DEPARTMENT OF COMMERCE
BUREAU OF STANDARDS
George K. Burgess, Director

TECHNOLOGIC PAPERS OF THE BUREAU OF STANDARDS, No. 332
[Part of Vol. 21]

STATICAL HYSTERESIS IN THE FLEXURE OF BARS

BY

G. H. KEULEGAN, Assistant Physicist
Bureau of Standards

November 4, 1926

PRICE 10 CENTS
$1.25 PER VOLUME ON SUBSCRIPTION
Sold only by the Superintendent of Documents, Government Printing Office
Washington, D. C.

WASHINGTON
GOVERNMENT PRINTING OFFICE
1926

STATICAL HYSTERESIS IN THE FLEXURE OF BARS

By G. H. Keulegan

Boltzmann's theory of elastic time effects ("elastische Nachwirkung") does not always account for the hysteresis observed in the deformation of an elastic body during a closed load cycle. In fact, there are materials for which the component of hysteresis which is due to the time effects constitutes only a small portion of the observed hysteresis. Thus other supplementary theories are required to account for that portion of the hysteresis which Boltzmann's theory does not explain.

In this paper is presented an additional theory of hysteresis on the assumption that the stress-strain relation of elastic bodies in a cyclic state for a stress cycle σ_1 to $\sigma_1+\sigma_m$ and back to σ_1, instead of being given by a straight line, is given by a symmetrical loop when the time effects are negligible. Hysteresis arising from this source is called statical hysteresis.

The theory is developed to show how the statical hysteresis of a rectangular bar, one end of which is clamped and the other end loaded, depends on the length of the bar and on the load.

CONTENTS

	Page
I. Introduction	145
II. A theory of statical hysteresis	148
III. Application of the theory to the flexure of bars	150
IV. Experiments and data	153
1. Description of bar and experiments	153
2. Determination of lengths of bar in flexure	155
V. Discussion of data	156
1. Energy loss due to hysteresis	156
2. Law of variation of width in hysteresis loops	158
VI. Determination of the statical hysteresis modulus	158
VII. Summary	162

I. INTRODUCTION

It appears that Vicat was the first to note, in 1834, the phenomenon of recovery in the residual deformation of a stressed body, which is generally observed subsequent to unloading. W. Weber discovered, in 1835, what may be regarded as a converse property of the above; that is, the phenomenon of drift in the deformation which is generally observed in a body under constant load.[1] Boltzmann[2] brought the

[1] A. E. H. Love, Mathematical Theory of Elasticity, 3d ed., p. 114.

[2] L. Boltzmann, Wien. K. Akad. d. Wiss. Stiz., **70**, 1874; or Wissenschaftliche Abhandlungen von L. Boltzmann, **1**, p. 616.

two above-mentioned phenomena into genetic correlation for the first time in 1874 by stating his well-known theory of "elastic afterworking," according to which drift and recovery are mutually dependent, so that when one of them exists the other exists also, and hysteresis will then necessarily be present in a deformation associated with a closed load cycle, which lasts any finite length of time. This correlation of drift, recovery, and hysteresis is effected by the introduction of a heredity function. Once the heredity function for a given body is known, let us say, from the study of drift, then the hysteresis can

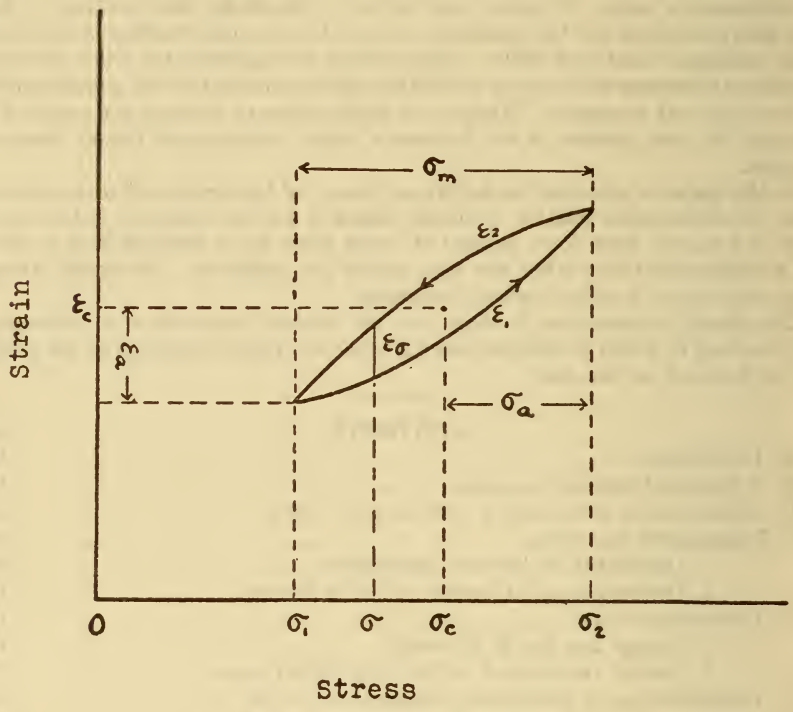

FIG. 1.—*Stress-strain diagram*

be evaluated for any prescribed closed load cycle. For the purpose of this present paper the author will refer to the hysteresis arising from this source as hereditary hysteresis.

Ewing [3] in describing some experiments on steel wires in tension gives an account of a type of hysteresis other than the hereditary one and arising from a specific departure from Hooke's law at stresses within the so-called proportional limit of bodies; that is, even though the heredity function of a body may be zero, the body may still exhibit hysteresis in a closed load cycle. In other words, the stress-strain curve for increasing load differs from that for decreasing load and forms with it a closed loop which is independent of the time occupied

[3] J. A. Ewing, Brit. Assn. Reports, p. 502; 1889.

by the load cycle. In order that this loop may be closed, it is necessary that a cyclic state be established between an upper and a lower limit of stress. In the present paper the author will refer to the hysteresis arising from this source as statical hysteresis. In the literature of the subject the commonly used term is "elastic hysteresis"; but this is rather an unfortunate use of the term.

