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Eigenset Generalizations of the Eigenvalue Concept

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For an *n-by-n* complex matrix A some generalizations of the eigenvalue-eigenvector equation

$$Ax = \lambda x, \; 0 \neq x \in C^n$$

are investigated. These take the form

$$AS = \lambda S \text{ or } AS \subseteq \lambda S$$

where S is a subset of C^n about which various assumptions are made. For example, it is shown that there exists a finite set $S \subset C^n$, the sum of whose elements is not 0, such that $AS = \lambda S$, if and only if λ is an eigenvalue of A in the usual sense. The requirement that the sum of the elements of S is not 0 should be viewed as a natural analog of the requirement $x \neq 0$ in the classical eigenvalue-eigenvector equation.

Key words: Bounded set; convex hull; eigenvalue-eigenvector equation; root of unity.

Let $M_n(C)$ denote the set of *n-by-n* complex matrices. For $A \in M_n(C)$, the complex number λ is said to be an *eigenvalue* of A if there exists an $x \in C^n$ such that

$$(1) \qquad Ax = \lambda x \quad \text{and} \quad x \neq 0.$$

We denote the set of all eigenvalues of A by $\sigma(A)$, the *spectrum* of A. For S an arbitrary nonempty subset of C^n, we define AS and αS, for $A \in M_n(C)$ and $\alpha \in C$, in the natural ways; that is, $AS = \{Ax : x \in S\}$ and $\alpha S = \{\alpha x : x \in S\}$. We may then consider two generalizations of the classical eigenvalue-eigenvector eq (1):

$$(2) \qquad AS = \lambda S$$

and

$$(3) \qquad AS \subseteq \lambda S,$$

and ask (analogously to the classical eigenvalue question) "for which complex numbers λ do there exist sets S satisfying (2) or (3)." Of course, if we allow $S = \{0\}$, then for any value of λ an S would exist satisfying (2) or (3), and, if we allow $S = C^n$, then, for non-singular A, any $\lambda \neq 0$ satisfies (2) and, for any $A \in M_n(C)$, any $\lambda \neq 0$ satisfies (3). Generally speaking, allowing S to be unbounded results in difficulty putting any restrictions on λ.

EXAMPLE 1: Let S be the eigenspace corresponding to any nonzero eigenvalue (if there is one) of $A \in M_n(C)$. Then, since any nonzero scalar multiple of S is identical to S, it follows that (2) and (3) hold for any $\lambda \neq 0$.

REMARK: There is a $\{0\} \neq S \subset C^n$ such that (2) or (3) holds for $\lambda = 0$, if and only if A is singular. This means that $\lambda = 0$ can satisfy (2) or (3) if and only if $0 \in \sigma(A)$, i.e. 0 is an eigenvalue in the usual sense.

EXAMPLE 2: Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\lambda = 1$ and $S = \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$.

Then (2) is satisfied despite the fact that neither of the two elements of S is an eigenvector.

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In this note we investigate some generalizations of (1) of the type (2) or (3). In each case some restriction regarding 0 is necessary, and these should be viewed as analogs of the second part of (1). (Of course, if $0 \in S$, then $0 \in AS$ and $0 \in \lambda S$. Thus, neither (2) nor (3) is disturbed by addition of 0, or deletion of 0 if S does not otherwise intersect the null space of A.) The classical case (1) may be viewed as a special case of (2) or (3) in which S has just one element. The first natural generalization of the one element case is the case in which S is finite; the next, the case in which S is bounded.

Remark: The case of S bounded in (3) might be viewed as a generalization of the instance in which A is a column stochastic matrix, $\lambda = 1$, and S is the unit simplex (the convex hull of the n coordinate vectors). A matrix is column stochastic if and only if it maps the unit simplex into itself, and, of course, such a matrix has an eigenvalue equal to 1.

We first consider the case in which S is finite.

Theorem 1: *Let* $A \in M_n(C)$. *There exists a finite set* $S \subset C^n$ *satisfying*

≡
$$AS = \lambda S$$

and $\sum_{v \in S} v \neq 0$, *if and only if* $\lambda \in \sigma(A)$.

