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Vector-Valued Entire Functions of Bounded Index Satisfying a Differential Equation*

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The concept of complex valued entire functions of bounded index is extended to C^n-valued entire functions by replacing the absolute value in the definition of an entire function of bounded index by the maximum of the absolute values of the components. If the components of a C^n-valued entire function are of bounded index, then the function is also of bounded index; however a C^n-valued function may be of bounded index without all of its components being of bounded index. Solutions of certain linear differential equations are related to C^n-valued functions of bounded index.

Key words: Bounded index; C^n-valued functions; entire functions; linear differential equations.

1. Introduction

If f is a complex-valued entire function, then f is said to be of bounded index of index N if N is the least non-negative integer such that

$$\frac{|f^{(j)}(z)|}{j!} \leq \max_{0\leq i\leq N} \frac{|f^{(i)}(z)|}{i!}$$

for all integers $j \geq 0$ and all $z \in C$([1],[1] [2]).

In this paper, we are concerned with one possible extension of this concept to vector-valued functions.

If f_i, $i = 1,2, \ldots, n$, are complex-valued entire functions, then $F(z) = [f_1(z) f_2(z) \cdots f_n(z)]$ is a C^n-valued entire function (we write C^n-valued functions in this manner for convenience throughout this paper regardless of whether they are to be interpreted as row vectors or column vectors) and we write $\|F(z)\| = \max\{ |f_i(z)| \,\big|\, i = 1,2, \ldots, n\}$ and $F'(z) = [f_1'(z)\, f_2'(z) \ldots f_n'(z)]$.

DEFINITION: A C^n-valued entire function F is said to be of bounded index of index N if N is the least non-negative integer such that

$$\frac{\|F^{(j)}(z)\|}{j!} \leq \max_{0\leq i\leq N} \frac{\|F^{(i)}(z)\|}{i!} \tag{1.1}$$

for all integers $j \geq 0$ and all $z \in C$.

2. Properties and Examples

We first relate the index of a C^n-valued (vector-valued) entire function of bounded index to the index of each of its components (scalars). We use lower case letters to denote scalar functions and upper case letters to denote vector-valued functions.

LEMMA 2.1: If f_k is an entire function of bounded index of index N_k for $k = 1,2, \ldots, n$, then $F = [f_1 f_2 \cdots f_n]$ is of bounded index of index $N \leq M = \max\{N_k | k = 1,2, \ldots, n\}$.

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[1] Figures in brackets indicate the literature references at the end of this paper.

PROOF: For each integer $j \geq 0$ and each $k = 1,2, \ldots, n$

$$\frac{|f_k^{(j)}(z)|}{j!} \leq \max_{0 \leq i \leq N_k} \frac{|f_k^{(i)}(z)|}{i!} = \max_{0 \leq i \leq M} \frac{|f_k^{(i)}(z)|}{i!} \leq \max_{0 \leq i \leq M} \frac{\|F^{(i)}(z)\|}{i!}$$

Therefore, (1.1) is satisfied.

REMARK 1: Both equality and inequality are attainable in Lemma 2.1.

EXAMPLE 1: Let $f_1(z) = e^{2z}, f_2(z) = e^z$, and $F = [f_1 f_2]$. Then $N_1 = 1, N_2 = 0$, and $N = 1$.

EXAMPLE 2: Let $f_1(z) = 2, f_2(z) = z$, and $F = [f_1 f_2]$. Then $N_1 = 0, N_2 = 1$, and $N = 0$.

REMARK 2: It is possible for $N < min\{N_k \,|\, k = 1,2, \ldots, n\}$ in Lemma 2.1.

EXAMPLE 3: Let $f_1(z) = z^2, f_2(z) = z - 8$, and $F = [f_1 f_2]$. Then $N_1 = 2$, $N_2 = 1$, and $N = 0$.

REMARK 3: If F is of bounded index, then its components may not be of bounded index.

EXAMPLE 4: Let f be of unbounded index such that $f - c$, where c is any non-zero constant, is of bounded index [7, Theorem 2, p. 128]. Let $f_1 = f$ and $f_2 = f - c$. Let N_2 be the index of f_2 and let $F = [f_1 f_2]$. Then F is of bounded index of index $N \leq N_2$.

PROOF: For any integer $j \geq 1$ and $z \epsilon C$, $f_1^{(j)}(z) = f_2^{(j)}(z)$ and so

$$\frac{\|F^{(j)}(z)\|}{j!} = \max \left\{ \frac{|f_1^{(j)}(z)|}{j!}, \frac{|f_2^{(j)}(z)|}{j!} \right\} = \frac{|f_2^{(j)}(z)|}{j!} \leq \max_{0 \leq i \leq N_2} \frac{|f_2^{(i)}(z)|}{i!} \leq \max_{0 \leq i \leq N_2} \frac{\|F^{(i)}(z)\|}{i!}.$$

Therefore (1.1) holds for all integers $j \geq 0$ and all $z \epsilon C$.