Therefore, in general, hysteresis as observed will consist of two parts—one the hereditary hysteresis and the other the statical hysteresis. Hersey[4] states that for a particular heredity function and for periodic deformations the hereditary hysteresis dies out quickly; that is, by repeated application of load or by the establishment of a cyclic state of regular period, the hereditary hysteresis will be eliminated to a great degree. Now, the relative magnitudes of these two types of hysteresis are not the same for all materials. If one proposes to study the laws of statical hysteresis in a statical test, and if the hereditary hysteresis is considerable, the latter can be evaluated and subtracted from the observed hysteresis. Such a procedure, of course, is not necessary when the observed hysteresis consists mainly of statical hysteresis, and such is the case if the residual deformation at zero load is small in comparison to the maximum hysteresis observed in the closed load cycle. Warburg and Heuse [5] have found that in a beam of hard German silver the statical is ten times as great as the hereditary hysteresis.

From the viewpoint of Ewing it is of great importance to know what the closed stress-strain curves are between the extremes of stress σ_1 and σ_2.[6] Of late some researches have been made in this direction. Hopkinson and Williams [7] found, in subjecting a rod of mild steel to alternating axial stresses and noting the temperature fall along the rod, that the energy loss due to hysteresis in a closed load cycle was proportional to the fourth power of the stress range ($\sigma_m{}^4$). In making a comparison between the energy loss as obtained from dynamic and static tests they assumed in the static tests that the loop between $+\frac{\sigma_m}{2}$ and $-\frac{\sigma_m}{2}$ was of a lenticular shape and therefore arrived at the energy value of the static loop by measuring the hysteresis difference at zero stress and noting the extreme stresses. In such a loop the maximum width is 1.5 times the average width of the loop. Rowett,[8] by causing a steel tube to vibrate torsionally, found that the energy loss per cycle was proportional to the third power of the stress range. He found from a static test that in a closed load cycle in which the maximum torsional loads were equal

[4] M. D. Hersey, J. Wash. Acad. Sci., **11**, p. 149; 1921.
[5] E. Warburg and W. Heuse, Verh. der Deut. Phy. Ges., **17**; 1915.
[6] σ's being either axial or shearing stresses.
[7] B. Hopkinson and G. T. Williams, London, Proc. R. Soc., **87**, p. 502; 1912.
[8] F. E. Rowett, London, Proc. Roy. Soc., **89**, p. 528; 1913–14.

and of opposite signs, the maximum width was twice the average width of the hysteresis loop.

Guest and Lea[9] in their paper on the torsional deformation of mild steel give considerable data on hysteresis. It appears from these data that for very small loads and with closed cycles in which the extreme loads are equal numerically but of opposite sign the maximum hysteresis is 1.5 times the average hysteresis.

The above researches, although very illuminating, do not provide a relation whereby, in torsion experiments, for example, the effect on the magnitude of the hysteresis of changing the length of rod can be predicted; or, if the hysteresis exhibited by a rod in compression is known, how the hysteresis which will be obtained with the same rod when in flexure can be computed. The author of this paper has made an attempt to formulate an approximate theory of statical hysteresis, applicable to small ranges of stress within the proportionality limit. For the experimental verification of the theory he has chosen as an appropriate method the flexure of a rectangular bar having one end free and the other end fixed. The limiting loads of the cycle are thus 0 and $+L_m$.

II. A THEORY OF STATICAL HYSTERESIS

Consider a cube of elastic material to be in a cyclic state between the extremes σ_1 and σ_2 of the axial stress, the loads being applied on any two opposite faces. The mechanism whereby the cyclic state is established consists in the repeated variation of stress from σ_1 to σ_2. The criterion for the establishment of the cyclic state may be taken as the constancy of ϵ or strain at the extreme stresses of the cycle (see fig. 1). Let the loop given there be the stress-strain diagram of the material after the establishment of the cyclic state. The lower and the upper limits of σ are $\sigma_1=\sigma_c-\sigma_a$ and $\sigma_2=\sigma_c+\sigma_a$; the corresponding strains are $\epsilon_c-\epsilon_a$ and $\epsilon_c+\epsilon_a$. σ_c and ϵ_c refer to the middle point of the straight line joining the extremities of the loop. Define σ_m as the range of the stress variation; then[10]

$$\sigma_m=|\sigma_1-\sigma_2|=2\sigma_a$$

Let ϵ_1 be the value of ϵ for increasing σ and ϵ_2 for decreasing σ. Define

$$\eta_1=\epsilon_1-\epsilon_c,$$

$$\eta_2=\epsilon_2-\epsilon_c,$$

and

$$S=\sigma-\sigma_c.$$

[9] J. J. Guest and F. C. Lea, London Proc. Roy. Soc., **93**, p. 313; 1916-17.

[10] σ_m and σ_a are positive quantities.

Then if there is no hereditary hysteresis

$$\eta_1 = f_1(\sigma_c, \sigma_a, s)$$

and

$$\eta_2 = f_2(\sigma_c, \sigma_a, s).$$

If it is assumed that the hysteresis loop has central symmetry about (σ_c, ϵ_c) as a center when σ_1 and σ_2 are far within the so-called proportional limit of the material, then

$$f_1(\sigma_c, \sigma_a, s) + f_2(\sigma_c, \sigma_a, -s) = 0 \tag{1}$$

We know that approximately

$$\eta_1 = \bigcirc = \eta_2 = \bigcirc = \kappa s$$

where κ is a constant. Hence, it is permissible to expand η in an ascending power series, as follows:

$$\eta_1 = f_1(\sigma_c, \sigma_a, s) = c + \alpha s + \beta s^2 + \gamma s^3 + \cdots \tag{2}$$

and according to (1)

$$f_2(\sigma_c, \sigma_a, -s) = -c - \alpha s - \beta s^2 - \gamma s^3 + \cdots$$

Hence

$$\eta_2 = f_2(\sigma_c, \sigma_a, s) = -c + \alpha s - \beta s^2 + \gamma s^3 + \cdots \tag{3}$$