Proof: If $\lambda \in \sigma(A)$, let $S = \{x\}$ where $x \neq 0$ is an eigenvector of A corresponding to λ. Then $AS = \lambda S$, $\sum_{v \in S} v \neq 0$ and $S \subset C^n$ is finite.

On the other hand, suppose $S \subset C^n$ is finite and satisfies (2) and that $x \equiv \sum_{v \in S} v \neq 0$. Then

$$Ax = \sum_{v \in S} Av = \sum_{v \in S} \lambda v = \lambda x$$

which means that $\lambda \in \sigma(A)$.

Example 3. The requirement "$\sum_{v \in S} v \neq 0$" is necessary in theorem 1.
If $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\lambda = 1$ and $S = \left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\}$, then $AS = \lambda S$ without $\lambda \in \sigma(A) = \{\pm i\}$. The trouble, of course, is that $\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, but it should be noted that $\lambda \in \sigma(A^4)$, and that $\lambda = 1 = (-i)i$ where one factor is a fourth root of one and the other lies in $\sigma(A)$ (cf. Theorem 2 below).

We next consider what happens if the requirement $\sum_{v \in S} v \neq 0$ of theorem 1 is deleted. This broadens the class of λ's which may occur. A complex number z is called a *root of unity* if there exists a positive integer k such that $z^k = 1$.

Theorem 2: *Let* $A \notin M_n(V)$. *There exists a finite set* $0 \notin S \subset C^n$ *satisfying*

$$AS \subseteq \lambda S$$

if and only if $\lambda = z\alpha$ *where* z *is a root of unity and* $\alpha \in \sigma(A)$.

Proof: First suppose $\lambda = z\alpha$ *where* $\alpha \in \sigma(A)$ and $z^k = 1$. Then let $S = \{x, zx, \cdots, z^{k-1}x\}$ where $0 \neq x \in C^n$ satisfies $Ax = \alpha x$. It follows that $zS = S$ so that $AS = \alpha S = \alpha(zS) = \lambda S$. Thus $0 \in S \subset C^n$ satisfies (3) as was to be shown.

On the other hand, suppose $0 \notin S$ C^n such that AS $\ddot{}\lambda S$ is given. Since S is finite there must exist a sequence of vectors $x_{i_1}, \cdots, x_{i_k} \in S$ such that

$$Ax_{i_j} = \lambda x_{i_{j+1}}, \; j = 1, \cdots, k - 1$$

and $Ax_{i_k} = \lambda x_{i_1}$. It then follows that

$$A^k x_{i_1} = \lambda^k x_{i_1}$$

and, therefore, that

$$\lambda^k = \alpha^k$$

for some $\alpha \in \sigma(A)$. If $\lambda = 0$, the desired consequence follows trivially, and, if not, this means that $z = \dfrac{\lambda}{\alpha}$ is a kth root of unity. Since $\lambda = z \cdot \alpha$, this completes the proof.

REMARK: If $0 \in S$ is allowed in the context of theorem 2, then any λ may be achieved by letting $S = \{0, x\}$ where x (possibly 0) is in the null space of A. However, if "$0 \notin S$" is replaced by "$\{0\} \neq S$" and "$AS \subseteq \lambda S$" is replaced by "$AS = \lambda S$" a valid alternative to theorem 2 results.

We call a set $S \subset C^n$ *bounded* if there is some finite number r (depending on S) such that the Euclidean length of each element of S is less than r. We next consider the case of S bounded in (2) or (3). We denote the *convex hull* of $T \subset C^n$ by $Co(T)$ and the closure of T by $\bar{T}$.

THEOREM 3: *Let* $A \in M_n(C)$. *There exists a bounded set* $S \subset C^n$ *satisfying*

$$AS \subseteq \lambda S$$

and $0 \notin \overline{Co(S)}$, *if and only if* $\lambda \in \sigma(A)$.

PROOF: If $\lambda \in \sigma(A)$, then we may choose $S = \{x\}$ where $0 \neq x \in C^n$ is an eigenvector of A corresponding to λ.

