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Cutting the d-Cube

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Some problems concerned with cutting faces of the cube with affine or linear spaces are considered. It is shown that through any d-3 points of R^d there passes a hyperplane which cuts all the facets of the d-cube. Furthermore, it is shown that if $m < d - 1$ and $d' < d - \left[(m + 1)/3 \right]$, then no m-dimensional affine subspace of R^d cuts all the d'-dimensional faces of the cube.

Key words: Cube; geometry; hyperplane.

1. Introduction

If K is a convex set in R^d and A is an affine subspace of R^d, we say that A *cuts* K if A intersects the relative interior of K, but A does not contain K.

Let C^d be the d-cube in R^d:

$$C^d = \left\{ (x_1, \cdots, x_d) \in R^d : -1 \le x_i \le 1, \quad \text{for} \quad 1 \le i \le d \right\}.$$

In section 2 we strengthen a result of Joel, Shier, and Stein [2][1] by showing that, if S is a set of at most d-3 points of R^d, then S is contained in a hyperplane H through the origin (i.e., a linear subspace of dimension $d - 1$) which cuts each facet of the cube C^d. We also characterize those sets S with $|S| = d - 2$ for which there is no such linear subspace.

In section 3 we prove that if $m < d - 1$ and $\left[1/3(m + 1) \right] < d - d'$, then no m-dimensional affine subspace of R^d cuts all the d'-dimensional faces of the cube. This sharpens a theorem of McMullen and Shephard [3], which may be regarded as asserting the nonexistence of such a *linear* subspace of R^d.

2. Cutting the Facets of the Cube with Planes and Hyperplanes

When $d \ge 3$, it is not difficult to construct planes in R^d which cut each facet of C^d. Suppose P is a convex polygon in the plane, R^2, symmetric about the origin, and bounded by 2d edges. Then there are linear functionals λ_i on R^2 with

$$P = \{u \in R^2 : -1 \le \lambda_i(u) \le 1, \quad \text{for} \quad 1 \le i \le d\}.$$

Consider the function $\lambda : R^2 \rightarrow R^d$ with

$$\lambda(u) = \left(\lambda_1(u), \cdots, \lambda_d(u)\right).$$

Let $L = \lambda(R^2)$, a plane in R^d. Note that $\lambda(P) = L \cap C^d$.

Furthermore, note that if u is a point of the polygon P on the relative interior of the edge of P given by $\lambda_i(x) = 1$ (or -1), then $\lambda(u)$ is on the relative interior of a corresponding facet $\left\{x \in C^d : x_i = 1 \text{ (or } -1) \right\}$ of C^d. It follows that L intersects the relative interior of each facet of C^d (and cuts each facet, when $d \ge 3$).

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[1] Figures in brackets refer to literature references at the end of this paper.

The following theorem is a strengthening of Theorem 2 of Joel, Shier, and Stein [2].

THEOREM 1. *Suppose* $d \geq 3$ *and* S *is a set of at most* $d - 3$ *points in* R^d. *There is a hyperplane* H, *containing* $S \cup \{0\}$, *which cuts each facet of* C^d.

PROOF. It is possible to find a plane L, through the origin, which cuts each facet of C^d. (We did this above!) The linear space spanned by $L \cup S$ has dimension at most $d - 1$, so there is a hyperplane H with $L \cup S \subset H$. This is the required hyperplane.

For which sets S of $d - 2$ points is there no such hyperplane? The following two theorems will provide the answer. With finitely many exceptions, each $(d - 2) -$ dimensional linear linear subspace of R^d is contained in a hyperplane which cuts each facet of C^d.

For $1 \leq i \leq j \leq d$, let

$$A(i, j) = \{x \in R^d : x_i = x_j = 0\}.$$

If F is a $(d - 3) -$ dimensional face of C^d, let $A(F)$ be the linear subspace of R^d spanned by F.

THEOREM 2. *Suppose* $d \geq 3$ *and* H *is a hyperplane in* R^d *which cuts each facet of* C^d. *Then* H *contains none of the linear subspaces* A(i, j), A(F).

PROOF. Let A be a plane orthogonal to $A(i, j)$. Let π be the orthogonal projection $\pi : R^d \rightarrow A$. If H contains $A(i, j)$ then $\pi(H)$ is a line in A which cuts all four edges of the square $\pi(C^d)$. There is no such line, so H does not contain $A(i, j)$.

Suppose H contains $A(F)$ (and, hence, F) for some $(d - 3) -$ face F of C^d. Let A' be the linear subspace of dimension three which is orthogonal to the *affine* span of F. The orthogonal projection $\pi' : R^d \rightarrow A'$ takes H to a plane $\pi'(H)$ which cuts each facet of the 3-cube $\pi'(C^d)$ and which contains the vertex $\pi'(F)$ of this 3-cube. No plane containing a vertex of the 3-cube cuts all the facets of that cube. (See, also, Joel, Shier and Stein [2], Theorem 3.) It follows that H cannot contain $A(F)$.

LEMMA. *Suppose* $d \geq 3$ *and* $\alpha = (\alpha_1, \cdots, \alpha_d)$ *is a point not on any of subspaces* A(i, j) *or* A(F). *Then there is a plane through the origin and* α *which cuts each facet of* C^d.

PROOF. Since α is on none of the subspaces $A(i, j)$, at most one of $\alpha_1, \cdots, \alpha_d$ is zero. Since it is on none of the subspaces $A(F)$, no three of the α_i's have the same absolute value. Let C be a circle centered at the origin in the plane, R^2, whose radius r is less than the minimum of the numbers $1/|\alpha_i|$, for $1 \leq i \leq d$ and $\alpha_i \neq 0$. Note that, if $\alpha_i \neq 0$, there are two lines in the plane through the point $(1/|\alpha_i|, 0)$ which are tangent to the circle; one of these has positive slope and the other has negative slope.