It will be further assumed that when σ_1 and σ_2 are far within the so-called proportional limit of the material powers of s higher than the second are negligible.[11] Now, when $s = \sigma_a$, $\eta_1 = \eta_2 = \epsilon_a$; this requires that

$$c = -\beta \sigma_a^2 \tag{4}$$

Therefore, finally,

$$\left.\begin{aligned} \eta_1 &= \alpha s + \beta(s^2 - \sigma_a^2) \\ \text{and} \quad \eta_2 &= \alpha s - \beta(s^2 - \sigma_a^2) \end{aligned}\right\} \tag{5}$$

Denote by ϵ_σ the difference $\epsilon_2 - \epsilon_1$; hence

$$\epsilon_\sigma = 2\,\beta(\sigma_a^2 - s^2) \tag{6}$$

The energy loss per unit volume δE_σ, due to the statical hysteresis for the load variation from σ to $\sigma_c + \sigma_a$ and back to σ, is

$$\begin{aligned} \delta E_\sigma &= \int_\sigma^{\sigma_a+\sigma_c} \epsilon_\sigma d\sigma = \int_{s+\sigma_c}^{\sigma_a+\sigma_c} \epsilon_\sigma d(s+\sigma_c) = \int^{\sigma_a} \epsilon_\sigma ds \\ &= \frac{4\beta\sigma_a^3}{3} - 2\beta\sigma_a^3\left[\frac{s}{\sigma_a} - \frac{s^3}{3\sigma_a^3}\right] \end{aligned} \tag{7}$$

[11] It is conceivable that there may be materials for which β vanishes or γs^3 is far more predominant than βs^2. Such materials are not considered here.

The energy loss per unit volume for the load variation from $\sigma_1 = \sigma_c - \sigma_a$ to $\sigma_2 = \sigma_c + \sigma_a$ and back to $\sigma_1 = \sigma_c - \sigma_a$ is

$$E_o = \frac{8\,\beta\,\sigma_a^3}{3} = \frac{\beta\,|\,\sigma_1 - \sigma_2\,|^3}{3} = \frac{\beta\sigma_m^3}{3} \tag{8}$$

Accordingly, energy loss per unit volume due to statical hysteresis depends merely on the range of the stress variation. To illustrate, the energy loss in the two cycles, one from 0 to $+\sigma_m$ and back to 0 and the other from 0 to $-\sigma_m$ and back to 0, are the same.

It is evident that the determination of β, which we shall call the statical hysteresis modulus, must be made experimentally. This done, the hysteresis can be evaluated for any closed load cycle for any form of body if the stress distribution is known.

Every uniform axial stress can be considered as a superposition of a uniform shearing stress plus a proportionate hydrostatic pressure. Since no hysteresis phenomena are known to be associated with hydrostatic pressure within the stresses corresponding to the limits of axial and shearing stresses which the material can withstand without severe deformation, the laws of this hysteresis loss must be the same whether the stresses are expressed as axial stresses or shearing stresses, neglecting the compensating hydrostatic pressure. However, for the actual value of the constant β it is necessary to specify whether the stress is axial or shearing. These constants will necessarily have the following relation[12]

$$\beta_{\text{shear}} = K\beta_{\text{axial}}$$

where K is a universal constant.

III. APPLICATION OF THE THEORY TO THE FLEXURE OF BARS

We next consider a bar of rectangular cross section, of width b, of thickness $2a$, and of length l. One end of the bar is fixed; the other end carries a load L. We adopt a system of rectangular axes (x–z), wherein the x axis coincides with the central axis of the bar, the z axis is in the plane of flexure, and the origin is at the fixed end of the bar. (See fig. 2.)

For a point P (x, z) the stress σ at the point corresponding to a load L is [13]

$$\sigma = \frac{xzL}{I} \tag{9}$$

[12] This was indicated to the author by Dr. L. B. Tuckerman.

[13] Applied Mechanics, Fuller and Johnston, 2, p. 183.

provided the bar is sufficiently long and thin, so that we can apply the simplification of the Euler-Bernoulli theory (that is, neglect local concentrations of stress and anticlastic curvature). The expression for I in (5) is

$$I=\frac{2}{3}\,ba^3$$

If the maximum load of the load cycle is L_m, the maximum stress σ_m is

$$\sigma_m=\frac{xzL_m}{I} \tag{10}$$

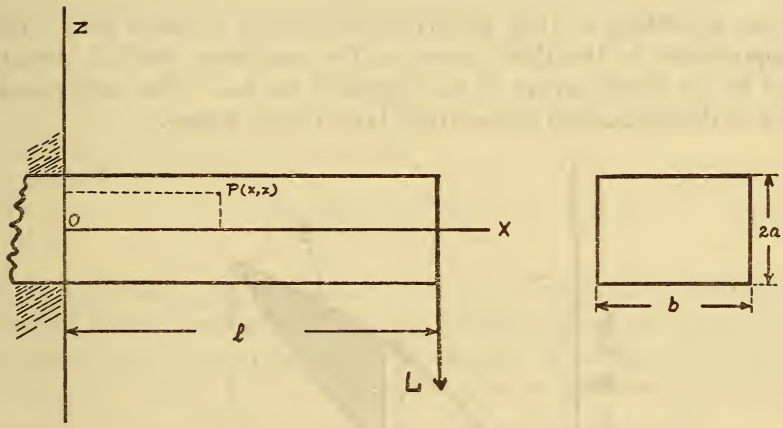

FIG. 2.—*Method of deflecting bar*

To calculate according to equation (7) the energy loss per unit volume, δE_L, at P when the load is increased from L to L_m and then brought back to L_m, we shall first note that in the present case

$$\left.\begin{aligned} s&=\sigma-\frac{\sigma_m}{2} \\ \text{and}\qquad \sigma_a&=\frac{\sigma_m}{2} \end{aligned}\right\} \tag{11}$$

as $\sigma_c-\sigma_a=0$. Thus from (9), (10), and (11) the expression for δE_L becomes after performing the necessary substitutions in equation (7)

$$\delta E_L=\frac{1}{3}\,\beta\,\frac{x^3z^3L_m^3}{I^3}-2\,\beta\,\frac{x^3z^3L_m^3}{I^3}\left[\frac{L^2}{2L_m^2}-\frac{L^3}{3L_m^3}\right] \tag{12}$$