On the other hand, suppose a bounded subset S of C^n such that $AS \subseteq \lambda S$ and $0 \notin \overline{Co(S)}$ is given. Then it is straighforward to check that $\hat{S} = \overline{Co(S)}$ satisfies $A\hat{S} \subseteq \lambda\hat{S}$. Now, if $\lambda = 0$, the desired result is immediate, and if not, $1/\lambda A$ is a continuous mapping which maps the compact, convex set $\hat{S}$ into itself. Therefore, by the Brouwer fixed point theorem $1/\lambda A$ has a fixed point $x \in \hat{S}$. Since $0 \notin \hat{S}$, this means that x is an eigenvector of A corresponding to λ and completes the proof.

REMARK: The assumption $0 \notin \overline{Co(S)}$ is, of course, crucial in the above argument. If $0 \in \overline{Co(S)}$, then 0 might well be the only fixed point of $1/\lambda A$.

If we relax the requirement $0 \notin \overline{Co(S)}$ the class of allowable λ's is again broadened.

REMARK: If (2) holds it follows in a straightforward way that for any positive integer t,

$$A^tS = \lambda^tS. \tag{4}$$

Similarly, if (3) holds it follows that

$$A^tS \subseteq \lambda^tS. \tag{5}$$

THEOREM 4: *Let* $A \in M_n(C)$. *There exists a bounded set* $\{0\} \neq S \subset C^n$ *satisfying*

$$AS \subseteq \lambda S$$

if and only if $|\lambda| \geq m = \min\limits_{\alpha\in\sigma(A)} |\alpha|$.

PROOF: If $|\lambda| \geqslant m$, let $x \neq 0$ be an eigenvector corresponding to an eigenvalue α of A for which $|\alpha| = m$ and let $S = \{ax{:}a \in C, |a| \leqslant 1\}$. Then $S \neq \{0\}$ and $AS = mS$ while $\lambda S = |\lambda|S$ so that (3) holds.

On the other hand, suppose that $\{0\} \neq S \subset C^n$ such that $AS \subseteq \lambda S$ is given, and select an element $0 \neq x \in S$. If λ were 0, then it would follow that $Ax = 0$ and that $0 \in \sigma(A)$ so that m would be 0. Thus $|\lambda| \geqslant m$ would be satisfied; and therefore we assume from here on that $\lambda \neq 0$. By continuity we may assume without loss of generality that A has a complete set of linearly independent eigenvectors $x_1, \cdots, x_n$ corresponding respectively to the eigenvalues $\alpha_1, \cdots, \alpha_n$. Then $x = \sum\limits_{i=1}^{n} a_ix_i$ for some complex numbers $a_1, \cdots, a_n$ and $A^tx = \sum\limits_{i=1}^{n} a_i\alpha_i^tx_i$. Since (3) implies (5), we obtain that $\sum\limits_{i=1}^{n} a_i\left(\dfrac{\alpha_i}{\lambda}\right)^t x_i \in S$ for all positive integers t. If $|\lambda|$ were less than m, then the boundedness of S would be contradicted so that we conclude that $|\lambda| \geqslant m$, as was to be shown.

Only slight modifications of the preceding argument yield the following.

THEOREM 5: *Let* $A \in M_n(C)$. *There exists a bounded set* $S \subset C^n$ *satisfying* $0 \notin \bar{S}$ *and*

$$AS \subseteq \lambda S$$

if and only if $|\lambda| = |\alpha|$ *for some* $\alpha \in \sigma(A)$.

THEOREM 6: *Let* $A \in M_n(C)$. *There exists a bounded set* $\{0\} \neq S \subset C^n$ *satisfying*

$$AS = \lambda S$$

if and only if $|\lambda| = |\alpha|$ *for some* $\alpha \in \sigma(A)$.

REMARK: Analogous to (3) we may also consider

$$AS \supseteq \lambda S. \tag{6}$$

Analysis of (6), however, is similar to that of (3) along the lines of this note. In fact for $\lambda \neq 0$ and A nonsingular, (6) holds if and only if (3) holds with A replaced by A^{-1} and λ by λ^{-1}. It is clear that $\lambda = 0$ satisfies (6) if and only if the intersection of S and the null space of A is nonempty. Also, in case S is finite and $\lambda \neq 0$, (6) implies (2).

We close by noting that many questions are suggested by the above observations. For example, what may be said about the geometric and algebraic properties of eigensets S in the senses of (2) and (3)?

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