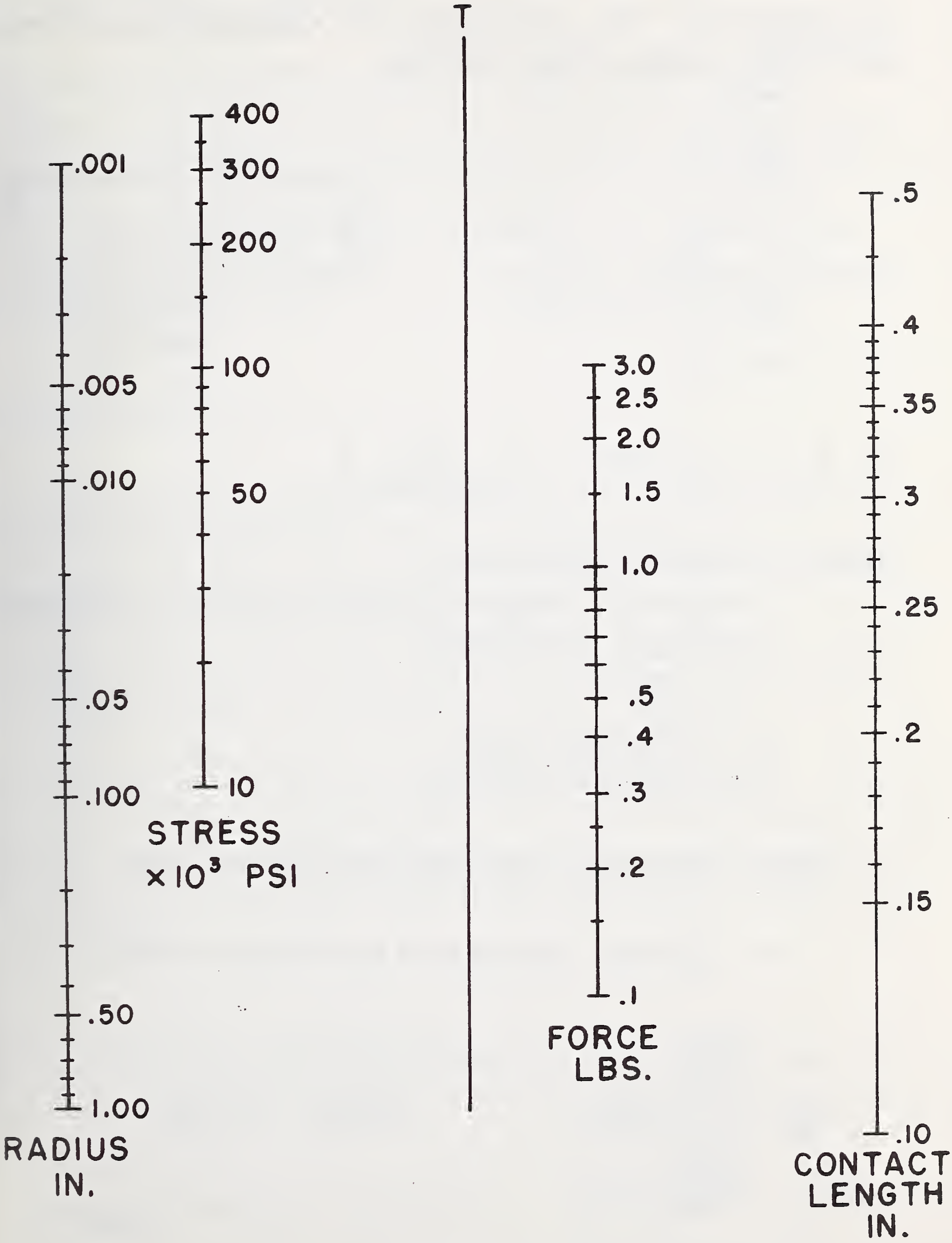
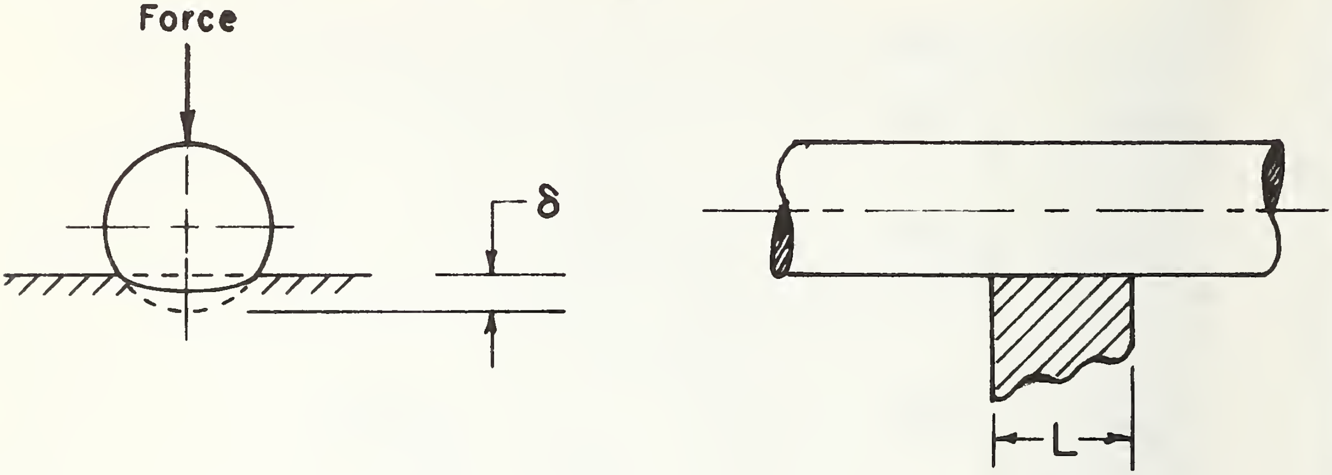