Hence, the energy loss in the whole bar E_L is

$$E_L=\frac{2}{3}\,\beta b\,\frac{L_m^3}{I^3}\int^a\!\!\int_0^1 x^3z^3dxdz-4\beta b\,\frac{L_m^3}{I^3}\left[\frac{L^2}{2L_m^2}-\frac{L^3}{3L_m^3}\right]\int_0^a\!\!\int_0^1 x^3z^3dxdz \tag{13}$$

Putting

$$H=\frac{2}{3}\beta b\frac{L_m^3}{I^3}\int_0^a\int_0^l x^3z^3dxdz \tag{14}$$

$$E_0=H-6H\left[\frac{L^2}{2L_m^2}-\frac{L^3}{3L_m^3}\right] \tag{15}$$

When we put $L=0$ in (15), we get the energy loss H due to the hysteresis in the whole bar for the load cycle 0 to L_m and back to 0. We have from (14)

$$H=\frac{9}{64}\frac{\beta}{b^2a^5}l^4L_m^3=C_1l^4L_m^3 \tag{16}$$

Thus, according to (16), the hysteresis loss in a closed load cycle is proportional to the third power of the maximum load of the cycle and to the fourth power of the length of the bar. The experimental test of this deduction is described later in this paper.

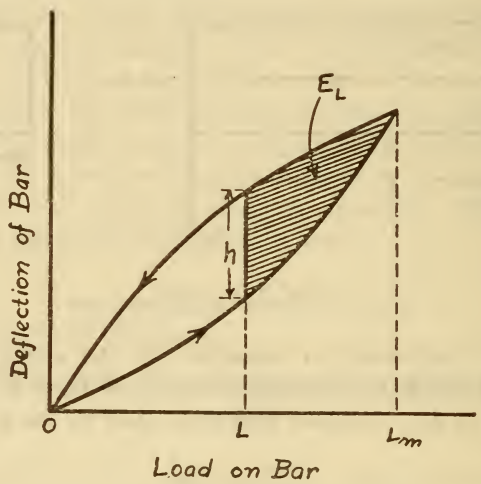

FIG. 3.—*Load-deflection curve of bar*

Next, let us consider the width h of the hysteresis loop of the bar at load L. h is the difference in deflection of the free end of the bar for a given load L for increasing and decreasing loads. Then E_L, the shaded area in Figure 3, will be given by the expression

$$E_L=\int_L^{L_m}hdL \tag{17}$$

Now, E_L can be also written after (15) as an integral of the form

$$E_L=\int_L^{L_m}6H\left[\frac{L}{L_m^2}-\frac{L^2}{L_m^3}\right]dL \tag{18}$$

Equating (17) and (18) we have the equation

$$\int_{L}^{L_m} h dL = \int_{L}^{L_m} 6H\left[\frac{L}{L_m^2} - \frac{L^2}{L_m^3}\right] dL \tag{19}$$

As equation (19) holds for any value of L

$$0 \leqq L \leqq L_m$$

then, necessarily

$$h = \frac{6H}{L_m}\left(1 - \frac{L}{L_m}\right)\frac{L}{L_m} \tag{20}$$

that is, the variation of hysteresis width is parabolic. If h_a is the average width of the hysteresis loop, then

$$h_a = \frac{H}{L_m} \tag{21}$$

so that instead of (16) we may write

$$\frac{h}{h_a} = 6\left(1 - \frac{L}{L_m}\right)\frac{L}{L_m} \tag{22}$$

Now, according to equation (22), the law of variation of width in hysteresis loops is independent of the length of the bar and of the maximum load of the cycle, where by the law of variation of width in hysteresis loops we understand the relation between the ratios $\frac{h}{h_a}$ and $\frac{L}{L_m}$. This deduction will be also tested by experiment.

IV. EXPERIMENTS AND DATA

1. DESCRIPTION OF BAR AND EXPERIMENTS (see fig. 4)

The straight portion of one of the prongs of a tuning fork made of Armco iron was used as a bar. The thickness and the width of the prong used were 0.830 and 1.395 cm, respectively. The chemical composition and the past history of the material are not known.

The prong P was clamped near its base in a holder H firmly attached to a heavy base plate B. On the free end E of the prong was mounted a rectangular collar R, which carries the load L. The deflection of the free end E was measured by a micrometer screw M. The cradle of the screw was fixed with respect to the base B by a heavy rigid arm A and the vertical support S. The micrometer was insulated from the remaining parts of the apparatus, so that the contact between the micrometer and the bar could be determined electrically. The contact points consisted of a gold-plated spherical cup soldered to the collar and a gold-plated cylindrical cup soldered to the micrometer head. The graduated drum D mounted on the

head was subdivided into 500 divisions, tenths of which could be estimated without difficulty. Since a complete revolution of the drum was equivalent to a vertical motion of the micrometer head of 0.5 mm, a motion of 0.0001 mm could be detected.

To study the possibility of errors arising from hysteresis in the measuring device tests were carried out in the following manner. Contact was established in the undisturbed state of the bar and five readings were taken. Next, the micrometer head was separated from the bar by a distance of 0.1 mm, was brought back to contact, and five more readings were taken. The difference of the averages of the contact readings before and after the separation did not exceed 0.0002 mm. The same procedure was carried out for separations amounting to 0.2, 0.3, 0.5, 1.0, and 1.5 mm. The average difference for two consecutive readings was never more than 0.0002 mm; that is, the measuring device had no hysteresis which was measurable.