3. Differential Equations With Constant Coefficients

If $A = [a_{ij}]$ is an $n \times n$ matrix, we use the norm $\|A\| = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|$. If F is a solution of the vector equation $F' = AF + Q$, where A is a matrix whose entries are entire functions and Q is a column vector whose entries are entire functions, then F is an entire function [4]. We will show that if A is a constant matrix, then any solution of $F' = AF$ is of bounded index and we will obtain an upper bound on its index.

THEOREM 3.1: If F is a solution of $F' = AF$ where A is a constant matrix, then F is an entire function of bounded index of index $N \leq M$ where

$$M = \min\{m \epsilon Z \,|\, m \geq 0 \text{ and } \|A\| \leq m + 1\}. \tag{3.1}$$

PROOF: Let m be a non-negative integer such that $\|A\| \leq m + 1$. Then $F^{(m+p)}(z) = AF^{(m+p-1)}(z)$ for all integers $p \geq 1$ and all $z \epsilon C$. Therefore

$$\frac{\|F^{(m+p)}(z)\|}{(m+p)!} \leq \frac{\|A\|}{m+p} \frac{\|F^{(m+p-1)}(z)\|}{(m+p-1)!} \leq \frac{\|F^{(m+p-1)}(z)\|}{(m+p-1)!}$$

and so

$$\frac{\|F^{(j)}(z)\|}{j!} \leq \max_{0 \leq i \leq m} \frac{\|F^{(i)}(z)\|}{i!}$$

for all integers $j \geq 0$ and all $z \epsilon C$.

REMARK: Both equality and inequality are attainable in Theorem 3.1.

EXAMPLE 5: Let $F(z) = [e^{2z} \; e^z]$ and $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$. Then $\|A\| = 2$ and $M = 1$ and $N = 1$ (see example 1).

EXAMPLE 6: Let $F(z) = [1/8\,z]$ and $A = \begin{bmatrix} 0 & 0 \\ 8 & 0 \end{bmatrix}$. Then $\|A\| = 8$ and $M = 7$; however $N = 1$.

Theorem 3.1 does not tell us whether or not the components of F are of bounded index; however every component of F satisfies a linear differential equation of the form

$$\alpha_0 y^{(n)} + \alpha_{1y}{}^{(n-1)} + \ldots + \alpha_{n-1} g' + \alpha_n \gamma = 0 \tag{3.2}$$

where $\alpha_k \epsilon\, C$, $k = 0,1,2, \ldots, n$ [3]. Therefore we have

THEOREM 3.2: If F is a solution of $F' = AF$ where A is a constant matrix, then each component of F is of bounded index.

PROOF: Since each component of F satisfies a differential equation of the form (3.2), each component is then of bounded index, by [5].

By the same type of argument as in Theorem 3.1 and Theorem 3.2, we have

THEOREM 3.3: If F is a solution of $F' = AF + Q$ where A is a constant matrix and $Q = [q_i]$ is a column vector whose entries are polynomials, then each component of F is of bounded index and F is of bounded index of index $N \leq \min\{m \epsilon Z^+ \,\big|\, \|A\| \leq m + 1 \text{ and } m - 1 \geq \max\limits_{1 \leq i \leq n} \deg q_i\}$.

PROOF: Let m be a non-negative integer such that $\|A\| \leq m + 1$ and $m - 1 \geq \max\limits_{1 \leq i \leq n} \deg q_i$. Then for any integer $p \geq 1$, we have $F^{(m+p)} = AF^{(m+p-1)} + Q^{(m+p-1)} = AF^{(m+p-1)}$ and

$$\frac{\|F^{(m+p)}(z)\|}{(m+p)!} \leq \frac{\|A\|}{m+p} \frac{\|F^{(m+p-1)}(z)\|}{(m+p-1)!} \leq \frac{\|F^{(m+p-1)}(z)\|}{(m+p-1)!}.$$

Therefore, $\dfrac{\|F^{(m+p)}(z)\|}{(m+p)!} \leq \dfrac{\|F^{(m)}(z)\|}{m!}$ for all $p \geq 1$ and all $z \epsilon C$. Hence (1.1) holds for all $j \geq 0$ and all $z \epsilon C$.

4. Differential Equations With Rational Coefficients

In [6], it is shown that if f is an entire solution of (3.2) where α_i are polynomials and $\deg \alpha_i \leq \deg \alpha_0$ for $i = 1, 2, \ldots, n$, then f is of bounded index. We will show a comparable result for vector equations.

THEOREM 4.1: If F is an entire solution of $F' = AF + Q$ where $A = [r_{ij}]$ is a matrix whose entries are rational functions which are bounded at infinity and Q is a vector whose entires are rational functions which are bounded at infinity, then F is a function of bounded index.