T
.001
.005
.010
.05
.100
.50
1.00
RADIUS IN.
400
300
200
100
50
10
STRESS $\times 10^3$ PSI
3.0
2.5
2.0
1.5
1.0
.5
.4
.3
.2
.1
FORCE LBS.
.5
.4
.35
.3
.25
.2
.15
.10
CONTACT LENGTH IN.

FIGURE 13 (b)

Equation For The Calculation Of Compression Between A Cylinder And Plane Surface



$$\delta = \frac{P}{L}(\lambda_1 + \lambda_2)\left[1.00 + \ln \frac{L^3}{(\lambda_1 + \lambda_2)PR}\right]$$

where P = measuring force

L = contact length between cylinder and plane

R = radius of cylinder

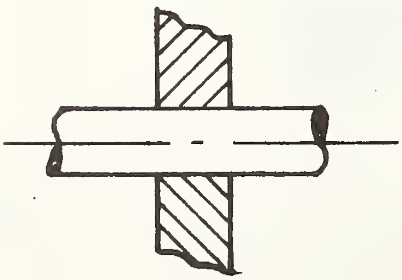
$\lambda_1 = \dfrac{1-\upsilon_1^2}{\pi E}$

υ_i = Poisson's ratio

E_i = Young's modulus

When material of cylinder and flat are same, $\lambda_1 = \lambda_2$

— Cylinder Between Two Parallel Planes —



$$\delta_{Total} = \delta_1 + \delta_2$$

FIGURE 14

FORM NBS-114A (1-71)

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16. ABSTRACT (A 200-word or less factual summary of most significant information. If document includes a significant bibliography or literature survey, mention it here.)

The measurement of a diameter of a cylinder has widespread application in the metrology field and industrial sector. Since the cylinder is usually placed between two flat parallel anvils, one needs to be able to apply corrections, to account for the finite measuring force used, for the most accurate determination of a diameter of the cylinder.

An extensive literature search was conducted to assemble the equations which have been developed for deformation of a cylinder to plane contact case. There are a number of formulae depending upon the assumptions made in the development. It was immediately evident that this subject has been unexplored in depth by the metrology community, and thus no coherent treatise for practical usage has been developed.

This report is an attempt to analyze the majority of these equations and to compare their results within the force range normally encountered in the metrology field. Graphs have been developed to facilitate easy computation of the maximum compressive stress encountered in the steel cylinder-steel plane contact case and the actual deformation involved.

Since the ultimate usefulness of any formula depends upon experimental verification, we have compiled results of pertinent experiments and various empirical formulae. A complete bibliography has been included for the cylinder-plane contact case for the interested reader.

17. KEY WORDS (Alphabetical order, separated by semicolons)

Contact stresses; cylinder measurement; deformation; Hertzian contact problem; line contact; mechanical measurement; metrology.

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