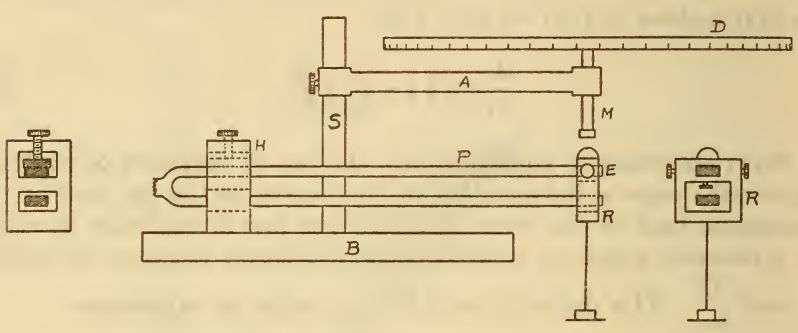

Fig. 4.—*Apparatus for measuring deflection*

Since in measuring the width of hysteresis loop only very small portions of the thread of the micrometer screw were involved for any one point, the micrometer was not calibrated.

Care was taken to insure that the material of the bar was in the cyclic state before hysteresis observations were made during the load cycles. To achieve this, the maximum load for a specified cycle was repeatedly applied to the bar until the residual deflection at zero load became constant or vanished. In the rested state of the bar the initial residual deflection is a measurable quantity, but it soon becomes reduced in value or becomes nil.

The cycle starts from zero, and the load is increased in equal steps (generally) to the maximum load L_m, and is then decreased by the same increments, but in the reverse order, to zero load. Each time an increment of load was applied or removed three separate readings were taken. Each load cycle was run four times, generally in succession. Apparently the rate at which the load cycles were carried out had no definite effect on the results. On the average, the time

that transpired between changes of load was two minutes. Experiments were conducted at room temperature (21 to 29° C.) in a basement laboratory. The temperature change during each test was never more than 0.2° C. and quite frequently it was 0.1° C. or less.

Hysteresis measurements were made with four different lengths of the bar, denoted later by the letters l_1, l_2, l_3, and l_4. The results of the tests are given in Tables 1 to 4. In column 3 are given the averages of the deflections for the four separate load cycles corresponding to the loads in column 2 for increasing loads. In column 4 are given the averages of the differences between the readings for ascending and descending loads for the four load cycles. On the whole, the correspondence of the observed individual maximum measurement with the average values given in column 4 is within 10 per cent of the averages.

2. DETERMINATION OF LENGTHS OF BAR IN FLEXURE

On account of the uncertainty of the clamping, the following formula [14] was used for determining the equivalent lengths of the bar in flexure:

$$l^3 = 3\,d\,E\,I \tag{23}$$

where l is the length of the bar, d is the deflection at the free end of the bar under unit load, E is Young's modulus of elasticity, and I is the moment of inertia of the cross section. In order to determine l accurately, it was necessary to measure E experimentally.

To determine E, the prong used in the deflection measurements was placed on two knife-edges, separated by a distance l', and the deflection d' of the midpoint of the bar between the knife-edges was observed under unit load. Three different values of l' chosen were $l'=28.0$, 26.0, and 24.0 cm. The corresponding deflections d' under 1 kg. load were $d'=0.0153$, 0.0122, and 0.0096 mm. The formula [14] for determining E is

$$E=\frac{1}{48I}\cdot\frac{l'^3}{d'} \tag{24}$$

which gives for the modulus

$$E=20.04\times10^{11} \text{ dynes per square centimeter}$$

It appears from Tables 1, 2, 3, and 4 that the deflection d of the bar for the lengths, $l=l_1$, l_2, l_3, and l_4 under 1 kg load are $d=0.487$, 0.300, 0.109, and 0.073 mm. We find from equation (23) for the equivalent lengths of the bar in flexure the following values: $l_1=27.09$ cm, $l_2=23.05$ cm, $l_3=16.45$ cm, $l_4=14.39$ cm.

[14] See footnote 13, p. 150.

V. DISCUSSION OF DATA

1. ENERGY LOSS DUE TO HYSTERESIS

The data in Tables 1, 2, 3, and 4 indicate that the residual deflections at zero load in the various cycles considered there are small in comparison with the maximum hysteresis observed; consequently the data give mainly the statical hysteresis. Hence the data as they stand can be used to test the various deductions made in connection with equations (16) and (22).

To start with equation (16), we first evaluate from the data the energy lost, H. In Figure 5 are plotted, for example, the hysteresis

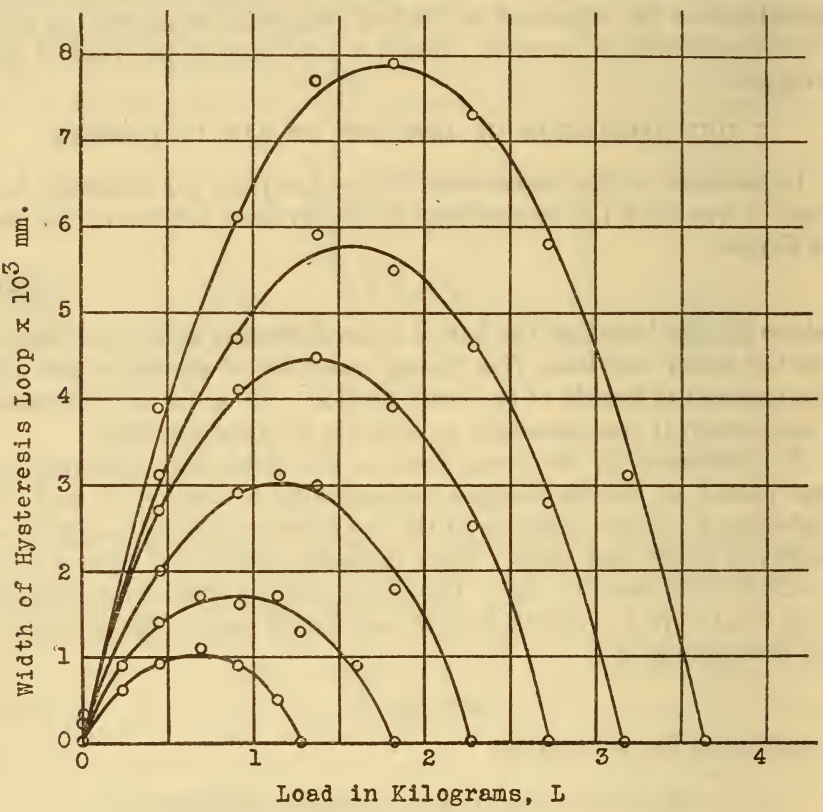

FIG. 5.—*Width of hysteresis loop in closed load cycles*

(Drawn for l=27.09 cm)