We construct d linear functionals on R^2. If $\alpha_i = 0$, let $\lambda_i(x_1, x_2) = x_2/r$ be the linear functional which has the value 1 at each point of the line parallel to and above the $x_1 -$ axis, tangent to the circle. If $\alpha_i \neq 0$ and there is no $j < i$ with $|\alpha_i| = |\alpha_j|$, let λ_i be the linear functional which has the value $sgn(\alpha_i)$ $(= \pm 1)$ at each point of the line tangent to the circle through $(1/|\alpha_i|, 0)$ with negative slope. If, on the other hand, there *is* a $j < i$ with $|\alpha_j| = |\alpha_i|$, let λ_i have the value $sgn(\alpha_i)$ on the tangent with *positive* slope through $(1/|\alpha_i|, 0)$.

Now, $P = \{x \in R^2 : -1 \leq \lambda_i(x) \leq 1 \text{ for } 1 \leq i \leq d\}$ is a convex polygon in the plane, symmetric about the origin, with 2d edges. The circle of radius r is inscribed in it. Consider the function $\lambda : R^2 \rightarrow R^d$ with $\lambda(x) = (\lambda_1(x), \cdots, \lambda_d(x))$. Let $L = \lambda(R^2)$. Then $\lambda(P) = L \cap C^d$, and L is a plane which cuts each facet of C^d. Furthermore, the point $\lambda(1, 0) = (\alpha_1, \cdots, \alpha_d)$ is on L, as required.

THEOREM 3. *Suppose* $d \geq 3$ *and* S *is a set of at most* $d - 2$ *points of* R^d. *Let* A *be the linear subspace spanned by* S. *Then there is a hyperplane* H *containing* S *which cuts each facet of* C^d *if and only if* A *is not one of the* $\binom{d}{2} + 4\binom{d}{3}$ *subspaces* A(i, j), A(F).

PROOF. Clearly, Theorem 2 implies that if A *is* one of the subspaces $A(i, j)$ or $A(F)$ then there is no such hyperplane H.

Suppose A is not such a subspace. If $\dim A \leq d - 3$, it follows from Theorem 1 that there is such a hyperplane. If $\dim A = d - 2$, there must be a point α in A not on any of the subspaces $A(i, j)$ or $A(F)$. By the lemma, there is a plane L, containing α and the origin, which cuts each facet of C^d. Since $A \cap L$ contains the line through 0 and α, $\dim (A \cup L) \leq d - 1$. Let H be a hyperplane containing $A \cup L$. H is the required hyperplane.

3. Cutting Cubes with Affine Spaces

Is there an affine subspace A of R^d of dimension $m = 2k$ which cuts each $(d - k) -$ face of C^d? We have seen that there is a plane which cuts each facet of C^d, so for $k = 1$, the answer is, "Yes." Also, if $d = 2k + 1$ the answer is again affirmative, since the hyperplane given by $x_1 + \cdots + x_d = 0$ cuts each $(k + 1) -$ face of C^d.

However, we assert that if $1 < k < (d - 1)/2$, then there is no such subspace A. That there is no such *linear* subspace follows by dualizing a theorem of McMullen and Shephard ([3], p. 130), and we use their result to prove the following theorem, from which the assertion follows.

THEOREM 4. *If* $0 \leq \mathrm{m} < \mathrm{d} - 1$, A *is an* m*-dimensional affine subspace of* $\mathrm{R^d}$, *and* A *cuts each* $\mathrm{d_0}$*-dimensional face of* $\mathrm{C^d}$, *then* $\mathrm{m} \geq 3(\mathrm{d} - \mathrm{d_0}) - 1$.

PROOF. Let x be an element of A. We show that the linear subspace $A' = A - x$ also cuts each $d_o -$ face of C^d. Suppose G is a $d_o -$ face of C^d. Let F be its relative interior. Then $-F$ is the relative interior of $-G$, so $A \cap (-F) \neq 0$. It follows that $(-A) \cap F \neq 0$. (Note that $-A = A - 2x$.) Let u be an element of $F \cap A$. Let $v = v_o - 2x$ be an element of $(-A) \cap F$, so that $v_o \in A$. Then $\frac{1}{2}(u + v) \in A' \cap F$, and A' cuts each $d_o -$ face of C^d.

Now, $A' \cap C^d$ is an m-dimensional centrally symmetric polytope with 2d facets, and any $d - d_o$ of these facets, no pair of which is opposite, intersect in a face of dimension $d_o - (d - m) = m - (d - d_o)$. Therefore, the dual of $A' \cap C^d$ (see [1], page 47) is an m-dimensional polytope with $2d \geq 2(m + 2)$ vertices, and any $d - d_o$ of these vertices, no pair of which is opposite, are the vertices of a $(d - d_o - 1) -$ simplex which is a face. By [3], p. 130, assertions (22) and (23), it follows that $d - d_o \leq [(\mathrm{m} + 1)/3]$; i.e., $m \geq 3(d - d_o) - 1$.

The following would be a consequence of the conjecture of McMullen and Shephard ([3], p. 133):

CONJECTURE. *For* $\mathrm{m} \geq 2$, *if* d *is larger than* 2m-3 *then there is no* $m -$ *dimensional affine subspace of* $\mathrm{R^d}$ *which cuts each* $(\mathrm{d} - 2)$ *– face of* $\mathrm{C^d}$.

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4. References

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