PROOF: Let p_0 be the least common denominator of the r_{ij}'s and q_i's where $Q = [q_i]$. Let $b_{ij} = p_0 r_{ij}$ and $p_i = p_0 q_i$. Then b_{ij} and p_i are polynomials of degree $\leq l = \deg p_0$. Let $B = [b_{ij}]$ and $P = [p_i]$ and so F is a solution of

$$p_0 F' = BF + P. \tag{4.1}$$

Differentiating N-times where $N \geq l + 1$, we obtain

$$\sum_{k=0}^{N} \binom{N}{k} p_0{}^{(k)} F^{(N+1-k)} = \sum_{k=0}^{N} \binom{N}{k} B^{(k)} F^{(N-k)}$$

which simplifies into

$$p_0 F^{(N+1)} = \sum_{k=0}^{l} \binom{N}{k} B^{(k)} F^{(N-k)} - \sum_{k=1}^{l} \binom{N}{k} p_0{}^{(k)} F^{(N+1-k)}$$

since $\deg b_{ij} \leq l = \deg p_0$.

Therefore

$$\frac{F^{(N+1)}}{(N+1)!} = \frac{1}{N+1} \sum_{k=0}^{l} \frac{1}{k!} \frac{B^{(k)}}{p_0} \frac{F^{(N=k)}}{(N-k)!} - \sum_{k=1}^{l} \frac{1}{k!} \left(1 - \frac{k}{N+1}\right) \frac{{p_0}^{(k)}}{p_0} \frac{F^{(N+1-k)}}{(N+1-k)!} \tag{4.2}$$

Since $\dfrac{{p_0}^{(k)}(z)}{p_0(z)} \rightarrow 0$ as $|z| \rightarrow \infty$ for $k = 1, 2, \ldots, l$, there exists $T > 0$ such that

$$\sum_{k=1}^{l} \frac{1}{k!} \left| \frac{{p_0}^{(k)}(z)}{p_0(z)} \right| < \frac{1}{2} \text{ if } |z| \geq T. \tag{4.3}$$

Since $deg\ b_{ij} \leq l$, $\displaystyle\sum_{k=0}^{l} \frac{1}{k!} \frac{\|B^{(k)}(z)\|}{|p_0(z)|} \leq H$, a constant, if $|z| \geq T$. Choose $N_1 \geq l + 1$ so that $\dfrac{1}{N_1 + 1} H < \dfrac{1}{2}$.

Then

$$\frac{1}{N+1} \sum_{k=0}^{l} \frac{1}{k!} \frac{\|B^{(k)}(z)\|}{|p_0(z)|} < \frac{1}{2} \text{ if } N \geq N_1 \text{ and } |z| \geq T. \tag{4.4}$$

Combining (4.2), (4.3), and (4.4) if $N \geq N_1$ and $|z| \geq T$, we have

$$\begin{aligned} \frac{\|F^{(N+1)}\|}{(N+1)!} &\leq \left[\frac{1}{N+1} \sum_{k=0}^{l} \frac{1}{k!} \frac{\|B^{(k)}\|}{p_0} + \sum_{k=1}^{l} \frac{1}{k!} \left| \frac{{p_0}^{(k)}}{p_0} \right| \right] \times \max_{0 \leq i \leq N} \frac{\|F^{(i)}\|}{i!} \\ &\leq \max_{0 \leq i \leq N} \frac{\|F^{(i)}\|}{i!}. \end{aligned}$$

Therefore $\dfrac{\|F^{(N+1)}(z)\|}{(N+1)!} \leq \displaystyle\max_{0 \leq i \leq N_1} \frac{\|F^{(i)}(z)\|}{i!}$ for all $N \geq N_1$ and $|z| \geq T$.

But every entire function is of bounded index on any compact set [2b, Theorem 16, p. 305]. Therefore, there is an integer $N_2 \geq 0$ such that

$$\frac{|f_k{}^{(j)}(z)|}{j!} \leq \max_{0 \leq i \leq N_2} \frac{|f_k{}^{(i)}(z)|}{i!} \text{ for all integers } j \geq 0,$$

$k = 1, 2, \ldots, n$, and all z such that $|z| \leq T$. Choosing $N \geq \max(N_1, N_2)$, we have (1.1) for all integers $j \geq 0$ and all $z \epsilon C$.

Theorem. 4.1 does not tell us whether or not the components of F are of bounded index. We conjecture that they are of bounded index; however, at present, we can prove this only for the case $n = 2$.

THEOREM 4.2: If $F = [f_1 f_2]$ is an entire solution of $F' = AF + Q$ where $A = [r_{ij}]$ is a matrix whose entries are rational functions which are bounded at infinity and Q is a column vector whose entries are rational functions which are bounded at infinity, then f_1 and f_2 are of bounded index.

PROOF: If $r_{12} \equiv 0$, then f_1 satisfies $f_1' = r_{11}f_1 + q_1$ and by [6], f_1 is of bounded index.

If $r_{12} \not\equiv 0$, then f_1 satisfies

$$f_1^{(2)} = \left(r_{11} + \frac{r_{12}'}{r_{12}} + r_{22} \right) f_1' + \left(r_{11}' + r_{12}r_{21} - r_{11}\frac{r_{12}'}{r_{12}} - r_{11}r_{22} \right) f_1 - \left(\frac{r_{12}'}{r_{12}} + r_{22} \right) q_1 + r_{12}q_2 + q_1'$$

and again by [6], f_1 is of bounded index.

Similarly, f_2 is of bounded index.

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