values against the load from Table 1. The areas under the curves in Figure 5 are proportional to the energy lost in the cycles corresponding to the curves. Table 5 is prepared from such plots. In it are given the lengths of the bar in flexure, the maximum loads of the cycles, the energy lost owing to hysteresis, the average values of hysteresis in a cycle, the ratio $\frac{H}{L_m^3}$, and the values of the maximum stress in the bar.

Now, write equation (16) as follows:

$$\left.\begin{aligned} H &= CL_m^3, \quad (1) \\ \text{where} \quad C &= C_1 l^4. \quad (2) \end{aligned}\right\} \qquad (16a)$$

Table 5 shows that the average values of C, obtained from the observations, are

$C = 38.6$ ergs per kg^3 when $l_1 = 27.09$ cm
$C = 22.0$ ergs per kg^3 when $l_2 = 23.05$ cm
$C = 5.4$ ergs per kg^3 when $l_3 = 16.45$ cm
$C = 3.7$ ergs per kg^3 when $l_4 = 14.39$ cm

Substituting these values in (16a 2), the following values of C_1 are obtained:

$C_1 = 0.73 \times 10^{-4}$ ergs per cm^4 per kg^3 when $l_1 = 27.09$ cm
$C_1 = .78 \times 10^{-4}$ ergs per cm^4 per kg^3 when $l_2 = 23.05$ cm
$C_1 = .74 \times 10^{-4}$ ergs per cm^4 per kg^3 when $l_3 = 16.45$ cm
$C_1 = .86 \times 10^{-4}$ ergs per cm^4 per kg^3 when $l_4 = 14.39$ cm

Thus, C_1 appears to be a constant having the average value 0.78×10^{-4}. In Figure 6 curves are plotted from the equation

$$H = 0.78 \times 10^{-4}\, L_m^3\, l^4 \qquad (16b)$$

In this figure the observed values of H are represented by circles. Evidently the experimental data conform with the theoretical deductions as to the dependence of energy loss of hysteresis (statical) upon the maximum load and the length of bar.

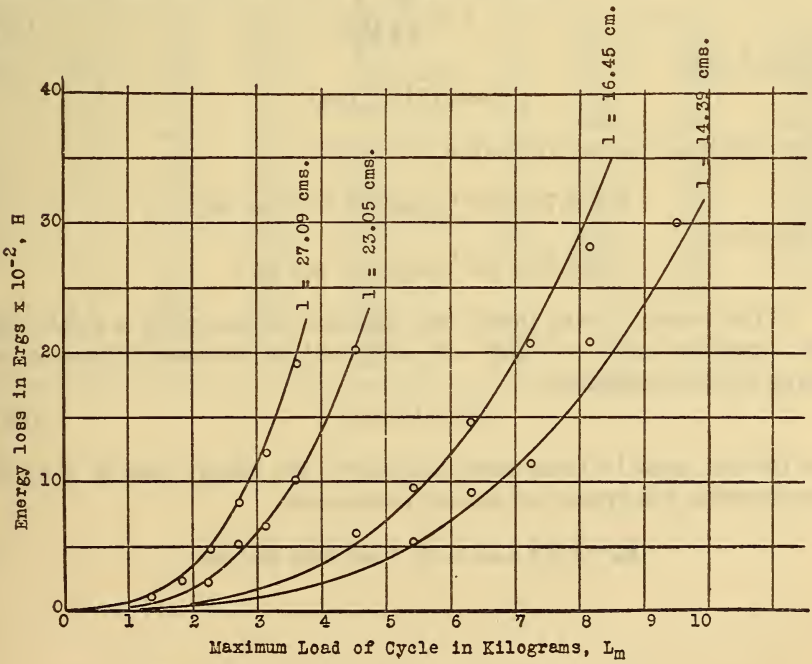

FIG. 6.—*Energy loss in closed load cycles for various lengths of bar*

2. LAW OF VARIATION OF WIDTH IN HYSTERESIS LOOPS

In order to see how closely the observed values of hysteresis width conform with the theoretical law of

$$\frac{h}{h_a}=6\left(1-\frac{L}{L_m}\right)\frac{L}{L_m} \tag{22a}$$

the ratios $\frac{h}{h_a}$ were determined for certain values of $\frac{L}{L_m}$ in all the load cycles for the length of bar $l=27.09$ cm. For a given $\frac{L}{L_m}$ and L_m, h is read from Figure 5, h_a are given in Table 5. The values of the ratio $\frac{h}{h_a}$ as observed and as calculated by the expression (22a) are given in Table 6. Considering the errors inherent in the experiments, the correspondence between the observed and calculated values of $\frac{h}{h_a}$ as shown by Table 6 is quite satisfactory. The expression (22a) also states that the maximum width of the hysteresis loop is 1.5 times as great as the average width.

VI. DETERMINATION OF THE STATICAL HYSTERESIS MODULUS

Introducing the constants of the bars, $b=1.395$ cm and $2a=0.830$ cm into the formula

$$C_1=\frac{9}{64}\frac{\beta}{b^2a^5} \tag{16}$$

we find that

$$\beta=0.171\ C_1\ \text{cm}^7$$

The experimental determination of C_1 gave

$$C_1=0.78\times10^{-4}\ \text{ergs per cm}^4\ \text{per kg}^3$$

Therefore

$$\beta=1.33\times10^{-5}\ \text{ergs cm}^3\ \text{per kg}^3$$

In the tneory it was found that the loss of energy in a cube, the two opposite faces of which are subjected to a stress difference, is given by the expression

$$E_0=1/3\beta\sigma_m^3 \tag{8a}$$

In the bar used in these tests, therefore, the energy loss is, if σ_m is expressed in kilograms per square centimeter,

$$E_0=1/3\times1.33\times10^{-5}\ \sigma_m^3\ \text{ergs per cm}^3$$

TABLE 1.—*Deflection and width of hysteresis loop of Armco iron bar length* $=l_1=27.09$ *cm (calculated)*

Maximum load in kilograms	Load	Deflection	Width of hysteresis loop	Maximum load in kilograms	Load	Deflection	Width of hysteresis loop
	kg	*mm*	*mm*		*kg*	*mm*	*mm*
	0.000	0.0000	0.0000		0.000	0.0000	0.0000
	.227	.1092	.0006		.454	.2204	.0020
	.454	.2202	.0009		.907	.4423	.0029
1.361	.680	.3317	.0011	2.268	1.137	.5522	.0031
	.907	.4423	.0009		1.361	.6640	.0030
	1.137	.5520	.0005		1.814	.8865	.0017
	1.361	.6634	.0000		2.268	1.1084	.0000
	.000	.0000	.0000		.000	.0000	.0000
	.227	.1096	.0009		.454	.2204	.0027
	.454	.2207	.0014		.907	.4424	.0041
	.680	.3323	.0017	2.722	1.361	.6636	.0045
1.814	.907	.4429	.0016		1.814	.8870	.0039
	1.137	.5527	.0017		2.268	1.1090	.0025
	1.361	.6640	.0013		2.722	1.3346	.0000
	1.588	.7757	.0009				
	1.814	.8867	.0000		.000	.0000	.0003
					.454	.2209	.0039
	.000	.0000	.0002		.907	.4427	.0061
	.454	.2209	.0031		1.361	.6644	.0077
	.907	.4427	.0047	3.629	1.814	.8872	.0079
	1.361	.6645	.0059		2.268	1.1084	.0073
3.175	1.814	.8874	.0055		2.722	1.3326	.0058
	2.268	1.1096	.0046		3.175	1.5558	.0031
	2.722	1.3325	.0028		3.629	1.7808	.0000
	3.175	1.5543	.0000				

TABLE 2.—*Deflection and width of hysteresis loop of Armco iron bar length* $=l_2=23.05$ *cm (calculated)*

Maximum load in kilograms	Load	Deflection	Width of hysteresis loop	Maximum load in kilograms	Load	Deflection	Width of hysteresis loop
	kg	*mm*	*mm*		*kg*	*mm*	*mm*
	0.000	0.0000	0.0000		0.000	0.0000	0.0002
	.427	.1284	.0009		.427	.1294	.0016
	.880	.2621	.0014		.880	.2630	.0024
2.241	1.334	.3957	.0014	2.695	1.334	.3963	.0028
	1.728	.5324	.0008		1.728	.5330	.0026
	2.241	.6688	.0000		2.241	.6694	.0016
					2.695	.8018	.0000
	.000	.0000	.0000				
	.427	.1292	.0015		.000	.0000	.0004
	.880	.2628	.0022		.427	.1297	.0020
3.145	1.334	.3959	.0030		.880	.2623	.0038
	1.728	.5325	.0031		1.334	.3972	.0041
	2.241	.6691	.0025	3.602	1.728	.5336	.0048
	2.695	.8021	.0016		2.241	.6701	.0041
	3.145	.9366	.0000		2.695	.8035	.0031
					3.145	.9380	.0018
	.000	.0000	.0006		3.602	1.0751	.0000
	.880	.2632	.0047				
	1.728	.5344	.0064				
4.509	2.695	.8041	.0063				
	3.602	1.0761	.0043				
	4.509	1.3451	.0000				

TABLE 3.—*Deflection and width of hysteresis loop of Armco iron bar length=l_3= 16.45 cm (calculated)*

Maximum load in kilograms	Load	Deflection	Width of hysteresis loop	Maximum load in kilograms	Load	Deflection	Width of hysteresis loop
	kg	*mm*	*mm*		*kg*	*mm*	*mm*
4.509	0.000	0.0000	0.0001	5.416	0.000	0.0000	0.0001
	.880	.0943	.0012		.880	.0944	.0014
	1.728	.1923	.0019		1.728	.1919	.0024
	2.695	.2913	.0019		2.695	.2911	.0028
	3.602	.3907	.0012		3.602	.3906	.0022
	4.509	.4895	.0000		4.509	.4887	.0014
					5.416	.5861	.0000
6.323	.000	.0000	.0002	7.231	.000	.0000	.0003
	.880	.0944	.0018		.880	.0944	.0019
	1.728	.1922	.0028		1.728	.1922	.0032
	2.695	.2914	.0034		2.695	.2913	.0041
	3.602	.3907	.0032		3.602	.3905	.0043
	4.509	.4886	.0028		4.509	.4885	.0040
	5.416	.5860	.0018		5.416	.5858	.0032
	6.323	.6844	.0000		6.323	.6841	.0019
					7.231	.7820	.0000
8.138	.000	.0000	.0005				
	1.334	.1430	.0029				
	2.695	.2916	.0046				
	4.055	.4398	.0053				
	5.416	.5866	.0045				
	6.877	.7335	.0028				
	8.138	.8826	.0000				

TABLE 4.—*Deflection and width of hysteresis loop of Armco iron bar length=l_4= 14.39 cm (calculated)*

Maximum load in kilograms	Load	Deflection	Width of hysteresis loop	Maximum load in kilograms	Load	Deflection	Width of hysteresis loop
	kg	*mm*	*mm*		*kg*	*mm*	*mm*
5.416	0.000	0.0000	0.0000	6.323	0.000	0.0000	0.0000
	.880	.0636	.0008		.880	.0640	.0011
	1.728	.1290	.0013		1.728	.1300	.0019
	2.695	.1935	.0014		2.695	.1946	.0021
	3.602	.2587	.0014		3.602	.2593	.0022
	4.509	.3246	.0009		4.509	.3255	.0017
	5.416	.3908	.0000		5.416	.3908	.0011
					6.323	.4570	.0000
7.231	.000	.0000	.0001	9 499	.000	.0000	.0002
	.880	.0644	.0016		1.334	.0982	.0024
	1.728	.1306	.0029		2.695	.1972	.0038
	2.695	.1961	.0033		4.055	.2953	.0044
	3.602	.2617	.0034		5.416	3930	.0046
	4.509	.3267	.0031		6.877	.4924	.0042
	5.416	.3924	.0025		8.138	.5932	.0022
	6.323	.4589	.0014		9.499	.6925	.0000
	7.231	.5251	.0000				
8.138	.000	.0000	.0000				
	1.334	.0984	.0022				
	2.695	.1968	.0034				
	4.055	.2953	.0037				
	5.416	.3933	.0033				
	6.877	.4929	.0022				
	8.138	.5934	.0000				

TABLE 5.—*Energy loss of hysteresis in Armco iron bar*

Length of bar in centimeters	Maximum load of load cycles or L_m	Energy loss of hysteresis or H	$\frac{H}{L_m^3}$	Average value of $\frac{H}{L_m^3}$ C	Average width of hysteresis loop or $h_a=\frac{H}{L_m}$	Maximum stress
	kg	*ergs*			*mm*	*kg/mm²*
l_1 or 27.09	1.36	91	36.2	38.6	7×10^{-4}	2.40
	1.81	216	36.5		12×10^{-4}	3.06
	2.27	466	39.7		21×10^{-4}	3.84
	2.72	831	41.4		31×10^{-4}	4.60
	3.18	1,218	37.9		39×10^{-4}	5.38
	3.63	1,908	39.8		53×10^{-4}	6.14
l_2 or 2305	2.24	207	18.4	22.0	9×10^{-4}	3.22
	2.70	511	26.0		19×10^{-4}	3.89
	3.14	653	21.1		21×10^{-4}	4.52
	3.60	1,035	22.2		29×10^{-4}	5.18
	4.51	2,050	22.3		46×10^{-4}	6.50
l_3 or 16.45	4.51	596	6.50	5.4	13×10^{-4}	4.63
	5.42	963	6.05		18×10^{-4}	5.56
	6.32	1,461	5.80		24×10^{-4}	6.48
	7.23	2,062	5.47		29×10^{-4}	7.43
	8.14	2,810	5.20		35×10^{-4}	8.35
l_4 or 14.39	5.42	532	3.34	3.7	10×10^{-4}	4.97
	6.32	922	3.65		15×10^{-4}	5.68
	7.23	1,630	4.34		23×10^{-4}	6.50
	8.14	2,090	3.87		26×10^{-4}	7.33
	9.50	3,000	3.50		32×10^{-4}	8.55

TABLE 6.—*Ratio of width of hysteresis loop to average width for Armco iron bar, $l=27.09$ cm*

$\frac{L}{L_m}$	$\frac{h}{h_a}$						
	L_m=1.36 kg	L_m=1.81 kg	L_m=2.27 kg	L_m=2.72 kg	L_m=3.18 kg	L_m=3.63 kg	Theoretical
0.0	0.00	0.00	0.00	0.00	0.05	0.06	0.00
.1	.57	.62	.57	.61	.56	.55	.54
.2	1.00	1.00	.95	.97	.99	.98	.96
.3	1.27	1.25	1.22	1.24	1.24	1.29	1.26
.4	1.40	1.37	1.38	1.39	1.43	1.44	1.44
.5	1.43	1.42	1.44	1.44	1.49	1.49	1.50
.6	1.40	1.37	1.39	1.39	1.42	1.41	1.44
.7	1.26	1.23	1.21	1.23	1.26	1.25	1.26
.8	.84	.96	.97	.98	.91	.96	.96
.9	.63	.58	.59	.61	.54	.52	.54
1.0	.00	.00	.00	.00	.00	.00	.00

VII. SUMMARY

1. When an elastic body is subjected to cyclic variations of axial stress σ between the extremes $\sigma=\sigma_1$ and $\sigma=\sigma_2$, $\sigma_1<\sigma_2$, the difference ϵ_σ of the magnitudes of the strain for increasing and decreasing σ is given by

$$\epsilon_\sigma=2\beta(\sigma_a^2-s^2)$$

where $\sigma_a=1/2\,(\sigma_2-\sigma_1)$, $s=\sigma-\sigma_1-\sigma_a$ and β is the statical hysteresis modulus.

2. Loss of energy E_o per unit volume due to statical hysteresis during a cyclic variation of load from σ_1 to σ_2 and back to σ_1 is given by

$$E_o=\frac{1}{3}\beta\sigma_m^3$$

where $\sigma_m=2\sigma_a=\sigma_2-\sigma_1$.

3. Loss of energy due to statical hysteresis in the flexure of a bar of length l, one end of which is clamped and the other end deflected under a load L, during the load variation from 0 to L_m and back to 0, is proportional to L_m^3 and l^4.

4. The ratio of the width of hysteresis loop to the average width is independent of the length of the bar and is given by the expression

$$\frac{h}{h_a}=6\left(1-\frac{L}{L_m}\right)\frac{L}{L_m}$$

where h is the difference in the deflection of the bar for increasing and decreasing L, h_a is the average value of h in the hysteresis loop, and L_m is the maximum value of the load in the load cycle.

5. Deductions (2), (3), and (4) are derived from (1). Deductions (3) and (4) have been verified for a rectangular bar of Armco iron.

6. If σ_m is measured in kilograms per square centimeters and centimeter is taken as a unit of length the value of β, for Armco iron is found to be

$$\beta=1.33\times10^{-5}\text{ ergs cm}^3\text{ per kg}^3$$

In conclusion the author wishes to thank Drs. L. J. Briggs and L. B. Tuckerman for their valuable criticisms and suggestions.

WASHINGTON January 13, 1926.